

Born's Rule and the Path Integral from a Phase-Consistency Condition

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Born's rule is the only probabilistic postulate of quantum mechanics that is not derived from its unitary dynamics. The phase-consistency condition (PCC) introduced here treats a quantum state's global phase as a beable — an intrinsic, particle-associated element of reality rather than an observable. Being individually inaccessible, this phase functions as a contextual (supplementary) hidden variable that deterministically maps each state onto a single eigenstate of the corresponding observable for a given measurement context. Over an ensemble, the detection frequencies yield the Born rule and its quadratic form, with no distribution over the phase assumed at any step. Applying Hamilton's principle of stationary action to the PCC yields a dimensionless-action formulation of the Feynman path integral. Therefore, both the probabilistic and dynamical contents of quantum mechanics follow from a single-phase condition.

Of the postulates of quantum mechanics (QM), only the Born rule—the identification of $|\langle\lambda|\psi\rangle|^2$ with the probability of outcome λ —has stubbornly resisted the reduction in the theory's unitary, deterministic core; whether it can be derived at all or must be imposed by fiat remains open. Existing programs purchase the rule at a price. Gleason's theorem and its POVM-based successors fix the quadratic form uniquely, but only after assuming a non-contextual, additive probability assignment over projections—precisely the structure whose origin is at issue [1, 2]. Decision-theoretic and self-locating arguments recover the rule within the Everett interpretation from axioms of rational preference [3–6], and Zurek's environment-assisted invariance derives the equiprobability of equal amplitudes from an exact symmetry [7]; both rest on contested premises—rational-agent norms and the uniqueness of the Schmidt decomposition once amplitudes are unequal. Operational reconstructions establish that the measurement postulates, Born's rule among them, are redundant given

unitarity and a finite-dimensionality constraint on the state spaces [8, 9]. The most physically transparent route is also the oldest: the frequency operator program of Finkelstein, Hartle, and Farhi–Goldstone–Gutmann, in which $|a_k|^2$ appears as the eigenvalue of a relative frequency operator on an infinite ensemble of copies [10–12]. This program, however, labors under a well-known circularity, made precise by Caves and Schack: the deviant component of the N -copy state has a vanishing Hilbert-space norm yet nonzero amplitude, so to declare it improbable in the limit is already to assume the Born measure one means to derive [13]. Subsequent measure-theoretic and cosmological refinements blunt, but do not remove, this objection, each importing an additional structure to convert the vanishing norm into a vanishing probability [14, 15]. Thus, confronting this circularity directly is a central obstacle to any frequency-based derivation. Here, Born’s rule and its quadratic form are derived without any need to postulate a probability measure over branches or equate the Hilbert space norm with probability, thus closing the gap that has defeated previous frequency-operator approaches.

Wave-Embedded Particle: This postulate extends de Broglie’s hypothesis [16], treating the particle as a physical constituent embedded within the quantum state vector $|\psi(\phi)\rangle \in \mathcal{H}$. Here, $\phi \in [0, 2\pi)$ is the global phase associated with $|\psi(\phi)\rangle$, denoted as $\phi = \text{ph}(|\psi(\phi)\rangle)$ and $\mathcal{H}$ is the Hilbert space. The notation $\text{ph}(\cdot)$ is for the phase of its argument. For measurement context $\mathcal{C}$, the completeness relation is $\sum_k |e_k^{\mathcal{C}}\rangle\langle e_k^{\mathcal{C}}| = I$. Therefore,

$$|\psi(\phi)\rangle = \sum_{k=0}^{\infty} |e_k^{\mathcal{C}}\rangle\langle e_k^{\mathcal{C}}|\psi(\phi)\rangle = \sum_{k=0}^{\infty} |e_k^{\mathcal{C}}\rangle |a_k^{\mathcal{C}}| e^{i\theta_k^{\mathcal{C}}}, \quad (1)$$

where $\langle e_j^{\mathcal{C}}|e_k^{\mathcal{C}}\rangle = \delta_{jk}$. The *phase consistency condition* (PCC) is defined as,

$$\phi = \text{ph}(|\psi(\phi)\rangle) = \text{ph}(\langle e_p^{\mathcal{C}}|\psi(\phi)\rangle) = \theta_p^{\mathcal{C}}. \quad (2)$$

Parameter ϕ is specifically linked to the eigenstate $|e_p^{\mathcal{C}}\rangle$, which is occupied by the particle, where p belongs to the set $\{0, 1, 2, \dots, \infty\}$ and simultaneously denotes the particle. PCC ensures that the quantity $\phi (= \theta_p^{\mathcal{C}})$ is a beable — an intrinsic, particle-associated element of reality. It transforms a particle from the state $|\psi(\phi)\rangle$ to the eigenstate $|e_p^{\mathcal{C}}\rangle$. Standard quantum mechanics treats ϕ as a gauge degree of freedom, and the physical states are rays in $\mathcal{H}$.

This study examined two scenarios: (i) the derivation of the relative frequency of detection yielding the Born rule and its quadratic form, and (ii) a phase-evolving single-particle state,

which results in the derivation of the Feynman path integral as a consequence of Hamilton's principle of stationary action.

Derivation of the Born Rule: Consider an ensemble of N identically prepared state vectors with the same norm $\|\psi(\phi)\rangle\| = |a|$, differing only in their global phase $\phi \in \{\phi_i \mid i \in \{1, 2, \dots, N\}\}$, while sharing the same moduli $|\langle e_k^{\mathcal{C}} | \psi(\phi) \rangle| = |a_k^{\mathcal{C}}|$. As a consequence of the PCC, let the detection yield $N_k^{\mathcal{C}}$ particles in the state $|e_k^{\mathcal{C}}\rangle$. Hence, each state $|e_k^{\mathcal{C}}\rangle$ is characterized by $N_k^{\mathcal{C}}$ and $a_k^{\mathcal{C}}$. Parseval's geometric identity [17] holds for each $|\psi(\phi)\rangle$,

$$|a|^2 = \sum_{k=0}^{\infty} (|a_k^{\mathcal{C}}|)^2. \quad (3)$$

The following unique combination,

$$\frac{|a|^2}{N} = \frac{|a_k^{\mathcal{C}}|^2}{N_k^{\mathcal{C}}} = \text{universal constant}; \forall \mathcal{C} \in \{\mathcal{C}\}. \quad (4)$$

satisfies the Eq. (3), which yields the conservation of the total number of particles $N = \sum_{k=0}^{\infty} N_k^{\mathcal{C}}$. In Eq. (4), $\{\mathcal{C}\}$ denotes the set of measurement contexts. By reformulating Eq. (4), the relative frequencies can be obtained, resulting in the Born rule of $N \rightarrow \infty$ and its quadratic form.

Dimensionless-Action Formulation of the Feynman Path Integral: Consider a set of state vectors $\{|\psi(\phi_i)\rangle \mid i \in \{0, 1, 2, \dots, N\}\}$ associated with a sequence of phases $\phi_0 < \phi_1 < \dots < \phi_N$, where ϕ_0 and ϕ_N represent the initial and final phases, respectively. The completeness relations are given by

$$\hat{I} = \sum_{k_i=0}^{\infty} |e_{k_i}^{\mathcal{C}}\rangle \langle e_{k_i}^{\mathcal{C}}|, \quad (5)$$

where the PCC at ϕ_i is defined as $\phi_i = \text{ph}(\langle e_{p_i}^{\mathcal{C}} | \psi(\phi_i) \rangle)$. The particle, originally in state $|e_{p_0}^{\mathcal{C}}\rangle$, undergoes transitions through a sequence of intermediate eigenstates, represented as $\{|e_{p_m}^{\mathcal{C}}\rangle \mid m \in \{1, 2, 3, \dots, N-1\}\}$, before reaching the final state $|e_{p_N}^{\mathcal{C}}\rangle$. The final state vector can be articulated in terms of the initial state vector using the completeness relations defined in Eq. (5),

$$\sum_{k_N=0}^{\infty} |e_{k_N}^{\mathcal{C}}\rangle \langle e_{k_N}^{\mathcal{C}} | \psi(\phi_N) \rangle = \sum_{k_N=0}^{\infty} |e_{k_N}^{\mathcal{C}}\rangle \sum_{k_{N-1}=0}^{\infty} \dots \sum_{k_0=0}^{\infty} \prod_{n=1}^N \langle e_{k_n}^{\mathcal{C}} | e_{k_{n-1}}^{\mathcal{C}} \rangle \langle e_{k_0}^{\mathcal{C}} | \psi(\phi_0) \rangle. \quad (6)$$

By applying PCC to the aforementioned equation, the following expression is obtained:

$$\text{ph}(\langle e_{p_N}^{\mathcal{C}} | \psi(\phi_N) \rangle) = \sum_{n=1}^N \text{ph}(\langle e_{p_n}^{\mathcal{C}} | e_{p_{n-1}}^{\mathcal{C}} \rangle) + \text{ph}(\langle e_{p_0}^{\mathcal{C}} | \psi(\phi_0) \rangle). \quad (7)$$

The variation of the above equation, while maintaining the end phases fixed, yields

$$\delta \text{ph}(\langle e_{p_n}^{\mathcal{C}} | e_{p_{n-1}}^{\mathcal{C}} \rangle) = 0. \quad (8)$$

Eq. (8) represents the quantum counterpart of Hamilton's principle of stationary action [18]. The general solution, defined up to any quantity $K_{n;n-1}^{\mathcal{C}}$ that satisfies the condition $\delta K_{n;n-1}^{\mathcal{C}} = 0$, is given by the equation

$$\text{ph}(\langle e_{p_n}^{\mathcal{C}} | e_{p_{n-1}}^{\mathcal{C}} \rangle) = \Phi_{p_n;p_{n-1}}^{\mathcal{C}} + K_{n;n-1}^{\mathcal{C}}, \quad (9)$$

where the function $\Phi_{p_n;p_{n-1}}^{\mathcal{C}}$ is dimensionless. Rewriting the above equation as follows:

$$\langle e_{p_n}^{\mathcal{C}} | e_{p_{n-1}}^{\mathcal{C}} \rangle = A_{p_n;p_{n-1}}^{\mathcal{C}} \exp \{ i \Phi_{p_n;p_{n-1}}^{\mathcal{C}} \}, \quad (10)$$

where $A_{p_n;p_{n-1}}^{\mathcal{C}} = |\langle e_{p_n}^{\mathcal{C}} | e_{p_{n-1}}^{\mathcal{C}} \rangle| \exp \{ i K_{n;n-1}^{\mathcal{C}} \}$, and using it in Eq. (6) results in:

$$\langle e_{k_N}^{\mathcal{C}} | \psi(\phi_N) \rangle = \sum_{k_{N-1}=0}^{\infty} \cdots \sum_{k_0=0}^{\infty} \prod_{n=1}^N A_{k_n;k_{n-1}}^{\mathcal{C}} \exp \left\{ i \sum_{n=1}^N \Phi_{k_n;k_{n-1}}^{\mathcal{C}} \right\} \langle e_{k_0}^{\mathcal{C}} | \psi(\phi_0) \rangle. \quad (11)$$

Eq. (11) delineates the dimensionless-action formulation of the Feynman path integral, which, when articulated in the continuous position basis, assumes a well-established form [19].

Discussions: Within the ontological-models framework of Harrigan and Spekkens [20], the PCC defines a ψ -ontic model: the quantum state is real but incomplete, supplemented by the global phase ϕ as an additional element of reality that determines the individual outcome. This places PCC-based QM outside the scope of the Pusey–Barrett–Rudolph theorem [21], which constrains ψ -epistemic models and distinguishes them from interpretations in which the state is purely informational.

Bohmian mechanics is formulated for individual systems via deterministic particle dynamics, whereas the usual Born rule probabilities enter through an ensemble-level quantum equilibrium assumption rather than the single-system equations of motion [22] or by dynamical relaxation [23]. Similar to Bohmian mechanics, PCC-based QM is a deterministic hidden-variable theory yielding a definite outcome for each event; in contrast, the hidden variable is the global phase rather than a configuration-space position, and the statistical weights follow without positing a distribution over it.

At some stage, the programs surveyed in the introduction import the probability structure that the PCC derives instead: an additive, non-contextual measure over projections [1, 2];

axioms of rational preference defined over agents rather than physical detection frequencies [3–6]; a measure that re-enters once equal-amplitude symmetry is extended to the unequal case [7]; or structural redundancy established without a constructive dynamical mechanism [8, 9]. The PCC differs in mapping a single particle in the quantum state to a single particle as the measurement outcome, yielding Born’s rule as the limiting relative frequency of detection. Because the Hilbert space norm is at no point equated with probability, the construction is not exposed to the circularity that Caves and Schack [13] identified in the frequency-operator program [11, 12] and that subsequent refinements [14, 15] could only mitigate by importing additional structures.

Conclusions: The contribution of the present study is the introduction of PCC, a foundational relation in the ontology of wave-embedded particles, which identifies the global phase of a quantum state vector as a beable associated with the particle. The derivation of Born’s rule in its quadratic form, which is independent of dimensionality and free of any prior distribution of ϕ , can be generalised straightforwardly to the spectrum of continuous eigenvalues. This derivation implies the conservation of the total particle number, which may extend to the conservation of the total energy–momentum for quantum states that do not individually conserve the particle number. Since the PCC involves only the global phase—trivially invariant under Lorentz transformations—it remains relativistically valid. The dimensionless-action formulation of the Feynman path integral follows Hamilton’s principle of stationary action applied to the PCC. The application of PCC to entangled states is left for future work.

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- [1] A. M. Gleason, Measures on the closed subspaces of a Hilbert space, *J. Math. Mech.* **6**, 885 (1957).
 - [2] P. Busch, Quantum states and generalized observables: A simple proof of Gleason’s theorem, *Phys. Rev. Lett.* **91**, 120403 (2003).
 - [3] D. Deutsch, Quantum theory of probability and decisions, *Proc. R. Soc. Lond. A* **455**, 3129 (1999).
 - [4] D. Wallace, Quantum probability from subjective likelihood: Improving on Deutsch’s proof of the probability rule, *Stud. Hist. Philos. Mod. Phys.* **38**, 311 (2007).

- [5] S. Saunders, Derivation of the Born rule from operational assumptions, *Proc. R. Soc. Lond. A* **460**, 1771 (2004).
- [6] C. T. Sebens and S. M. Carroll, Self-locating uncertainty and the origin of probability in Everettian quantum mechanics, *Brit. J. Philos. Sci.* **69**, 25 (2018).
- [7] W. H. Zurek, Probabilities from entanglement, Born's rule $p_k = |\psi_k|^2$ from envariance, *Phys. Rev. A* **71**, 052105 (2005).
- [8] L. Masanes, T. D. Galley, and M. P. Müller, The measurement postulates of quantum mechanics are operationally redundant, *Nat. Commun.* **10**, 1361 (2019).
- [9] T. D. Galley and L. Masanes, Classification of all alternatives to the Born rule in terms of informational properties, *Quantum* **1**, 15 (2017).
- [10] D. Finkelstein, The logic of quantum physics, *Trans. N.Y. Acad. Sci.* **25**, 621 (1963).
- [11] J. B. Hartle, Quantum mechanics of individual systems, *Am. J. Phys.* **36**, 704 (1968).
- [12] E. Farhi, J. Goldstone, and S. Gutmann, How probability arises in quantum mechanics, *Ann. Phys. (N.Y.)* **192**, 368 (1989).
- [13] C. M. Caves and R. Schack, Properties of the frequency operator do not imply the quantum probability postulate, *Ann. Phys. (N.Y.)* **315**, 123 (2005).
- [14] N. P. Landsman, Macroscopic observables and the Born rule, i: Long run frequencies, *Rev. Math. Phys.* **20**, 1173 (2008).
- [15] A. Aguirre and M. Tegmark, Born in an infinite universe: A cosmological interpretation of quantum mechanics, *Phys. Rev. D* **84**, 105002 (2011).
- [16] L. de Broglie, *Recherches sur la théorie des quanta*, *Annales de Physique* **3**, 22 (1925), doctoral thesis defended in 1924.
- [17] W. Rudin, *Real and Complex Analysis*, 3rd ed. (McGraw-Hill, New York, 1987).
- [18] H. Goldstein, C. P. Poole, and J. L. Safko, *Classical Mechanics*, 3rd ed. (Addison-Wesley, San Francisco, 2002).
- [19] R. P. Feynman, Space-time approach to non-relativistic quantum mechanics, *Rev. Mod. Phys.* **20**, 367 (1948).
- [20] N. Harrigan and R. W. Spekkens, Einstein, incompleteness, and the epistemic view of quantum states, *Found. Phys.* **40**, 125 (2010).
- [21] M. F. Pusey, J. Barrett, and T. Rudolph, On the reality of the quantum state, *Nat. Phys.* **8**, 475 (2012).

- [22] D. Dürr, S. Goldstein, and N. Zanghì, Quantum equilibrium and the origin of absolute uncertainty, *J. Stat. Phys.* **67**, 843 (1992).
- [23] A. Valentini, Signal-locality, uncertainty, and the subquantum H-theorem. I, *Phys. Lett. A* **156**, 5 (1991).