Aliquot spectral basis

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Abstract

The author observed belatedly that $\{9, 4\}$ is the spectral basis of $n = \sigma(9) - 1 = 12$, where $\sigma'(9) = \sigma(9) - 9 = 4$, the aliquot sum of 9. The author provides examples of m such that $\{m, \sigma'(m)\}$ is the spectral basis of $n = \sigma(m) - 1$, called the aliquot spectral basis of n.

Duodecimal declaration: [4] The author uses duodecimal, base 12, throughout this document, except for dates and document counters, with X for 10 and Ɛ for 11.

1 Review

1.1 Number theory

Let n be a positive integer and let d be a divisor of n. Define $\sigma_k(n)$ is the sum of the kth powers of the divisors of n, where k is nonnegative, that is,

$$\sigma_k(n) \triangleq \sum_{d|n} d^k, \quad k \geq 0. \tag{1}$$

Definition 1.1 (Number of divisors [7, 12]). *Define $\tau(n)$ to be the number of divisors of n, that is,*

$$\tau(n) \triangleq \sigma_0(n) = \sum_{d|n} 1. \tag{2}$$

Theorem 1.2 (Number of divisors [7]). *Let $\tau(n)$ denote the number of divisors of n. Let p be prime, let $p_1, \ldots, p_j$ be distinct primes, and let $k, k_1, \ldots, k_j$ be positive integers. Then*

$$\tau(p^k) = k + 1. \tag{3}$$

Furthermore, τ is multiplicative:

$$\tau(p_1^{k_1} \cdots p_j^{k_j}) = \tau(p_1^{k_1}) \cdots \tau(p_j^{k_j}). \tag{4}$$

Proof. An exercise for the reader. □

Definition 1.3 (Sum of divisors [7, 13], aliquot sum [8, 15]). *Define $\sigma(n)$ to be the sum of the divisors of n, that is,*

$$\sigma(n) \triangleq \sigma_1(n) = \sum_{d|n} d. \tag{5}$$

Further, we shall define $\sigma'(n) = \sigma(n) - n$ to be the aliquot sum of n.

Theorem 1.4 (Sum of divisors [7]). *Let $\sigma(n)$ denote the sum of the divisors of n. Let p be prime, let $p_1, \dots, p_j$ be distinct primes, and let $k, k_1, \dots, k_j$ be positive integers. Then*

$$\sigma(2^k) = 2^{k+1} - 1, \tag{6}$$

$$\sigma(p^k) = \frac{p^{k+1} - 1}{p - 1}. \tag{7}$$

Furthermore, σ is multiplicative:

$$\sigma(p_1^{k_1} \cdots p_j^{k_j}) = \sigma(p_1^{k_1}) \cdots \sigma(p_j^{k_j}). \tag{8}$$

Proof. An exercise for the reader. □

1.2 The spectral basis

Let us consider the isomorphism $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{a_1}} \oplus \mathbb{Z}_{p_2^{a_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{a_k}}$, where $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is the prime factorization of n. We shall assume for our purpose that n has at least two prime factors. Otherwise, we would consider a p-adic expansion with respect to $n = p^a$.

Definition 1.5 (Spectral basis [6]). *Given $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, define*

$$h_i \triangleq n/p_i^{a_i},$$
$$b_i \triangleq h_i^{-1} \mod p_i^{a_i},$$
$$s_i \triangleq b_i h_i,$$

so that, in $\mathbb{Z}_n$, we have the decomposition of the identity into mutually orthogonal idempotents

$$s_1 + s_2 + \cdots + s_k = 1$$

such that

$$s_i^2 = 0 \quad \textit{and} \quad s_i s_j = 0$$

for $1 \le i, j \le k$ with $i \ne j$. Each s_i acts as an orthogonal projections onto its primary part $\mathbb{Z}_{p_i^{a_i}}$ of $\mathbb{Z}_n$.

If $x \in \mathbb{Z}_n$, then we have

$$x = x_1 s_1 + x_2 s_2 + \cdots + x_k s_k,$$

where $x \equiv x_i \pmod{p_i^{a_i}}$. Furthermore, we have

$$x^r = x_1^r s_1 + x_2^r s_2 + \cdots + x_k^r s_k.$$

If x is relatively prime to n, then r can be negative.

We shall often say "spectral basis of n" instead of "spectral basis of $\mathbb{Z}_n$."

Definition 1.6 (Power spectral basis [3]). *The integer n has a* power spectral basis *if and only if the elements of the basis are prime or a (possibly composite) power.*

The following theorems on power spectral bases are from [3] and are stated without proof.

Theorem 1.7 (Mersenne [10]). *Let M_p be a Mersenne prime with Mersenne exponent p. Then*

1. *$2M_p$ has spectral basis $\{M_p, 2^p\}$.*
2. *$2^p M_p$ has spectral basis $\{M_p^2, 2^p\}$.*
3. *$2^{p+1}M_p$ has spectral basis $\{M_p^2, 2^{2p}\}$ or, equivalently, $\{M_p^2, (M_p+1)^2\}$.*
4. *$2^{2p+1}M_p^2$ has spectral basis $\{M_p^2(M_p+2)^2, (M_p^2-1)^2\}$.*

Both 1 and 2 of Theorem 1.7 occur as examples of an aliquot spectral basis. See Examples 3.1 and 3.3, respectively.

Theorem 1.8 (Fermat [11]). *If $F_i = 2^{f_i} + 1$ is a Fermat prime with exponent $f_i = 2^i$, $i \geq 0$, then*

1. *$2^{f_i}F_i$ has spectral basis $\{F_i, 2^{2f_i}\}$.*
2. *$2^{f_i+1}F_i$ has spectral basis $\{F_i^2, 2^{2f_i}\}$.*
3. *$2^{2f_i+1}F_i^2$ has spectral basis $\{(F_i-2)^2F_i^2, (F_i^2-1)^2\}$.*

None of the examples of Theorem 1.8 are known to occur as an aliquot spectral basis.

Theorem 1.9 (Semiprime [16]). *Given p an odd prime and $k \geq 3$ an integer, the integer pq has power-spectral basis $\{q, p^k\}$ if and only if $q = (p^k-1)/(p-1)$ is prime.*

See Example 3.1.

Theorem 1.10 (Cyclotomic). *Let p, q, r be primes and e a nonnegative integer. Then the integer $n = p^{r^e}q$ is power spectral with spectral basis $\{q, p^{r^{e+1}}\}$ if and only if $q = \Phi_{r^{e+1}}(p)$ is prime, where $\Phi_{r^{e+1}}(x)$ is the r^{e+1}-th cyclotomic polynomial [9].*

None of the examples of Theorem 1.10 are known to occur as an aliquot spectral basis.

2 The aliquot spectral basis

In investigating which numbers n have power spectral basis [3], the author discovered that $\{9, 4\}$ is the spectral basis of $10 = 12|_{12}$, namely Theorem 1.7.2. However, in thinking about another project, the author noticed that $\sigma(9) = 11$ and that $\sigma'(9) = 4$, so that the pair $\{9, \sigma'(9)\}$ is the spectral basis of $10 = \sigma(9) - 1$. Of course, this gave the author an interesting idea: Are there any other integers n such that there exists an m such that $n = \sigma(m) - 1$, $m' = \sigma'(m)$, and $\{m, m'\}$ is the spectral basis of n.

Definition 2.1 (Aliquot spectral basis). *The positive integer n will have an* aliquot spectral basis *if there exists an integer m such that $n = \sigma(m) - 1$, $m' = \sigma'(m)$, and $\{m, m'\}$ is the spectral basis of n. See Table 1.*

If n is to have an aliquot spectral basis, it must have only two factors, namely, it is necessary to have $n = p^s q^t$, $p < q$, $s, t \geq 1$. The examples below will use Table 1 as a guide.

3 Examples of the aliquot spectral basis

The following example is Theorem 1.9 established as an aliquot spectral basis.

Example 3.1 (Semiprime [16]). *If $q = (p^k - 1)/(p - 1)$ is prime, $k \geq 1$, then $\{q, p^k\}$ is the aliquot spectral basis of $n = pq$.*

Remark 3.2. *See entries* 1, 2, 4, 5 *of Table 1 as illustrations of Example 3.1.*

Proof. Observe that, if $m = p^k$, then $\sigma(m) = (p^{k+1} - 1)/(p - 1)$ and $n = \sigma(m) - 1 = p \cdot q$, where $q = (p^k - 1)/(p - 1)$. If q is prime, then $\{q, p^k\}$ is the spectral basis of $n = pq$. □

The following example is Theorem 1.7.2 established as an aliquot spectral basis.

Example 3.3 (Mersenne). *Let M_p be a Mersenne prime with Mersenne exponent p. Then $2^p M_p$ has aliquot spectral basis $\{M_p^2, 2^p\}$.*

Remark 3.4. *See entries* 3, 6, Ɛ, 15 *of Table 1 as illustrations of Example 3.3.*

Proof. Let $m = M_p^2$. Then $\sigma(m) = 2^{2p} - 2^p + 1$, $n = 2^{2p} - 2^p = 2^p M_p$, and $\sigma'(m) = 2^p$. Verification of spectral basis is left as an exercise for the reader. □

The following example is new to the author.

Example 3.5 (Six-eleven numbers [14]). *If $m = 6 \cdot Ɛ^k$, $k \geq 1$, and $q = 2(m - 1)/(Ɛ - 1)$ is prime, then $\{6q, m\}$ is the spectral basis of $n = Ɛq$.*

Remark 3.6. *See entries* 7, 13, 19, 24, 30 *of Table 1 as illustrations of Example 3.5. See also Table 2.*

Proof. Observe that if we assume

$$\sigma(6 \cdot Ɛ^k) = Ɛq + 1,$$

then

$$q = \frac{10 \cdot \dfrac{Ɛ^{k+1} - 1}{Ɛ - 1} - 1}{Ɛ}$$

$$= \frac{10 \cdot \mathcal{E}^{k+1} - 10 - \mathcal{E} + 1}{\mathcal{E}(\mathcal{E} - 1)}$$
$$= \frac{10 \cdot \mathcal{E}^{k+1} - 2\mathcal{E}}{\mathcal{E}(\mathcal{E} - 1)}$$
$$= \frac{10 \cdot \mathcal{E}^{k} - 2}{\mathcal{E} - 1}$$
$$q = 2 \cdot \frac{6 \cdot \mathcal{E}^{k} - 1}{\mathcal{E} - 1}$$

It is an exercise for the reader to verify that, if q is prime, then $\{6q, m\}$ is the spectral basis of $n = \mathcal{E}q$. □

Example 3.7 (Ljunggren–Nagell [2, 5, 1]). *Entry* X *of Table 1,* $n = 3 \cdot \mathcal{E}^2$*, with spectral basis* $\{3^5, \mathcal{E}^2\}$*, is a consequence of the fact that the only two solutions to the equation*

$$\frac{x^m - 1}{x - 1} = y^2$$

are $m = 4$, $x = 7$, $y = 18$, *and* $m = 5$, $x = 3$, $y = \mathcal{E}$. *The only known solution to*

$$\frac{x^m - 1}{x - 1} = y^p$$

with $m, p > 2$ *is* $m = p = 3$, $x = 16$, $y = 7$, *and* $m = p = 3$, $x = -17$, $y = 7$, *and it is conjectured there are no others.*

4 Table of aliquot spectral bases

i	$n = \sigma(m) - 1$	m	$m' = \sigma'(m)$	#PF	$\gcd(m, m')$
1.	$(2)(3)$	$(2)^2$	(3)	[1,1]	1
2.	$(2)(7)$	$(2)^3$	(7)	[1,1]	1
3.	$(2)^2(3)$	$(3)^2$	$(2)^2$	[1,1]	1
4.	$(3)(11)$	$(3)^3$	(11)	[1,1]	1
5.	$(2)(27)$	$(2)^5$	(27)	[1,1]	1
6.	$(2)^3(7)$	$(7)^2$	$(2)^3$	[1,1]	1
7.	$(\mathcal{E})(11)$	$(2)(3)(\mathcal{E})$	$(2)(3)(11)$	[3,3]	6
8.	$(5)(27)$	$(5)^3$	(27)	[1,1]	1
9.	$(2)(X7)$	$(2)^7$	$(X7)$	[1,1]	1
X.	$(3)(\mathcal{E})^2$	$(3)^5$	$(\mathcal{E})^2$	[1,1]	1
$\mathcal{E}$.	$(2)^5(27)$	$(27)^2$	$(2)^5$	[1,1]	1
10.	$(3)(771)$	$(3)^7$	(771)	[1,1]	1

▽

i	$n = \sigma(m) - 1$	m	$m' = \sigma'(m)$	#PF	$\gcd(m, m')$
11.	(7)(775)	(3)(7)(Ɛ)(17)	(3)(775)	[4,2]	3
12.	(15)(217)	$(15)^3$	(217)	[1,1]	1
13.	(Ɛ)(Ɛ11)	$(2)(3)(Ɛ)^3$	(2)(3)(Ɛ11)	[3,3]	6
14.	(2)(48X7)	$(2)^{11}$	(48X7)	[1,1]	1
15.	$(2)^7$(X7)	$(X7)^2$	$(2)^7$	[1,1]	1
16.	(7)(1755)	$(7)^5$	(1755)	[1,1]	1
17.	(35)(ƐƐ7)	$(35)^3$	(ƐƐ7)	[1,1]	1
18.	(5)(Ɛ377)	$(5)^7$	(Ɛ377)	[1,1]	1
19.	(Ɛ)(X201)	$(2)(3)(Ɛ)^4$	(2)(3)(X201)	[3,3]	6
1X.	(2)(63X27)	$(2)^{15}$	(63X27)	[1,1]	1
1Ɛ.	(4Ɛ)(2071)	$(4Ɛ)^3$	(2071)	[1,1]	1
20.	(5Ɛ)(2Ɛ61)	$(5Ɛ)^3$	(2Ɛ61)	[1,1]	1
21.	(11)(15XX5)	$(11)^5$	(15XX5)	[1,1]	1
22.	(2)(2134X7)	$(2)^{17}$	(2134X7)	[1,1]	1
23.	(75)(4777)	$(75)^3$	(4777)	[1,1]	1
24.	(Ɛ)(93X11)	$(2)(3)(Ɛ)^5$	(2)(3)(93X11)	[3,3]	6
25.	(85)(5Ɛ67)	$(85)^3$	(5Ɛ67)	[1,1]	1
26.	(15)(43431)	$(15)^5$	(43431)	[1,1]	1
27.	(3)(3253X1)	$(3)^{11}$	(3253X1)	[1,1]	1
28.	(XƐ)(X011)	$(XƐ)^3$	(X011)	[1,1]	1
29.	(11Ɛ)(142X1)	$(11Ɛ)^3$	(142X1)	[1,1]	1
2X.	(125)(15507)	$(125)^3$	(15507)	[1,1]	1
2Ɛ.	(1Ɛ)(121381)	$(1Ɛ)^5$	(121381)	[1,1]	1
30.	(Ɛ)(866301)	$(2)(3)(Ɛ)^6$	(2)(3)(866301)	[3,3]	6
31.	(25)(2Ɛ3Ɛ11)	$(25)^5$	(2Ɛ3Ɛ11)	[1,1]	1
32.	(205)(41X27)	$(205)^3$	(41X27)	[1,1]	1
33.	(5)(4108307)	$(5)^{Ɛ}$	(4108307)	[1,1]	1
34.	(27Ɛ)(71141)	$(27Ɛ)^3$	(71141)	[1,1]	1
35.	(11)(1902097)	$(11)^7$	(1902097)	[1,1]	1
36.	$(2)^{11}$(48X7)	$(48X7)^2$	$(2)^{11}$	[1,1]	1
37.	(17)(14479Ɛ1)	(17)(271)(5ƐƐ1)	(14479Ɛ1)	[3,1]	1
38.	(37)(12096Ɛ5)	$(37)^5$	(12096Ɛ5)	[1,1]	1
39.	(320Ɛ)(35311)	(5)(15)(46Ɛ)(320Ɛ)	$(5)^2$(3Ɛ)(35311)	[4,3]	5
3X.	(485)(1X1767)	$(485)^3$	(1X1767)	[1,1]	1

▽

i	$n = \sigma(m) - 1$	m	$m' = \sigma'(m)$	#PF	$\gcd(m, m')$
3Ɛ.	(4X5)(1Ɛ8947)	$(4X5)^3$	(1Ɛ8947)	[1,1]	1
40.	(51Ɛ)(227XX1)	$(51Ɛ)^3$	(227XX1)	[1,1]	1
41.	(15)(8709647)	$(15)^7$	(8709647)	[1,1]	1
42.	(535)(23Ɛ6Ɛ7)	$(535)^3$	(23Ɛ6Ɛ7)	[1,1]	1
43.	(545)(24X2X7)	$(545)^3$	(24X2X7)	[1,1]	1
44.	(58Ɛ)(290331)	$(58Ɛ)^3$	(290331)	[1,1]	1
45.	(59Ɛ)(29ƐX21)	$(59Ɛ)^3$	(29ƐX21)	[1,1]	1
46.	(5Ɛ5)(2Ɛ5637)	$(5Ɛ5)^3$	(2Ɛ5637)	[1,1]	1
47.	(63Ɛ)(340981)	$(63Ɛ)^3$	(340981)	[1,1]	1
48.	(5)(86252431)	$(5)^{11}$	(86252431)	[1,1]	1
49.	(76Ɛ)(495551)	$(76Ɛ)^3$	(495551)	[1,1]	1
4X.	(775)(4X1077)	$(775)^3$	(4X1077)	[1,1]	1
4Ɛ.	(80Ɛ)(5534Ɛ1)	$(80Ɛ)^3$	(5534Ɛ1)	[1,1]	1
50.	(825)(573X07)	$(825)^3$	(573X07)	[1,1]	1
51.	(835)(5883Ɛ7)	$(835)^3$	(5883Ɛ7)	[1,1]	1
52.	(855)(5Ɛ5997)	$(855)^3$	(5Ɛ5997)	[1,1]	1
53.	(61)(9786505)	$(61)^5$	(9786505)	[1,1]	1
54.	(2)(4ƐƐ2308X7)	$(2)^{27}$	(4ƐƐ2308X7)	[1,1]	1
55.	(965)(76Ɛ887)	$(965)^3$	(76Ɛ887)	[1,1]	1
56.	(9XƐ)(823311)	$(9XƐ)^3$	(823311)	[1,1]	1
57.	(67)(11265655)	$(67)^5$	(11265655)	[1,1]	1
58.	(X3Ɛ)(8X8581)	$(X3Ɛ)^3$	(8X8581)	[1,1]	1
59.	(X9Ɛ)(993521)	$(X9Ɛ)^3$	(993521)	[1,1]	1
5X.	(6Ɛ)(141072X1)	$(6Ɛ)^5$	(141072X1)	[1,1]	1
5Ɛ.	(XƐƐ)(X0Ɛ101)	$(XƐƐ)^3$	(X0Ɛ101)	[1,1]	1
60.	(ƐX5)(Ɛ8Ɛ247)	$(ƐX5)^3$	(Ɛ8Ɛ247)	[1,1]	1

Table 1: Integers $m < 10^9$ such that $n = \sigma(m) - 1$ has aliquot spectral basis $\{m, m'\}$.

5 Table of m

k	$m = 6 \cdot Ɛ^k$	q
1	(2)(3)(Ɛ)	(11)
3	(2)(3)$(Ɛ)^3$	(Ɛ11)
4	(2)(3)$(Ɛ)^4$	(X201)
5	(2)(3)$(Ɛ)^5$	(93X11)
6	(2)(3)$(Ɛ)^6$	(866301)
13	(2)(3)$(Ɛ)^{13}$	(3XX25X500050511)
25	(2)(3)$(Ɛ)^{25}$	(11X352X8321X90042572101832X11)
56	(2)(3)$(Ɛ)^{56}$	(67935509409099841211 2Ɛ677784Ɛ389925Ɛ70487352XƐ187X394702X7Ɛ42901)
6Ɛ	(2)(3)$(Ɛ)^{6Ɛ}$	(162110650XX99165X4420Ɛ80182Ɛ240X722Ɛ1114521157ƐX49175803741X439184X2XX179Ɛ8139711)
Ɛ6	(2)(3)$(Ɛ)^{Ɛ6}$	(19X2453724ƐƐ19203XX01Ɛ19921Ɛ111X634X8X23368X03758725848658Ɛ087150Ɛ1X843451841749X6Ɛ76539794829X3Ɛ65202593658532ƐX25X15X89Ɛ373384375901)

Table 2: Primes q of the form $q = 2(6 \cdot Ɛ^k - 1)/(Ɛ - 1)$ for $k < 100$. See Example 3.5.

6 Anomalous examples

i	n	m	$m' = \sigma'(m)$	#PF	$\gcd(m, m')$
11.	(7)(775)	(3)(7)(Ɛ)(17)	(3)(775)	[4,2]	3
37.	(17)(14479Ɛ1)	(17)(271)(5ƐƐ1)	(14479Ɛ1)	[3,1]	1
39.	(320Ɛ)(35311)	(5)(15)(46Ɛ)(320Ɛ)	$(5)^2$(3Ɛ)(35311)	[4,3]	5

Table 3: Anomalous examples of aliquot spectral basis from Table 1: they work because they work. The author was trying to find another anomalous example when he searched out to $m < 10^9$.

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