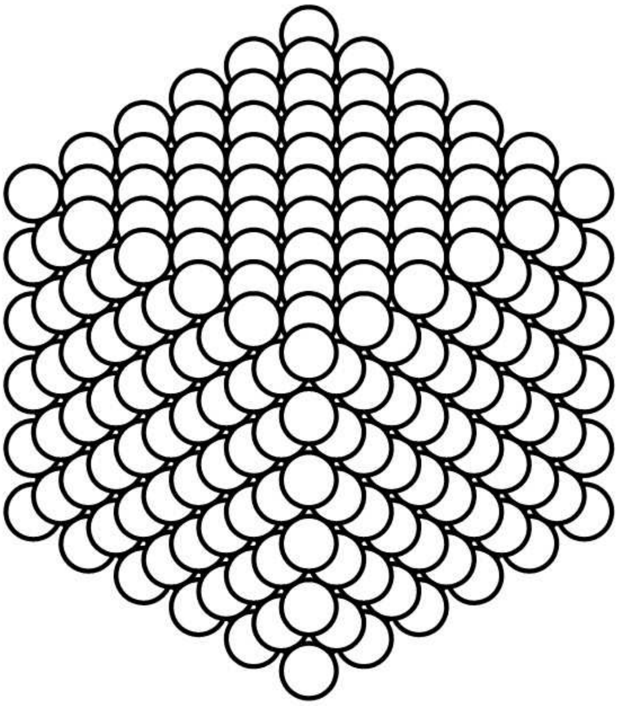
Secrets of Sphere Packings and Figurate Numbers



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Keywords: Sphere packings; figurate numbers; polytopal numbers; sequences; number planes; dimensions; outermost shells; axes; pyramidal caps; difference sequences; negative numbers; lattices; philosophy

0 Introduction

This paper is about the arrangement and stacking of equal-sized spheres into regular geometric figures, and the sequences and remarkable yet largely unknown insights that can be obtained as a result.

In “Über einige Strategien des mathematischen Denkens” (About some Strategies of Mathematical Thinking) by Ernst Kleinert, a study on the structure of mathematics, figurate numbers are referred to as a pastime in a footnote [5, p. 64].

Oddly enough, not entirely unknown mathematicians have been engaged with this pastime for thousands of years, as can be seen from the preface of the book “Figurate Numbers” by Elena Deza and Michel Marie Deza [2].

“Figurate numbers, as well as a majority of classes of special numbers, have long and rich history. They were introduced in Pythagorean school (VI-th century BC) as an attempt to connect Geometry and Arithmetic. Pythagoreans, following their credo ‘‘all is number’’, considered any positive integer as a set of points on the plane. (...)

The theory of figurate numbers does not belong to the central domains of Mathematics, but the beauty of these numbers attracted the attention of many scientists during thousands years. The list (not full) of famous mathematicians, who worked in this domain, consists of Pythagoras of Samos (ca. 582 BC–ca. 507 BC), Hypsicles of Alexandria (190 BC–120 BC), Plutarch of Chaeronea (ca. 46–ca. 122), Nicomachus of Gerasa (ca. 60–ca. 120), Theon of Smyrna (70–135), Diophantus of Alexandria (ca. 210–ca. 290), Leonardo of Pisa, also known as Leonardo Fibonacci (ca. 1170–ca. 1250), Michel Stifel (1487–1567), Gerolamo Cardano (1501–1576), Claude Gaspard Bachet de Meziriac (1581–1638), Rene Descartes (1596–1650), Pierre de Fermat (1601–1665), John Pell (1611–1685), Blaise Pascal (1623–1662), Leonhard Euler (1707–1783), Joseph-Louis Lagrange (1736–1813), Adrien-Marie Legendre (1752–1833), Carl Friedrich Gauss (1777–1855), Augustin-Louis Cauchy (1789–1857), Carl Gustav Jacob Jacobi (1804–1851), Waclaw Franciszek Sierpinski (1882–1969), Barnes Wallis (1887–1979).”

And even more oddly and actually hard to believe is that none of these outstanding mathematicians is said to have discovered the simple and beautiful connections outlined in this work.

When I myself began intuitively stacking spheres and cubes on my search for universal truth, the first discovery that seemed relevant to me was that the summation of consecutive centered hexagonal numbers leads to cubic numbers. So the cube is formed not only by stacking equal-sized squares, but also by increasing hexagons. As a result of this discovery, I began to study intensely sphere packings and figurate numbers.

At first, I only found helpful informations and many inspirations for my research area in the book “Syneregtics: Explorations in the Geometry of Thinking” by R. Buckminster Fuller [1].

Later on, N. J. A. Sloane´s “The On-Line Encyclopedia of Integer Sequences” (OEIS) [8] helped me to check if my key discoverys had already been published somewhere.

And when I had already found everything I present here, I came across to Boon K. Teo´s and N. J. A. Sloane´s remarkable paper “Magic Numbers in Polygonal and Polyhedral Clusters” [10], which includes, among other things, generalized formulas for sphere packings and some of the things I had been working on after all.

The basics of my research area should therefore have been known for a long time, even though hardly anyone knows about it. However, strangely enough, in all of the publications I found on sphere packings and figurate numbers, the most interesting figures are consistently ignored.

This paper aims to show that study of sphere packings and figurate numbers is not just a pastime, but leads to significant mathematical insights.

1 ONE-DIMENSIONAL FUNDAMENTALS

Everything arises from one. Everything arises from unity. All sequences of regular geometric figures generated on the basis of sphere packings start with one. One is a triangular number, a square number, a hexagonal number, a tetrahedral number, an octahedral number, a cubic number etc. Everything is one. But a single sphere does not yet form a unit cell of a geometric figure. That is one of the reasons why the formulas in this work are set up so that inserting zero leads to one and inserting one leads to the respective unit cell. At first, this may be confusing, if the formulas for natural numbers, square numbers and cubic numbers are not n, n^2, and n^3, but $n+1$, $(n+1)^2$, and $(n+1)^3$. However, later it becomes evident, also in the generalized formulas of Teo and Sloane [10], that it has to be exactly like that.

The number one forms the simplest and most elementary sequence:

Natural Numbers				*n+1*
v_0=1	v_1=2	v_2=3	v_3=4	A000027 (OEIS-Nr.)
1	1+1	1+1+1	1+1+1+1	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31,

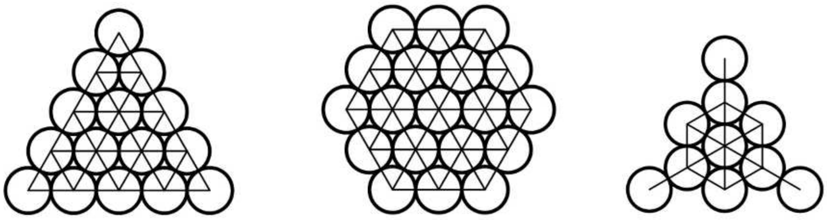
The sequence of odd numbers also plays a role:

Odd Numbers				*2n+1*
v_0=1	v_1=3	v_2=5	v_3=7	A005408 (OEIS-Nr.)
1	1+2	1+2+2	1+2+2+2	1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 49, 51, 53, 55, 57,

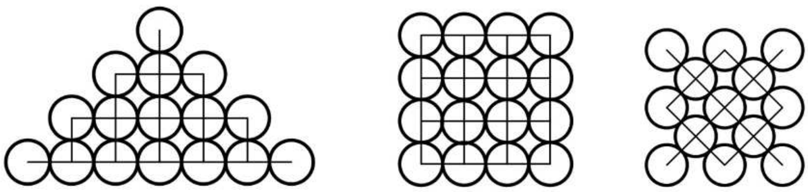
The most important things are already shown by this. Whole numbers and addition. Everything else follows from this. “v_n” means frequency *n*.

2 TWO-DIMENSIONAL FIGURES

In the plane, there are only two relevant possibilities for the arrangement of spheres (or rather circles), with spheres also being able to be considered as nodes of lattices.



60°, hexagonal arrangement, 6 neighbors (contacts), 3 axes

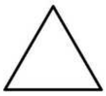
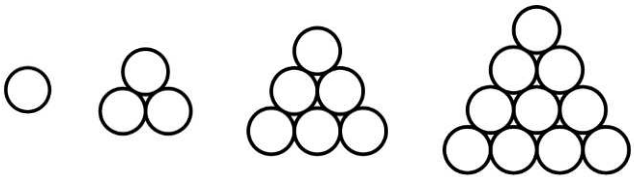


or 90°, orthogonal arrangement, 4 neighbors (contacts), 2 axes.

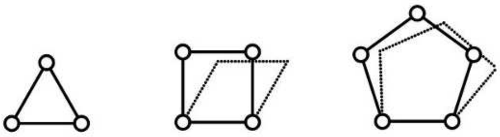
Teo and Sloane give the following generalized formula for the total number of spheres (or rather circles) G_n for a *p*-sided regular polygon, which can be decomposed in *F* triangles [10, p. 4547]:

$$G_n = \tfrac{1}{2}Fn^2 + \tfrac{1}{2}pn + 1$$

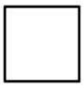
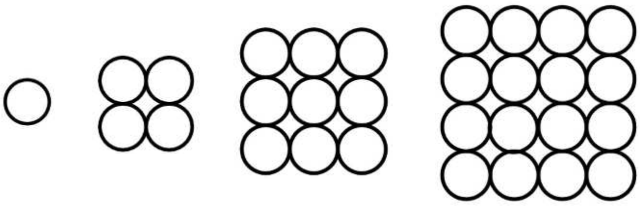
The symplex or simplest figure in the plane is the triangle. The corresponding triangle numbers are formed from the summation (∑) of the natural numbers.

 Triangle	$\sum (n+1) = ½(n^2+3n+2)$
$v_0=1$ $v_1=3$ $v_2=6$ $v_3=10$	A000217 (OEIS-Nr.)
	1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 630, 666, 703, 741, 780,
	Triangles *F*: 1△, Sides *p*: 3
1 1+2 1+2+3 1+2+3+4	Hexagonal Lattice

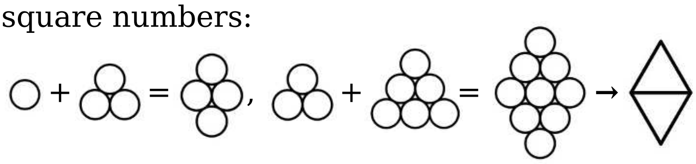
According to Buckminster Fuller, only triangles are stable [1, sec. 609.01]:



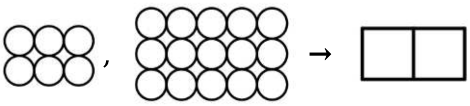
The summation of odd numbers leads to square numbers:

 Square	$\sum (2n+1) = n^2+2n+1$
$v_0=1$ $v_1=4$ $v_2=9$ $v_3=16$	A000290 (OEIS-Nr.)
	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961, 1024, 1089, 1156, 1225, 1296, 1369
	Triangles *F*: 2△, Sides *p*: 4
1 1+3 1+3+5 1+3+5+7	Orthogonal Lattice

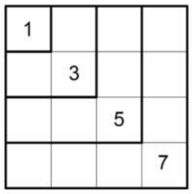
A square can be divided diagonally into two triangles. Likewise, two triangles form a rhombus or rather diamond, which corresponds to the square numbers:

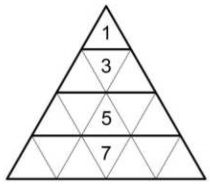
 = 1, 4, 9, 16, 25, 36, 49, ...

And the doubling of a square leads to the odd triangle numbers:

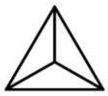
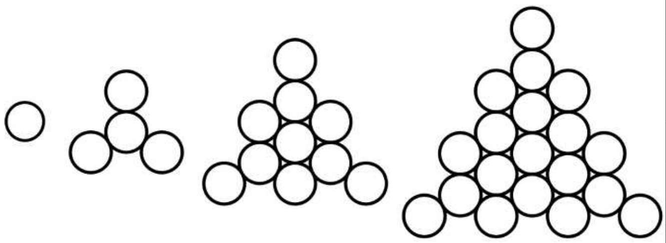
 = 1, 6, 15, 28, 45, 66, 91, ...

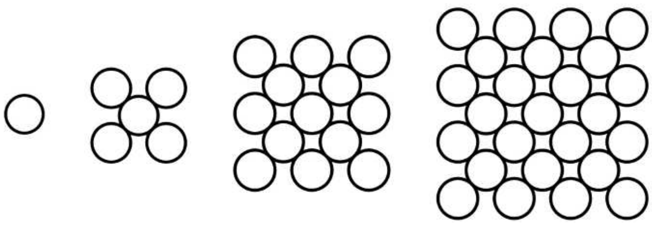
Square numbers are formed by adding odd numbers. As Buckminster Fuller shows, it is even the case that the area growth of all regular figures corresponds to the summation of odd numbers [1, sec. 990.01]:



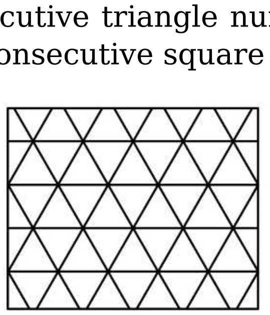
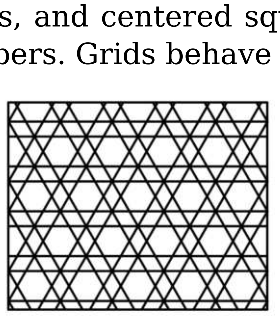
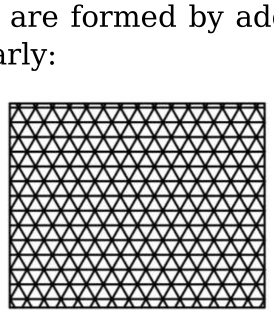


Triangle and square can also be formed with spheres as centered polygons:

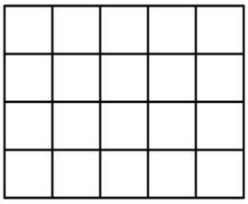
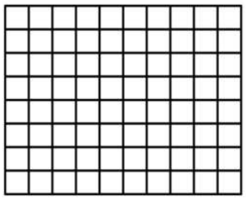
 Centered Triangle	$\frac{1}{2}(3n^2+3n+2)$
v_0=1 v_1=4 v_2=10 v_3=19	A005448 (OEIS-Nr.)
	1, 4, 10, 19, 31, 46, 64, 85, 109, 136, 166, 199, 235, 274, 316, 361, 409, 460, 514, 571, 631, 694, 760, 829, 901, 976, 1054, 1135, 1219, 1306, 1396, 1489, 1585, 1684,
	Triangles F: 3△, Sides p: 3
1 1+3 1+3+6 3+6+10	Hexagonal Lattice

 Centered Square	$2n^2+2n+1$
v_0=1 v_1=5 v_2=13 v_3=25	A001844 (OEIS-Nr.)
	1, 5, 13, 25, 41, 61, 85, 113, 145, 181, 221, 265, 313, 365, 421, 481, 545, 613, 685, 761, 841, 925, 1013, 1105, 1201, 1301, 1405, 1513, 1625, 1741, 1861, 1985, 2113,
	Triangles F: 4△, Sides p: 4
1 1+4 4+9 9+16	Orthogonal Lattice

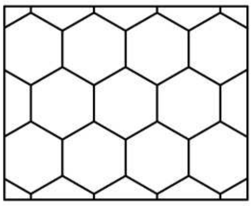
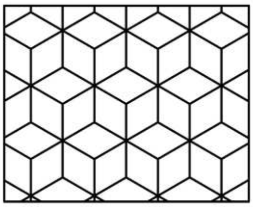
Centered triangles are formed (as of frequency 2) by adding three consecutive triangle numbers, and centered squares are formed by adding two consecutive square numbers. Grids behave similarly:

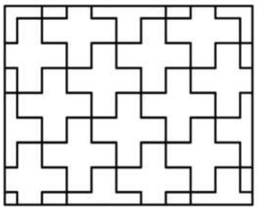
If two triangular grids are centered on top of each other, a grid of hexagrams is formed. Only when a 3rd triangular grid is added, does a complete triangular grid reappear. But if two square grids are centered on top of each other, a square grid immediately reappears:

If two hexagonal grids are centered on top of each other, a grid of diamonds is formed, reminiscent of cubes:

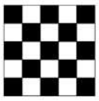



Adding a third hexagonal grid would lead to a triangular grid. And a grid of crosses immediately leads to swastikas:

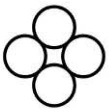
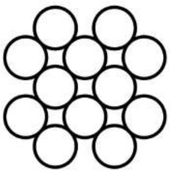
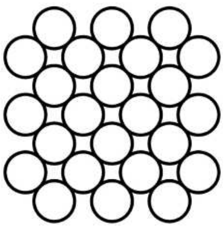


The essential regular shapes that fill the plane seamlessly are shown with triangles, squares, hexagons, and crosses.

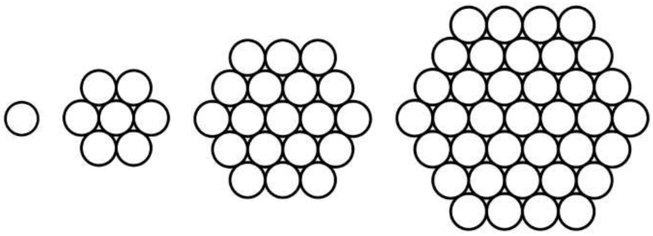
The centered squares are comparable to the black fields of odd-numbered “chessboards”:



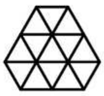
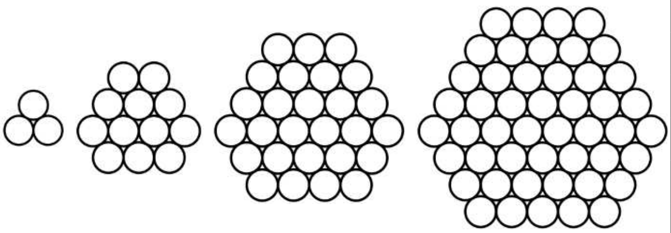
Later, layers corresponding to the white fields will also be required:

	„Centered“ Square (-1)			$2n^2+2n$
$v_0=0$	$v_1=4$	$v_2=12$	$v_3=24$	A046092 (OEIS-Nr.)
				1, 4, 12, 24, 40, 60, 84, 112, 144, 180, 220, 264, 312, 364, 420, 480, 544, 612, 684, 760, 840, 924, 1012, 1104, 1200, 1300, 1404, 1512, 1624, 1740, 1860, 1984, 2112,
				Triangles *F*: 4△, Sides *p*: 4
0	4∗1	4∗3	4∗6	Orthogonal Lattice

By enclosing a central sphere, hexagons are formed which also belong to the centered polygons:

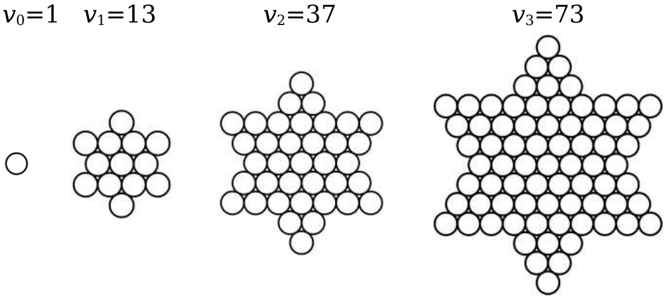
 Centered Hexagon	$3n^2+3n+1$
$v_0=1$ $v_1=7$ $v_2=19$ $v_3=37$	A003215 (OEIS-Nr.)
	1, 7, 19, 37, 61, 91, 127, 169, 217, 271, 331, 397, 469, 547, 631, 721, 817, 919, 1027, 1141, 1261, 1387, 1519, 1657, 1801, 1951, 2107, 2269, 2437, 2611, 2791, 2977,
	Triangles F: 6△, Sides p: 6
1 1+6 1+6+12 1+6+12+18	Hexagonal Lattice

Irregular hexagons are formed by enclosing a central triangle:

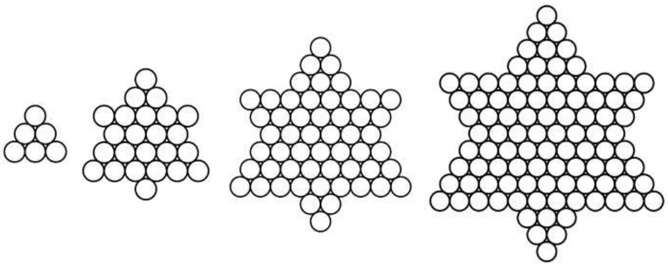
 Irregular Hexagon	$3\,\square = 3n^2+6n+3$
$v_0=3$ $v_1=12$ $v_2=27$ $v_3=48$	A033428 (OEIS-Nr.)
	3, 12, 27, 48, 75, 108, 147, 192, 243, 300, 363, 432, 507, 588, 675, 738, 867, 972, 1083, 1200, 1323, 1452, 1587, 1728, 1875, 2028, 2187, 2352, 2523, 2700, 2883,
	Triangles F: 13△, Sides p: 6
3∗1 3∗4 3∗9 3∗16	Hexagonal Lattice

Later on, irregular hexagons of various types appear frequently in layers of sphere packings.

As can be seen in the triangular grid, two intersecting triangles form hexagrams, which are also of interest for sphere packings:

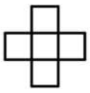
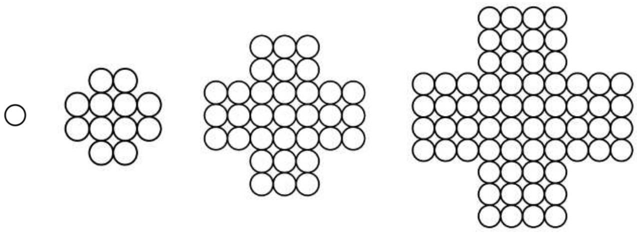
 Hexagram	$6n^2+6n+1$
$v_0=1$ $v_1=13$ $v_2=37$ $v_3=73$	A003154 (OEIS-Nr.)
	1, 13, 37, 73, 121, 181, 253, 337, 433, 541, 661, 793, 937, 1093, 1261, 1441, 1633, 1837, 2053, 2281, 2521, 2773, 3037, 3313, 3601, 3901, 4213, 4537, 4873,
	Triangles F: 12△, Sides p: 12
1 1+12 1+12+24 1+12+24+36	Hexagonal Lattice

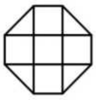
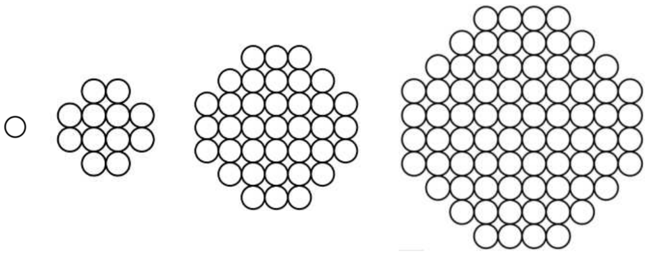
Irregular hexagrams consisting of two consecutive triangles will also be needed as layers in sphere packings:

 Irregular Hexagram	$6\,\square = 6n^2+12n+6$
v_0=6 v_1=24 v_2=54 v_3=96	A033581 (OEIS-Nr.)
	6, 24, 54, 96, 150, 216, 294, 384, 486, 600, 726, 864, 1014, 1176, 1350, 1536, 1734, 1944, 2166, 2400, 2646, 2904, 3174, 3456, 3750, 4056, 4374, 4704, 5046,
	Triangles F: 28$\triangle$, Sides p: 12
6∗1 6∗4 6∗9 6∗16	Hexagonal Lattice

As previously shown, the generalized formula by Teo and Sloane (see page 3) is not applicable to irregular polygons. It only works when the sequence begins with one.

Just as with crosses and semi-regular octagons, which are shown here as additional planar figures:

 Cross	$5n^2+6n+1$
V_0=1 v_1=12 v_2=33 v_3=64	A051624 (OEIS-Nr.)
	1, 12, 33, 64, 105, 156, 217, 288, 369, 460, 561, 672, 793, 924, 1065, 1216, 1377, 1548, 1729, 1920, 2121, 2332, 2553, 2784, 3025, 3276, 3537, 3808, 4089, 4380,
	Triangles F: 10$\triangle$, Sides p: 12
1 4+4∗2 9+4∗6 16+4∗12	Orthogonal Lattice

 Octagon	$7n^2+4n+1$
v_0=1 v_1=12 v_2=37 v_3=76	A005892 (OEIS-Nr.)
	1, 12, 37, 76, 129, 196, 277, 372, 481, 604, 741, 892, 1057, 1236, 1429, 1636, 1857, 2092, 2341, 2604, 2881, 3172, 3477, 3796, 4129, 4476, 4837, 5212, 5601,
	Triangles F: 14$\triangle$, Sides p: 8
1 16−4∗1 49−4∗3 100−4∗6	Orthogonal Lattice

With this the essential two-dimensional figures are shown. They provide the layers from which the following sphere packings are composed, or are otherwise useful as objects of demonstration.

3 THREE-DIMENSIONAL FIGURES

For the arrangement of spheres in spatial geometric figures, there are different types of sphere packings, by which this section is divided into.

Teo and Sloane give the following generalized formula for the total number of spheres G_n in regular polyhedra [10, p. 4548]:

$$G_n = \alpha n^3 + \tfrac{1}{2}\beta n^2 + \gamma n + 1$$

whereby

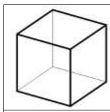
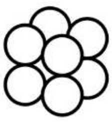
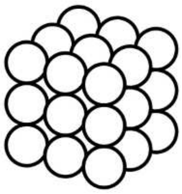
$$\alpha = \tfrac{1}{6}C,\ \beta = \tfrac{1}{2}F_s,\ \gamma = \tfrac{1}{4}F_s + I_1 + 1 - \tfrac{1}{6}C$$

Here, C is the number of tetrahedral cells into which the respective polyhedron is divided, F_s is the number of triangular faces into which the surface can be decomposed, and I_1 is the number of spheres in the interior of the unit cell.

3.1 PRIMITIVE CUBIC PACKING

In the primitive cubic sphere packing, inner spheres have 6 neighbors (contacts). This packing corresponds to the common 3-axis spatial imagination.

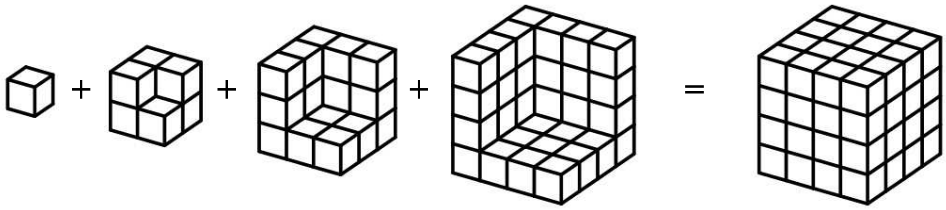
The easiest way to demonstrate this type of packing is with the conventional cube:

 Primitive Cube			$\sum$  $= n^3+3n^2+3n+1$
v_0=1	v_1=8	v_2=27	A000578 (OEIS-Nr.)
			1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, 1728, 2197, 2744, 3375, 4096, 4913, 5832, 6859, 8000, 9261, 10648, 12167, 13824, 15625, 17576, 19683, 21952, 24389,
1∗1	2∗4	3∗9	Tetrahedral Cells C: 6 , Triangles F_S: 12 (6 ), Core I_1: 0
1	1+7	1+7+19	Primitive Cubic Packing

However, building these primitive cubes with single spheres is a rather unstable matter. Although it is possible with magnetized spheres, it is easier to construct this packing using building blocks made of smaller cubes instead of spheres. It can therefore also be called a “cube packing”.

The tetrahedral cells into which the polyhedra are divided do not have to be regular tetrahedra. Teo and Sloane show how the primitive cube is divided into 6 tetrahedra [10, p. 4551]. This division is easier to determine for most other polyhedra.

As mentioned in the introduction, the summation of consecutive centered hexagonal numbers leads to the sequence of primitive cubic numbers:



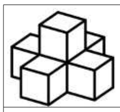
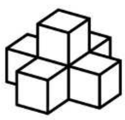
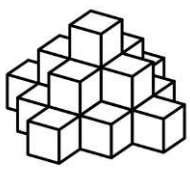
1 7 19 37 64

This should not come as a surprise, as the hexagon is the two-dimensional projection of the three-dimensional cube:

3 Axes

Since the sequence of primitive cubic numbers also reappears in the case of figures in other packing types, I will come back to this connection more often.

The following sequence, which arises from stacking consecutive centered squares, will also play a role. For a more illustrative presentation, the spheres are replaced by cubes:

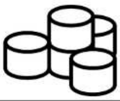
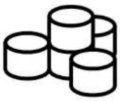
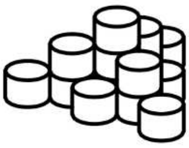
<table>
<tr><td colspan="3"> Centered
Square Step Pyramid</td><td>$\sum\boxtimes = \frac{1}{3}(2n^3+6n^2+7n+3)$</td></tr>
<tr><td rowspan="3">v_0=1
</td><td rowspan="3">v_1=6
</td><td rowspan="3">v_2=19
</td><td>A005900 (OEIS-Nr.)</td></tr>
<tr><td>1, 6, 19, 44, 85, 146, 231, 344, 489, 670, 891, 1156, 1469, 1834, 2255, 2736, 3281, 3894, 4579, 5340, 6181, 7106, 8119, 9224, 10425, 11726, 13131, 14644, 16269, 18010,</td></tr>
<tr><td>Tetrahedral Cells C: 4 ,
Triangles F_S: 8 △ , Core I_1: 0</td></tr>
<tr><td>1</td><td>1+5</td><td>1+5+13</td><td>Primitive Cubic Packing</td></tr>
</table>

3.2 PRIMITIVE HEXAGONAL PACKING

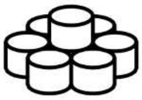
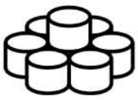
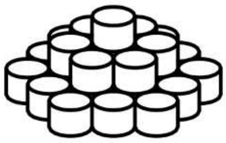
In the primitive hexagonal packing, inner spheres have 8 neighbors (contacts), so this packing has 4 axes (three in the plane and one perpendicular). While this packing is also quite unstable with spheres, it can be easily constructed as a “cylinder packing”.

I will only show the figures as examples whose sequences appear later in other packing types.

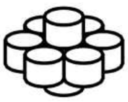
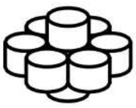
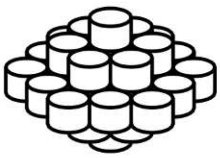
Stacking consecutive centered triangles creates a triangular step pyramid or rather a step tetrahedron:

 Centered Triangular Step Pyramid			$\sum$  $= ½(n^3+3n^2+4n+2)$
v_0=1	v_1=5	v_2=15	A006003 (OEIS-Nr.)
			1, 5, 15, 34, 65, 111, 175, 260, 369, 505, 671, 870, 1105, 1379, 1695, 2056, 2465, 2925, 3439, 4010, 4641, 5335, 6095, 6924, 7825, 8801, 9855, 10990, 12209,13515,
			Tetrahedral Cells C: 3 , Triangles F_S: 6  , Core I_1: 0
1	1+4	1+4+10	Primitive Hexagonal Packing

Stacking consecutive centered hexagons creates a hexagonal step pyramid and once again the sequence of primitive cubic numbers appears:

 Hexagonal Step Pyramid			$\sum$  = n^3+3n^2+3n+1
v_0=1	v_1=8	v_2=27	A000578 (OEIS-Nr.)
			1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, 1728, 2197, 2744, 3375, 4096, 4913, 5832, 6859, 8000, 9261, 10648, 12167, 13824, 15625, 17576, 19683, 21952, 24389,
			Tetrahedral Cells C: 6 , Triangles F_S: 12 , Core I_1: 0
1	1+7	1+7+19	Primitive Hexagonal Packing

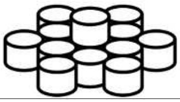
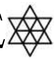
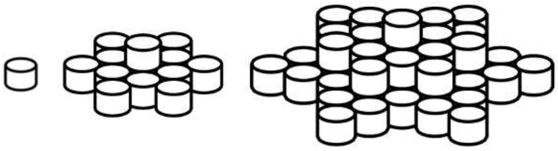
From two consecutive hexagonal step pyramids, a hexagonal step bipyramid is formed:

 Hexagonal Step Bipyramid			$Bi\sum$  $= 2n^3+3n^2+3n+1$
v_0=1	v_1=9	v_2=35	A 005898 (OEIS-Nr.)
			1, 9, 35, 91, 189, 341, 559, 855, 1241, 1729, 2331, 3059, 3925, 4941, 6119, 7471, 9009, 10745, 12691, 14859, 17261, 19909, 22815, 25991, 29449, 33201, 37259, 41635, 46341,
			Tetrahedral Cells C: 12 , Triangles F_S: 12 , Core I_1: 1
1	1+7+1	1+7+19+7+1	Primitive Hexagonal Packing

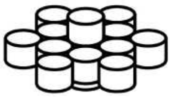
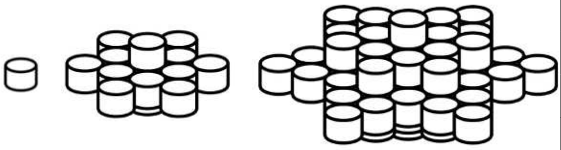
As such a "mirrored" stacking or summation will occur frequently later on, the following definition of the "bi-summation" is given:

$$Bi\sum a_n = a_0+a_1+a_2+...+a_{n-1}+a_n+a_{n-1}+...+a_2+a_1+a_0$$

Consecutive hexagrams can also be stacked, resulting in a hexagrammic step pyramid:

 Hexagrammic Step Pyramid	$\sum$  $= 2n^3+6n^2+5n+1$
v_0=1 v_1=14 v_2=51	A007588 (OEIS-Nr.)
	1, 14, 51, 124, 245, 426, 679, 1016, 1449, 1990, 2651, 3444, 4381, 5474, 6735, 8176, 9809, 11646, 13699, 15980, 18501, 21274, 24311, 27624, 31225, 35126, 39339, 43876,
	Tetrahedral Cells C: 12 , Triangles F_S: 24 △, Core I_1: 0
1 1+13 1+13+37	Primitive Hexagonal Packing

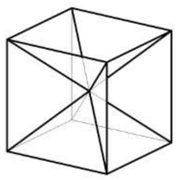
From two consecutive hexagrammic step pyramids, a hexagrammic step bipyramid is formed:

 Hexagrammic Step Bipyramid	$Bi\sum$  $= 4n^3+6n^2+4n+1$
v_0=1 v_1=15 v_2=65	A 005917 (OEIS-Nr.)
	1, 15, 65, 175, 369, 671, 1105, 1695, 2465, 3439, 4641, 6095, 7825, 9855, 12209, 14911, 17985, 21455, 25345, 29679, 34481, 39775, 45585, 51935, 58849, 66351, 74465,
	Tetrahedral Cells C: 24 , Triangles F_S: 24 △, Core I_1: 1
1 1+13+1 1+13+37+13+1	Primitive Hexagonal Packing

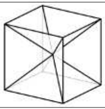
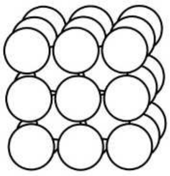
The graphical representation of the hexagrammic step bipyramid is unsatisfactory, but the resulting sequence is quite significant.

3.3 BODY-CENTERED CUBIC PACKING

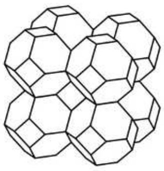
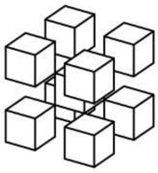
In the body-centered cubic Packing, inner spheres also have 8 neighbors (contacts), so this packing also has 4 axes corresponding to the 4 space diagonals of the cube.



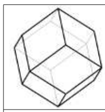
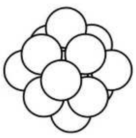
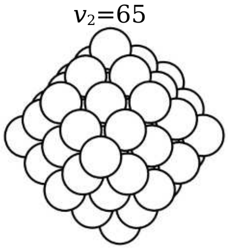
The naming figure of this packing type is the body-centered cube, which corresponds to the sequence of the hexagonal step bipyramid:

 Body-Centered Cube			$Bi\sum = 2n^3+3n^2+3n+1$
v_0=1	v_1=9	v_2=35	A 005898 (OEIS-Nr.)
			1, 9, 35, 91, 189, 341, 559, 855, 1241, 1729, 2331, 3059, 3925, 4941, 6119, 7471, 9009, 10745, 12691, 14859, 17261, 19909, 22815, 25991, 29449, 33201, 37259, 41635, 46341,
1	8+1	27+8	Tetrahedral Cells C: 12 (6), Triangles F_S: 12 , Core I_1: 1
1	1+7+1	1+7+19+7+1	Body-Centered Cubic Packing

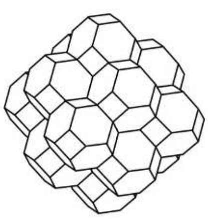
Similar to the centered square consisting of two consecutive squares, the body-centered cube consists of two consecutive primitive cubes. At the intersection of the 4 space diagonals of the unit cell, another sphere is arranged, so the 8 corner spheres do not touch each other, but only have contact with the central sphere inside. As a result, this packing arrangement with spheres is a rather unstable matter. Using building blocks made of truncated octahedra, this packing is easily constructible, but in that case, also the corner blocks touch each other. A comparable body is also a cube cluster, in which the individual cubes only touch at the corners.

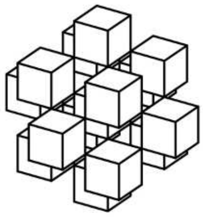
 

The 4 space diagonals of the body-centered cube form 6 interior pyramids. If these 6 pyramids are inverted and attached from the outside to the cube, a rhombic dodecahedron is formed, whose sequence corresponds to that of the hexagrammic step bipyramid (see page 12):

 Rhombic Dodecahedron			$Bi\sum = 4n^3+6n^2+4n+1$
v_0=1	v_1=15	v_2=65	A 005917 (OEIS-Nr.)
			1, 15, 65, 175, 369, 671, 1105, 1695, 2465, 3439, 4641, 6095, 7825, 9855, 12209, 14911, 17985, 21455, 25345, 29679, 34481, 39775, 45585, 51935, 58849, 66351, 74465,
1	9+6∗1	35+6∗5	Tetrahedral Cells C: 24 (6), Triangles F_S: 24 (12), Core I_1: 1
1	1+13+1	1+13+37+13+1	Body-Centered Cubic Packing

The triangular faces of two adjacent inverted pyramids lie in the same plane, forming the 12 rhombuses that give this figure its name.





In addition to the fact that clusters of rhombic dodecahedra fill space without gaps, it is noteworthy that the summation of the consecutive rhombic dodecahedral numbers leads to n^4 or rather to $(n+1)^4$:

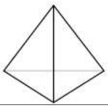
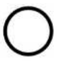
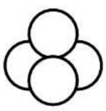
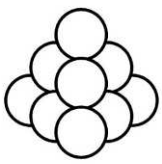
$\sum$  $= 1 + 15 + 65 + 175 + \dots = (n+1)^4$

Later, another figure follows with the same sequence, where it will be easier to gain an idea of the 4th dimension.

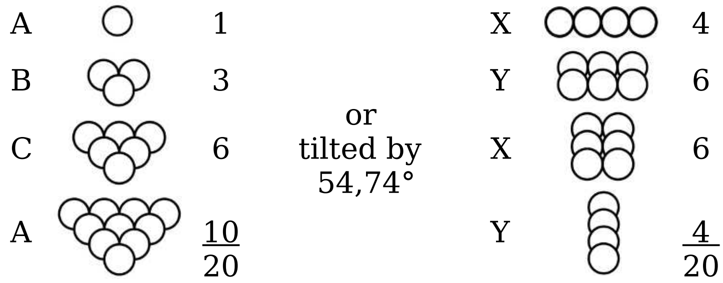
3.4 CUBIC CLOSE-PACKED PACKING (ABC)

In close-packed sphere packings, inner spheres have 12 neighbors (contacts). There are various close-packed sphere packings that differ in their hexagonal layer sequences. In analogy to the triangular grid (see page 5), the close-packed stacking of hexagonal layers has three different positions for spheres (A, B, and C). In the cubic close-packed sphere packing, these three positions are always occupied alternately in the same order (ABC ABC, etc.), so spheres always lie directly on top of each other after three hexagonal layers. As such a close-packing of equal-sized spheres also produces tilted orthogonal layers and regular cubes, it is called cubic close-packed or face-centered cubic packing. I prefer to call it the “ABC packing”. It has 6 axes.

The simplest figure of this packing type and the simplest figure (simplex) in space at all is the tetrahedron. It is formed by stacking consecutive triangles on top of each other:

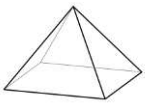
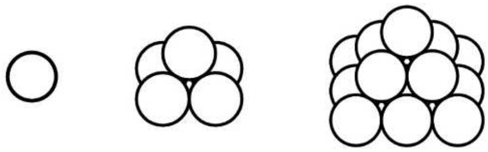
 Tetrahedron			$\sum \triangle = \frac{1}{6}(n^3+6n^2+11n+6)$
$v_0=1$	$v_1=4$	$v_2=10$	A000292 (OEIS-Nr.)
			1, 4, 10, 20, 35, 56, 84, 120, 165, 220, 286, 364, 455, 560, 680, 816, 969, 1140, 1330, 1540, 1771, 2024, 2300, 2600, 2925, 3276, 3654, 4060, 4495, 4960, 5456, 5984, 6545,
			Tetrahedral Cells C: 1 , Triangles F_S: 4 △ , Core I_1: 0
1	1+3	1+3+6	Cubic Close-Packed Packing (ABC)

Example Layer Sequence Tetrahedron of Frequency 3:

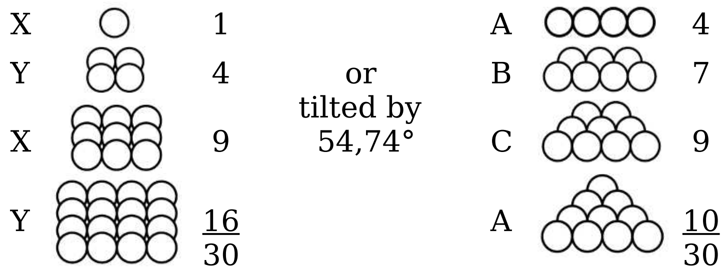


According to Buckminster Fuller, the regular tetrahedron is used as volumetric unity with a volume equal to one [1, sec. 222.30, 990.00]. Then, since the volume of the regular square pyramid is equal to two, and all figures in close-packed packings are composed of tetrahedra and square pyramids (or octahedra), the volume in the ABC packing can be easily read off from the respective figure. The volume of unit cells related to the simplex tetrahedron corresponds to the number of tetrahedral cells, and can be expressed for all figures in this packing type in whole numbers.

By stacking consecutive squares on top of each other, pyramids are formed:

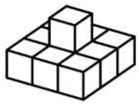
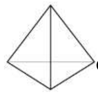
 Square Pyramid	$\sum \square = \frac{1}{6}(2n^3+9n^2+13n+6)$
v_0=1 v_1=5 v_2=14	A000330 (OEIS-Nr.)
	1, 5, 14, 30, 55, 91, 140, 204, 285, 385, 506, 650, 819, 1015, 1240, 1496, 1785, 2109, 2470, 2870, 3311, 3795, 4324, 4900, 5525, 6201, 6930, 7714, 8555, 9455, 10416, 11440
	Tetrahedral Cells C: 2 (1), Triangles F_S: 6 (4 △, 1 □), Core I_1: 0
1 1+4 1+4+9	Cubic Close-Packed Packing (ABC)

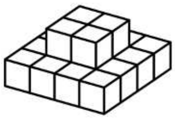
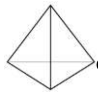
Example Layer Sequence Square Pyramid of Frequency 3:



In analogy to the square grid (see page 6), in the close-packed stacking of orthogonal layers, there are only two different positions for spheres (X and Y). And as is even more apparent in the case of the octahedron, tetrahedron and pyramid actually have the same sphere packing despite having different bases, but tilted by 54.74°.

If only every other square is stacked on top of each other, square step pyramids (primitive cubic) are formed, which correspond to the tetrahedral numbers:

 1+9+25+49+… →  $_{\text{odd}}$ = 1, 10, 35, 84, …

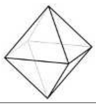
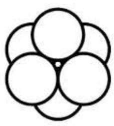
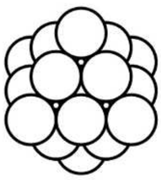
 4+16+36+64+… →  $_{\text{even}}$ = 4, 20, 56, 120, …

And the stacking of every other centered square leads to the square pyramidal numbers.

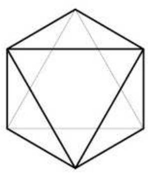
As stacking oranges to form tetrahedra or pyramids shows, the close-packing of equal-sized spheres is actually the most stable way to form clusters of spheres. Figures in the close-packed packings can also be easily built using steel balls and magnet rods. Such models greatly facilitate the analysis of sphere packings, and thus lead to a haptic mathematics.

As mentioned above, all close-packed packings are composed of the unit cells of tetrahedra and square pyramids, with square pyramids predominantly occurring as square bipyramids or rather octahedra, and only as individual square pyramids, if there are squares in the surface. In the ABC packing, it should be noted that two tetrahedral unit cells or two octahedral unit cells never touch at their triangular faces.

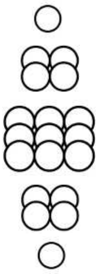
Two consecutive square pyramids together form an octahedron:

 Octahedron			$Bi\sum\square = \sum\boxtimes = \frac{1}{3}(2n^3+6n^2+7n+3)$
v_0=1	v_1=6	v_2=19	A005900 (OEIS-Nr.)
			1, 6, 19, 44, 85, 146, 231, 344, 489, 670, 891, 1156, 1469, 1834, 2255, 2736, 3281, 3894, 4579, 5340, 6181, 7106, 8119, 9224, 10425, 11726, 13131, 14644, 16269, 18010,
1	5+1	14+5	Tetrahedral Cells C: 4 (1 ), Triangles F_S: 8 △ , Core I_1: 0
1	1+4+1	1+4+9+4+1	Cubic Close-Packed Packing (ABC)

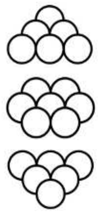
The octahedron, beside the tetrahedron and the square pyramid, is one of the most elementary figures for close-packed packings. Its 12 edges form three mutually perpendicular squares:



Example Layer Sequence Octahedron of Frequency 2:

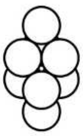
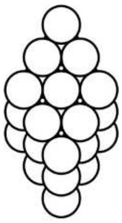


X	1			
Y	4		A	6
X	9	or tilted by 54,74°	B	7
X	4		C	6
Y	1			19
	19			

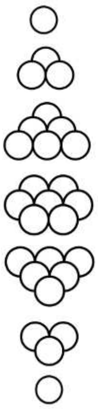
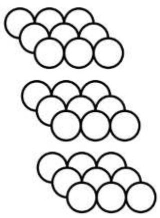
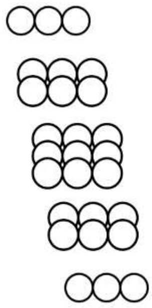


The octahedral numbers result from both the summation of consecutive centered squares and the bi-summation of the square numbers.

By placing two opposing tetrahedra on the octahedron, it becomes a rhombohedron:

 Rhombohedron			$\sum$  = n^3+3n^2+3n+1
$v_0=1$	$v_1=8$	$v_2=27$	A000578 (OEIS-Nr.)
			1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, 1728, 2197, 2744, 3375, 4096, 4913, 5832, 6859, 8000, 9261, 10648, 12167, 13824, 15625, 17576, 19683, 21952, 24389,
1	6+2∗1	19+2∗4	Tetrahedral Cells C: 6 (1 , 2 ), Triangles F_S: 12 (6 ), Core I_1: 0
1	1+7	1+7+19	Cubic Close-Packed Packing (ABC)

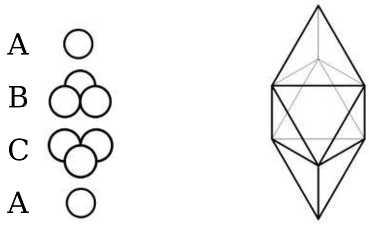
Example Layer Sequence Rhombohedron of Frequency 2:

A	1							
B	3						X	3
C	6		A	9			Y	6
A	7	or	B	9	or tilted by 54,74°		X	9
B	6		C	9			Y	6
C	3			27			X	3
A	1							27
	27							

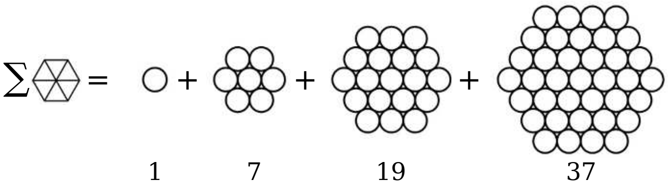
The rhombohedron has the particularity that it has two differing hexagonal layer sequences, since not all vertices have the same distance from the center due to the opposing tetrahedra placed on the octahedron. In the other polyhedra considered here in the cubic close-packed packing, the hexagonal layers will always be congruent.

Due to the layer sequence ABC ABC, etc., all solids in the cubic close-packed packing are composed of unit cells of the rhombohedron in their interior:



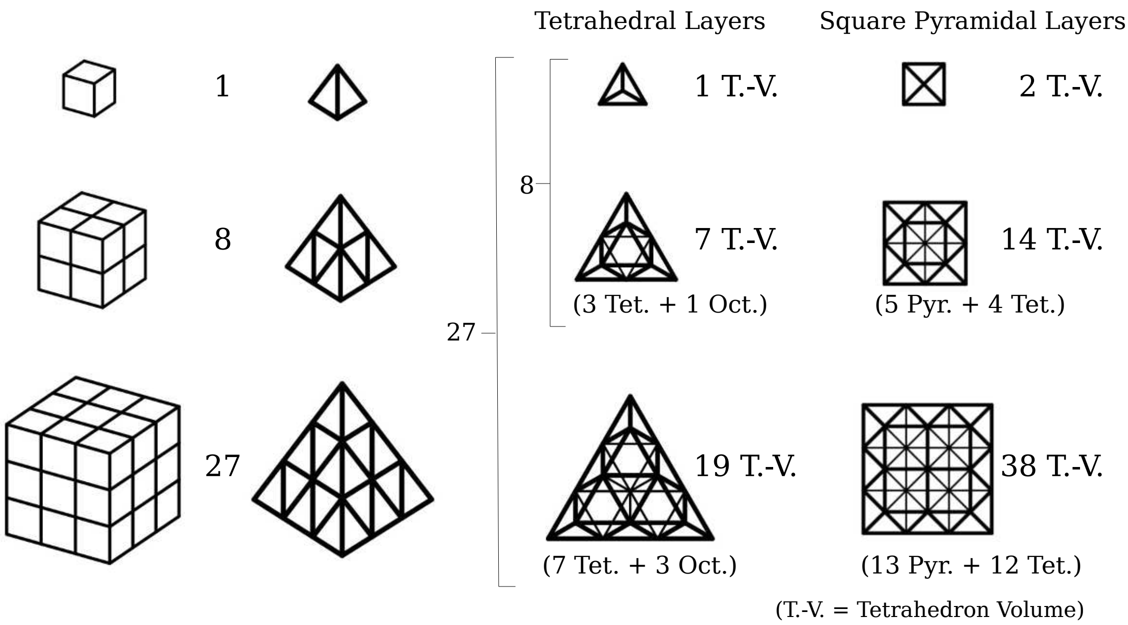
In the book “Anschauliche Geometrie” (Illustrative Geometry) by David Hilbert and Stephan Cohn-Vossen, the cubic close-packed packing is also referred to as the “Kugellagerung des Rhomboëdergitters” (sphere packing of the rhombohedral lattice) [3, p. 41].

Rhombohedra fill space without gaps. It is a distorted primitive cube in which the 6 square faces are deformed into rhombuses. Therefore, the rhombohedral numbers correspond to the primitive cubic numbers and the summation of the centered hexagonal numbers:

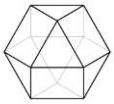
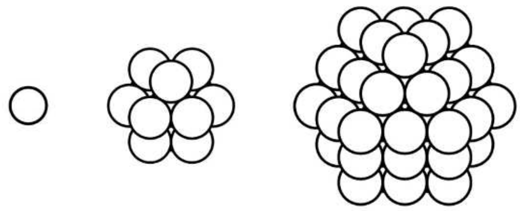


+ … = 1, 8, 27, 64, … = $(n+1)^3$

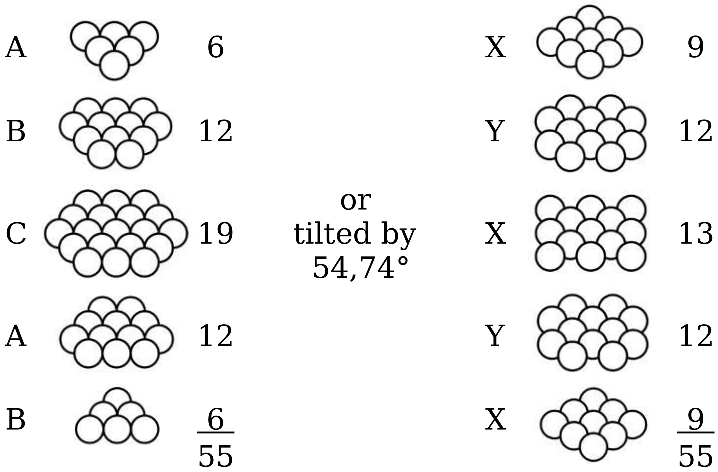
Just as the area growth of all regular plane figures follows the summation of odd numbers (see page 5), Buckminster Fuller demonstrates that the volume growth of all regular spatial figures follows the summation of centered hexagonal numbers [1, sec. 990.01]:



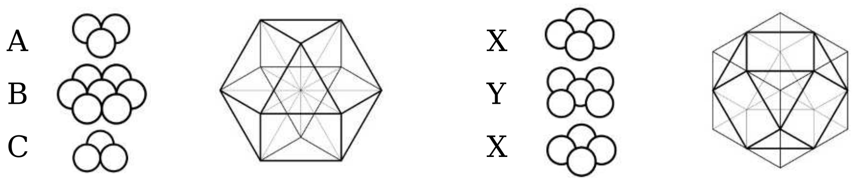
Just as the hexagon is to the plane, the cuboctahedron is to space. The unit cell of the cuboctahedron is formed by completely enveloping a central sphere:

 Cuboctahedron	*⅓(10n³+15n²+11n+3)*
v_0=1 v_1=13 v_2=55	A005902 (OEIS-Nr.)
	1, 13, 55, 147, 309, 561, 923, 1415, 2057, 2869, 3871, 5083, 6525, 8217, 10179, 12431, 14993, 17885, 21127, 24739, 28741, 33153, 37995, 43287, 49049, 55301, 62063,
	Tetrahedral Cells C: 20 (8, 6), Triangles F_S: 20 (8, 6), Core I_1: 1
1 1+12 1+12+42	Cubic Close-Packed Packing (ABC)

Example Layer Sequence Cuboctahedron of Frequency 2:



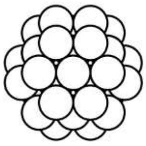
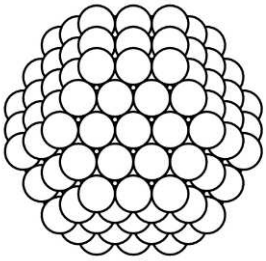
The unit cell of the cuboctahedron is the simplest figure to demonstrate the 12 neighbors of a central sphere as well as the resulting 6 axes in cubic close-packed packing:



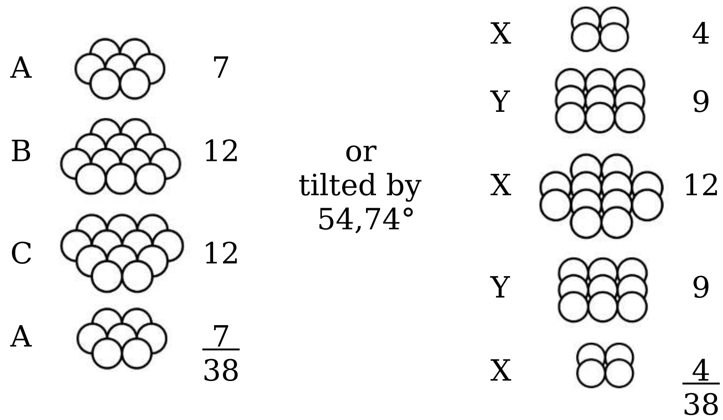
In the hexagonal layers, a sphere in the plane has 6 neighbors, 3 above and 3 below. In the orthogonal layers, a sphere in the plane has 4 neighbors, 4 above and 4 below. By placing another shell of spheres around it, the next larger cuboctahedron is formed.

The cuboctahedron is composed of 8 tetrahedra and 6 pyramids. The 24 outer edges form 4 independent hexagons. The unit cell corresponds to a cube with ¼-pyramids cut off at its 8 corners. Buckminster Fuller has given the name "Vector Equilibrium" to the cuboctahedron [1, sec. 430.00].

When the unit cell of an octahedron is completely enveloped by spheres, the truncated octahedron is formed:

 Truncated Octahedron	*16n³+15n²+6n+1*
v_0=1 v_1=38 v_2=201	A005910 (OEIS-Nr.)
  	1, 38, 201, 586, 1289, 2406, 4033, 6266, 9201, 12934, 17561, 23178, 29881, 37766, 46929, 57466, 69473, 83046, 98281, 115274, 134121, 154918, 177761, 202746,
	Tetrahedral Cells C: 96 (32, 13, 6), Triangles F_S: 60 (8, 6), Core I_1: 6
1 44−6∗1 231−6∗5	Cubic Close-Packed Packing (ABC)

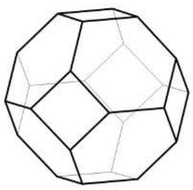
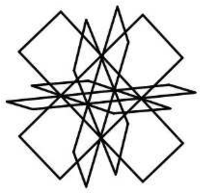
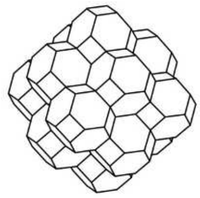
Example Layer Sequence Truncated Octahedron of Frequency 1:



The truncated octahedron is the giant among the regular solids, with a tetrahedron volume of the unit cell of 96. This is because it has an octahedron and thus 6 spheres as its core, while all other unit cells of regular solids either have no core or a core of a single sphere. It is composed of a total of 32 tetrahedra, 13 octahedra, and 6 pyramids, and has on its surface 8 centered hexagons and 6 squares.

According to its name, the truncated octahedron is also formed by removing spheres at the corners of an octahedron corresponding to the square pyramidal numbers, resulting in regular hexagons and squares in the surface. Since only every 3rd triangle can be reduced to a regular hexagon, only every 3rd octahedron can be reduced to a regular truncated octahedron.

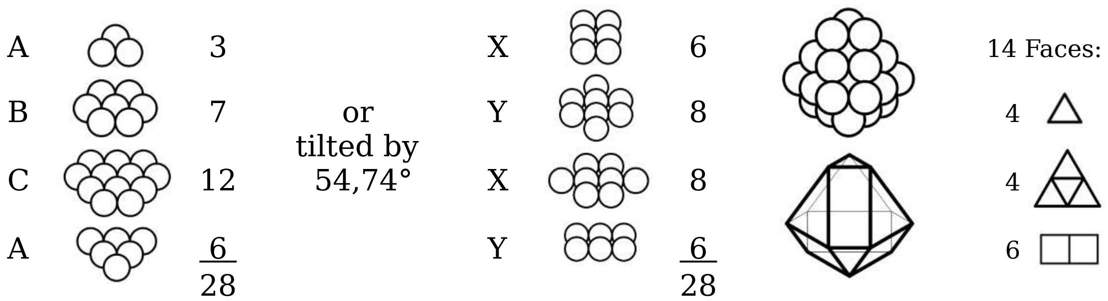
And just as an octahedron consists of three squares that are perpendicular to each other, the truncated octahedron consists of three crosses or rather octagons that are perpendicular to each other. Here, the central octahedron is formed by the center squares of the crosses, and the 3∗8 outer corner spheres of the crosses form the 24 corners of the truncated octahedron. The center spheres of the 8 outer centered hexagons, in turn, together with the central octahedron, form a stellated octahedron consisting of two intersecting tetrahedra (see page 23).

As a lattice, the truncated octahedron is formed analogously to the cuboctahedron by enveloping a central octahedron with 12 octahedra. And as shown in Section 3.3, the space can be filled without gaps using building blocks made of truncated octahedra. Continuously enveloping a central truncated octahedron with truncated octahedra leads to the sequence of rhombic dodecahedral numbers.

Cuboctahedron and truncated octahedron are formed by enveloping a central sphere or rather a central octahedron, and both have 14 faces. Enveloping other geometric figures with a shell of spheres leads to irregular solids.

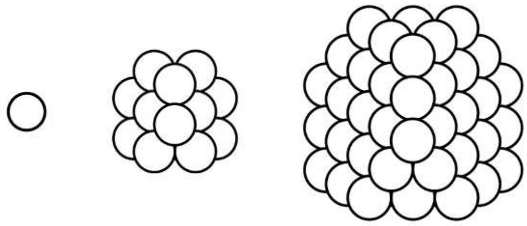
For Example “Enveloped Tetrahedron”:



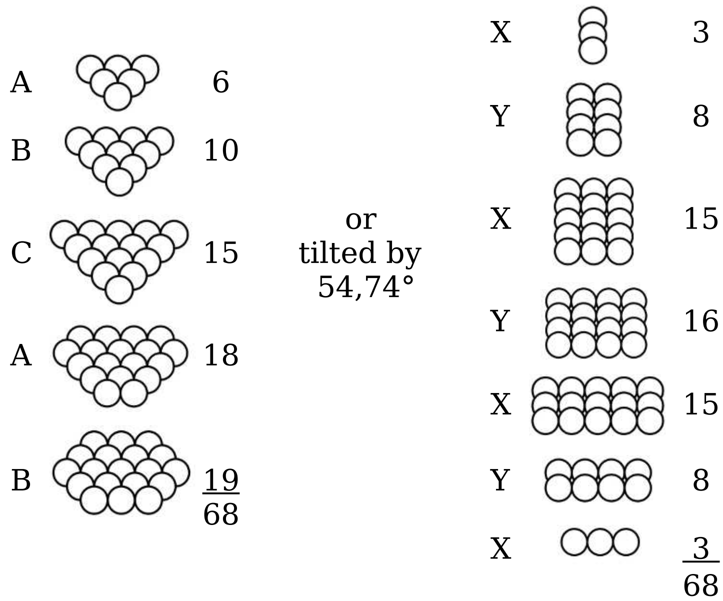
Using steel balls and magnetic rods, it can be quickly demonstrated that in the cubic close-packed packing, the envelopments of all convex regular geometric figures (single sphere, spheres in a row, triangle, rhombus, square, centered square, centered hexagon, octagon, tetrahedron, pyramid, octahedron, rhombohedron, cuboctahedron, truncated octahedron, truncated tetrahedron, close-packed cube) lead to solids with 14 faces.

This regularity seems to have a correspondence in the animate nature. Buckminster Fuller points out in “Synergetics” in connection with his “Isotropic Vector Matrix” made up of tetrahedra and octahedra, that bubbles inside foam as well as internal biological cells also always have 14 facets [1, sec. 1025.13].

In addition to the truncated octahedron, the truncated tetrahedron can also be formed in the cubic close-packed packing:

 Truncated Tetrahedron	$\frac{1}{6}(23n^3+42n^2+25n+6)$
$v_0=1$ $v_1=16$ $v_2=68$	A005906 (OEIS-Nr.)
	1, 16, 68, 180, 375, 676, 1106, 1688, 2445, 3400, 4576, 5996, 7683, 9660, 11950, 14576, 17561, 20928, 24700, 28900, 33551, 38676, 44298, 50440, 57125, 64376, 72216,
	Tetrahedral Cells C: 23 (7, 4), Triangles F_S: 28 (4, 4), Core I_1:0
1 20−4∗1 84−4∗4	Cubic Close-Packed Packing (ABC)

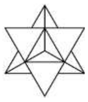
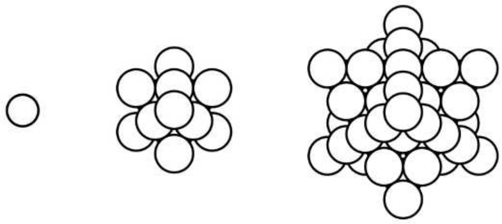
Example Layer Sequence Truncated Tetrahedron of Frequency 2:



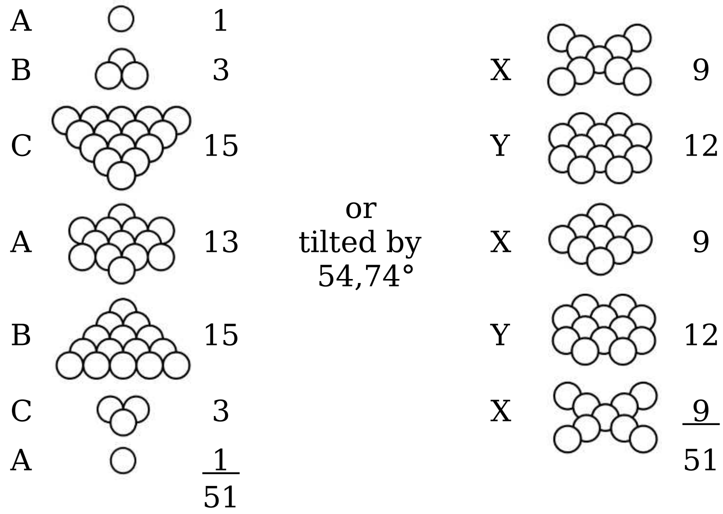
The truncated tetrahedron is formed when spheres are removed from the corners of a tetrahedron in such a way that regular hexagons and triangles are emerged on the surface. For this purpose, smaller tetrahedra are removed. Since only every 3rd triangle can be reduced to a regular hexagon, only every 3rd tetrahedron can be reduced to a regular truncated tetrahedron. The unit cell is composed of 7 tetrahedra and 4 octahedra, and has 4 centered hexagons and 4 triangles on its surface. It has a tetrahedron volume of 23.

The truncated tetrahedron is also formed when 4 octahedra are placed on the triangular faces of a central tetrahedron. The central tetrahedron thereby provides the central spheres for the 4 centered hexagons in the outer shell.

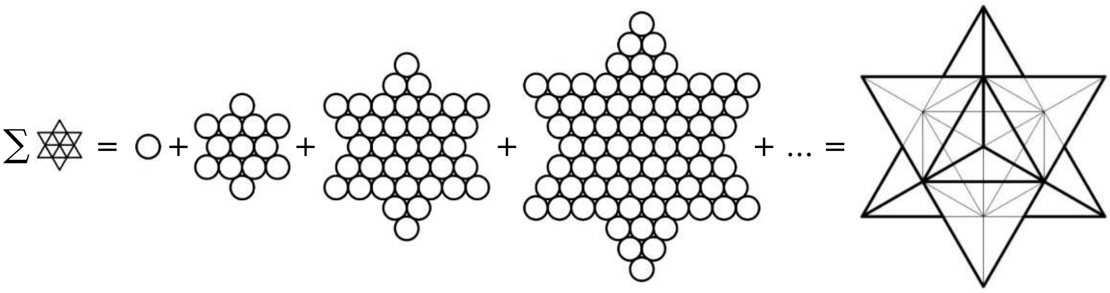
If, on the other hand, 8 tetrahedra are placed on the triangular faces of a central octahedron, a stellated octahedron is formed:

 Stellated Octahedron			$\sum$ = $2n^3+6n^2+5n+1$
v_0=1	v_1=14	v_2=51	A007588 (OEIS-Nr.)
			1, 14, 51, 124, 245, 426, 679, 1016, 1449, 1990, 2651, 3444, 4381, 5474, 6735, 8176, 9809, 11646, 13699, 15980, 18501, 21274, 24311, 27624, 31225, 35126, 39339, 43876,
1	6+8∗1	19+8∗4	Tetrahedral Cells C: 12 (8, 1), Triangles F_S: 24, Core I_1: 0
1	1+13	1+13+37	Cubic Close-Packed Packing (ABC)

Example Layer Sequence Stellated Octahedron of Frequency 2:

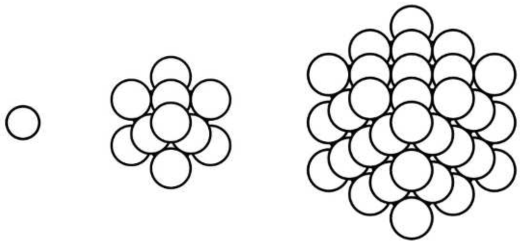


Since this solid consists of two intersecting tetrahedra, it is also called a star tetrahedron. However, I prefer to name star polyhedra after their central polyhedron. And this star polyhedron is formed by placing 8 tetrahedra on the triangular faces of a central octahedron. And as shown in section 3.2, stacking consecutive hexagrams also leads to the sequence of stellated octahedral numbers:

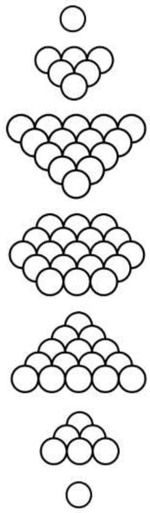
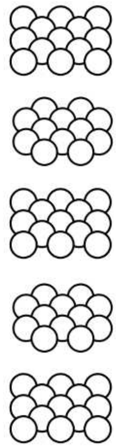


The summation of intersecting triangles leads to intersecting tetrahedra. It is not for nothing that the stellated octahedron is also considered as a three-dimensional hexagram. And the intersections of both figures have 6 vertices each with an octahedron and a hexagon. And yet, I have not found this numerical relationship explicitly published anywhere. In “Figurate Numbers” by Deza and Deza it is mentioned briefly in connection with centered dodecagonal pyramidal numbers [2, p. 140].

The 8 outer vertices of the stellated octahedron form a regular cube. And the unit cell of the stellated octahedron is also the unit cell of the cube in the cubic close-packed packing:

 Close-Packed Cube	$4n^3+6n^2+3n+1$
v_0=1 v_1=14 v_2=63	A050492 (OEIS-Nr.)
	1, 14, 63, 172, 365, 666, 1099, 1688, 2457, 3430, 4631, 6084, 7813, 9842, 12195, 14896, 17969, 21438, 25327, 29660, 34461, 39754, 45563, 51912, 58825, 66326, 74439,
	Tetrahedral Cells C: 24 (8, 1, 12 ½), Triangles F_S: 24 (6 $\boxtimes$), Core I_1: 0
1 2∗5+1∗4 3∗13+2∗12	Cubic Close-Packed Packing (ABC)

Example Layer Sequence Close-Packed Cube of Frequency 2:

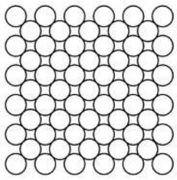



A	1			
B	6		X	13
C	15		Y	12
A	19	or tilted by 54,74°	X	13
B	15		Y	12
C	6		X	13
A	1			63
	63			

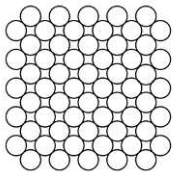
Two intersecting tetrahedra form the unit cell of the cube in the close-packed packing. Thus, the close-packed cube can be formed not only by stacking the two types of centered squares, but also by filling the 12 “edges” of the stellated octahedron with half pyramids or rather small tetrahedra. Due to this solid, the ABC packing is called the cubic close-packed packing or the face-centered cubic packing. Nevertheless, there is not a single reference for the thus most elementary sequence of this packing type in “The On-Line Encyclopedia of Integer Sequences” [8]. Even in the current standard work “Figurate Numbers” by Deza and Deza, the face-centered cube and its sequence cannot be found [2].

I myself became aware of this solid at Buckminster Fuller. In “Synergetics“ the unit cell is shown, and the next larger close-packed cube is described in text as a cuboctahedron with 55 spheres, to each of whose 8 triangular faces another sphere is attached [1, sec. 416.03].

And as already with the triangular numbers, the branded number 666 also appears in the sequence of the close-packed cube:



61

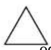
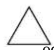
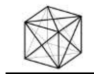


60

$$\begin{aligned} 6*61 &= 366 \\ 5*60 &= \underline{300} \\ &\ 666 \end{aligned}$$

(6 spheres per edge)

Since I happened to know that 365 is 555 in the octal numeral system (positional numeral system with a base of 8), I arrived at the following remarkable table:

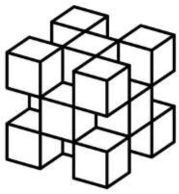
k	Number of (XY-) Layers	 odd		 / k	Base of Numeral System	Number of (XY-) Layers	 odd		 / k
1	1	1	1	1					
2	3	6	14	7	2	11	11+11	111+111	111
3	5	15	63	21	4	11	33	333	111
4	7	28	172	43	6	11	44	444	111
5	9	45	365	73	8	11	55	555	111
6	11	66	666	111	10	11	66	666	111
7	13	91	1099	157	12	11	77	777	111
8	15	120	1688	211	14	11	88	888	111
9	17	153	2457	273	16	11	99	999	111
10	19	190	3430	343	18	11	XX	XXX	111

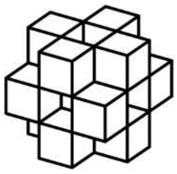
k = Number of Spheres in Outer Edge of Close-Packed Cube

In the positional numeral system with a base of 2, there is no digit 2. And how to fill in the row *k=1* on the right-hand side remains to be investigated. The odd triangular numbers in this table correspond to the largest triangle in the hexagonal layer sequences of the respective close-packed cube.

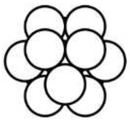
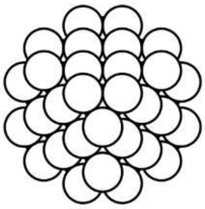
In which positional numeral system does the universe calculate? Computing machines require positional numeral systems whose base is a power of 2. On the other hand, humans prefer to calculate in the decimal system. This highlights the number 666 at the most prominent position in this structure. A structure that is too beautiful not to have any relevance.

Just as centered squares correspond to the black squares of odd-sized “chessboards” (see page 6), close-packed cubes can be compared to odd-sized “chess spaces” or rather a cube clusters made up of small cubes and cube-shaped cavities:

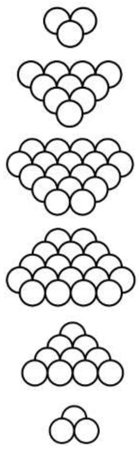




If cubes and cavities are interchanged in the reference object on the left, a close-packed cube without defined corners is formed:

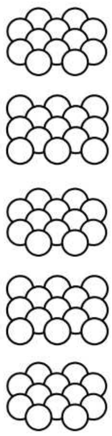
	Close-Packed Cube (−1)		$4n^3+6n^2+3n$
v_0=0	v_1=13	v_2=62	A268201 (OEIS-Nr.)
			0, 13, 62, 171, 364, 665, 1098, 1687, 2456, 3429, 4630, 6083, 7812, 9841, 12194, 14895, 17968, 21437, 25326, 29659, 34460, 39753, 45562, 51911, 58824, 66325, 74438,
			Tetrahedral Cells C: 24 (8, 6, 8 ¼), Triangles F_S: 24 (6), Core I_1: 1
0	2∗4+1∗5	3∗12+2∗13	Cubic Close-Packed Packing (ABC)

Example Layer Sequence Close-Packed Cube (−1) of Frequency 2:



A	3
B	10
C	18
A	18
B	10
C	3
	62

or tilted by 54,74°



X	12
Y	13
X	12
Y	13
X	12
	62

The unit cell of the close-packed cube without defined corners corresponds to the unit cell of the cuboctahedron (see page 19). At each of the 8 corners a ¼-square pyramid is missing. It is formed by interchanging the orthogonal XY-layers of the regular close-packed cube and has only one sphere less. The close-packed cube without defined corners provides the cores of the regular close-packed cube and vice versa.

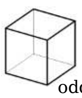
And also with the close-packed cube without defined corners, it is worthwhile to have a look at other positional numeral systems:

k	Number of (XY-) Layers			 / k	Base of Numeral System	Number of (XY-) Layers			 / k
0	1	0	0		1				
1	3	4	13	13	3	10	11	111	111
2	5	12	62	31	5	10	22	222	111
3	7	24	171	57	7	10	33	333	111
4	9	40	364	91	9	10	44	444	111
5	11	60	665	133	11	10	55	555	111
6	13	84	1098	183	13	10	66	666	111
7	15	112	1687	241	15	10	77	777	111
8	17	144	2456	307	17	10	88	888	111
9	19	180	3429	381	19	10	99	999	111

k = Number of Spheres in Outer Edge of Close-Packed Cube (−1)

How to completely fill in row $k=0$ remains to be investigated. The “centered” squares (−1) in this table correspond to the faces of the close-packed cubes (−1).

The two types of Close-packed cubes add up to the odd primitive cubic numbers. When these close-packed cubes are divided by the number of spheres in the outer edge, two sequences are created, which can be found as diagonals in the Ulam spiral [9, p. 43-44]:

s			 odd
1	1	0	1
3	14	13	27
5	63	62	125
7	172	171	343
9	365	364	729
11	666	665	1331
13	1099	1098	2197

s = Number of XY-Layers

196	195	194	193	192	191	190	189	188	187	186	185	184	183
145	144	143	142	141	140	139	138	137	136	135	134	133	182
146	101	100	99	98	97	96	95	94	93	92	91	132	181
147	102	65	64	63	62	61	60	59	58	57	90	131	180
148	103	66	37	36	35	34	33	32	31	56	89	130	179
149	104	67	38	17	16	15	14	13	30	55	88	129	178
150	105	68	39	18	5	4	3	12	29	54	87	128	177
151	106	69	40	19	6	1	2	11	28	53	86	127	176
152	107	70	41	20	7	8	9	10	27	52	85	126	175
153	108	71	42	21	22	23	24	25	26	51	84	125	174
154	109	72	43	44	45	46	47	48	49	50	83	124	173
155	110	73	74	75	76	77	78	79	80	81	82	123	172
156	111	112	113	114	115	116	117	118	119	120	121	122	171
157	158	159	160	161	162	163	164	165	166	167	168	169	170

Ulam Spiral

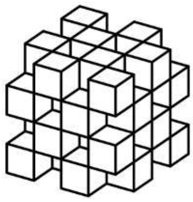
Only the “3” doesn't seem to fit into this scheme (?). Or one arrives at the following absurd conclusion:

$$0/0 = 3$$

In connection with the Pythagorean Lambdoma, “*0/0*” is considered a symbol for the highest deity [4, § 25a]. I will return to this in section 6.

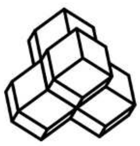
At any rate, for the two types of close-packed cubes shown, there always exists a suitable positional number system, such that the number of spheres of the respective close-packed cube, if it has a core, can always be expressed by three equal digits, where the digit corresponds to the number of spheres in an outer edge.

With an even number of layers, irregular close-packed cubes are formed, in which only 4 corners are occupied with spheres corresponding to an inscribed tetrahedron. The sequence corresponds to half of the even primitive cubic numbers (→ 4, 32, 108, 256, 500, 864, 1372, ...). If these “cubes” are also divided by the number of spheres in an outer edge, the sequence of even square numbers (→ 4, 16, 36, 64, 100, 144, 196, ...) occurs, which is also a diagonal in the Ulam spiral.



All solids in the cubic close-packed packing can be represented as a “chess space” or rather a cube cluster of small cubes and cube-like cavities. The 12 edges of the cube correspond to the contact points of spheres.

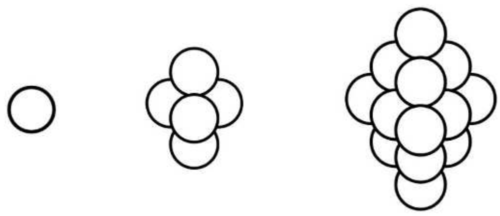
Replacing the cubes with rhombic dodecahedra results in gapless packings without cavities, where the 12 faces of the rhombic dodecahedra correspond to the contact points of the spheres.

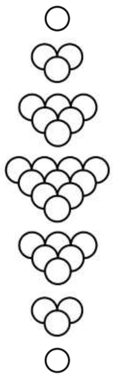
But it is not possible to add a fifth cube to a tetrahedron made up of four small cubes in order to form a triangular bipyramid or a solid with a hexagonal AB AB layer sequence.

3.5 “MIRRORED” CUBIC CLOSE-PACKED PACKING

The “mirrored” cubic close-packed packing represents a special case for which I have only found one regular solid with the triangular bipyramid, consisting of two tetrahedra:

Triangular Bipyramid			$Bi\sum\triangle = \sum\Diamond$ = $\frac{1}{6}(2n^3+9n^2+13n+6)$
v_0=1	v_1=5	v_2=14	A000330 (OEIS-Nr.)
			1, 5, 14, 30, 55, 91, 140, 204, 285, 385, 506, 650, 819, 1015, 1240, 1496, 1785, 2109, 2470, 2870, 3311, 3795, 4324, 4900, 5525, 6201, 6930, 7714, 8555, 9455, 10416, 11440
			Tetrahedral Cells C: 2, Triangels F_S: 6 $\triangle$, Core I_1: 0
1	1+3+1	1+3+6+3+1	“Mirrored” Cubic Close-Packed Packing

Example Layer Sequence Triangular Bipyramid of Frequency 3:



A	1
B	3
C	6
A	10
C	6
B	3
A	1
	30

The triangular bipyramid is composed of two tetrahedra that touch each other on a triangular face. The “mirroring” of the ABC packing gives each consecutive triangular bipyramid its own layer sequence (→ ABA, ABCBA, ABCACBA, ABCABACBA, …). As a result, there are no longer any orthogonal layers for the entire solid. And just as two consecutive triangular numbers result in square numbers, two consecutive tetrahedral numbers lead to square pyramidal numbers. Thus, triangular bipyramid and square pyramid have the same sequence.

The triangular bipyramid plays an important role in the following section.

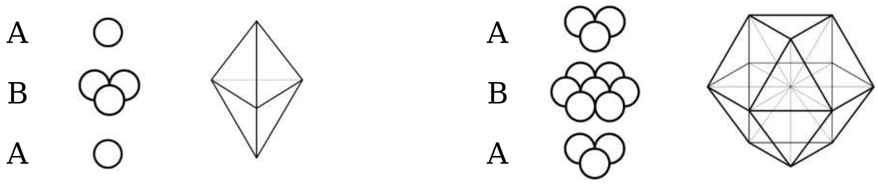
4 FIGURES WITH AN ADDITIONAL DIMENSION

The figures in this section are also solids that are undoubtedly three-dimensional according to conventional definition. However, while all previously shown solids as sphere packings produce only a sequence, the solids treated in this section even generate number planes as sphere packings. In this respect, it seems justified to claim that these figures have an additional hidden dimension.

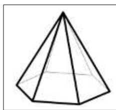
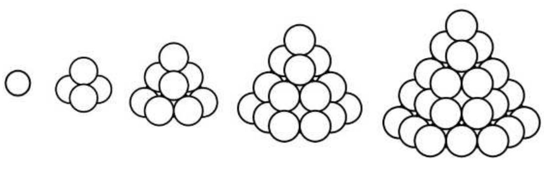
The following shows first in section 4.1 some solids in the hexagonal close-packed packing, and then in section 4.2 their higher levels leading to number planes.

4.1 HEXAGONAL CLOSE-PACKED PACKING (AB AB)

In close-packed packings, inner spheres have 12 neighbors (contacts). In contrast to the cubic close-packed packing shown in Section 3.4, in hexagonal close-packed packing only two of the three possible positions for spheres (A, B, and C) are occupied when stacking hexagonal layers. Position C remains unoccupied throughout, resulting in an AB AB layer sequence. As a consequence, in the hexagonal close-packed packing, both the unit cells of tetrahedra and the unit cells of octahedra touching at their own triangular faces perpendicular to the hexagonal layer sequence. While the cubic close-packed packing is internally built from unit cells of rhombohedra, the hexagonal close-packed packing is internally built from unit cells of triangular bipyramids. The difference can also be demonstrated with a cuboctahedron, where one half is rotated by 60°:

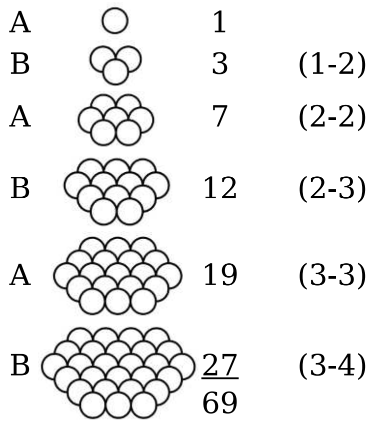


Probably the simplest solid in the hexagonal close-packed packing is the hexagonal pyramid. A primitive hexagonal step pyramid is shown in Section 3.2 (see page 11). In order to create a stable close-packed packing, intermediate layers of irregular hexagons are needed, which are created by enclosing a central triangle of 3 spheres (see page 7):

 Hexagonal Pyramid	regular: ½$(4n^3+9n^2+7n+2)$ irregluar: ½$(4n^3+15n^2+19n+8)$
v_0=1 v_1=4 v_2=11 v_3=23 v_4=42	A019298 (OEIS-Nr.)
	1, 4, 11, 23, 42, 69, 106, 154, 215, 290, 381, 489, 616, 763, 932, 1124, 1341, 1584, 1855, 2155, 2486, 2849, 3246, 3678, 4147, 4654, 5201, 5789, 6420, 7095, 7816, 8584, 9401,
	Tetrahedral Cells C: 1½ , Triangles F_S: 4 , Core I_1: 0
1 1+3 1+3+7 11+12 23+19	Hexagonal Close-Packed Packing (AB AB)

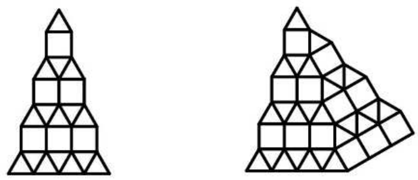
At the tip of the hexagonal pyramid, the sphere packing is still a tetrahedron, but in the layers below it becomes hexagonal, alternating regularly and irregularly.

Example Layer Sequence Hexagonal Pyramid of Frequency 5:



The layers marked “A” contain regular hexagons, and the layers marked “B” contain irregular hexagons. The number of spheres in two adjacent sides of the hexagons is shown in parentheses.

The lateral faces of the hexagonal pyramid have the shape shown below left, with triangles and squares alternating from top to bottom. Since the tip of the hexagonal pyramid is still tetrahedral, two adjacent lateral faces meet as shown below right, always offset by one layer. And two adjacent lateral faces of the hexagonal pyramid add up to triangular numbers.



Due to the alternation of triangles and squares, the lateral faces do not lie exactly in one plane. In the interior of the hexagonal pyramid it behaves accordingly. In the hexagonal close-packed packing there are no consistently orthogonal layers.

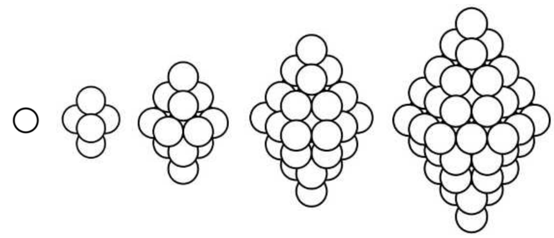
Since the tip of the hexagonal pyramid is not yet hexagonal as a sphere packing, the volume cannot be read directly from the unit cell as in the ABC packing. It is already questionable what the unit cell actually is. Also, the volume growth of each layer does not seem to correspond to the hexagonal numbers, but rather to every 3rd triangular number (→ 1, 10, 28, 55, 91, …). However, if half a tetrahedron volume is added to each layer, the calculation works again:

1½, 10½ (7∗1½), 28½ (19∗ 1½), 55½ (37∗ 1½), …

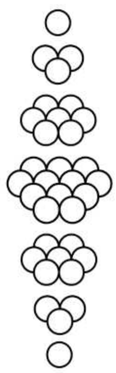
In this paper, the hexagonal pyramidal numbers are not an arithmetic sequence for the first time. Since the lowest difference sequence alternates (→ 1, 2, 1, 2, …), there is no formula that can be represented as a polynomial for the entire solid. There are different formulas for regular and irregular hexagonal pyramids, whereby this differentiation is based on the lowest layer. Only the formula for the regular hexagonal pyramid corresponds to the generalized formula of Teo and Sloane [10] (see page 9), provided that a tetrahedron volume of 12 (1½+10½) is assumed for the regular unit cell (11 spheres) (see also section 5.4).

In the book “Figurate Numbers” by Deza and Deza, this hexagonal pyramid cannot be found [2].

If two consecutive hexagonal pyramids are combined to form a hexagonal bipyramid, an arithmetic sequence is created again with a formula that can be represented as a polynomial:

 Hexagonal Bipyramid	$\sum \triangle = \frac{1}{2}(n^3+3n^2+4n+2)$
v_0=1 v_1=5 v_2=15 v_3=34 v_4=65	A006003 (OEIS-Nr.)
	1, 5, 15, 34, 65, 111, 175, 260, 369, 505, 671, 870, 1105, 1379, 1695, 2056, 2465, 2925, 3439, 4010, 4641, 5335, 6095, 6924, 7825, 8801, 9855, 10990, 12209,13515,
	Tetrahedral Cells C: 3, Triangles F_S: 6, Core I_1: 0
1 4+1 11+4 23+11 42+23	Hexagonal Close-Packed Packing (AB AB)

Example Layer Sequence Hexagonal Bipyramid of Frequency 3:



A	1	
B	3	(1-2)
A	7	(2-2)
B	12	(2-3)
A	7	(2-2)
B	3	(1-2)
A	1	
	34	

The sequence of hexagonal bipyramidal numbers also arises from the summation of centered triangular numbers and is already shown in section 3.2 for the primitive hexagonal centered triangular step pyramid (see page 11). The hexagonal bipyramidal numbers are also the magic sums of so-called magic squares. And just as “5” is only the theoretical magic sum of a non-existent magic square of the form 2∗2, here “5” is also not yet a hexagonal bipyramid either, but a triangular bipyramid. Thus, an exact match.

This sequence has no name in „The On-Line Encyclopedia of Integer Sequences“. Although there are 28 comments on this sequence, there is no reference to the hexagonal pyramid [8]. And I have not found the following published anywhere before.

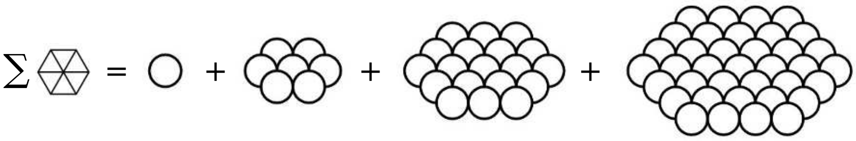
The summation of hexagonal bipyramids leads to the sequence of the triangular numbers of the triangular numbers, i.e. to the 1st, 3rd, 6th, 10th, 15th, ... triangular number:

$$\sum = 1 + 5 + 15 + 34 + 65 + \ldots = 1, 6, 21, 55, 120, \ldots =$$

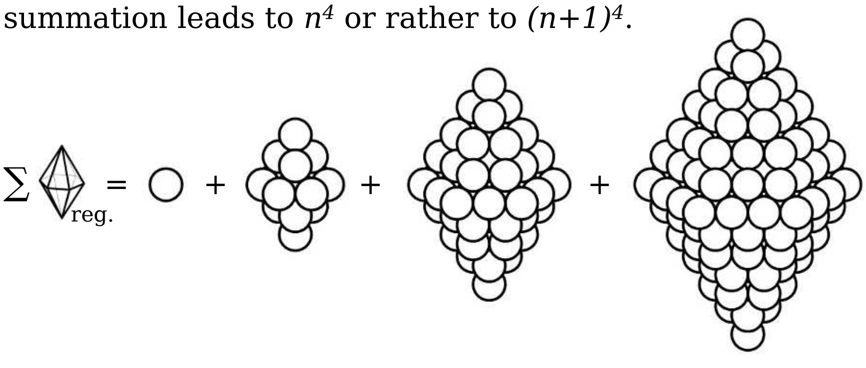
The summation of regular hexagonal bipyramids leads to the sequence of the square numbers of the square numbers, i.e. to the 1st, 4th,9th, 16th, 25th, ... square number:

$$\sum_{\text{reg.}} = 1 + 15 + 65 + 175 + \ldots = 1, 16, 81, 256, \ldots =$$

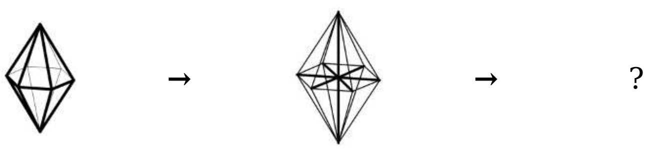
By regular are meant those hexagonal bipyramids whose central layer is a regular centered hexagon. These regular hexagonal bipyramids correspond to the rhombic dodecahedral numbers (see page 13). Thus, the summation of the regular hexagonal bipyramidal numbers also leads to n^4 or rather to $(n+1)^4$. To gain at least a quantitative understanding of the 4th dimension, the following comparison may be helpful:


$+ \ldots = (n+1)^3$

The summation of consecutive centered hexagons leads to n^3 or rather to $(n+1)^3$. If these hexagons are now "pyramidized" in both directions, their summation leads to n^4 or rather to $(n+1)^4$.


$+ \ldots = (n+1)^4$

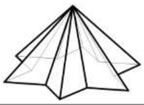
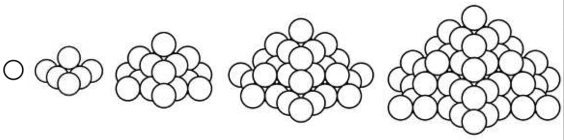
Just as the hexagon is the two-dimensional projection of the three-dimensional cube (see page 10), the regular hexagonal bipyramid appears to be the three-dimensional projection of a four-dimensional “cube”.



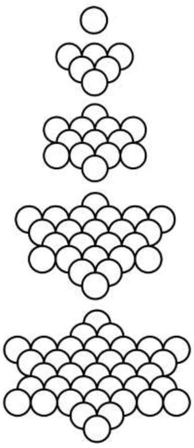
4 Axes

The central layers of the hexagonal bipyramid form the familiar 3rd dimension. A possible higher-dimensional reality cannot be perceived by the human sensory apparatus.

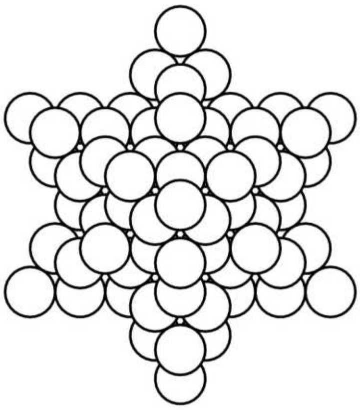
Hexagrams can also be stacked to form a stable close-packed packing. A primitive hexagonal hexagrammic step pyramid is shown in Section 3.2 (see page 12). In order to achieve a close-packed packing, intermediate layers of irregular hexagrams made up of two intersecting consecutive triangles (see page 8) are needed:

 Hexagrammic Pyramid	regular: $4n^3+9n^2+6n+1$ irregular: $4n^3+15n^2+18n+7$
v_0=1 v_2=20 v_3=44 v_4=81	A011934 (OEIS-Nr.)
v_1=7 	1, 7, 20, 44, 81, 135, 208, 304, 425, 575, 756, 972, 1225, 1519, 1856, 2240, 2673, 3159, 3700, 4300, 4961, 5687, 6460, 7344, 8281, 9295, 10388, 11564, 12825, 14175,
	Tetrahedral Cells C: 3, Triangles F_S: 10, Core I_1: 0
1 1+6 1+6+13 20+24 44+37	Hexagonal Close-Packed Packing (AB AB)

Example Layer Sequence Hexagrammic Pyramid Frequency 4:



A	1	
B	6	(2-3)
A	13	(4-4)
B	24	(5-6)
A	37	(7-7)
	81	



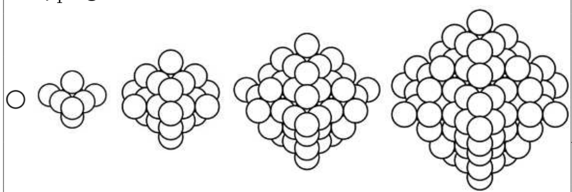
The regular hexagrams are located in the layers marked „A“, while the irregular ones are in the layers marked „B“. The number of spheres in the edges of the two intersecting triangles is given in parentheses. A top view is shown on the right.

The tip of the hexagrammic pyramid corresponds to a corner of a close-packed cube (see page 24) and is not yet hexagrammic as a sphere packing. The layers below then contain hexagrams, alternating regularly and irregularly. The distinction between regular and irregular hexagrammic pyramids is again made based on the lowest layer.

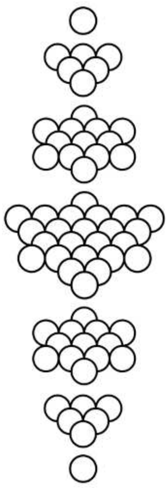
And as with the hexagonal pyramid, two different formulas are also needed here, because for the entire solid the lowest difference sequence alternates again (→ 4, 2, 4, 2, …). The formula for the regular hexagrammic pyramid corresponds to the generalized formula of Teo and Sloane [10], if a tetrahedron volume of 24 and 36 triangular faces in the surface were assumed for the regular unit cell (20 spheres). However, volume determination and volume growth are still not fully understood and need to be investigated. The formula for the irregular solid does not correspond to the generalized formula.

The sequence of the hexagrammic pyramid is found in “The On-Line Encyclopedia of Integer Sequences“, but without a name and without reference to this solid [8]. I have also not found an hexagrammic pyramid published anywhere yet.

Nevertheless, I am showing this solid, because when two consecutive hexagrammic pyramids are combined to form an hexagrammic bipyramid, a prominent sequence is created:

 Hexagrammic Bipyramid	$\sum$  $= n^3+3n^2+3n+1$
v_0=1 v_2=27 v_3=64 v_4=125	A000578 (OEIS-Nr.)
v_1=8 	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, 1728, 2197, 2744, 3375, 4096, 4913, 5832, 6859, 8000, 9261, 10648, 12167, 13824, 15625, 17576, 19683, 21952, 24389,
	Tetrahedral Cells C: 6, Triangles F_S: 12, Core I_1: 0
1 7+1 20+7 44+20 81+44	Hexagonal Close-Packed Packing (AB AB)

Example Layer Sequence Hexagrammic Bipyramid of Frequency 3:



A	1	
B	6	(2-3)
A	13	(4-4)
B	24	(5-6)
A	13	(4-4)
B	6	(2-3)
A	1	
	64	

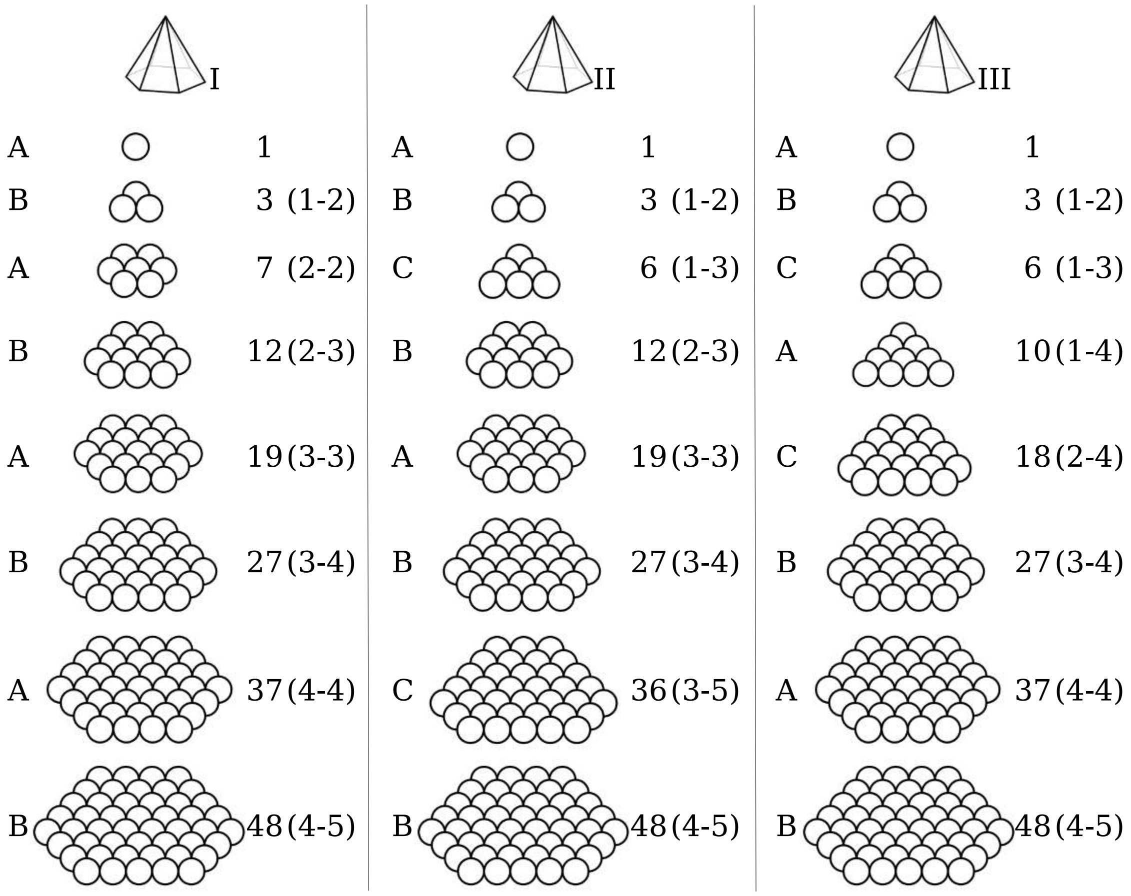
The hexagrammic bipyramidal numbers correspond to the primitive cubic numbers, since two consecutive layers always add up to centered hexagonal numbers. The irregular unit cell (8 spheres) is depicted in another paper by Teo and Sloane and is referred to as “edge-capped triangular bipyramid” [11, p. 2317].

4.2 HIGHER LEVELS OF THE HEXAGONAL CLOSE-PACKED PACKING

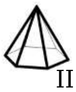
The solids shown in the hexagonal close-packed packing are composed of triangular bipyramids in their interiors. Since each of the consecutive triangular bipyramids (→ 5, 14, 30, 55, …) generates its own layer sequence (→ ABA, ABCBA, ABCACBA, ABCABACBA, …), and the solids shown in section 4.1 can be constructed from each of these increasingly larger triangular bipyramids, these solids not only produce a sequence, but even a plane of numbers.

The solids in the ABC packing could also be constructed with increasingly larger tetrahedra or rhombohedra. The difference is that this would only produce solids and numbers that are already present in the sequence. This is different for the solids constructed from triangular bipyramids. Previously non-existent variations and numbers are created this way.

Layer Sequences Hexagonal Pyramid, Levels I, II, and III:



The following table shows for levels I, II, and III the individual layers as well as the total number of spheres in the corresponding hexagonal pyramids and hexagonal bipyramids:

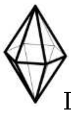
 I				 I	 II				 II	 III				 III
A	1		1	1	A	1		1	1	A	1		1	1
B	3	(1-2)	4	5	B	3	(1-2)	4		B	3	(1-2)	4	
A	7	(2-2)	11	15	C	6	(1-3)	10	14	C	6	(1-3)	10	
B	12	(2-3)	23	34	B	12	(2-3)	22		A	10	(1-4)	20	30
A	19	(3-3)	42	65	A	19	(3-3)	41	63	C	18	(2-4)	38	
B	27	(3-4)	69	111	B	27	(3-4)	68		B	27	(3-4)	65	
A	37	(4-4)	106	175	C	36	(3-5)	104	172	A	37	(4-4)	102	167
B	48	(4-5)	154	260	B	48	(4-5)	152		B	48	(4-5)	150	
A	61	(5-5)	215	369	A	61	(5-5)	213	365	C	60	(4-6)	210	
B	75	(5-6)	290	505	B	75	(5-6)	288		A	73	(4-7)	283	493
A	91	(6-6)	381	671	C	90	(5-7)	378	666	C	90	(5-7)	373	
B	108	(6-7)	489	870	B	108	(6-7)	486		B	108	(6-7)	481	
A	127	(7-7)	616	1105	A	127	(7-7)	613	1099	A	127	(7-7)	608	1089
B	147	(7-8)	763	1379	B	147	(7-8)	760		B	147	(7-8)	755	
A	169	(8-8)	932	1695	C	168	(7-9)	928	1688	C	168	(7-9)	923	
B	192	(8-9)	1124	2056	B	192	(8-9)	1120		A	190	(7-10)	1113	2036
A	217	(9-9)	1341	2465	A	217	(9-9)	1337	2457	C	216	(8-10)	1329	
B	243	(9-10)	1584	2925	B	243	(9-10)	1580		B	243	(9-10)	1572	
A	271	(10-10)	1855	3439	C	270	(9-11)	1850	3430	A	271	(10-10)	1843	3415

For the hexagonal pyramids of level II, always two layers are needed until the next complete specimen is reached (→ 1, 10, 41, 104, 213, …), for level III always three layers (→ 1, 20, 102, 283, 608, …), etc.

The sequences of the higher levels of the hexagonal pyramid are not found in “The On-Line Encyclopedia of Integer Sequences” (OEIS) [8] and do not seem to be published anywhere else. The same applies to the sequences of the higher levels of the hexagonal bipyramid, with one exception. Surprisingly, level II exactly corresponds to the sequence of the regular close-packed cube (see page 24). This is the sequence with the branded number 666, for which there is not a single reference in the OEIS.

In the following, I limit myself to the hexagonal bipyramids that seem most relevant to me.

The following table shows the number plane of the hexagonal bipyramid:

n	 I	 II	 III	 IV	 V	 VI	 VII	 VIII	 IX
0	1	1	1	1	1	1	1	1	1
1	5	14	30	55	91	140	204	285	385
2	15	63	167	349	631	1035	1583	2297	3199
3	34	172	493	1075	1996	3334	5167	7573	10630
4	65	365	1089	2425	4561	7685	11985	17649	24865
5	111	666	2036	4591	8701	14736	23066	34061	48091
6	175	1099	3415	7765	14791	25135	39439	58345	82495
7	260	1688	5307	12139	23206	39530	62133	92037	130264
8	369	2457	7793	17905	34321	58569	92177	136673	193585
9	505	3430	10954	25255	48511	82900	130600	193789	274645
10	671	4631	14871	34381	66151	113171	178431	264921	375631
11	870	6084	19625	45475	87616	150030	236699	351605	498730
12	1105	7813	25297	58729	113281	194125	306433	455377	646129

In row *n=1*, there are the triangular bipyramidal or rather square pyramidal numbers (see pages 15 and 29). All horizontal sequences below them are also searched in vain in “The On-Line Encyclopedia of Integer Sequences“ [8].

In this number plane, both horizontally and vertically sequences are arithmetic sequences:

	Formulas horizontal		Formulas vertical
1	$\frac{1}{6}(2n^3+9n^2+13n+6)$	I	$\frac{1}{2}(n^3+3n^2+4n+2)$
2	$\frac{1}{6}(22n^3+36n^2+26n+6)$	II	$\frac{1}{2}(8n^3+12n^2+6n+2)$
3	$\frac{1}{6}(78n^3+81n^2+39n+6)$	III	$\frac{1}{2}(27n^3+27n^2+4n+2)$
4	$\frac{1}{6}(188n^3+144n^2+52n+6)$	IV	$\frac{1}{2}(64n^3+48n^2-4n+2)$
5	$\frac{1}{6}(370n^3+225n^2+65n+6)$	V	$\frac{1}{2}(125n^3+75n^2-20n+2)$
6	$\frac{1}{6}(642n^3+324n^2+78n+6)$	VI	$\frac{1}{2}(216n^3+108n^2-46n+2)$
7	$\frac{1}{6}(1022n^3+441n^2+91n+6)$	VII	$\frac{1}{2}(343n^3+147n^2-84n+2)$
8	$\frac{1}{6}(1528n^3+576n^2+104n+6)$	VIII	$\frac{1}{2}(512n^3+192n^2-136n+2)$
9	$\frac{1}{6}(2178n^3+729n^2+117n+6)$	IX	$\frac{1}{2}(729n^3+243n^2-204n+2)$
10	$\frac{1}{6}(2990n^3+900n^2+130n+6)$	X	$\frac{1}{2}(1000n^3+300n^2-290n+2)$
11	$\frac{1}{6}(3982n^3+1089n^2+143n+6)$	XI	$\frac{1}{2}(1331n^3+363n^2-396n+2)$
12	$\frac{1}{6}(5172n^3+1296n^2+156n+6)$	XII	$\frac{1}{2}(1728n^3+432n^2-524n+2)$

The formulas of the horizontal sequences all correspond to the generalized formula of Teo and Sloane [10]. The formulas of the vertical sequences differ because the unit cells (row *n=1*) are not yet hexagonal bipyramids, but triangular bipyramids (see page 31). The coefficients of n^3 here correspond to the tetrahedron volume of the triangular bipyramid divided by 4. The coefficients of n^2 match in the horizontal and vertical formulas when they are put on the same denominator. It is striking that the coefficients of *n* in the vertical formulas are negative from level IV onwards: 2, 3, 2, −2, −10, −23, −42, … (→ $-1/6n^3+13/6n$).

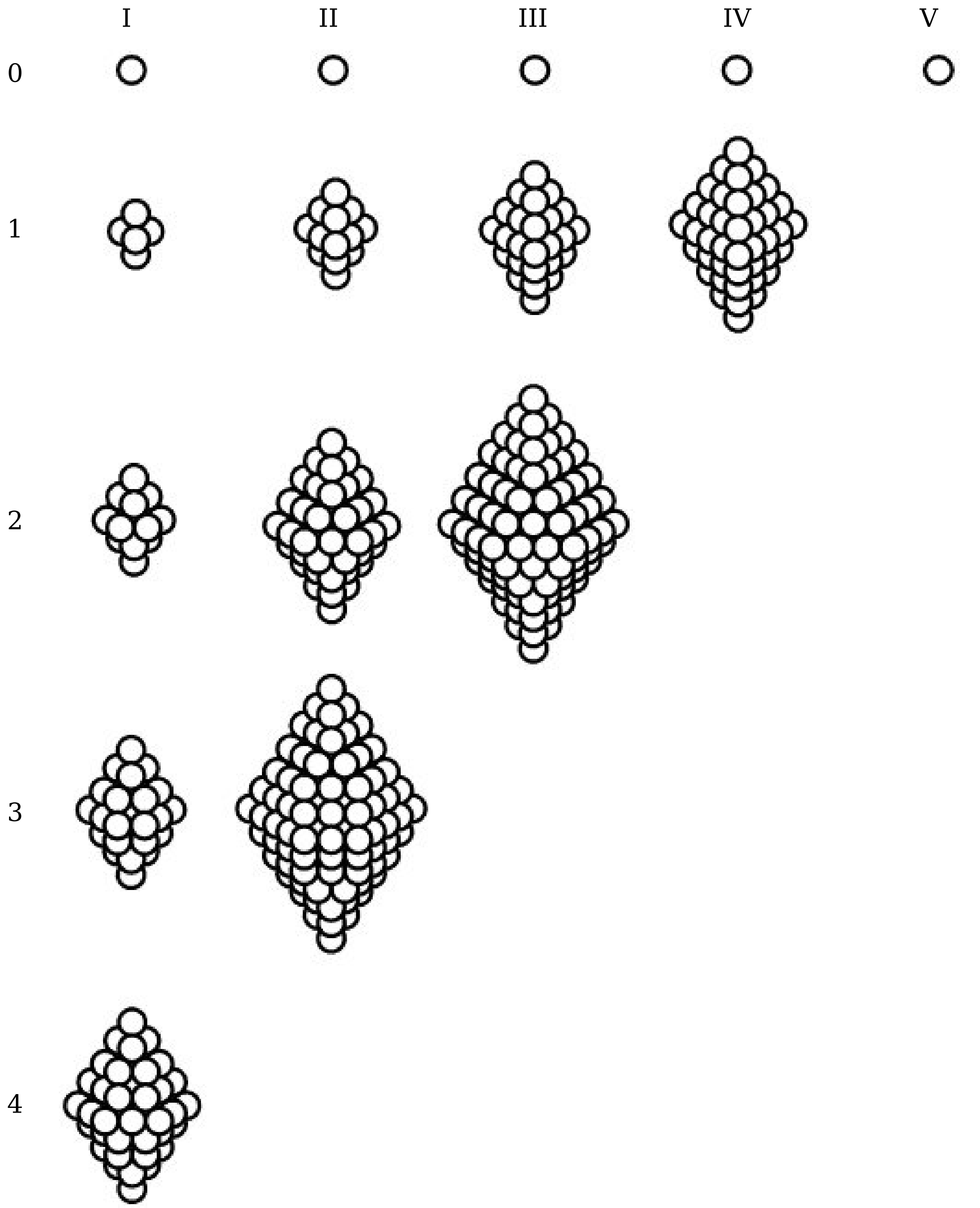
The following table is sorted by the number of layers *s*:

s	I	II *Δ1*	III *Δ4*	IV *Δ10*	V *Δ20*	VI *Δ35*	VII *Δ56*	VIII *Δ84*	IX *Δ120*	X *Δ165*	XI *Δ220*	XII *Δ286*	XIII *Δ364*
1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	5												
5	15 1^3	14											
7	34		30										
9	65	63	2^3	55									
11	111				91								
13	175	172	167	3^3		140							
15	260						204						
17	369	365		349		4^3		285					
19	505		493						385				
21	671	666			631		5^3			506			
23	870										650		
25	1105	1099	1089	1075		1035			6^3			819	
27	1379												1015
29	1695	1688					1583			7^3			
31	2056		2036		1996								
33	2465	2457		2425				2297				8^3	

Compared to level I, the following higher levels decrease according to multiples of increasing tetrahedral numbers (*Δ1∗n, Δ4∗n, Δ10∗n, Δ20∗n, Δ35∗n*, …). And the primitive cubic numbers n^3 also appear as distances in this table.

The hexagonal bipyramid and all other solids in the hexagonal close-packed packing increase in size in two directions, thus having one more direction or rather dimension than solids in other types of sphere packings.

The following excerpt shows the growth of the hexagonal bipyramid in these two directions:



The further right one goes in the number plane of the hexagonal bipyramid, the larger the sections with ABC layer sequences become. In this sense, this number plane can also be understood as a superordinate structure into which the cubic close-packed packing is embedded.

The hexagrammic pyramid and the hexagrammic bipyramid can also be built from increasingly larger triangular bipyramids and thus also form number planes. Their tables are shown in Appendix III.

5 FURTHER EXPLORATIONS

In this section, further aspects of sphere packings will be examined, which will lead to interesting insights here and there.

5.1 OUTERMOST SHELLS

The sequences for the number of spheres in the outermost shell or the outer boundary of the regular geometric figures can be easily determined through simple generalized formulas.

A row of spheres is bounded at its two ends. Therefore, for the number of outer spheres S_n in one dimension

$$S_n = 2$$

where $n \geq 1$, since $n = 0$ leads to a single sphere.

For regular two-dimensional figures, Teo and Sloane give the following generalized formula for the number of outer spheres S_n [10, p. 4547]

$$S_n = pn$$

where p corresponds to the number of sides of the polygon and again $n \geq 1$ holds, since $n = 0$ also leads to a single sphere in two dimensions.

For the two-dimensional figures shown in Section 2, whose number sequences do not start with “1”, v_0 (number of spheres for $n = 0$) must be added. The formula then applies for $n \geq 0$ and is

$$S_n = pn + v_0$$

For the first two dimensions, these formulas may be sufficient. (Sequences can be found in the table in Appendix I.)

The formulas for the outer shells of the tetrahedron, octahedron, primitive cube, cuboctahedron, and rhombic dodecahedron can be found in “Synergetics” by Buckminster Fuller [1, sec. 415.03]. Since these can be easily generalized, I already knew the following generalized formula for the number of spheres in outer shells S_n in spatial figures before finding it in the paper by Teo and Sloane [10, p. 4548]

$$S_n = \beta n^2 + 2$$

where

$$\beta = \tfrac{1}{2} F_s$$

Here, F_s again represents the number of triangular faces into which the surface of the respective solid can be decomposed. And again, the formula only applies for $n \geq 1$, since $n = 0$ also usually leads to a single sphere in three dimensions.

The factor β also arises by simply subtracting two spheres from the outer shell spheres of the respective unit cell.

This gives the following formulas for the number of spheres in the outer shells of spatial figures:

	Tetrahedron	4 △	$2n^2 + 2$
	Pyramid	4 △, 1 □	$3n^2 + 2$
	Triangular Bipyramid	6 △	$3n^2 + 2$
	Hexagonal Bipyramid	6 △	$3n^2 + 2$
	Octahedron	8 △	$4n^2 + 2$
	Primitive Cube	6 □	$6n^2 + 2$
	Body-Centered Cube	6 □	$6n^2 + 2$
	Rhombohedron	6 ⟠	$6n^2 + 2$
	Hexagrammic Bipyramid	12 △	$6n^2 + 2$
 reg.	Hexagonal Pyramid (regular)	12 △, 3 □	$9n^2 + 2$
	Coboctahedron	8 △, 6 □	$10n^2 + 2$
	Rhombic Dodecahedron	12 ⟠	$12n^2 + 2$
	Stellated Octahedron	24 △	$12n^2 + 2$
 reg.	Close-Packed Cube (regular)	6 ⊠	$12n^2 + 2$
	Truncated Tetrahedron	28 △	$14n^2 + 2$
 reg.	Hexagrammic Pyramid (regular)	36 △	$18n^2 + 2$
	Truncated Octahedron	48 △, 6 □	$30n^2 + 2$

Squares can be divided into two triangles, centered squares into 4 triangles. In the hexagonal bipyramid and the hegagrammic bipyramid, two adjacent lateral faces form a triangular number.

The generalized formula does not work for irregular figures. Different formulas are needed for hexagonal and hexagrammic pyramids whose lowest layer is an irregular hexagon or hexagram, as well as for the irregular close-packed cubes shown in Section 3.4 with an even number of layers or without defined corners. The following formulas are valid for $n \geq 0$:

 irr.	Hexagonal Pyramid (irregular)	4 △	$9n^2 + 9n + 4$
 irr.	Hexagrammic Pyramid (irregular)	10 △	$18n^2 + 18n + 7$
 irr.	Close-Packed Cube (irregular)	6 ⊠	$12n^2 + 12n + 4$
	Close-Packed Cube (−1)	6 ⧈	$12n^2 + 0$

Buckminster Fuller interprets the “2” in the outer shell formulas as poles of an axis [1, sec. 223.11]. A closer examination of the different close-packed cubes supports this interpretation.

The various outer shell formulas for the three variants of the close-packed cube can be combined into the single formula

$$3n^2 + P$$

where P is the number of poles and n ≥ 1. The coefficient of n^2 is reduced from *12* to *3* because not only cubes with an odd number of layers are considered.

If a space diagonal is assumed as the axis, then for the regular close-packed cube, $P = 2$, since all 8 corners are occupied by spheres. For the close-packed cube without defined corners, $P = 0$. And for the irregular close-packed cube with an even number of layers, $P = 1$, since with only 4 developed corners only one end of the space diagonal is occupied by a sphere.

Thus, the “2” in the outer shell formulas can actually be interpreted as the two poles of an axis.

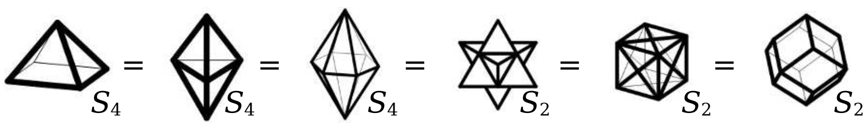
Outer shells create hollow spatial figures. And there is a regular solid, the icosahedron, that can be built stable from equal-sized spheres, but only as a hollow spatial figure:

	Icosahedron	20△	$10n^2 + 2$

Again, the formula only applies for $n \geq 1$. The sequence of the icosahedron corresponds to the outer shells of the cuboctahedron (→ 1, 12, 42, 92, …). In contrast to the cuboctahedron, there is not enough space inside the icosahedron for another equal-sized sphere.

From the list on the previous page it can be seen that different solids often have the same number of spheres in their outer shells (see also table in Appendix II):

e.g. 50 spheres in the outer shell:



Cores: 5 5 15 1 13 15

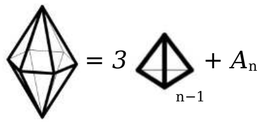
5.2 AXES

The existence of polar spheres leads to the question whether the axis spheres of a solid are also significant.

Therefore, in the following, the polar spheres of a solid are connected by a line and the axis spheres lying on this line A_n are subtracted from the total number of spheres G_n of the solid:

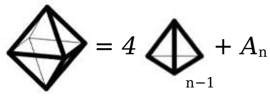
G_n	5	15	34	65	111	175	...
A_n	2	3	4	5	6	7	...
$G_n - A_n$	3	12	30	60	105	168	...
$3*$	1	4	10	20	35	56	...

Hexagonal Bipyramid



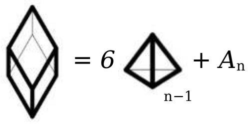
G_n	6	19	44	85	146	231	...
A_n	2	3	4	5	6	7	...
$G_n - A_n$	4	16	40	80	140	224	...
$4*$	1	4	10	20	35	56	...

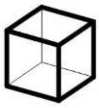
Octahedron



G_n	8	27	64	125	216	343	...
A_n	2	3	4	5	6	7	...
$G_n - A_n$	6	24	60	120	210	336	...
$6*$	1	4	10	20	35	56	...

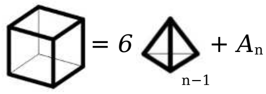
Rhombohedron





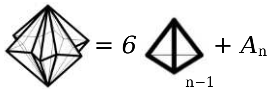
G_n	8	27	64	125	216	343	...
A_n	2	3	4	5	6	7	...
$G_n - A_n$	6	24	60	120	210	336	...
$6*$	1	4	10	20	35	56	...

Primitive Cube



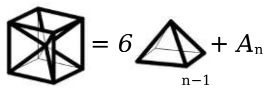
G_n	8	27	64	125	216	343	...
A_n	2	3	4	5	6	7	...
$G_n - A_n$	6	24	60	120	210	336	...
$6*$	1	4	10	20	35	56	...

Hexagrammic Bipyramid



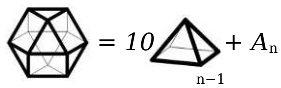
G_n	9	35	91	189	341	559	...
A_n	3	5	7	9	11	13	...
$G_n - A_n$	6	30	84	180	330	546	...
$6*$	1	5	14	30	55	91	...

Body-Centered Cube



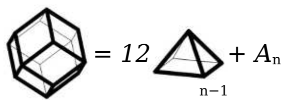
G_n	13	55	147	309	561	923	...
A_n	3	5	7	9	11	13	...
$G_n - A_n$	10	50	140	300	550	910	...
$10*$	1	5	14	30	55	91	...

Cuboctahedron



G_n	15	65	175	369	671	1105	...
A_n	3	5	7	9	11	13	...
$G_n - A_n$	12	60	168	360	660	1092	...
$12*$	1	5	14	30	55	91	...

Rhombic Dodecahedron




reg.

G_n	14	63	172	365	666	1099	...	Close-Packed Cube (reg.)
A_n	2	3	4	5	6	7	...	
$G_n - A_n$	12	60	168	360	660	1092	...	
12∗	1	5	14	30	55	91	...	


reg. $= 12$  $+ A_n$
n−1

G_n	14	51	124	245	426	679	...	Stellated Octahedron
A_n	2	3	4	5	6	7	...	
$G_n - A_n$	12	48	120	240	420	672	...	
12∗	1	4	10	20	35	56	...	

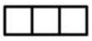

$= 12$  $+ A_n$
n−1

If only the regular hexagonal bipyramids are considered, the factor of *3* increases to *12*, and tetrahedrons become pyramids. In the case of the close-packed cube, it is exactly the opposite if the irregular ones with an even number of layers are also taken into account.

Multiples of Pyramids seem to remain after subtracting the axis spheres when the unit cell of the respective solid has a core of one sphere, as in the case of the cuboctahedron, body-centered cube and rhombic dodecahedron, or when only every other or rather the regular solids are considered, as in the case of the close-packed cube.

The factors of the remaining tetrahedral or pyramidal numbers after subtracting the axis spheres correspond to the coefficients in the outer shell formulas (see page 42). In addition, the factors, together with the tetrahedron or pyramid, always exactly reflect the tetrahedron volume (or the number of tetrahedral cells) of the respective unit cell, since a pyramid has a volume of two tetrahedra (or can be divided into two tetrahedral cells).

The subtraction of axis spheres, however, does not result in multiples of tetrahedral or pyramidal numbers for the following solids: tetrahedron, square pyramid, hexagonal pyramid, hexagrammic pyramid, triangular bipyramid, truncated tetrahedron, truncated octahedron. With the exception of the triangular bipyramid, there are no axes in the sphere arrangement of these solids that connect two vertices and the center of the respective solid.

In the truncated octahedron, although there is no axis going through the center of the sphere arrangement, there is a bar made up of three squares (). Subtracting this "barred axis" BA_n from the total number of spheres G_n leads to the following:

G_n	38	201	586	1289	2406	4033	...	Abgestumpftes Oktaeder
BA_n	8	21	40	65	96	133	...	
$G_n - BA_n$	30	180	546	1224	2310	3900	...	
6∗	5	30	91	204	385	650	...	


$= 6$  $+ BA_n$
2n−1

5.3 PYRAMIDAL CAPS

At the various pyramids, it is worth taking a look at “caps”. By caps I mean hollow pyramids that are open at the bottom, i.e. quasi outer shells without a bottom.

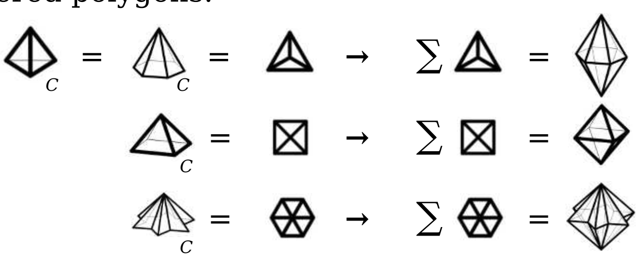
These caps are formed by stacking only the outer spheres (see page 41) of the layers that make up the pyramids. The sequences are also result from subtracting those spheres of the lowest layer which do not lie in the plane of the lateral faces from the spheres in the outer shells.

Thus, the following sequences are found for the pyramidal caps C_n:

C	Tetrahedron	1	4	10	19	31	46	64	85	109
C	Hexagonal Pyramid	1	4	10	19	31	46	64	85	109
C	Pyramid	1	5	13	25	41	61	85	113	145
C	Hexagrammic Pyramid	1	7	19	37	61	91	127	169	217

The caps of the hexagonal pyramid correspond to the caps of the tetrahedron, since two adjacent lateral faces of the hexagonal pyramid form a triangular number (see page 31). Likewise, the lateral faces of the hexagrammic pyramid lying between two outer corner points form a triangular number.

It is noticeable that the sequences of the pyramidal caps correspond to those of the centered polygons:



The generalized formulas for the total number of spheres in regular polygons as well as for the number of spheres in outer shells of solids are shown on pages 3 and 41. Since for centered polygons the number of triangular faces F and the number of sides p are the same, and two consecutive pyramidal caps lead to the outer shells of the corresponding bipyramids, it is proven by

$$\tfrac{1}{2}pn^2 + \tfrac{1}{2}pn + 1 + \tfrac{1}{2}p(n-1)^2 + \tfrac{1}{2}p(n-1) + 1 = pn^2 + 2$$

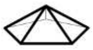
that the sequences of pyramidal caps and corresponding centered polygons coincide.

Surprisingly, it turns out that the summation of consecutive tetrahedral caps leads to hexagonal bipyramids and not, as one would expect, to triangular bipyramids. And here the full justification for the hexagrammic pyramid also becomes evident. Its caps demonstrate how the centered hexagon can be raised up out of the plane by folding in its sides. Whether this is also possible for other centered polygons with a number of sides $p > 6$ remains to be investigated.

In connection with caps and outer shells, the pentagon is mentioned for the first time in this paper.

While triangles, squares, and hexagons fill the plane without gaps, and can be stacked with spheres to form close-packed pyramids, neither can be done with pentagons.

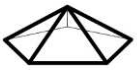
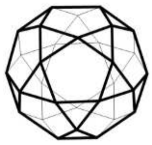
Only caps, hollow pyramids without a bottom, can be formed from pentagons with spheres:

	Pentagonal Cap	1	6	16	31	51	76	106	141	181

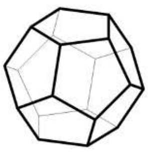
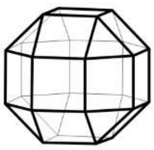
The number sequence of the pentagonal caps corresponds to that of a centered pentagon. Therefore, the formula is: $\frac{1}{2}(5n^2+5n+2)$

All spatial figures with pentagonal faces are not stable with spheres (see page 4). However, these figures become stable when the pentagons are “pyramidized” into caps, which turns the respective figures into other figures. But that's exactly what caps are useful for. They stabilize outer shells or rather hollow spatial figures.

On page 43, the icosahedron is briefly introduced. It consists of overlapping pentagonal caps. However, an icosidodecahedron (below right) can only be built stably with spheres if a cap or rather another sphere is placed on each of the 12 pentagonal faces, resulting in an icosahedron made up of 42 spheres.

The pentagon dodecahedron (below left) or the rhombicuboctahedron (below center) are formed into star polyhedra when the 12 pentagonal or the 18 square faces are stabilized by placing caps on them. As an example, the star polyhedron of the rhombicuboctahedron formed from 42 steel balls and 120 magnetic rods is shown below right.

With the snub cube, the snub dodecahedron, the truncated icosahedron, and the rhombicosidodecahedron, further archimedean solids can be built with spheres as stable hollow figures by "pyramidizing" the square and/or pentagonal faces.

5.4 DIFFERENCE SEQUENCES

The following demonstrates the significance of the lowest difference sequences of the regular geometric figures shown in this paper.

Difference Sequences for Polygons and Caps:


```
1  3  6  10  15  21
 2  3  4  5  6
  1  1  1  1
```

```
1  4  9  16  25  36
 3  5  7  9  11
  2  2  2  2
```

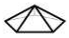
```
1  4  10  19  31  46
 3  6  9  12  15
  3  3  3  3
```

```
1  5  13  25  41  61
 4  8  12  16  20
  4  4  4  4
```

```
1  7  19  37  61  91
 6  12  18  24  30
  6  6  6  6
```

```
1  12  33  64  105  156
 11  21  31  41  51
  10  10  10  10
```

```
1  13  37  73  121  181
 12  24  36  48  60
  12  12  12  12
```

```
1  12  37  76  129  196
 11  25  39  53  67
  14  14  14  14
```

```
1  6  16  31  51  76
 5  10  15  20  25
  5  5  5  5
```

In regular polygons and caps, the value of the lowest difference sequence corresponds to the number of triangles F into which the figure can be decomposed.

Difference Sequences for Outer Shells ($n \geq 1$):


```
5  14  29  50  77  110
 9  15  21  27  33
  6  6  6  6
```

```
6  18  38  66  102  146
 12  20  28  36  44
  8  8  8  8
```

```
8  26  56  98  152  218
 18  30  42  54  66
  12  12  12  12
```

```
12  42  92  162  252  362
 30  50  70  90  110
  20  20  20  20
```

```
14  50  110  194  302  434
 36  60  84  108  132
  24  24  24  24
```

```
16  58  128  226  352  506
 42  70  98  126  154
  28  28  28  28
```

Also for the outer shells of regular solids, the value of the lowest difference sequence corresponds to the number of triangular faces F_S on the respective surface. This also works for the regular solids not shown here.

Difference Sequences for Solids in ABC Packung:


```
1  4  10  20  35  56
 3  6  10  15  21
  3  4  5  6
   1  1  1
```

```
1  5  14  30  55  91
 4  9  16  25  36
  5  7  9  11
   2  2  2
```

```
1  6  19  44  85  146
 5  13  25  41  61
  8  12  16  20
   4  4  4
```

```
1  8  27  64  125  216
 7  19  37  61  91
  12  18  24  30
   6  6  6
```

```
1  14  51  124  245  426
 13  37  73  121  181
  24  36  48  60
   12  12  12
```

```
1  13  55  147  309  561
 12  42  92  162  252
  30  50  70  90
   20  20  20
```

```
1  16  68  180  375  676
 15  52  112  195  301
  37  60  83  106
   23  23  23
```

```
1  14  63  172  365  666
 13  49  109  193  301
  36  60  84  108
   24  24  24
```

```
1  38  201  586  1289  2406
 37  163  385  703  1117
  126  222  318  414
   96  96  96
```

For all solids in the cubic close-packed packing, the value of the lowest difference sequence corresponds to the tetrahedron volume of the respective unit cell or rather to the number of tetrahedral cells C into which the respective solid is divided.

Difference Sequences for Solids in Other Packings:

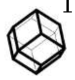
1 4 11 23 42 69 106
3 7 12 19 27 37
4 5 7 8 10
1 2 1 2

1 7 20 44 81 135 208
6 13 24 37 54 73
7 11 13 17 19
4 2 4 2

1 5 15 34 65 111
4 10 19 31 46
6 9 12 15
3 3 3

1 8 27 64 125 216
7 19 37 61 91
12 18 24 30
6 6 6

1 9 35 91 189 341
8 26 56 98 152
18 30 42 54
12 12 12



1 15 65 175 369 671
14 50 110 194 302
36 60 84 108
24 24 24

The difficulty of determining the volume of the hexagonal and hexagrammic pyramids has already been mentioned (see pages 32 and 35). Their unit cells do not yet reflect the actual solid. From the values of the corresponding bipyramids, it can be concluded that interpolation can be used for the two alternating lowest difference sequences. For the body-centered cube and the rhombic dodecahedron, the values of the lowest difference sequences each correspond to the number of tetrahedral cells *C* into which the two solids can be divided. I suspect that this is also the case with the hexagonal and hexagrammic bipyramids, since the values there also correspond to the coefficient of n^3 of the respective formula (see pages 9, 32, and 35) multiplied by *6*.

The correspondences of the lowest difference sequences with the number of triangular faces or tetrahedral cells support Buckminster Fuller´s demand to refer to the symplexes triangle and tetrahedron for area and volume data [1, sec. 990.00].

5.5 FORMULAS AND NEGATIVE NUMBERS

When I still used the formulas for the sequences of the total number of spheres in geometric figures in the form where $n = 1$ resulted in one, I was always bothered by the fact that the coefficients of the formulas could be read off at the 10th element for some figures and at the 11th element for others:

	Natural Numbers	n	10th Element:	10
	Odd Numbers	$2n-1$	11th Element:	21
	Square	n^2	10th Element:	100
	Centered Square	$2n^2-2n+1$	11th Element:	221
	Centered Hexagon	$3n^2-3n+1$	11th Element:	331
	Hexagram	$6n^2-6n+1$	11th Element:	661
	Primitive Cube	n^3	10th Element:	1000
	Body-Centered Cube	$2n^3-3n^2+3n-1$	11th Element:	2331
	Rhombic Dodecahedron	$4n^3-6n^2+4n-1$	11th Element:	4641

The substitution of n in the formulas with $n + 1$ while simultaneously extending the natural numbers to include the "number" zero solves this problem and offers further advantages. The negative signs thus disappear from the formulas. And in all formulas expressible without fractions with single-digit coefficients, these coefficients can be read off directly at the element $n = 10$. And even where this is not immediately possible, the coefficients can be relatively easily found through trial and error via the element $n = 10$.

However, the greatest advantage of this modification of the formulas is revealed when negative numbers are inserted into the formulas:

	Formula	-5	-4	-3	-2	-1	0	1	2	3	4	5
	$n+1$	-4	-3	-2	-1	0	1	2	3	4	5	6
	$2n+1$	-9	-7	-5	-3	-1	1	3	5	7	9	11
	$\tfrac{1}{2}(n^2+3n+2)$	6	3	1	0	0	1	3	6	10	15	21
	n^2+2n+1	16	9	4	1	0	1	4	9	16	25	36
	$\tfrac{1}{2}(3n^2+3n+2)$	31	19	10	4	1	1	4	10	19	31	46
	$2n^2+2n+1$	41	25	13	5	1	1	5	13	25	41	61
	$3n^2+3n+1$	61	37	19	7	1	1	7	19	37	61	91
	$5n^2+6n+1$	96	57	28	9	0	1	12	33	64	105	156
	$6n^2+6n+1$	121	73	37	13	1	1	13	37	73	121	181
	$7n^2+4n+1$	156	97	52	21	4	1	12	37	76	129	196
	$\tfrac{1}{6}(n^3+6n^2+11n+6)$	-4	-1	0	0	0	1	4	10	20	35	56
	$\tfrac{1}{6}(2n^3+9n^2+13n+6)$	-14	-5	-1	0	0	1	5	14	30	55	91
	$\tfrac{1}{2}(n^3+3n^2+4n+2)$	-34	-15	-5	-1	0	1	5	15	34	65	111
	$\tfrac{1}{3}(2n^3+6n^2+7n+3)$	-44	-19	-6	-1	0	1	6	19	44	85	146
	n^3+3n^2+3n+1	-64	-27	-8	-1	0	1	8	27	64	125	216
	$2n^3+3n^2+3n+1$	-189	-91	-35	-9	-1	1	9	35	91	189	341
r.	$\tfrac{1}{2}(4n^3+9n^2+7n+2)$	-154	-69	-23	-4	0	1	11	42	106	215	381
	$2n^3+6n^2+5n+1$	-124	-51	-14	-1	0	1	14	51	124	245	426
	$\tfrac{1}{3}(10n^3+15n^2+11n+3)$	-309	-147	-55	-13	-1	1	13	55	147	309	561
	$4n^3+6n^2+3n+1$	-364	-171	-62	-13	0	1	14	63	172	365	666
	$4n^3+6n^2+4n+1$	-369	-175	-65	-15	-1	1	15	65	175	369	671
	$\tfrac{1}{6}(23n^3+42n^2+25n+6)$	-324	-149	-52	-10	0	1	16	68	180	375	676
r.	$4n^3+9n^2+6n+1$	-304	-135	-44	-7	0	1	20	81	208	425	756
	$16n^3+15n^2+6n+1$	-1654	-807	-314	-79	-6	1	38	201	586	1289	2406
	$n^4+4n^3+6n^2+4n+1$	256	81	16	1	0	1	16	81	256	625	1296

Inserting positive numbers into the formulas provides the total number of spheres G_n in the respective geometric figure. Inserting negative numbers into the same formulas provides the corresponding number of spheres inside the respective figure I_n, the enclosed spheres, the cores, the total number of spheres minus the outer shell. Here, the negative sign is to be ignored for odd dimensions.

And if, as in the case of the tetrahedron, three zeros are needed, the formula also produces three zeros. And if the enclosed spheres inside deviate from the actual figure, as in the case of the cross or the truncated octahedron, inserting negative numbers into the formulas still produces the correct result. Negative numbers lead to the interior of structures.

When I found this, I thought it was a sensational discovery. However, Teo and Sloane already found this, as well as the following remarkable formula for d-dimensional figures

$$I_n = (-1)^d G_{-n}$$

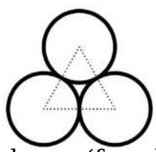
for which even more references are given [10, p. 4549].

Although this has been known for a long time and has been published several times, this particularly philosophically significant insight is hardly appropriately appreciated.

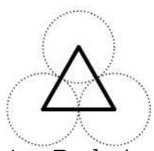
Zero leads to one. One leads to the form. Ten leads to the formula. Negative numbers lead to the interior.

5.6 LATTICES

Geometric figures in close-packed packings can be easily built and visualized using steel balls and magnetic rods. The magnetic rods correspond to the points of contact between the spheres. These contact points are also called kissing points. However, the spheres can also be interpreted as nodes of lattices. Then only the lattice of magnetic rods remains. And whether spheres and contact points or nodes and lattice rods, both aspects are always present simultaneously.



Spheres (female)



Lattice Rods (male)

And counting the magnetic or rather lattice rods of geometric figures also leads to sequences, formulas, and further insights.

The following table shows the total number of lattice rods G_n in geometric figures. For solids, it also shows the number of lattice rods in their outer shells S_n, and for pyramids, the number of lattice rods in their caps C_n:

	n	0	1	2	3	4	5	6	7	8	9	10
	G_n	0	3	9	18	30	45	63	84	108	135	165
	G_n	0	3	15	36	66	105	153	210	276	351	435
	G_n	0	4	12	24	40	60	84	112	144	180	220
	G_n	0	4	16	36	64	100	144	196	256	324	400
	G_n	0	5	16	33	56	85	120	161	208	261	320
	G_n	0	7	22	45	76	115	162	217	280	351	430
	G_n	0	12	42	90	156	240	342	462	600	756	930
	G_n	3	24	63	120	195	288	399	528	675	840	1023
	G_n	0	16	52	108	184	280	396	532	688	864	1060
	G_n	0	16	60	132	232	360	516	700	912	1152	1420
	G_n	0	24	84	180	312	480	684	924	1200	1512	1860
	G_n	9	51	129	243	393	579	801	1059	1353	1683	2049
	G_n	0	6	24	60	120	210	336	504	720	990	1320
	S_n	0	6	24	54	96	150	216	294	384	486	600
	C_n	0	6	21	45	78	120	171	231	300	378	465
	G_n	0	6	27	72	150	270	441	672	972	1350	1815
	S_n	0	6	24	54	96	150	216	294	384	486	600
	C_n	0	6	18	39	66	102	144	195	252	318	390
	G_n	0	8	36	96	200	360	588	896	1296	1800	2420
	S_n	0	8	32	72	128	200	288	392	512	648	800
	C_n	0	8	28	60	104	160	228	308	400	504	620
	G_n	0	12	54	144	300	540	882	1344	1944	2700	3630
	S_n	0	12	48	108	192	300	432	588	768	972	1200
	C_n	0	9	36	75	132	201	288	387	504	633	780
	G_n	0	9	39	102	210	375	609	924	1332	1845	2475
	S_n	0	9	36	81	144	225	324	441	576	729	900
	G_n	0	9	42	120	258	477	792	1224	1788	2505	3390
	S_n	0	9	30	69	120	189	270	369	480	609	750
	G_n	0	12	60	168	360	660	1092	1680	2448	3420	4620
	S_n	0	12	48	108	192	300	432	588	768	972	1200
	G_n	0	15	84	237	516	951	1584	2445	3576	5007	6780
	S_n	0	12	60	132	240	372	540	732	960	1212	1500

Continuation:

	n	0	1	2	3	4	5	6	7	8	9	10
	G_n	0	12	54	144	300	540	882	1344	1944	2700	3630
	S_n	0	12	48	108	192	300	432	588	768	972	1200
	G_n	0	18	90	252	540	990	1638	2520	3672	5130	6930
	S_n	0	18	72	162	288	450	648	882	1152	1458	1800
	G_n	0	8	64	216	512	1000	1728	2744	4096	5832	8000
	S_n	0	0	0	0	0	0	0	0	0	0	0
	G_n	0	32	184	552	1232	2320	3912	6104	8992	12672	17240
	S_n	0	24	96	216	384	600	864	1176	1536	1944	2400
	G_n	0	36	180	504	1080	1980	3276	5040	7344	10260	13860
	S_n	0	36	144	324	576	900	1296	1764	2304	2916	3600
	G_n	0	36	240	756	1728	3300	5616	8820	13056	18468	25200
	S_n	0	24	96	216	384	600	864	1176	1536	1944	2400
	G_n	0	36	216	660	1488	2820	4776	7476	11040	15588	21240
	S_n	0	24	96	216	384	600	864	1176	1536	1944	2400
	G_n	0	48	276	822	1824	3420	5748	8946	13152	18504	25140
	S_n	0	42	168	378	672	1050	1512	2058	2688	3402	4200
	G_n	0	144	948	2988	6840	13080	22284	35028	51888	73440	100260
	S_n	0	84	336	756	1344	2100	3024	4116	5376	6804	8400
	C_n	0	10	35	75	130	200	285	385	500	630	775
	S_n	0	30	120	270	480	750	1080	1470	1920	2430	3000

In “The On-Line Encyclopedia of Integer Sequences“, only the above sequences of triangle, square, rhombus, hexagon, hexagram, as well as tetrahedron and primitive cube, are found in this context as the total number of lattice rods G_n or rather contact points in geometric figures [8].

Even with lattice rods, the summation of consecutive centered hexagons leads to the sequence of primitive cubes. The primitive cube and the hexagrammic pyramid correspond, while with spheres, the primitive cube and the hexagrammic bipyramid match. Likewise, both the two types of centered squares and the two types of close-packed cubes with an odd number of layers correspond. The summation of the tetrahedral caps leads to the sequence of the hexagonal pyramid, and not as with spheres to that of the hexagonal bipyramid. Similarly, the summation of the pyramidal caps leads to the sequence of the pyramid and not as with spheres to that of the octahedron. And some solids have the same number of lattice rods in the outer shell. Overall, there are significantly fewer correspondences than with spheres.

A look at formulas is also worthwhile for lattice rods. These can be easily found via the sequence element $n = 10$ and can be easily generalized with knowledge of the generalized formulas of Teo and Sloane for sphere packings [10].

The generalized formula for the total number of lattice rods G_n in regular two-dimensional geometric figures is

$$G_n = (\tfrac{1}{2}S_1 + I_1)n^2 + \tfrac{1}{2}S_1 n$$

Here, S_1 is the number of outer boundary lattice rods and I_1 is the number of internal lattice rods of the unit cell of the respective polygon.

The formulas of the irregular hexagon ($9n^2+12n+3$) and the irregular hexagram ($18n^2+24n+9$), as well as surprisingly the formula of the centered triangle ($\tfrac{1}{2}(9n^2-3n)$), deviate from the generalized formula for the two-dimensional figures in the table on page 52. Apart from the numerical coincidence with the tetrahedral caps (see page 46), the centered triangle does not appear in any stable solid as a layer, and therefore, in my opinion, can be neglected as a geometrical entity.

The generalized formula for plane figures can also be applied to regular pyramidal caps C_n, if it is noted that I_1 is the number of inner lattice rods that rise from the plane of the respective unit cell. However, for irregular hexagonal pyramidal caps ($15n^2-12n+3$) and irregular hexagrammic pyramidal caps ($30n^2-24n+3$), different formulas are needed.

For solids, the following very simple generalized formula is found for the number of lattice rods S_n in outer shells:

$$S_n = R_S n^2$$

Here, R_S is the number of lattice rods in the outer shell of the unit cell of the respective solid.

Different formulas are needed for the outer shells of irregular hexagonal bipyramids ($30n^2-30n+9$) and irregular hexagrammic bipyramids ($60n^2-60n+12$).

The generalized formula for the total number of lattice rods G_n in polyhedra is as follows:

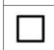
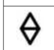
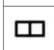
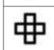
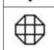
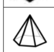
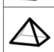
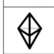
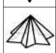
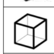
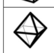
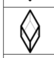
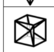
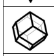
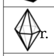
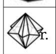
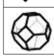
$$G_n = \tfrac{1}{12}kCn^3 + \tfrac{1}{2}R_S n^2 + (\tfrac{1}{2}R_S + I_1 - \tfrac{1}{12}kC)n$$

where C is the number of tetrahedral cells of the polyhedron, k is the number of lattice rods meeting at an inner node, R_S is the number of lattice rods in the outer shell, and I_1 is the number of lattice rods in the interior of the respective unit cell.

For irregular hexagonal bipyramids ($24n^3-21n^2+6n$) and irregular hexagrammic bipyramids ($48n^3-42n^2+12n-3$), different formulas are also needed for the total number of lattice rods. For their simple pyramids, on the other hand, formulas can be found without distinguishing between regular and irregular ones. This is exactly the opposite of what is observed for spheres.

The factor k is 6 ($\frac{1}{12}k = \frac{1}{2}$) in the primitive cubic packing, 8 ($\frac{1}{12}k = \frac{2}{3}$) in the body-centered cubic packing, and 12 ($\frac{1}{12}k = 1$) in the close-packed packings.

The following table shows the formulas for the total number of lattice rods G_n for regular figures as well as the first five positive and negative elements of the respective sequences:

	Formula	-5	-4	-3	-2	-1	0	1	2	3	4	5
	$\frac{1}{2}(3n^2+3n)$	30	18	9	3	0	0	3	9	18	30	45
	$2n^2+2n$	40	24	12	4	0	0	4	12	24	40	60
	$3n^2+2n$	65	40	21	8	1	0	5	16	33	56	85
	$4n^2$	100	64	36	16	4	0	4	16	36	64	100
	$4n^2+3n$	85	52	27	10	1	0	7	22	45	76	115
	$9n^2+3n$	210	132	72	30	6	0	12	42	90	156	240
	$10n^2+6n$	220	136	72	28	4	0	16	52	108	184	280
	$18n^2+6n$	420	264	144	60	12	0	24	84	180	312	480
	$14n^2+2n$	340	216	120	52	12	0	16	60	132	232	360
	n^3+3n^2+2n	-60	-24	-6	0	0	0	6	24	60	120	210
	$\frac{1}{2}(3n^3+6n^2+3n)$	-120	-54	-18	-3	0	0	6	27	72	150	270
	$2n^3+4n^2+2n$	-160	-72	-24	-4	0	0	8	36	96	200	360
	$\frac{1}{2}(4n^3+9n^2+5n)$	-150	-66	-21	-3	0	0	9	39	102	210	375
	$3n^3+6n^2+3n$	-240	-108	-36	-6	0	0	12	54	144	300	540
	$3n^3+6n^2+3n$	-240	-108	-36	-6	0	0	12	54	144	300	540
	$4n^3+6^2+2n$	-360	-168	-60	-12	0	0	12	60	168	360	660
	$6n^3+9n^2+3n$	-540	-252	-90	-18	0	0	18	90	252	540	990
	$8n^3$	-1000	-512	-216	-64	-8	0	8	64	216	512	1000
	$12n^3+18n^2+6n$	-1080	-504	-180	-36	0	0	36	180	504	1080	1980
	$16n^3+12n^2+4n$	-1720	-848	-336	-88	-8	0	32	184	552	1232	2320
	$20n^3+12n^2+4n$	-2220	-1104	-444	-120	-12	0	36	216	660	1488	2820
	$24n^3+12n^2$	-2700	-1344	-540	-144	-12	0	36	240	756	1728	3300
	$24n^3+15n^2+3n$	-2640	-1308	-522	-138	-12	0	42	258	792	1788	3390
	$23n^3+21n^2+4n$	-2370	-1152	-444	-108	-6	0	48	276	822	1824	3420
	$48n^3+30n^2+6n$	-5280	-2616	-1044	-276	-24	0	84	516	1584	3576	6780
	$96n^3+42n^2+6n$	-10980	-5496	-2232	-612	-60	0	144	948	2988	6840	13080

As with spheres, inserting negative numbers into the formulas for the total number of lattice rods G_n also leads to the corresponding number of internal lattice rods I_n for the respective figure, i.e. the total number of lattice rods minus the outer shell. Again, the negative sign is to be ignored for odd dimensions.

The following is immediately evident from the generalized formulas for lattice rods for d-dimensional geometric figures, as already for spheres:

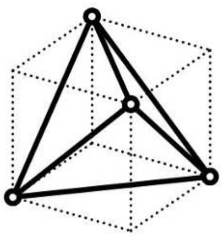
$$I_n = G_n - S_n = (-1)^d G_{-n}$$

Thus, negative numbers not only lead to the interior of structures in the case of spheres.

Zero remains zero. One leads to the form. Ten leads to the formula. Negative numbers lead to the interior.

As a conclusion to the examination of lattices, a closer look at close-packed cubes is worthwhile. In the tables above, only the regular close-packed cube with an odd number of layers is represented, where number of layers referring to nodes or rather to spheres. Three different formulas were required for the total number of spheres for the various close-packed cubes (see pages 24, 26, and 28). For lattice rods, on the other hand, a single formula is sufficient.

The unit cell of the irregular close-packed cube is a cube with an inscribed tetrahedron:



The tetrahedron volume of the unit cell is 3, and there are 6 lattice rods in the outer shell and no lattice rods in the interior. From this, the following formula arises for close-packed cubes ($\frac{1}{12}k = 1$)

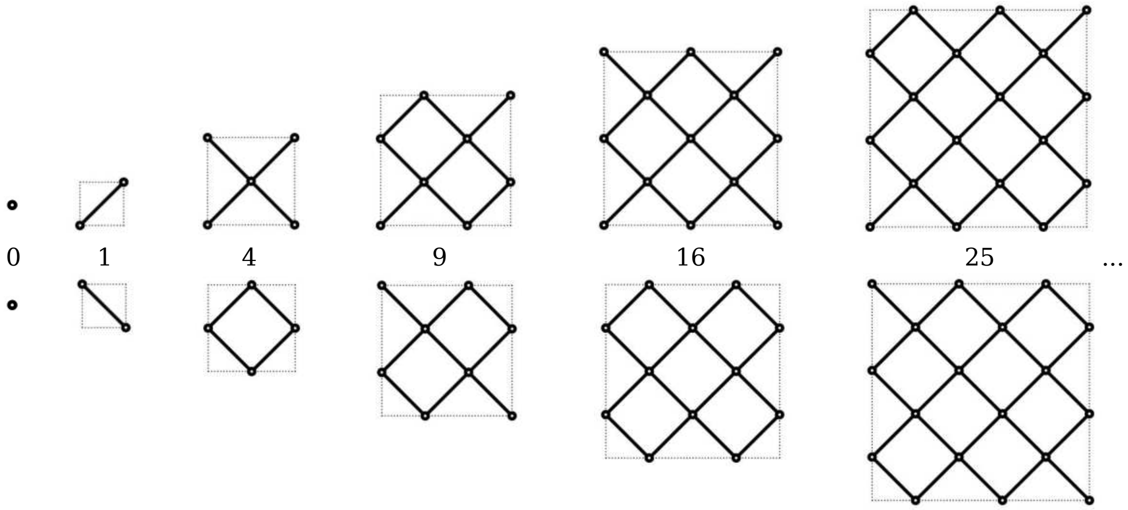
$$G_n = 3n^3 + 3n^2$$

and the sequence: 6, 36, 108, 240, 450, 756, 1176, 1728, 2430, 3300, ...

As can be easily confirmed by simple recounting, this sequence represents the total number of lattice rods in irregular and regular close-packed cubes. With lattice rods, no distinction between the different close-packed cubes is actually necessary anymore.

This can also be illustrated by looking at the square layers of the various close-packed cubes. In the case of the planar figures generated from spheres or rather from circles, I have shown in Section 2 only the "centered" squares comparable to odd "chessboards" (see pages 5 and 6).

In the case of lattices, a complete picture only emerges when the squares comparable with even "chessboards" are also taken into account:



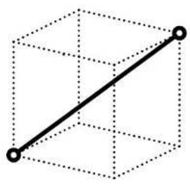
The unit cell of the above "centered" squares is a single diagonal lattice rod.

One inner lattice rod in the unit cell and no bounding outer lattice rods lead to the following formula:

$$G_n = n^2$$

The total number of lattice rods in the square layers required for close-packed cubes corresponds to the sequence of conventional square numbers. A distinction, as in the case of spheres, between irregular and regular is not necessary.

The shown can be easily transferred to body-centered cubes. The unit cell is then a single lattice rod that connects two opposite corners along a space diagonal:



Since $\frac{1}{12}kC$ is one for such a unit cell and there are no lattice rods in the outer shell and one lattice rod inside the unit cell, the total number of lattice rods in a body-centered cubic lattice leads to the sequence of primitive cubic numbers with the formula:

$$G_n = n^3$$

As a lattice, the centered square and the body-centered cube only have inner lattice rods. Thus, square and cubic numbers with the conventional formulas can be found by counting these inner lattice rods.

6. CONCLUSION AND PHILOSOPHICAL SPECULATIONS

The present work demonstrates that sphere packings and figurate numbers are not just pastime, but that significant mathematical insights can be gained from them.

This paper does not claim to have proven everything that is shown. With sphere packings and figurate numbers, the focus is not primarily on formalism, but on intuitive clarity. The easiest way to understand what is shown is by using models of steel balls and magnetic rods. Thus, it is a mathematics that can be physically grasped. An ancient mathematics that does not require calculating machines or brain acrobatics, and for which only whole numbers are needed. One may recall the famous saying by Leopold Kronecker [12, p. 19]:

„Die ganzen Zahlen hat der liebe Gott gemacht,
alles andere ist Menschenwerk.“

(The integers were made by dear God, everything else is the work of man.)

However, due to the generalized formulas of Teo und Sloane [10], much of what is demonstrated in this work has been proven. The equations with summation symbol in this paper can also be easily proven using mathematical induction (see example proof in Appendix IV). The space here was too precious for that. Proofs for the generalized formulas for lattice rods are missing, but these should be easy to find for trained mathematicians.

Despite its simplicity, this kind of mathematics provides fundamental insights into the structure and building blocks of the universe, into dimensions and negative numbers, and much more. For example, in the cubic close-packed packing, not only are there the most solids, but also the tetrahedron volume of solids is accurately represented as the coefficient of n^3 in the formulas for lattice rods. However, if the primitive cube is used as the volumetric unity and the basic building block of mathematics, all other volume measurements become irrational [1, sec. 990.00].

Several of the discoveries shown in this work, I have not found published anywhere else so far. This does not necessarily mean that these discoveries are actually not to be found anywhere else. On the other hand, they are often such beautiful and simple connections that they should be more widely known if they were already publicly accessible somewhere.

In particular, I have searched in vain for the following:

- The truncated octahedron is composed of three crosses that are perpendicular to each other (pages 20 f.).
- In the ABC packing, enveloping all convex regular geometric figures with a shell of spheres leads to solids with 14 faces (page 21).

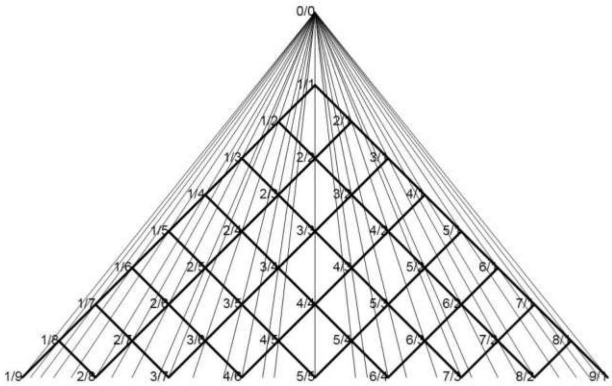
- The summation of consecutive hexagrammic numbers leads to the sequence of the stellated octahedral numbers (page 23).
- Regularity of sequences of close-packed cubes in other positional numeral systems (pages 25 and 27).
- Quotient of close-packed cubes and number of spheres in an outer edge leads to diagonals of the Ulam spiral (pages 27 f.).
- Summation of consecutive hexagonal bipyramidal numbers leads to the sequence of triangular numbers of triangular numbers (page 33).
- Summation of consecutive regular hexagonal bipyramidal numbers leads to the sequence of square numbers of square numbers or rather to $(n+1)^4$ (page 33).
- Existence of the hexagrammic pyramid (page 34).
- Sequence of hexagrammic bipyramidal numbers corresponds to the (primitive) cubic numbers $(n+1)^3$ (page 35).
- Higher levels of the solids built from triangular bipyramids and resulting number planes (pages 36 ff.).
- Sequence of the hexagonal bipyramidal numbers of level II corresponds to that of the regular close-packed cube (page 37).
- For many solids, subtracting the axis spheres results in multiples of tetrahedral or pyramidal numbers (see pages 44 ff.).
- Sequences of pyramidal caps correspond to those of centered polygons (page 46).
- The lowest difference sequence corresponds to the number of triangular faces or tetrahedral cells of the respective figure (pages 48 f.).
- Generalized formulas for the number of contact points or rather lattice rods (page 54).
- Negative numbers also lead to the interior of the structures in the case of contact points or rather lattice rods (pages 55 f.).

The shown are likely just the tip of the iceberg. Further explorations of this area of mathematics will be helpful in solving many mysteries of the universe. And in addition to the purely mathematical insights presented in this work, the employment with sphere packings and figurate numbers is also suitable for philosophical speculations:

- $0/0 = 3$

How to complete the row $k = 0$ in the table on page 27? Even an absurd conclusion drawn from the Ulam spiral, such as $0/0 = 3$ (see page 28), should not be dismissed so easily.

In the Pythagorean Lambdoma, all so-called “equal tone lines” converge at the point marked with “*0/0*” outside the actual system. In harmonic symbolism, this point represents both everything and nothing and is therefore identified with the highest deity [4, § 25a]:



In this sense, *0/0 = 3* could also be interpreted as a reference to the divine trinity (becoming, maintaining, passing away).

On the other hand, it could be concluded from $0^0 = 0^{(1-1)} = 0^1 * 0^{-1} = 0/0$ that $0/0 = 1$. Thus, one and the same expression would lead to different results in different contexts. In the embodied world at least, there is no perfection anyway. Perfect circles or exactly straight lines are nowhere to be found in nature. If everything were perfectly and evenly distributed, existence as a whole would cancel itself out. The embodied world only arises through deviation from perfection. It is a world of imperfection. Errors are inherent in the system. And examples in mathematics are also conceivable for this.

- Negative Numbers

Negative numbers in sphere packings are not just a mere reflection of positive numbers. They are related to the interior. They lead to the inside. And the most essential interior in humans is their mental world of thoughts.

As is easy to see, the product of all (real) numbers (except zero) leads to minus one. Here, zero is not considered as a number, but rather as the absence of a number. The real numbers are not countable, but since each number has exactly one reciprocal, it is possible (as with particles and antiparticles) to form pairs whose product is always one. After complete pair production, two numbers remain without a reciprocal different from themselves. One and minus one. Therefore, the product of all (real) numbers (except zero) is minus one, or expressed using Euler´s identity for the angle π (*180*°):

$$\prod_{a=-\infty}^{\infty} a = e^{i\pi}$$

Since one is the symbol of unity, of the whole, minus one can be seen as the symbol for the inner unity, for the spiritual world. And since the connection of everything leads to minus one (and not to one), it could be postulated that the world is a spiritual world.

- 666

The most interesting figures from my point of view are the triangle, the close-packed cube, and the hexagonal bipyramid. The triangle is the first and simplest geometric figure, often a symbol for God. According to Buckminster Fuller, only triangles are stable, and all stable figures are made up of triangles [1, sec. 609.01]. This reminds one of verse 42 of the Chinese “Dàodéjīng” [6, ch. 42]:

“The Tâo produced One;
One produced Two;
Two produced Three;
Three produced All things.”

The close-packed cube, on the other hand, stands out simply because of the regularity of its sequence in other positional numeral systems. And the hexagonal bipyramid leads to an additional hidden dimension through its higher levels.

And the branded number 666 appears in all three figures. In the close-packed cube even in the most prominent position. Therefore, if you deal with sphere packings and figurate numbers, it is impossible to ignore the number 666. So, is this the mathematics of the devil? The exact opposite is true. Everything here screams universal truth and divine glory. And in a spiritual world, nothing is higher than truth. What is true is word of God. But a world in which truth is suppressed will undoubtedly destroy itself.

The question remains why many of the simple and fundamental mathematical connections shown in this paper are not published anywhere. It is hard to believe that I should be the first to have stumbled upon it. My book “666 oder Die Rückkehr der Hermaphroditen” (666 or The Return of the Hermaphrodites) provides an explanation for this question as well as other mathematical discoveries [7].

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APPENDIX I_1 - TABLE POLYGONS

The following table shows the total number of spheres G_n in polygons arranged such that the number on the left indicates the number of inner spheres of the polygon with the number to the right. For the outer spheres S_n, each number indicates the number of outer spheres that enclose all the numbers or rather the outer spheres to its left.

Polygon		1	2	3	4	5	6	7	8	9	10	11	12
Triangle	G_n	1	10	28	55	91	136	190	253	325	406	496	595
	G_n	3	15	36	66	105	153	210	276	351	435	528	630
	G_n	6	21	45	78	120	171	231	300	378	465	561	666
Outer Spheres	S_n	1	9	18	27	36	45	54	63	72	81	90	99
	S_n	3	12	21	30	39	48	57	66	75	84	93	102
	S_n	6	15	24	33	42	51	60	69	78	87	96	105
Square	G_n	1	9	25	49	81	121	169	225	289	361	441	529
	G_n	4	16	36	64	100	144	196	256	324	400	484	576
Outer Spheres	S_n	1	8	16	24	32	40	48	56	64	72	80	88
	S_n	4	12	20	28	36	44	52	60	68	76	84	92
Ctr. Triangle	G_n	1	4	10	19	31	46	64	85	109	136	166	199
Outer Spheres	S_n	1	3	6	9	12	15	18	21	24	27	30	33
Ctr. Square	G_n	1	5	13	25	41	61	85	113	145	181	221	265
Outer Spheres	S_n	1	4	8	12	16	20	24	28	32	36	40	44
Ctr. Square (-1)	G_n	0	4	12	24	40	60	84	112	144	180	220	264
Outer Spheres	S_n	0	4	8	12	16	20	24	28	32	36	40	44
Ctr. Hexagon	G_n	1	7	19	37	61	91	127	169	217	271	331	397
Outer Spheres	S_n	1	6	12	18	24	30	36	42	48	54	60	66
Irreg. Hexagon	G_n	3	12	27	48	75	108	147	192	243	300	363	432
Outer Spheres	S_n	3	9	15	21	27	33	39	45	51	57	63	69
Hexagram	G_n	1	13	37	73	121	181	253	337	433	541	661	793
Outer Spheres	S_n	1	12	24	36	48	60	72	84	96	108	120	132
Irreg. Hexagram	G_n	6	24	54	96	150	216	294	384	486	600	726	864
Outer Spheres	S_n	6	18	30	42	54	66	78	90	102	114	126	138

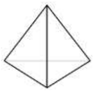
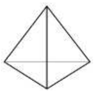
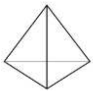
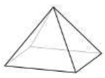
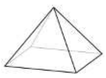
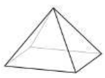
APPENDIX I_2 - TABLE POLYGONS

For the double square and the triple square as well as for the cross and the octagon, the enclosed inner spheres deviate from the actual shape of the figure.

Polygon		1	2	3	4	5	6	7	8	9	10	11	12
Double Square	G_n	1	6	15	28	45	66	91	120	153	190	231	276
Outer Spheres	S_n	1	6	12	18	24	30	36	42	48	54	60	66
Tripel Square	G_n	1	8	21	40	65	96	133	176	225	280	341	408
Outer Spheres	S_n	1	8	16	24	32	40	48	56	64	72	80	88
Cross	G_n	1	12	33	64	105	156	217	288	369	460	561	672
Outer Spheres	S_n	1	12	24	36	48	60	72	84	96	108	120	132
Octagon	G_n	1	12	37	76	129	196	277	372	481	604	741	892
Outer Spheres	S_n	1	8	16	24	32	40	48	56	64	72	80	88

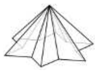
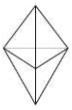
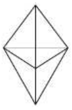
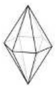
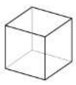
APPENDIX II$_1$ - TABLE POLYHEDRA

The following table shows the total number of spheres G_n in polyhedra arranged such that the number on the left indicates the core of the polyhedron with the number to the right. For outer shells S_n and caps C_n, each number indicates the outer shell or cap that encloses all the numbers or rather the outer shells and caps to its left.

Polyhedron		1	2	3	4	5	6	7	8
Tetrahedron 	G_n	1	35	165	455	969	1771	2925	4495
	G_n	4	56	220	560	1140	2024	3276	4960
	G_n	10	84	286	680	1330	2300	3654	5456
	G_n	20	120	364	816	1540	2600	4060	5984
Outer Shell 	S_n	1	34	130	290	514	802	1154	1570
	S_n	4	52	164	340	580	884	1252	1684
	S_n	10	74	202	394	650	970	1354	1802
	S_n	20	100	244	452	724	1060	1460	1924
Cap 	C_n	1	19	64	136	235	361	514	694
	C_n	4	31	85	166	274	409	571	760
	C_n	10	46	109	199	316	460	631	829
Pyramid 	G_n	1	30	140	385	819	1496	2470	3795
	G_n	5	55	204	506	1015	1785	2870	4324
	G_n	14	91	285	650	1240	2109	3311	4900
Outer Shell 	S_n	1	29	110	245	434	677	974	1325
	S_n	5	50	149	302	509	770	1085	1454
	S_n	14	77	194	365	590	869	1202	1589
Cap 	C_n	1	13	41	85	145	221	313	421
	C_n	5	25	61	113	181	265	365	481
Hexagonal Pyramid 	G_n	1	23	106	290	616	1124	1855	2849
	G_n	4	42	154	381	763	1341	2155	3246
	G_n	11	69	215	489	932	1584	2486	3678
Outer Shell 	S_n	1	22	83	184	326	508	731	994
	S_n	4	38	112	227	382	578	814	1091
	S_n	11	58	146	274	443	652	902	1192
Cap 	C_n	1	10	31	64	109	166	235	316
	C_n	4	19	46	85	136	199	274	361

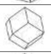
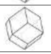
APPENDIX II_2 - TABLE POLYHEDRA

Continuation:

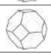
	Polyhedron		1	2	3	4	5	6	7	8
	Hexagrammic Pyramid	G_n	1	44	208	575	1225	2240	3700	5687
		G_n	7	81	304	756	1519	2673	4300	648
		G_n	20	135	425	972	1856	3159	4961	7344
	Outer Shell	S_n	1	43	164	367	650	1015	1460	1987
		S_n	7	74	223	452	763	1154	1627	2180
		S_n	20	115	290	547	884	1303	1802	2383
	Cap	C_n	1	19	61	127	217	331	469	631
		C_n	7	37	91	169	271	397	547	721
	Triangular Bipyramid	G_n	1	30	140	385	819	1496	2470	3795
		G_n	5	55	204	506	1015	1785	2870	4324
		G_n	14	91	285	650	1240	2109	3311	4900
	Outer Shell	S_n	1	29	110	245	434	677	974	1325
		S_n	5	50	149	302	509	770	1085	1454
		S_n	14	77	194	365	590	869	1202	1589
	Oktaeder	G_n	1	19	85	231	489	891	1469	2255
		G_n	6	44	146	341	670	1156	1834	2736
	Outer Shell	S_n	1	18	66	146	258	402	578	786
		S_n	6	38	102	198	326	486	678	902
	Heyagonal Bipyramid	G_n	1	15	65	175	369	671	1105	1695
		G_n	5	34	111	260	505	870	1379	2056
	Outer Shell	S_n	1	14	50	110	194	302	434	590
		S_n	5	29	77	149	245	365	509	677
	Hexagrammic Bipyramid	G_n	1	27	125	343	729	1331	2197	3375
		G_n	8	64	216	512	1000	1728	2744	4096
	Outer Shell	S_n	1	26	98	218	386	602	866	1178
		S_n	8	56	152	296	488	728	1016	1352
	Primitive Cube	G_n	1	27	125	343	729	1331	2197	3375
		G_n	8	64	216	512	1000	1728	2744	4096
	Outer Shell	S_n	1	26	98	218	386	602	866	1178
		S_n	8	56	152	296	488	728	1016	1352

APPENDIX II_3 - TABLE POLYHEDRA

Continuation:

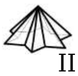
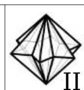
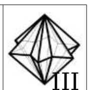
	Polyhedron		1	2	3	4	5	6	7	8
	Rhombohedron	G_n	1	27	125	343	729	1331	2197	3375
		G_n	8	64	216	512	1000	1728	2744	4096
	Outer Shell	S_n	1	26	98	218	386	602	866	1178
		S_n	8	56	152	296	488	728	1016	1352
	Stellated Octahedron	G_n	1	51	245	679	1449	2651	4381	6735
		G_n	14	124	426	1016	1990	3444	5474	8176
	Outer Shell	S_n	1	50	194	434	770	1202	1730	2354
		S_n	14	110	302	590	974	1454	2030	2702
	Cuboktahedron	G_n	1	13	55	147	309	561	923	1415
	Outer Shell	S_n	1	12	42	92	162	252	362	492
	Body-Centered Cube	G_n	1	9	35	91	189	341	559	855
	Outer Shell	S_n	1	8	26	56	98	152	218	296
	Rhombic dodecahedron	G_n	1	15	65	175	369	671	1105	1695
	Outer Shell	S_n	1	14	50	110	194	302	434	590
	Icosahedron	S_n	1	12	42	92	162	252	362	492

For the close-packed cubes with an odd number of layers as well as for the truncated tetrahedron and the truncated octahedron, the enveloped inner spheres deviate from the actual shape of the solid.

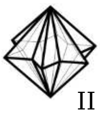
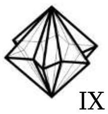
	Polyeder		1	2	3	4	5	6	7	8
	Close-Packed Cube	G_n	1	13	63	171	365	665	1099	1687
		G_n	0	14	62	172	364	666	1098	1688
	Outer Shell	S_n	1	12	50	108	194	300	434	588
		S_n	0	14	48	110	192	302	432	590
	Truncated Tetrahedron	G_n	1	16	68	180	375	676	1106	1688
	Outer Shell	S_n	1	16	58	128	226	352	506	688
	Truncated Octahedron	G_n	1	38	201	586	1289	2406	4033	6266
	Outer Shell	S_n	1	32	122	272	482	752	1082	1472

APPENDIX III_1 - HIGHER LEVELS OF THE HEXAGRAMMIC BIPYRAMID

The following table shows the layers as well as the total number of spheres in the corresponding hexagrammic pyramids and hexagrammic bipyramids of levels I, II, and III:

 I		 I	 II		 II	 III		 III
A 1	1	1	A 1	1	1	A 1	1	1
B 6 (2-3)	7	8	B 3	4		B 3	4	
A 13 (4-4)	20	27	C 15 (3-5)	19	23	C 6	10	
B 24 (5-6)	44	64	B 21	40		A 28 (4-7)	38	48
A 37 (7-7)	81	125	A 37 (7-7)	77	117	C 36	74	
B 54 (8-9)	135	216	B 51	128		B 45	119	
A 73 (10-10)	208	343	C 75 (9-11)	203	331	A 73 (10-10)	192	311
B 96 (11-12)	304	512	B 93	296		B 93	285	
A 121 (13-13)	425	729	A 121 (13-13)	417	713	C 114	399	
B 150 (14-15)	575	1000	B 147	564		A 154 (13-16)	553	952
A 181 (16-16)	756	1331	C 183 (15-17)	747	1311	C 180	733	
B 216 (17-18)	972	1728	B 213	960		B 207	940	
A 253 (19-19)	1225	2197	A 253 (19-19)	1213	2173	A 253 (19-19)	1193	2133

Number Plane of the Hexagrammic Bipyramid:

n	 I	 II	 III	 IV	 V	 VI	 VII	 VIII	 IX
0	1	1	1	1	1	1	1	1	1
1	8	23	48	85	136	203	288	393	520
2	27	117	311	649	1171	1917	2927	4241	5899
3	64	331	952	2077	3856	6439	9976	14617	20512
4	125	713	2133	4753	8941	15065	23493	34593	48733
5	216	1311	4016	9061	17176	29091	45536	67241	94936
6	343	2173	6763	15385	29311	49813	78163	115633	163495
7	512	3347	10536	24109	46096	78527	123432	182841	258784
8	729	4881	15497	35617	68281	116529	183401	271937	385177
9	1000	6823	21808	50293	96616	165115	260128	385993	547048
10	1331	9221	29631	68521	131851	225581	355671	528081	748771

APPENDIX III$_2$ - HIGHER LEVELS OF THE HEXAGRAMMIC BIPYRAMID

Formulas of Number Plane of the Hexagrammic Bipyramid:

	Formulas horizontal		Formulas vertical
1	$\frac{1}{3}(n^3+9n^2+11n+3)$	I	n^3+3n^2+3n+1
2	$\frac{1}{3}(20n^3+36n^2+22n+3)$	II	$8n^3+12n^2+2n+1$
3	$\frac{1}{3}(75n^3+81n^2+33n+3)$	III	$27n^3+27n^2-7n+1$
4	$\frac{1}{3}(184n^3+144n^2+44n+3)$	IV	$64n^3+48n^2-28n+1$
5	$\frac{1}{3}(365n^3+225n^2+55n+3)$	V	$125n^3+75n^2-65n+1$
6	$\frac{1}{3}(636n^3+324n^2+66n+3)$	VI	$216n^3+108n^2-122n+1$
7	$\frac{1}{3}(1015n^3+441n^2+77n+3)$	VII	$343n^3+147n^2-203n+1$
8	$\frac{1}{3}(1520n^3+576n^2+88n+3)$	VIII	$512n^3+192n^2-312n+1$
9	$\frac{1}{3}(2169n^3+729n^2+99n+3)$	IX	$729n^3+243n^2-453n+1$
10	$\frac{1}{3}(2980n^3+900n^2+110n+3)$	X	$1000n^3+300n^2-630n+1$

APPENDIX IV - EXAMPLE PROOF

Hypothesis: The summation of the regular hexagonal bipyramidal numbers from 0 to n is equal to the 4th power of $(n+1)$.
More precisely:

$$\sum_{n=0}^{n}(4n^3+6n^2+4n+1) \;=\; (n+1)^4$$

Proof by Mathematical Induction

To prove the formula, the following two points must be satisfied:

1. $\displaystyle\sum_{n=0}^{0}(4n^3+6n^2+4n+1) \;=\; (0+1)^4$

2. If $\displaystyle\sum_{n=0}^{n}(4n^3+6n^2+4n+1) \;=\; (n+1)^4$,

   then $\displaystyle\sum_{n=0}^{n+1}(4n^3+6n^2+4n+1) \;=\; ((n+1)+1)^4$

Since $\displaystyle\sum_{n=0}^{0}(4n^3+6n^2+4n+1) \;=\; 1 \;=\; (0+1)^4$, the first point is satisfied.

Since $\displaystyle\sum_{n=0}^{n+1}(4n^3+6n^2+4n+1) \;=$

$$\sum_{n=0}^{n}(4n^3+6n^2+4n+1)+4(n+1)^3+6(n+1)^2+4(n+1)+1 \;=$$

$$(n+1)^4+4(n+1)^3+6(n+1)^2+4(n+1)+1 \;=$$

$$n^4+4n^3+6n^2+4n+1+4n^3+12n^2+12n+4+6n^2+12n+6+4n+4+1 \;=$$

$$n^4+8n^3+24n^2+32n+16 \;=\; (n+2)^4 \;=\; ((n+1)+1)^4 \;,$$

the second point is also satisfied.

Thus $\displaystyle\sum_{n=0}^{n}(4n^3+6n^2+4n+1) \;=\; (n+1)^4$ is proven.