KINEMATIC ANALYSIS OF THE MOTION OF AN ELLIPSOGRAPH POINT UNDER VARIOUS VELOCITY LAWS

V. F. Strohm *Independent Researcher* e-mail: vfstrohm@yahoo.de

DOI: https://doi.org/10.5281/zenodo.20557970

Abstract This paper investigates the kinematics of a fixed point on the connecting bar of an ellipsograph (Archimedes' trammel) under different driving motion regimes: uniform, uniformly accelerated, and Keplerian (elliptical). Based on differential constraint equations, analytical expressions for the velocity and acceleration vectors of the trajectory point are derived. A numerical approach, modeling the division of the orbital quadrant into equal time intervals, is utilized to calculate the corresponding areal velocities. It is rigorously demonstrated that the constancy of the areal velocity and the fulfillment of Kepler's second law occur exclusively under one specific law of angular velocity—the Keplerian regime—whereas under uniform and uniformly accelerated motions, the law of areas is violated. This work establishes a link between the mechanical modeling of trajectories and the author's fundamental theoretical research in the focal and central kinematics of the ellipse.

Keywords: ellipsograph, Archimedes' trammel, kinematics, trajectory, areal velocity, Kepler's laws, numerical simulation, acceleration vectors.

Introduction

Historically, Kepler's laws were derived from astronomical observations of the motion of Mars and are traditionally explained within the framework of Newton's dynamics of inverse-square fields. However, these laws are fundamentally kinematic in nature, describing the geometry and temporal parameters of a point moving along a curve.

To gain a profound understanding of the nature of elliptical motion, its classical mechanical modeling serves as a highly productive approach. A universal instrument for reproducing such trajectories is the ellipsograph (Archimedes' drawing tool). Any point on its connecting bar that does not lie on the axes of the sliders traces a strictly elliptical trajectory.

The objective of this paper is to use the equations of motion of an ellipsograph to investigate how precisely the law of variation of the mechanism's input velocity affects the fulfillment or violation of the law of areas (Kepler's second law). This study serves as a numerical and experimental basis that develops and confirms the analytical conclusions presented by the author in papers [1–4].

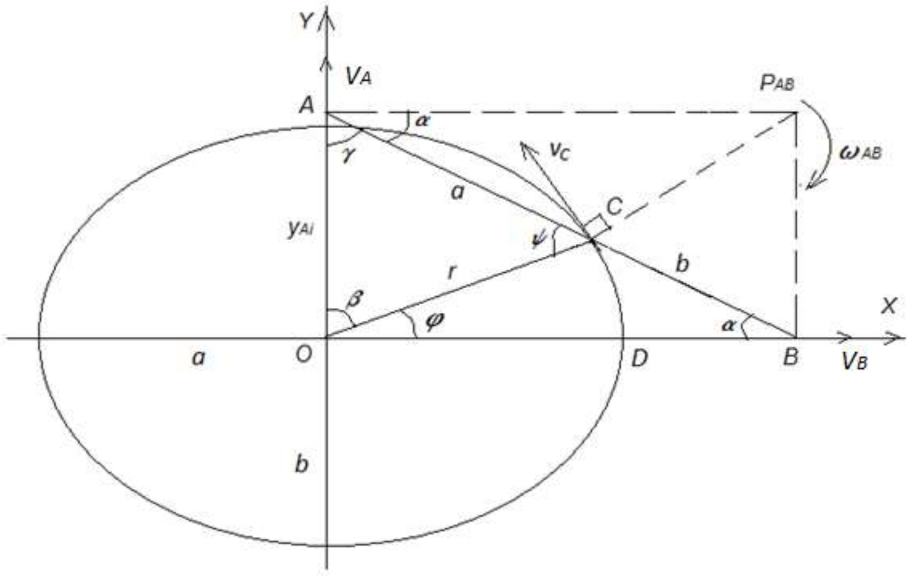

Fig. 1

1. Geometric Foundations and Kinematic Constraint Equations

Consider the schematic diagram of a classical ellipsograph. The bar *AB* moves in such a way that slider *A* shifts along the vertical *Y*-axis, while slider *B* moves along the horizontal *X*-axis (Fig. 1). A point *C* is rigidly fixed on the bar, describing an ellipse within a quadrant of the coordinate plane during the motion.

Let the length of the segment *AC* = *b* (the semi-minor axis of the generated ellipse) and the segment *BC* = a (the semi-major axis of the ellipse). The angle of inclination of the bar relative to the horizontal *X* - axis is denoted by α.

The coordinates of the moving point *C* in a Cartesian coordinate system centered at the intersection of the guides (*O*) are expressed by the trigonometric constraint equations:

$$x(t) = a \cdot \cos \alpha(t) \tag{1}$$

$$y(t) = b \cdot \sin \alpha(t) \tag{2}$$

The linear velocity of slider *A*, moving along the *Y*-axis, determines the instantaneous angular velocity of the bar's rotation ω_{AB} relative to the instantaneous center of zero velocity P_{AB}:

$$\omega_{AB} = \frac{v_A}{AP_{AB}} \tag{3}$$

Accordingly, the components of the linear velocity vector of point *C* are obtained via the first time derivative of equations (1) and (2):

$$v_x = \dot{x} = -a \cdot \sin \alpha(t) \cdot \dot{\alpha} \tag{4}$$

$$v_y = \dot{x} = -b \cdot \cos \alpha(t) \cdot \dot{\alpha} \tag{5}$$

where $\dot{\alpha} = \omega$ is the instantaneous angular velocity of the bar's inclination angle. The total velocity magnitude of point *C* is:

$$v_y = \sqrt{v_x^2 + v_y^2} = \dot{\alpha}\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha} \quad (6)$$

The components of the total acceleration vector of point *C* are found by repeatedly differentiating with respect to time:

$$a_x = \ddot{x} = -a(\ddot{\alpha}\sin\alpha + \dot{\alpha}^2\cos\alpha) \quad (7)$$

$$a_y = \ddot{y} = -b(\ddot{\alpha}\cos\alpha - \dot{\alpha}^2\sin\alpha) \quad (8)$$

where $\ddot{\alpha} = \varepsilon$ is the angular acceleration of the bar.

2. Modeling Various Laws of the Connecting Bar Motion

Within the framework of the numerical experiment, the period of a complete traverse of the elliptical quadrant *T* is divided into *N* equal time intervals. Each calculation step corresponds to a specific moment in time:

$$t_n = n\frac{T}{N}, (n = 0, 1, 2, \dots, N) \quad (9)$$

We investigate three fundamentally different motion regimes of the bar, defined by the law of variation of the angle $\alpha(t)$:

Regime 1: Uniform variation of the bar's angle

In this case, the angular velocity is constant, and the angular acceleration is zero:

$$\alpha(t) = \omega_0 \cdot t,\ \dot{\alpha} = \omega_0 = \text{const},\ \ddot{\alpha} = 0 \quad (10)$$

Substituting (10) into the acceleration equations (7)-(8) reduces them to pure centripetal components proportional to the coordinates, which corresponds to motion in a central field governed by Hooke's law.

Regime 2: Uniformly accelerated variation of the bar's angle

The angle varies with a constant angular acceleration ε_0:

$$\alpha(t) = \frac{\varepsilon_0 t^2}{2}, \dot{\alpha} = \varepsilon_0 \cdot t,\ \ddot{\alpha} =\ \varepsilon_0 = 0 = \text{const} \quad (11)$$

Regime 3: Keplerian (elliptical) motion

In this regime, the law of variation of the angle $\alpha(t)$ is specified non-linearly and is computed from the theoretical condition of a constant areal velocity relative to the focus of the ellipse. As rigorously proven by the author in [1], this law requires the fulfillment of a kinematic analogue of the Binet formula, where the angular velocity is strictly linked to the current geometric position of the point. In polar coordinates relative to the center of the ellipse, this law takes the form:

$\frac{a}{b}\tan\varphi = \tan\left(\frac{2\pi\cdot t}{T}\right)$ (12)

where φ is the polar angle of point C, related to the bar's inclination angle α by the relation $\varphi = \frac{a}{b}\tan\alpha$. This yields the exact analytical expression for the angular step in the Keplerian regime:

$\alpha_n = \arctan\left(\tan\left(\frac{2\pi\cdot t_n}{T}\right)\right)$ (13)

3. Numerical Areal Velocity Calculation and Result Analysis

To verify the fulfillment of Kepler's second law, the simulation program calculated the areas of two sectors of the ellipse traced by the radius-vector of point *C* over identical time intervals $\Delta t = t_{end} - t_{start}$ at different phases of the motion.

The area of an arbitrary sector of the ellipse bounded by angles φ_1 and φ_2 was evaluated via a definite integral in the polar coordinate system:

$S = \frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2(\varphi)d\varphi$ (14)

Results of the Numerical Experiment:

• **Under Input Regimes 1 (uniform) and 2 (uniformly accelerated):** The calculations of the areas of two sectors S_1 and S_2, taken over equal time intervals Δt at the initial and final stages of the quarter-period, showed a significant discrepancy:

$$\Delta S = |S_1 - S_2| \gg 0$$

• This numerically confirms that during arbitrary mechanical rotation of the bar, Kepler's law of areas **is not satisfied**. The areal velocity experiences periodic fluctuations, which are described in detail in the author's analytical paper on the central ellipse [2].

• **Under Input Regime 3 (Keplerian):** When the angular motion of the bar is forcibly subjected to law (13), numerical integration of the sector areas yields strictly identical results (up to the computer rounding error):

$$S_1 = 0.027487, \qquad S_2 = 0.0274891 \Rightarrow \Delta S \approx 0$$

- The equality of the sector areas is satisfied flawlessly.

Conclusion

1. By modeling the motion of an ellipsograph bar, it is experimentally proven that the elliptical shape of a trajectory does not inherently guarantee the fulfillment of Kepler's laws. Kepler's laws are a property not of the trajectory as a geometric object, but of a strictly defined kinematic regime of motion along it.
2. Numerical calculations confirmed that the constancy of the areal velocity is achieved exclusively when the variation of the angular velocity is governed by the specific non-linear law (13).

3. Archimedes' mechanical ellipsograph serves as an excellent visual and computational model for demonstrating the differences between a center-symmetric description of motion (Hooke's law) and a focal-oriented description (Newtonian gravitation), successfully uniting the author's early applied research with the current series of theoretical publications [1–4].

References

1. Strohm, V. F. (2026). *From Elliptical Motion Kinematics to the Gravitational Force*. Zenodo. Version 2. DOI: 10.5281/zenodo.20546389
2. Strohm, V. F. (2026). *Differential Equation of Motion of a Material Point Along an Ellipse Relative to its Geometric Center*. Zenodo. DOI: 10.5281/zenodo.20546429
3. Strohm, V. F. (2026). *A Dynamic Model of Tidal Deformations of the Earth's Elastic Crust in the Central Force Field of the Earth-Moon System*. Zenodo. DOI: 10.5281/zenodo.20521795
4. Strohm, V. F. (2026). *Mathematical Description of the Oscillations of the Radius Difference Modulus at Symmetrical Orbital Points*. Zenodo. DOI: 10.5281/zenodo.20069135