Ramanujan polynomial expansion for Hurwitz zeta function

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Abstract

Polynomial expansions of zeta functions provide a natural way to connect analytic continuation, regularized summation, Mellin analysis, and orthogonal polynomial theory. In this paper we try to develop a shifted Ramanujan–Mellin expansion for the Hurwitz zeta function in the critical strip. The construction combines Abel–Plana regularization over the nonnegative integers, Ramanujan summation for shifted Dirichlet terms, the Cayley transform of the right half-plane, and Mellin transforms of Laguerre functions. The main result proves that the Hurwitz zeta function admits a locally uniformly convergent expansion in a universal polynomial basis that is independent of the shift parameter. The shift appears only through explicit coefficients involving the digamma function and shifted Hurwitz zeta values. The Riemann zeta function is obtained as a special case. On the critical line, the normalized basis forms a complete orthonormal system with respect to a hyperbolic weight, and every zero of each basis polynomial lies on the critical line. The final result gives an exact zero-free compact criterion equivalent to the Riemann Hypothesis.

Keywords: Hurwitz zeta function; Riemann zeta function; Ramanujan summation; Abel–Plana formula; Mellin transform.

1. Introduction

The Riemann zeta function is first defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \Re(s) > 1, \qquad (1)$$

and it extends meromorphically to the complex plane with a single simple pole at $s = 1$. Riemann's 1859 memoir connected this function to the distribution of prime numbers and made the location of its zeros a central problem of modern number theory [1].

The Hurwitz zeta function introduces a positive shift parameter. For $\alpha > 0$, it is given in its half-plane of absolute convergence by

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}, \Re(s) > 1. \qquad (2)$$

It has a meromorphic continuation in s, with only a simple pole at $s = 1$, and the specialization

$$\zeta(s,1) = \zeta(s) \qquad (3)$$

recovers the Riemann zeta function [2,3].

The Hurwitz family is not merely a notational enlargement. It allows one to study zeta phenomena through a continuous shift parameter, and it connects naturally with the Gamma function, the digamma function, generalized Bernoulli polynomials, and Dirichlet L-functions. One of its fundamental representations is the Mellin integral

$$\Gamma(s)\zeta(s,\alpha) = \int_0^{\infty} \frac{x^{s-1}e^{-\alpha x}}{1-e^{-x}}\, dx, \Re(s) > 1, \qquad (4)$$

which becomes a tool for continuation once the singular term at $x = 0$ is separated [4].

Regularized summation supplies a complementary viewpoint. Instead of treating a divergent series as meaningless outside its region of convergence, Ramanujan summation extracts the constant term left after the dominant integral contribution is removed. This principle is closely related to the Euler–Maclaurin and Abel–Plana formulas and is treated systematically in the classical theory of divergent series [5,6].

For shifted zeta functions, Abel–Plana regularization is particularly natural. It converts sums over integer shifts into vertical-line integrals and gives a canonical regularized value of

$$\sum_{n\geq 0} (n+\alpha)^{-s} \qquad (5)$$

after subtracting the integral term

$$\int_0^{\infty} (x+\alpha)^{-s}\, dx = \frac{\alpha^{1-s}}{s-1}. \qquad (6)$$

This is precisely the term responsible for the pole of $\zeta(s,\alpha)$at $s = 1$.

A third component of the construction is the Mellin transform of Laguerre functions. The Laguerre polynomials form a classical orthogonal system on $(0,\infty)$, and their Mellin transforms produce polynomial systems in the spectral variable s. This connection places zeta expansions into an orthogonal Hilbert-space setting [7,8].

Polynomial and hypergeometric expansions of zeta functions have appeared in several forms through the studies. Maślanka introduced a hypergeometric-type expansion for the Riemann zeta function [9]. Báez-Duarte studied and clarified the convergence of Maślanka's representation [10]. Flajolet and Vepstas developed asymptotic tools for finite differences of zeta values and related Newton-type expansions [11]. Rubinstein gave identities involving the Hurwitz zeta function and related L-functions [12], and Adam studied generalized hypergeometric expansions connected with the Hurwitz zeta function [13]. These works illustrate that polynomial and hypergeometric representations are not only formal identities; they often encode analytic continuation, special values, and computational structure.

In this paper we develop a shifted Ramanujan–Mellin expansion for the entire Hurwitz family. The polynomial basis is constructed from the generating function

$$\frac{2}{1-\xi}\left(\frac{1-\xi}{1+\xi}\right)^{s} = \sum_{m=0}^{\infty} P_m(s)\xi^m, |\xi| < 1. \qquad (7)$$

After the Cayley substitution

$$\xi = \frac{z-1}{z+1}, \qquad (8)$$

one obtains the identity

$$z^{-s} = \sum_{m=0}^{\infty} \frac{1}{z+1}\left(\frac{z-1}{z+1}\right)^{m} P_m(s), \Re(z) > 0. \qquad (9)$$

The main result proves that, for every $\alpha > 0$,

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} x_m(\alpha) P_m(s), 0 < \Re(s) < 1, \qquad (10)$$

with locally uniform convergence. The coefficient sequence is explicit:

$$x_0(\alpha) = -\Psi(\alpha+1), \qquad (11)$$

and, for $m \geq 1$,

$$x_m(\alpha) = -\Psi(\alpha+1) + \sum_{k=1}^{m} \binom{m}{k} (-2)^k \left(\zeta(k+1, \alpha+1) - \frac{1}{k}\right), \qquad (12)$$

where $\Psi = \Gamma'/\Gamma$denotes the digamma function.

We actually do not aim at proving the Riemann Hypothesis in this work. It remains the assertion that all nontrivial zeros of $\zeta(s)$have real part ½ [14]. The final theorem records a precise zero-free compact criterion equivalent to RH for the polynomial approximants associated with $\alpha = 1$. The criterion is included because it identifies the exact zero-localization property required of the approximants.

2. Ramanujan Summation over the Nonnegative Integers

First, let

$$\mathbb{H} = \{z \in \mathbb{C} : \Re(z) > 0\} \qquad (13)$$

and let $\bar{\mathbb{H}}$ denote its closure. We use Abel–Plana summation for functions analytic in a neighborhood of $\bar{\mathbb{H}}$.

Let $\mathcal{A}$ be the class of functions $f$analytic in a neighborhood of $\bar{\mathbb{H}}$such that there are constants $a < 2\pi$and $b > 0$with the following property: for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$satisfying

$$| f(z) | \leq C_\varepsilon e^{b\Re(z)} e^{a|\Im(z)|}, \Re(z) > -\varepsilon. \qquad (14)$$

For $f \in \mathcal{A}$, define

$$\mathcal{R}_0(f) = \sum_{n\geq 0}^{\mathcal{R}} f(n) = \frac{1}{2} f(0) + i \int_0^\infty \frac{f(it) - f(-it)}{e^{2\pi t} - 1}\, dt, \qquad (15)$$

whenever the integral is convergent. This is the Ramanujan value of the series over the nonnegative integers.

Theorem 1. Let $f \in \mathcal{A}$. Suppose that $\sum_{n\geq 0} f(n)$and $\int_0^\infty f(x)\,dx$both converge, and suppose that the Abel–Plana remainder tends to zero. Then

$$\mathcal{R}_0(f) = \sum_{n=0}^{\infty} f(n) - \int_0^\infty f(x)\,dx. \qquad (16)$$

Proof of Theorem 1. Abel–Plana summation over the interval $[0, N]$gives

$$\sum_{n=0}^{N} f(n) = \frac{f(0)+f(N)}{2} + \int_0^N f(x)\,dx + i\int_0^\infty \frac{f(it)-f(-it)}{e^{2\pi t}-1}\,dt - i\int_0^\infty \frac{f(N+it)-f(N-it)}{e^{2\pi}\ -1}\,dt. \qquad (17)$$

Subtracting $\int_0^N f(x)\,dx$from both sides gives

$$\sum_{n=0}^{N} f(n) - \int_0^N f(x)\,dx = \frac{f(0)+f(N)}{2} + i\int_0^\infty \frac{f(it)-f(-it)}{e^{2\pi}\ -1}\,dt - i\int_0^\infty \frac{f(N+it)-f(N-it)}{e^{2\pi t}-1}\,dt. \qquad (18)$$

Letting $N \to \infty$, the hypotheses give $f(N) \to 0$, and the last integral tends to zero. The right-hand side tends to (15), proving (16).

Proposition 2. Let $\alpha > 0$. For $s \neq 1$,

$$\sum_{n\geq 0}^{\mathcal{R}} \frac{1}{(n+\alpha)^s} = \zeta(s,\alpha) - \frac{\alpha^{1-s}}{s-1}. \qquad (19)$$

Proof of Proposition 2. Assume first that $\Re(s) > 1$, and put

$$f(z) = (z+\alpha)^{-s}, \qquad (20)$$

where the principal branch is used. Because $\alpha > 0$, the function $f$is analytic in a neighborhood of $\bar{\mathbb{H}}$. By Theorem 1,

$$\sum_{n\geq 0}^{\mathcal{R}} \frac{1}{(n+\alpha)^s} = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s} - \int_0^{\infty} (x+\alpha)^{-s}\, dx. \qquad (21)$$

The series equals $\zeta(s,\alpha)$, and the integral is

$$\int_0^{\infty} (x+\alpha)^{-s}\, dx = \frac{\alpha^{1-s}}{s-1}. \qquad (22)$$

Thus (19) holds for $\Re(s) > 1$. Both sides have analytic continuation in $s$after the pole contribution is removed, so the identity theorem extends (19) to $s \neq 1$.

Lemma 3. Let $\alpha > 0$. For every integer $k \geq 1$,

$$\sum_{n\geq 0}^{\mathcal{R}} \frac{1}{(n+\alpha+1)^{k+1}} = \zeta(k+1,\alpha+1) - \frac{(\alpha+1)^{-k}}{k}. \qquad (23)$$

Moreover,

$$\sum_{n\geq 0}^{\mathcal{R}} \frac{1}{n+\alpha+1} = \log(\alpha+1) - \Psi(\alpha+1). \qquad (24)$$

Proof of Lemma 3. Formula (23) is Proposition 2 with αreplaced by $\alpha+1$and $s = k+1$. For the logarithmic case, take the finite part as $s \to 1$in

$$\zeta(s,\alpha+1) - \frac{(\alpha+1)^{1-s}}{s-1}. \qquad (25)$$

The Laurent expansion at $s = 1$is

$$\zeta(s,\alpha+1) = \frac{1}{s-1} - \Psi(\alpha+1) + O(s-1), \qquad (26)$$

and

$$\frac{(\alpha+1)^{1-s}}{s-1} = \frac{1}{s-1} - \log(\alpha+1) + O(s-1). \qquad (27)$$

Subtracting (27) from (26) gives (24).

3. The Cayley–Laguerre Polynomial Basis

For $m = 0,1,2,\ldots$, we define

$$\psi_m(z) = \frac{1}{z+1}\left(\frac{z-1}{z+1}\right)^m, \Re(z) > 0. \qquad (28)$$

Since

$$\mid z-1 \mid^2 = (\Re z - 1)^2 + (\Im z)^2 \qquad (29)$$

and

$$\mid z+1 \mid^2 = (\Re z + 1)^2 + (\Im z)^2, \qquad (30)$$

one has

$$\mid \frac{z-1}{z+1} \mid < 1 \qquad (31)$$

for every $\Re(z) > 0$.

Define polynomials $P_m(s)$ by

$$\frac{2}{1-\xi}\left(\frac{1-\xi}{1+\xi}\right)^s = \sum_{m=0}^{\infty} P_m(s)\xi^m, \mid \xi \mid < 1. \qquad (32)$$

Taking

$$\xi = \frac{z-1}{z+1} \qquad (33)$$

gives

$$z^{-s} = \sum_{m=0}^{\infty} \psi_m(z) P_m(s), \Re(z) > 0. \qquad (34)$$

Proposition 4. For every $m \geq 0$,

$$P_m(s) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k 2^{k+1} \frac{(s)_k}{k!}, \qquad (35)$$

where

$$(s)_0 = 1, (s)_k = s(s+1) \cdots (s+k-1) (k \geq 1). \qquad (36)$$

Proof of Proposition 4. We Put

$$\xi = \frac{x}{1+x}. \qquad (37)$$

Then

$$1 - \xi = \frac{1}{1+x}, 1 + \xi = \frac{1+2x}{1+x}, \qquad (38)$$

and therefore

$$\frac{1-\xi}{1+\xi} = \frac{1}{1+2x}. \qquad (39)$$

Multiplying (32) by ξgives

$$\sum_{m=0}^{\infty} P_m(s) \left(\frac{x}{1+x}\right)^{m+1} = 2x(1+2x)^{-s}. \qquad (40)$$

Expanding the right-hand side,

$$2x(1+2x)^{-s} = \sum_{k=0}^{\infty} (-1)^k 2^{k+1} \frac{(s)_k}{k!} x^{k+1}. \qquad (41)$$

The binomial-transform identity

$$\sum_{m=0}^{\infty} b_m \left(\frac{x}{1+x}\right)^{m+1} = \sum_{k=0}^{\infty} a_k\, x^{k+1} \Leftrightarrow b_m = \sum_{k=0}^{m} \binom{m}{k} a_k \qquad (42)$$

then yields (35).

The first polynomials are

$$P_0(s) = 2, \qquad (43)$$

$$P_1(s) = 2 - 4s, \qquad (44)$$

$$P_2(s) = 4s^2 - 4s + 2, \qquad (45)$$

$$P_3(s) = -\frac{8}{3}s^3 + 4s^2 - \frac{16}{3}s + 2. \qquad (46)$$

Let $L_m(x)$denote the Laguerre polynomial

$$L_m(x) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k \frac{x^k}{k!}. \qquad (47)$$

Lemma 5. For $\Re(s) > 0$,

$$P_m(s) = \frac{2}{\Gamma(s)} \int_0^{\infty} e^{-x}\, L_m(2x) x^{s-1}\, dx. \qquad (48)$$

Proof of Lemma 5. Expanding $L_m(2x)$, we get

$$\int_0^{\infty} e^{-x}\, L_m(2x) x^{s-1}\, dx = \sum_{k=0}^{m} \binom{m}{k} (-1)^k 2^k \frac{\Gamma(s+k)}{k!}. \qquad (49)$$

Since

$$\Gamma(s+k) = \Gamma(s)(s)_k, \qquad (50)$$

equation (48) follows from (35).

Lemma 6. If $0 < \Re(s) < 1$, then

$$P_m(s) = \frac{2\sin(\pi s)}{\pi} \int_0^1 t^{s-1} (1-t)^{-s} (1-2t)^m \, dt. \qquad (51)$$

Proof of Lemma 6. For $0 < \Re(s) < 1$,

$$\frac{(s)_k}{k!} = \frac{1}{\Gamma(s)\Gamma(1-s)} \int_0^1 t^{s+k-1} (1-t)^{-s} \, dt. \qquad (52)$$

With substituting (52) into (35) gives

$$P_m(s) = \frac{2}{\Gamma(s)\Gamma(1-s)} \int_0^1 t^{s-1} (1-t)^{-s} \sum_{k=0}^{m} \binom{m}{k} (-2t)^k \, dt. \qquad (53)$$

The finite sum equals $(1-2t)^m$. Euler's reflection formula,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, \qquad (54)$$

then gives (51).

Lemma 7. Let $s = \sigma + i\tau$, where $0 < \sigma < 1$. For $m \geq 1$,

$$| P_m(s) | \leq e^{\pi|\tau|} \left(\frac{\Gamma(\sigma)}{m^{\sigma}} + \frac{\Gamma(1-\sigma)}{m^{1-\sigma}} \right). \qquad (55)$$

Proof of Lemma 7. From (51),

$$| P_m(s) | \leq \frac{2 \, | \sin(\pi s) |}{\pi} \int_0^1 t^{\sigma-1} (1-t)^{-\sigma} \, | 1-2t |^m \, dt. \qquad (56)$$

The bound

$$| \sin(\pi s) | \leq e^{\pi|\tau|} \qquad (57)$$

and the change of variables $u = 1-2t$give

$$| P_m(s) | \leq e^{\pi|\tau|} \int_{-1}^{1} (1-u)^{\sigma-1} (1+u)^{-\sigma} \, | u |^m \, du. \qquad (58)$$

Splitting the integral at $u = 0$,

$$\int_{-1}^{1} (1-u)^{\sigma-1}(1+u)^{-\sigma} \mid u \mid^m \; du \leq \int_0^1 (1-u)^{\sigma-1} u^m \, du + \int_0^1 (1-u)^{-\sigma} u^m \, du. \tag{59}$$

For $0 < \beta < 1$,

$$\int_0^1 (1-u)^{\beta-1} u^m \, du \leq \int_0^\infty t^{\beta-1} \, e^{-mt} \, dt = \frac{\Gamma(\beta)}{m^\beta}. \tag{60}$$

Taking $\beta = \sigma$and $\beta = 1 - \sigma$proves (55).

Lemma 8. For every $m \geq 0$,

$$P_m(1-s) = (-1)^m P_m(s). \tag{61}$$

Proof of Lemma 8. Replace $t$by $1 - t$in (51). Since

$$1 - 2(1-t) = -(1-2t), \tag{62}$$

and

$$\sin(\pi(1-s)) = \sin(\pi s), \tag{63}$$

we obtain (61).

4. The Shifted Ramanujan–Mellin Expansion

Fix $\alpha > 0$. Define

$$z_m(\alpha) = \sum_{n\geq 0}^{\mathcal{R}} \psi_m \, (n+\alpha). \tag{64}$$

Since $n + \alpha > 0$, equation (34) gives

$$(n+\alpha)^{-s} = \sum_{m=0}^{\infty} \psi_m \, (n+\alpha) P_m(s). \tag{65}$$

Theorem 9. Let $\alpha > 0$. For $0 < \Re(s) < 1$,

$$\zeta(s,\alpha) = \frac{\alpha^{1-s}}{s-1} + \sum_{m=0}^{\infty} z_m(\alpha) P_m(s). \qquad (66)$$

The convergence is locally uniform in the strip $0 < \Re(s) < 1$.

Proof of Theorem 9. By Proposition 2,

$$\sum_{n\geq 0}^{\mathcal{R}} (n+\alpha)^{-s} = \zeta(s,\alpha) - \frac{\alpha^{1-s}}{s-1}. \qquad (67)$$

It remains to justify termwise Ramanujan summation in (65).

From (15),

$$z_m(\alpha) = \frac{1}{2}\psi_m(\alpha) + i\int_0^{\infty} \frac{\psi_m(\alpha+it) - \psi_m(\alpha-it)}{e^{2\pi} \; - 1}\, dt. \qquad (68)$$

For $t \geq 0$,

$$|\, \psi_m(\alpha+it)\, | = \frac{1}{\sqrt{(\alpha+1)^2+t^2}} \left(\frac{\sqrt{(\alpha-1)^2+t^2}}{\sqrt{(\alpha+1)^2+t^2}} \right)^m . \qquad (69)$$

Put

$$r_\alpha(t) = \frac{\sqrt{(\alpha-1)^2+t^2}}{\sqrt{(\alpha+1)^2+t^2}}. \qquad (70)$$

Since $\alpha > 0$, one has $0 \leq r_\alpha(t) < 1$. Hence

$$|\, \psi_m(\alpha+it) - \psi_m(\alpha-it)\, | \leq \frac{2 r_\alpha(t)^m}{\sqrt{(\alpha+1)^2+t^2}}. \qquad (71)$$

Let K be compact in $0 < \Re(s) < 1$. Choose $\eta \in (0,1/2]$such that

$$\eta \leq \Re(s) \leq 1-\eta, s \in K. \qquad (72)$$

By Lemma 7, there exists $C_K > 0$satisfying

$$\mid P_m(s) \mid \leq C_K m^{-\eta}, m \geq 1, s \in K. \qquad (73)$$

The endpoint contribution is harmless because

$$\mid \psi_m(\alpha) \mid = \frac{1}{\alpha+1} \mid \frac{\alpha-1}{\alpha+1} \mid^m, \qquad (74)$$

and the ratio is strictly smaller than 1.

For the integral contribution, the absolute value is controlled by

$$\frac{2C_K}{\sqrt{(\alpha+1)^2+t^2}} \sum_{m=1}^{\infty} r_\alpha(t)^m m^{-\eta}. \qquad (75)$$

Near $t=0$, the ratio $r_\alpha(t)$is bounded away from 1, and

$$\psi_m(\alpha+it) - \psi_m(\alpha-it) = O(t). \qquad (76)$$

Since

$$e^{2\pi t} - 1 \sim 2\pi t, t \to 0, \qquad (77)$$

the integrand remains bounded near the origin.

For large t,

$$1 - r_\alpha(t) = \frac{(\alpha+1)^2 - (\alpha-1)^2}{\sqrt{(\alpha+1)^2+t^2}\left(\sqrt{(\alpha+1)^2+t^2}+\sqrt{(\alpha-1)^2+t^2}\right)} = O(t^{-2}). \qquad (78)$$

The standard estimate

$$\sum_{m=1}^{\infty} r^m m^{-\eta} = O((1-r)^{\eta-1}), r \to 1^-, \qquad (79)$$

shows that the m-sum grows at most polynomially in t. The factor $(e^{2\pi t}-1)^{-1}$then gives exponential decay at infinity. Dominated convergence applies, locally uniformly on K.

Therefore

$$\sum_{m=0}^{\infty} z_m(\alpha)P_m(s) = \sum_{n\geq 0}^{\mathcal{R}} \sum_{m=0}^{\infty} \psi_m(n+\alpha)P_m(s) = \sum_{n\geq 0}^{\mathcal{R}} (n+\alpha)^{-s}. \qquad (80)$$

Combining (80) with (67) proves (66).

Define

$$y_m(\alpha) = \int_0^{\alpha} \psi_m(x)\,dx. \qquad (81)$$

Lemma 10. For $0 < \Re(s) < 1$,

$$\frac{\alpha^{1-s}}{s-1} = -\sum_{m=0}^{\infty} y_m(\alpha)P_m(s). \qquad (82)$$

The convergence is locally uniform in the strip.

Proof of Lemma 10. For $x > 0$, equation (34) gives

$$x^{-s} = \sum_{m=0}^{\infty} \psi_m(x)P_m(s). \qquad (83)$$

Integrating over $0 < x < \alpha$, we get formally

$$\int_0^{\alpha} x^{-s}\,dx = \sum_{m=0}^{\infty} \left(\int_0^{\alpha} \psi_m(x)\,dx \right) P_m(s). \qquad (84)$$

We justify the interchange. If $0 < x \leq 1$, then

$$\left| \frac{x-1}{x+1} \right| = \frac{1-x}{1+x} \leq e^{-x}. \qquad (85)$$

Therefore

$$\int_0^{\min(\alpha,1)} |\psi_m(x)|\ dx \le \int_0^1 e^{-mx}\ dx \le \frac{1}{m+1}. \qquad (86)$$

On any interval $[\delta, \alpha]$with $\delta > 0$, the Cayley factor is bounded by a number strictly smaller than 1, so that part decays exponentially in m. Thus there is $C_\alpha > 0$such that

$$\int_0^{\alpha} |\psi_m(x)|\ dx \le \frac{C_\alpha}{m+1}. \qquad (87)$$

Together with Lemma 7, this gives

$$\sum_{m=1}^{\infty} |P_m(s)| \int_0^{\alpha} |\psi_m(x)|\ dx < \infty \qquad (88)$$

locally uniformly in $0 < \Re(s) < 1$. Hence (84) is justified. Since

$$\int_0^{\alpha} x^{-s}\ dx = \frac{\alpha^{1-s}}{1-s}, \qquad (89)$$

equation (82) follows.

Theorem 11. Let $\alpha > 0$. For $0 < \Re(s) < 1$,

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} x_m(\alpha) P_m(s), \qquad (90)$$

where the convergence is locally uniform. The coefficients are

$$x_0(\alpha) = -\Psi(\alpha+1), \qquad (91)$$

and, for $m \ge 1$,

$$x_m(\alpha) = -\Psi(\alpha+1) + \sum_{k=1}^{m} \binom{m}{k} (-2)^k \left(\zeta(k+1, \alpha+1) - \frac{1}{k}\right). \qquad (92)$$

Proof of Theorem 11. By Theorem 9 and Lemma 10,

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} (z_m(\alpha) - y_m(\alpha))P_m(s). \qquad (93)$$

It remains to compute $z_m(\alpha) - y_m(\alpha)$.

From (28),

$$\psi_m(n+\alpha) = \frac{1}{n+\alpha+1}\left(\frac{n+\alpha-1}{n+\alpha+1}\right)^m. \qquad (94)$$

Since

$$\frac{n+\alpha-1}{n+\alpha+1} = 1 - \frac{2}{n+\alpha+1}, \qquad (95)$$

the binomial theorem gives

$$\psi_m(n+\alpha) = \sum_{k=0}^{m} \binom{m}{k} (-2)^k (n+\alpha+1)^{-k-1}. \qquad (96)$$

Using Lemma 3,

$$z_m(\alpha) = \log(\alpha+1) - \Psi(\alpha+1) + \sum_{k=1}^{m} \binom{m}{k} (-2)^k \left(\zeta(k+1,\alpha+1) - \frac{(\alpha+1)^{-k}}{k}\right). \qquad (97)$$

Next,

$$y_m(\alpha) = \int_0^{\alpha} \frac{1}{x+1}\left(1 - \frac{2}{x+1}\right)^m dx. \qquad (98)$$

Expanding the integrand,

$$y_m(\alpha) = \log(\alpha+1) + \sum_{k=1}^{m} \binom{m}{k} (-2)^k \int_0^{\alpha} (x+1)^{-k-1}\, dx. \qquad (99)$$

For $k \geq 1$,

$$\int_0^{\alpha} (x+1)^{-k-1}\, dx = \frac{1-(\alpha+1)^{-k}}{k}. \qquad (100)$$

Thus

$$y_m(\alpha) = \log(\alpha+1) + \sum_{k=1}^{m} \binom{m}{k} (-2)^k \frac{1-(\alpha+1)^{-k}}{k}. \qquad (101)$$

Subtracting (101) from (97) gives (91) and (92).

Remark 1. The basis $P_m(s)$is universal. The parameter αappears only through the coefficient functional $x_m(\alpha)$.

Example 1. For $\alpha = 1$, equation (90) becomes

$$\zeta(s) = \sum_{m=0}^{\infty} x_m(1) P_m(s), 0 < \Re(s) < 1. \qquad (102)$$

Since

$$\Psi(2) = 1 - \gamma, \qquad (103)$$

the first coefficient is

$$x_0(1) = \gamma - 1. \qquad (104)$$

For $m \geq 1$,

$$x_m(1) = \gamma - 1 + \sum_{k=1}^{m} \binom{m}{k} (-2)^k \left(\zeta(k+1) - 1 - \frac{1}{k}\right). \qquad (105)$$

Equivalently,

$$x_m(1) = \gamma + \sum_{k=1}^{m} \binom{m}{k} (-2)^k \zeta(k+1) + (-1)^{m+1} + \sum_{k=1}^{m} \frac{1 + (-1)^{k-1}}{k}. \qquad (106)$$

The specialization $\alpha = 1$agrees with the known Riemann-zeta member of this polynomial construction.

The first values are

$$x_1(1) = \gamma - 2\zeta(2) + 3, \qquad (107)$$
$$x_2(1) = \gamma - 4\zeta(2) + 4\zeta(3) + 1, \qquad (108)$$
$$x_3(1) = \gamma - 6\zeta(2) + 12\zeta(3) - 8\zeta(4) + \frac{11}{3}, \qquad (109)$$

and

$$x_4(1) = \gamma - 8\zeta(2) + 24\zeta(3) - 32\zeta(4) + 16\zeta(5) + \frac{5}{3}. \qquad (110)$$

5. Orthogonality on the Critical Line

We define

$$\phi_m(x) = \sqrt{2}\, e^{-x} L_m(2x), x > 0. \qquad (111)$$

The Laguerre orthogonality relation gives

$$\int_0^\infty \phi_m(x) \phi_n(x)\, dx = \delta_{mn}. \qquad (112)$$

Hence $\{\phi_m\}_{m \geq 0}$is an orthonormal basis of $L^2(0, \infty)$.

By Lemma 5,

$$\mathcal{M}\phi_m(s) = \int_0^\infty \phi_m(x) x^{s-1}\, dx = \frac{1}{\sqrt{2}} \Gamma(s) P_m(s). \qquad (113)$$

Then define

$$Q_m(t) = \frac{1}{2} P_m\left(\frac{1}{2}+it\right). \qquad (114)$$

Theorem 12. The sequence $\{Q_m\}_{m\geq 0}$is a complete orthonormal system in

$$L^2\left(\mathbb{R}, \frac{dt}{\cosh(\pi t)}\right), \qquad (115)$$

with inner product

$$\langle F, G\rangle = \int_{-\infty}^{\infty} F(t)\, \bar{G}(t) \frac{dt}{\cosh(\pi t)}. \qquad (116)$$

Proof of Theorem 12. Mellin–Parseval gives

$$\int_0^{\infty} \phi_m(x)\phi_n(x)\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{M}\,\phi_m\left(\frac{1}{2}+it\right) \mathcal{M}\,\bar{\phi_n}\left(\frac{1}{2}+\imath t\right) dt. \qquad (117)$$

Using (113),

$$\delta_{mn} = \frac{1}{4\pi}\int_{-\infty}^{\infty} \mid \Gamma\left(\frac{1}{2}+it\right)\mid^2 P_m\left(\frac{1}{2}+it\right) P_n\left(\bar{\frac{1}{2}}+\imath t\right) dt. \qquad (118)$$

The identity

$$\mid \Gamma\left(\frac{1}{2}+it\right)\mid^2 = \frac{\pi}{\cosh(\pi t)} \qquad (119)$$

turns (118) into

$$\delta_{mn} = \int_{-\infty}^{\infty} Q_m(t)\bar{Q_n}(t)\frac{dt}{\cosh(\pi t)}. \qquad (120)$$

This proves orthonormality.

For completeness, let $H \in L^2(\mathbb{R}, dt/\cosh\ \pi t)$be orthogonal to every Q_m. Then

$$H(t)\Gamma\left(\frac{1}{2}+it\right) \in L^2(\mathbb{R}, dt). \qquad (121)$$

By (113), orthogonality to every Q_m is equivalent to orthogonality of the function in (121) to each Mellin transform

$$\mathcal{M}\phi_m\left(\frac{1}{2}+it\right). \qquad (122)$$

The Mellin transform is unitary on the critical line after the logarithmic change of variables $x = e^u$, and $\{\phi_m\}$ is complete in $L^2(0,\infty)$. Therefore the function in (121) vanishes almost everywhere. Hence $H = 0$, proving completeness.

Corollary 13. For every $m \geq 1$, all zeros of $P_m(s)$ lie on

$$\Re(s) = \frac{1}{2}. \qquad (123)$$

Proof of Corollary 13. By Lemma 8,

$$P_m\left(\frac{1}{2}-it\right) = (-1)^m P_m\left(\frac{1}{2}+it\right). \qquad (124)$$

Therefore

$$R_m(t) = i^m Q_m(t) \qquad (125)$$

is a real polynomial in t of degree m. The polynomials R_m are orthogonal with respect to the positive measure

$$d\mu(t) = \frac{dt}{\cosh(\pi t)}. \qquad (126)$$

The standard theorem on real orthogonal polynomials states that R_m has m real simple zeros. Since

$$R_m(t) = i^m \frac{1}{2} P_m\left(\frac{1}{2}+it\right), \qquad (127)$$

every zero of P_m has the form

$$s = \frac{1}{2} + it, t \in \mathbb{R}. \qquad (128)$$

Thus all zeros of $P_m$lie on $\Re(s) = 1/2$.

Example 2. The first zero sets are

$$P_1(s) = 0 \Leftrightarrow s = \frac{1}{2}, \qquad (129)$$
$$P_2(s) = 0 \Leftrightarrow s = \frac{1}{2} \pm \frac{i}{2}, \qquad (130)$$

and

$$P_3(s) = 0 \Leftrightarrow s = \frac{1}{2}, s = \frac{1}{2} \pm i\sqrt{\frac{5}{4}}. \qquad (131)$$

6. Hilbert-Space Interpretation of the Coefficients

For $\alpha > 0$, define

$$F_\alpha(x) = \frac{e^{-\alpha x}}{1 - e^{-x}} - \frac{1}{x}, x > 0. \qquad (132)$$

Near $x = 0$,

$$\frac{e^{-\alpha x}}{1 - e^{-x}} = \frac{1}{x} + \left(\frac{1}{2} - \alpha\right) + O(x), \qquad (133)$$

so

$$F_\alpha(x) = \frac{1}{2} - \alpha + O(x). \qquad (134)$$

As $x \to \infty$,

$$F_\alpha(x) = -\frac{1}{x} + O(e^{-\alpha}\ ). \qquad (135)$$

Therefore $F_\alpha \in L^2(0,\infty)$.

Proposition 14. For $0 < \Re(s) < 1$,

$$\mathcal{M}F_\alpha(s) = \int_0^\infty F_\alpha(x)x^{s-1}\,dx = \Gamma(s)\zeta(s,\alpha). \qquad (136)$$

Proof of Proposition 14. For $\Re(s) > 1$, the Mellin representation (4) holds. The expansion at the origin shows that subtracting $1/x$ removes the nonintegrable singular part. The resulting integral converges in the strip $0 < \Re(s) < 1$and gives the analytic continuation of $\Gamma(s)\zeta(s,\alpha)$there. Hence (136) follows.

Proposition 15. The coefficients in Theorem 11 satisfy

$$x_m(\alpha) = \frac{1}{\sqrt{2}}\int_0^\infty \phi_m(x)F_\alpha(x)\,dx. \qquad (137)$$

Proof of Proposition 15. By (111),

$$\frac{1}{\sqrt{2}}\int_0^\infty \phi_m(x)F_\alpha(x)\,dx = \int_0^\infty e^{-x}L_m(2x)F_\alpha(x)\,dx. \qquad (138)$$

Using (47),

$$\int_0^\infty e^{-x}L_m(2x)F_\alpha(x)\,dx = \sum_{k=0}^{m}\binom{m}{k}(-2)^k A_k(\alpha), \qquad (139)$$

where

$$A_k(\alpha) = \int_0^\infty e^{-x}\frac{x^k}{k!}F_\alpha(x)\,dx. \qquad (140)$$

For $k \geq 1$,

$$A_k(\alpha) = \int_0^\infty e^{-x}\frac{x^k}{k!}\frac{e^{-\alpha}}{1-e^{-x}}\,dx - \int_0^\infty e^{-x}\frac{x^{k-1}}{k!}\,dx. \qquad (141)$$

The first integral equals

$$\sum_{n=0}^{\infty}\int_0^{\infty}\frac{x^k}{k!}e^{-(n+\alpha+1)x}\,dx=\sum_{n=0}^{\infty}\frac{1}{(n+\alpha+1)^{k+1}}=\zeta(k+1,\alpha+1), \qquad (142)$$

and the second integral equals

$$\int_0^{\infty}e^{-x}\frac{x^{k-1}}{k!}\,dx=\frac{\Gamma(k)}{k!}=\frac{1}{k}. \qquad (143)$$

Therefore

$$A_k(\alpha)=\zeta(k+1,\alpha+1)-\frac{1}{k}, k\geq 1. \qquad (144)$$

For $k=0$, the classical digamma integral gives

$$A_0(\alpha)=\int_0^{\infty}\left(\frac{e^{-(\alpha+1)x}}{1-e^{-x}}-\frac{e^{-x}}{x}\right)dx=-\Psi(\alpha+1). \qquad (145)$$

Substituting (144) and (145) into (139) gives precisely (91) and (92).

Since $\{\phi_m\}_{m\geq 0}$is an orthonormal basis of $L^2(0,\infty)$, Proposition 15 gives

$$F_\alpha(x)=\sqrt{2}\sum_{m=0}^{\infty}x_m\,(\alpha)\phi_m(x) \qquad (146)$$

in $L^2(0,\infty)$. Taking Mellin transforms and using (113) yields

$$\zeta\left(\frac{1}{2}+it,\alpha\right)=2\sum_{m=0}^{\infty}x_m\,(\alpha)Q_m(t) \qquad (147)$$

in

$$L^2\left(\mathbb{R}, \frac{dt}{\cosh\,(\pi t)}\right). \qquad (148)$$

The coefficient formula is

$$x_m(\alpha) = \frac{1}{2}\int_{-\infty}^{\infty} Q_m(t)\,\zeta\left(\frac{1}{2} + it, \alpha\right)\frac{dt}{\cosh(\pi t)}. \qquad (149)$$

7. The Classical Case and a Zero-Free Compact Criterion

Set $\alpha = 1$, and define

$$S_N(s) = \sum_{m=0}^{N} x_m(1)P_m(s). \qquad (150)$$

By Theorem 11,

$$S_N(s) \longrightarrow \zeta(s) \qquad (151)$$

locally uniformly in

$$0 < \Re(s) < 1. \qquad (152)$$

Let

$$\Omega = \left\{s \in \mathbb{C}: 0 < \Re(s) < 1, \Re(s) \neq \frac{1}{2}\right\}. \qquad (153)$$

Theorem 16. The Riemann Hypothesis is equivalent to the following zero-free compact property: for every compact set $K \subset \Omega$, there exists $N_K$such that

$$S_N(s) \neq 0 \text{for all } s \in K \text{ and all } N \geq N_K. \qquad (154)$$

Proof of Theorem 16. Assume the Riemann Hypothesis. Then $\zeta(s) \neq 0$on every compact set $K \subset \Omega$. Since $K$is compact,

$$\min_{s\in K} \mid \zeta(s) \mid > 0. \qquad (155)$$

By locally uniform convergence, there is $N_K$such that

$$\sup_{s\in K} \mid S_N(s) - \zeta(s) \mid < \min_{s\in K} \mid \zeta(s) \mid \quad (156)$$

for all $N \geq N_K$. Hence $S_N(s) \neq 0$on K.

Conversely, assume the zero-free compact property. If the Riemann Hypothesis were false, then there would be a zero $\rho$of ζsuch that

$$0 < \Re(\rho) < 1, \Re(\rho) \neq \frac{1}{2}. \quad (157)$$

Choose a closed disk $D \subset \Omega$, centered at ρ, whose boundary contains no zero of ζ. Since $S_N \to \zeta$locally uniformly, Hurwitz's theorem implies that $S_N$has a zero in $D$for all sufficiently large N. This contradicts (154), applied with $K = D$. Therefore no such zero exists, and the Riemann Hypothesis follows.

Remark 2. Theorem 16 is an equivalent formulation, not a proof of the Riemann Hypothesis. The open problem is to establish the zero-free compact property for the explicit approximants S_N.

The critical-line geometry of the polynomial basis is illustrated in **Figure 1**. Each zero of $P_m(s)$is written as $s = 1/2 + it$, so the plot records only the ordinate t. The symmetry about $t = 0$reflects the identity $P_m(1 - s) = (-1)^m P_m(s)$.

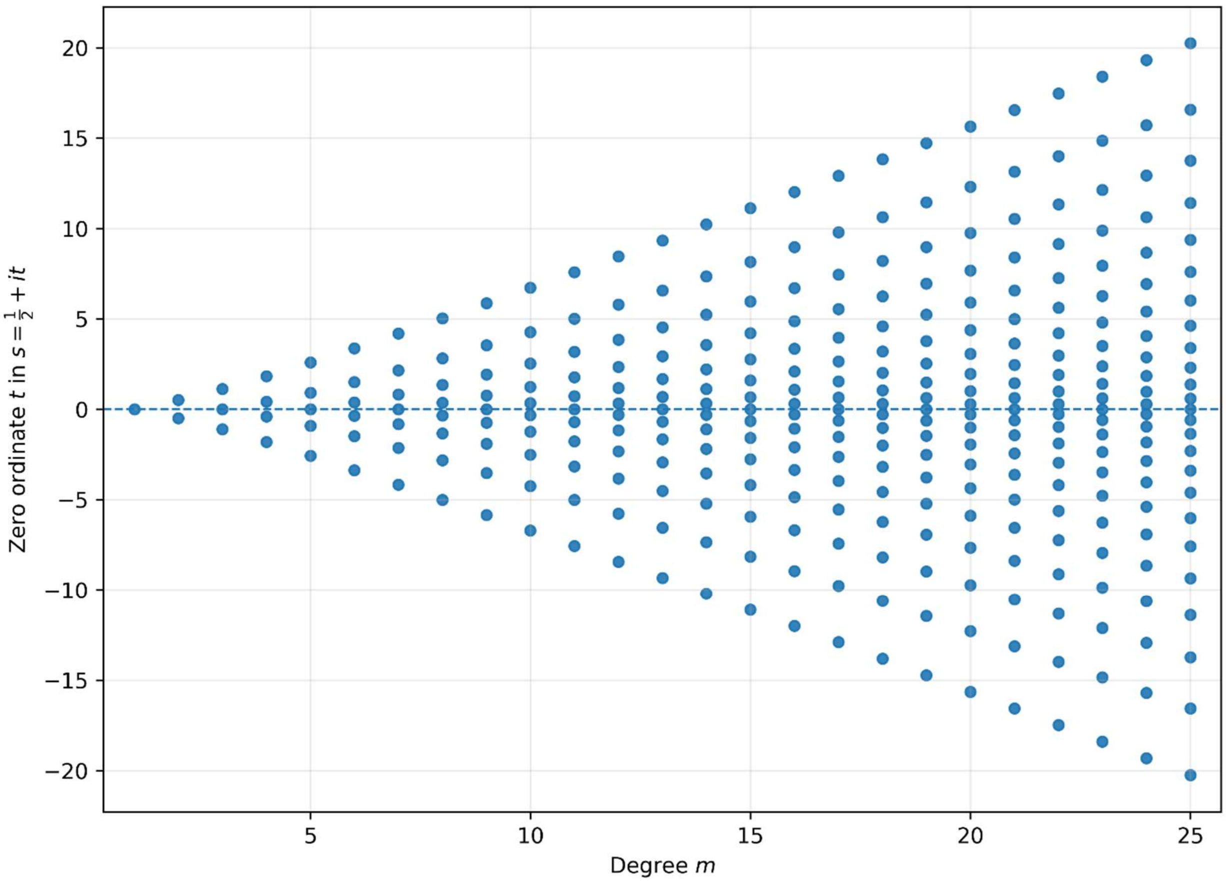

Figure 1. The critical-line geometry of $P_m(s)$

This agrees with the orthogonality theorem: after normalization, $Q_m(t) = P_m(1/2 + it)/2$ belongs to a real orthogonal polynomial system with respect to $dt/\cosh(\pi t)$, and therefore its zeros are real in the t-variable. Equivalently, the zeros of $P_m(s)$lie on $\Re(s) = 1/2$.

As can be seen in **Figure 2,** the shift parameter α enters the expansion through the coefficients $x_m(\alpha)$. The polynomial basis $P_m(s)$is fixed, while the Hurwitz parameter changes the coefficient sequence. The first coefficient is especially sensitive to the shift, since $x_0(\alpha) = -\Psi(\alpha + 1)$.

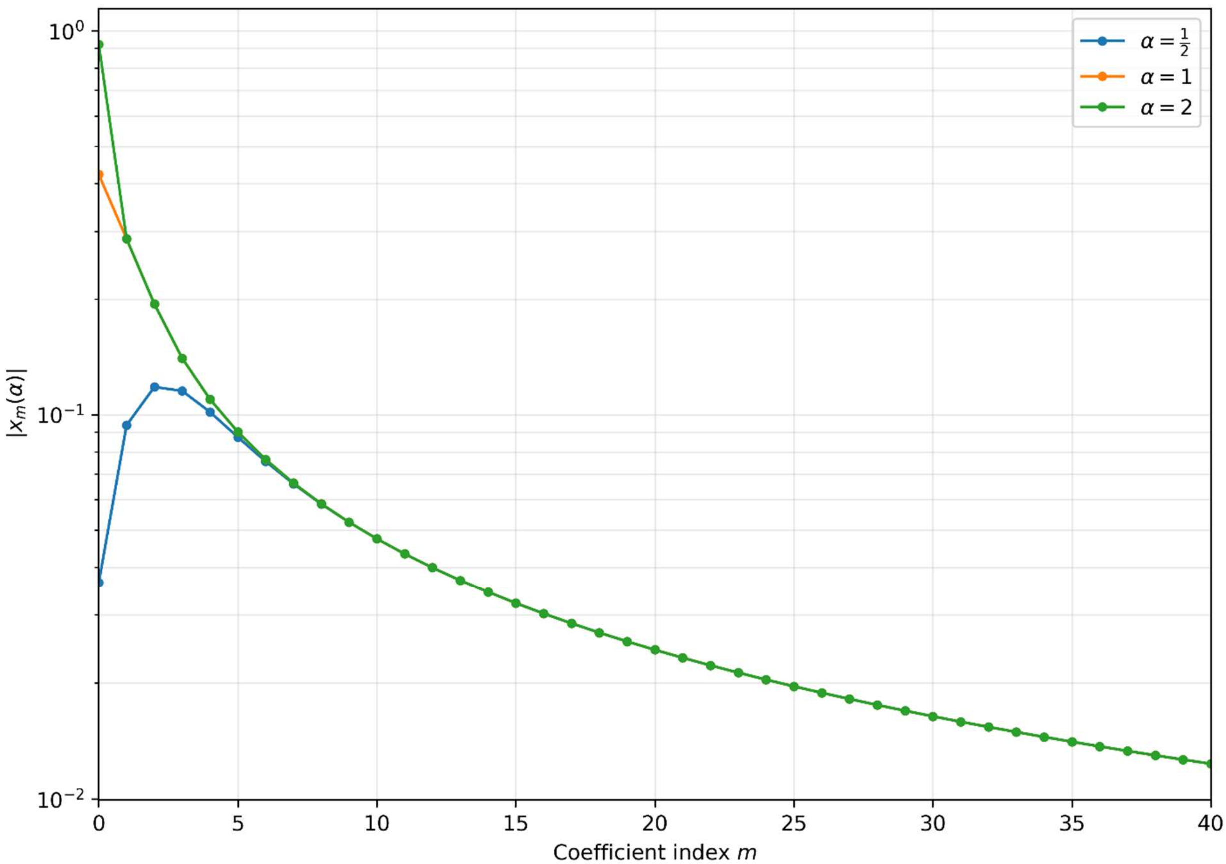

Figure 2. Decay of shifted Ramanujan-Mellin coefficients

After the first few indices, the three plotted profiles become close, indicating that the higher-order coefficient decay is largely governed by the common Ramanujan–Mellin structure. The logarithmic scale highlights this decay.

8. Conclusion

This work proves a shifted Ramanujan–Mellin expansion for the Hurwitz zeta function. The main formula is

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} x_m(\alpha) P_m(s), \alpha > 0, 0 < \Re(s) < 1, \quad (163)$$

with locally uniform convergence. The polynomial basis is independent of α, while the coefficients are explicit finite expressions involving the digamma function and shifted Hurwitz zeta values.

The construction is built from three connected structures. Ramanujan summation removes the pole-producing integral contribution from the shifted Dirichlet series. The Cayley transform produces the polynomial basis $P_m(s)$. Mellin transforms of Laguerre functions give the orthogonality on the critical line.

For

$$Q_m(t) = \frac{1}{2} P_m\left(\frac{1}{2}+it\right), \qquad (164)$$

the sequence $\{Q_m\}_{m\geq 0}$is a complete orthonormal basis of

$$L^2\left(\mathbb{R}, \frac{dt}{\cosh(\pi t)}\right). \qquad (165)$$

Consequently, every zero of P_m lies on $\Re(s) = 1/2$. This statement concerns the basis polynomials, not arbitrary linear combinations of them. In the special case $\alpha = 1$, the expansion becomes a polynomial expansion of the Riemann zeta function. The zero-free compact criterion in Theorem 16 is equivalent to the Riemann Hypothesis. The criterion identifies the precise zero-localization property that remains to be established for the polynomial approximants.

Dedication

To ***Srinivasa Ramanujan***, whose mathematics still feels alive: fearless, luminous, and far ahead of its time. His work on infinite series, summation, special functions, and number theory continues to remind us that beauty in mathematics often begins where ordinary intuition hesitates.

And to my daughter, ***Artemisia***, whose presence brought quiet, steadiness, and peace while this manuscript was being completed. Her calm gave me the strength to continue.

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