Generalization of A New Home for Bivectors

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Abstract

The impetus for the work is this quote:

> *"...as shown by Gel'fand's approach, we can only abstract a unique manifold if our algebra is commutative."[1]*

Geometric algebra is non-commutative. Components of different grades can be staged on different manifolds. As operations on those elements proceed, they can effect the promotion and/or demotion of components to higher and/or lower grades, and thus to different manifolds. This paper includes imagery that visually displays bivector addition and rotation on a sphere.

David Hestenes interpreted the vector product or rotor in two-dimensions:

> *"as a directed arc of fixed length that can be rotated at will on the unit circle, just as we interpret a vector* $\mathbf{a}$ *as a directed line segment that can be translated at will without changing its length or direction…"[2]*

Rotors can be used to develop addition and multiplication of bivectors on a sphere. For those rotational dynamics, rotors of length $\pi/2$ are the basis elements. The geometric algebra of bivectors – Hamilton's "*pure quaternions*" – is thus shown to transparently operate on a spherical manifold.

This paper also explores the possible generalizations that emerge from the placement of the graded elements which make up a geometric algebra onto separate manifolds.

Keywords: Bivectors/Visualization/Rotors/Spherical Manifold/Quaternions/Non-Commutative Algebra

Statements and Declarations

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Introduction

Hiley's quest for an algebra of process led him to develop rules that quickly take shape in one form as a quaternion algebra[3]. In his textbook[4], Alan Macdonald redefines the pure quaternions, which exclude the scalar part, for use as the bivectors of geometric algebra. Those sit comfortably on a spherical manifold. In three dimensions, that manifold hosts visual representations for both bivector multiplication, and bivector addition, representations which transparently subsume the algebraic formulation of those operations. The algebra is a symbolic analog for rotational transformations in 3D space, highlighting the power of the geometric algebra developed by David Hestenes[5].

In the case formulated by Hiley which led him to the quaternions, the quest for the process in question implicates rotational dynamics on a spherical manifold in three dimensions. That process further equipped with the dynamics of time for those operations, will encompass swirling movements both expansive and contractive throughout the spherical domain. The trajectory of those movements will be a geometrical mirror for the dynamical process in question.

An algebra that embeds into multiple geometries

In a follow-up paper[1], Hiley and Callaghan expand on the reasoning for pursuing an algebra for that structure process:

> *Most of the work in[3] was to establish how it is possible to produce an algebraic description of this structure process. Having demonstrated how this was possible, we went on to show that this algebra had enfolded in it a series of what we called 'shadow manifolds'. We deliberately choose the plural 'manifolds' because* ***we have a non-commutative algebra and as shown by Gel'fand's approach, we can only abstract a unique manifold if our algebra is commutative****.*
>
> ***Notice that our approach stands the conventional approach on its head, as it were, because we start with the algebra and then abstract the geometry. We do not start with a a priori given manifold and then build an algebra on that.*** *[emphases added]*

Geometric algebra is non-commutative. The implication is that algebraic components of different grades must be staged on different geometric manifolds. As operations on those elements proceed, they will effect the promotion and/or demotion of components to higher and/or lower grades and to different manifolds.

In his Oersted Medal Lecture[2], Hestenes introduced the key to bivectors in two dimensions, one that translates to three dimensions. He established a definition for rotors that parallels the definition for vectors in the plane. In his words:

> *...we should interpret* $\boldsymbol{U}_{\boldsymbol{\theta}}$ [1] *as a directed arc of fixed length that can be rotated at will on the unit circle, just as we interpret a vector* $\boldsymbol{a}$ *as a directed line segment that can be translated at will without changing its length or direction.*

Corresponding to those -dimensional bivectors, basis components of a three-dimensional bivector algebra are staged and rotated on a spherical manifold, each translated without changing its length or direction. Hamilton's pure quaternions are revealed to be homologous to the coordinate-free rendering of bivectors on that manifold.

The notation used in what follows is shown in (1) which defines the relationship of bivectors to Hamilton's pure quaternions $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ [2]:

$$\begin{aligned} \boldsymbol{i}_1&=\boldsymbol{e}_2\boldsymbol{e}_3=\boldsymbol{j}\\ \boldsymbol{i}_2&=\boldsymbol{e}_3\boldsymbol{e}_1=\boldsymbol{k}\\ \boldsymbol{i}_3&=\boldsymbol{e}_1\boldsymbol{e}_2=\boldsymbol{i} \end{aligned} \tag{1}$$

1 $\boldsymbol{U}_{\boldsymbol{\theta}}$ is Hestenes' notatation for the bivector product of two vectors.
2 The basis bivectors in Macdonald[4] are negatives of these.

Bivector addition on a spherical manifold

The $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ form an orthonormal basis of unit vectors for axes $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ in three dimensions (Figure 1).

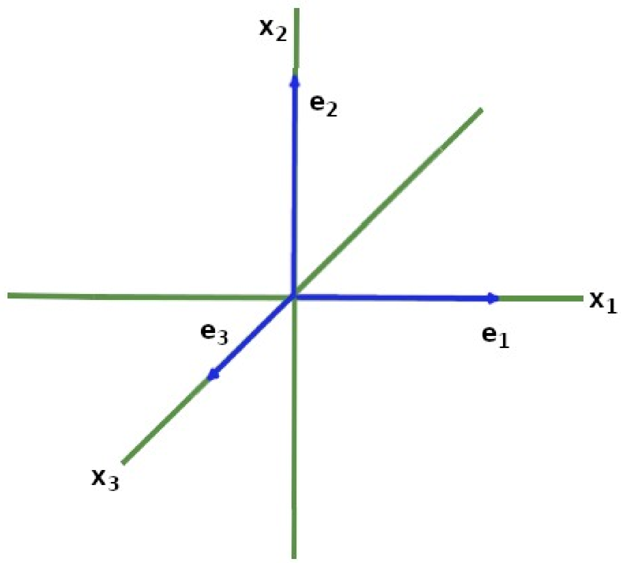

Figure 1: Standard basis for Euclidean space

Bivector $\boldsymbol{i}_3$ is derived from the multiplication of corresponding unit vectors $\boldsymbol{e}_1 \boldsymbol{e}_2$ (Figure 2) with $\pi/2$ taking on the role of the basis element on the sphere[3]. The bivectors are rotational elements corresponding to rotors in two dimensions. They can be considered as existing anywhere on the great circle they inhabit. The images displaying bivector addition present a visual extension of that motion.

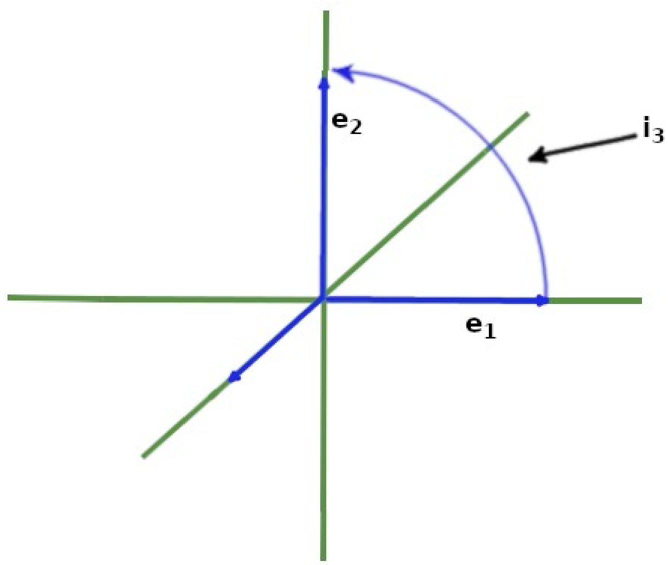

Figure 2: Product of unit basis vectors produces bivector basis $\mathbf{i}_3$

3 For explanatory purposes, images will posit it as twice that length for the sake of displaying bivector addition.

Near the end of one of his online tutorials[6] and using the same example as in his textbook[4], Macdonald works through a calculation using geometric algebra to derive a result that can be visualized on the sphere. Explaining how rotations compose in the algebra, he starts with the composition of rotations:

$$\exp(-\boldsymbol{i}_2\theta_2/2)\exp(-\boldsymbol{i}_1\theta_1/2)\boldsymbol{u}\exp(\boldsymbol{i}_1\theta_1/2)\exp(\boldsymbol{i}_2\theta_2/2) \tag{2}$$

from which he derives the equivalent normalized unit bivector in the 3D geometric algebra G^3 :

$$\exp(\boldsymbol{i}\,\theta/2)=\exp\left(\frac{\boldsymbol{e}_1\boldsymbol{e}_2+\boldsymbol{e}_2\boldsymbol{e}_3+\boldsymbol{e}_1\boldsymbol{e}_3}{\sqrt{3}}\frac{\pi}{3}\right) \tag{3}$$

The two-dimensional plane where θ operates is defined by the bracketed pseudoscalar $\boldsymbol{e}_1\boldsymbol{e}_2+\boldsymbol{e}_2\boldsymbol{e}_3+\boldsymbol{e}_1\boldsymbol{e}_3$. This equation specifies how the pseudoscalar defining the two-dimensional hosting plane can be described using the bivector bases in the three-dimensional space where it is located.

The normal to that plane in the dual space can be obtained through right multiplication by $\boldsymbol{e}_3\boldsymbol{e}_2\boldsymbol{e}_1$, the reversion of the three dimensional pseudoscalar $\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$ so that:

$$(\boldsymbol{e}_1\boldsymbol{e}_2+\boldsymbol{e}_2\boldsymbol{e}_3+\boldsymbol{e}_1\boldsymbol{e}_3)\boldsymbol{e}_3\boldsymbol{e}_2\boldsymbol{e}_1=\boldsymbol{e}_3+\boldsymbol{e}_2-\boldsymbol{e}_1 \tag{4}$$

With this algebraic formulation for the plane and the normal to it, we can move forward to develop its visualization on the sphere.

As in Figure 2 we proceed by developing bivectors on the sphere – defined by both the $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, and by the $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ axes – in the same fashion, extending both to a half-circle. (Figure 3).

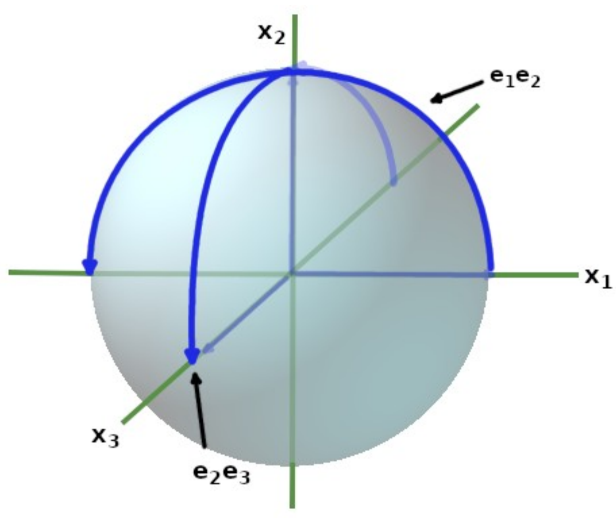

Figure 3: The two rotors $\boldsymbol{e}_1\boldsymbol{e}_2$ and $\boldsymbol{e}_2\boldsymbol{e}_3$ on the sphere

The logical route to the addition of bivectors is to assume their position on the sphere mediates the resulting bivector. The operation should, therefore, result in a bivector that is halfway between them (Figure 4). The basis for this logic is the coordinate free nature of vector and bivector addition. In two dimensions, the positional relationship is irrelevant. While the addition is often shown with the vectors placed head to tail, they can be displayed joined at the base as well. The vector sum is the same in either case. Adding two bivectors in the same way gives the placement shown in Figure 4.

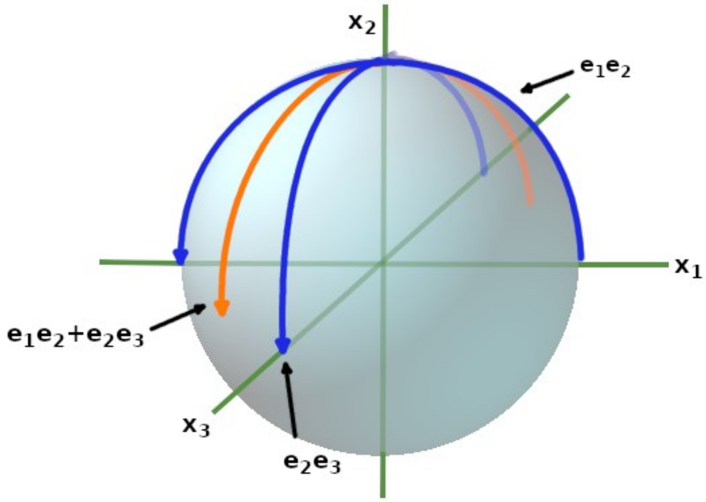

Figure 4: The addition of two bivectors halfway between them

This intermediate placement yields a commutative addition of bivectors: $\boldsymbol{e}_1\boldsymbol{e}_2+\boldsymbol{e}_2\boldsymbol{e}_3$ gives the same result as $\boldsymbol{e}_2\boldsymbol{e}_3+\boldsymbol{e}_1\boldsymbol{e}_2$. Note that the bivector sum is displaced by $\pm\pi/4$ from its parents, which informs the next calculation.

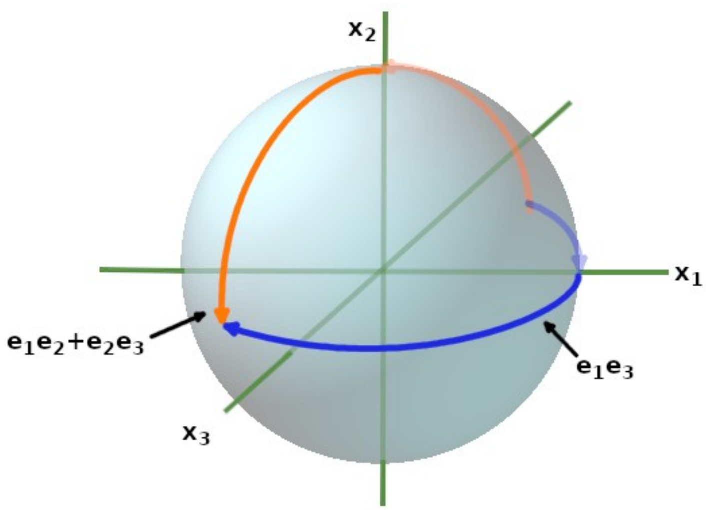

Figure 5: Bivector $\boldsymbol{e}_1\boldsymbol{e}_3$ overlain on the sphere

The bivector in the third plane defined by $\boldsymbol{x}_3$ and $\boldsymbol{x}_1$ can next be displayed on the sphere as we proceed (Figure 5). We now have a framework in place to complete the summation. Adding $\boldsymbol{e}_1\boldsymbol{e}_2+\boldsymbol{e}_2\boldsymbol{e}_3$ to $\boldsymbol{e}_1\boldsymbol{e}_3$

splits the difference once again, with the bivector positioned halfway between the two. The bivector sum, itself a new bivector, is angled between them. The plane the bivector rotates on is shown in the image to offer perspective. That plane runs through the center of the sphere, as do all the different planes hosting bivectors on the sphere.

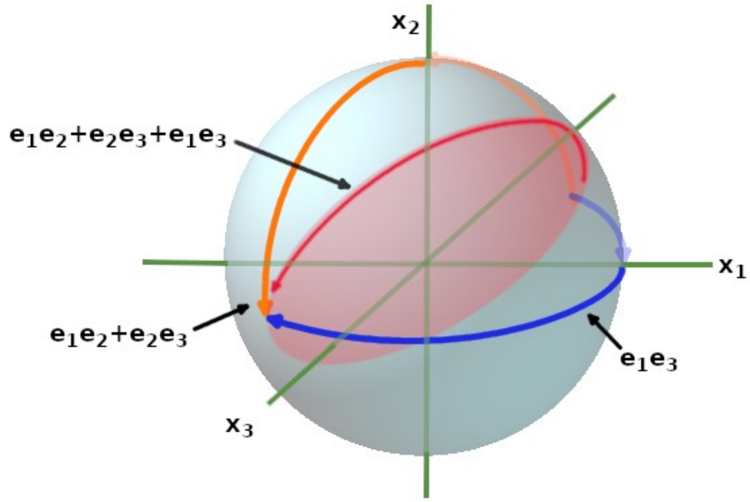

Figure 6: The bivector sum $\boldsymbol{e}_1\boldsymbol{e}_2+\boldsymbol{e}_2\boldsymbol{e}_3+\boldsymbol{e}_1\boldsymbol{e}_3$

The normal $\boldsymbol{e}_1+\boldsymbol{e}_3-\boldsymbol{e}_2$ to the plane of the bivector sum, projects downward into the $(\boldsymbol{x}_1, -\boldsymbol{x}_2, \boldsymbol{x}_3)$ portion of the sphere (Figure 7).

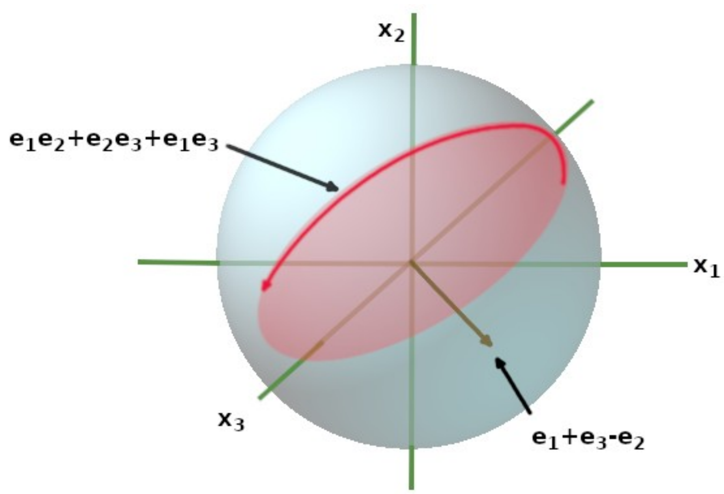

Figure 7: Bivector sum, its plane and normal to it: $\boldsymbol{e}_1+\boldsymbol{e}_3-\boldsymbol{e}_2$

This mirrors the algebraic calculation with bivector addition residing comfortably on a sphere in this example. This provides the impetus for mobilizing spherical geometry to derive properly weighted bivectors emerging from calculations.

Bivector multiplication on the sphere

Multiplication of bivectors can be properly understood as the rotation of one bivector, when operated on by a second, into the third axial plane. The spin direction reverses as the bivectors being multiplied switch places. The result is that anti-commutative multiplication of basis bivectors has a representation on a sphere in three dimensions. Once again, the starting point is a set of orthonormal basis vectors for axes $\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}$ in three-dimensions (Figure 1).

This model for bivector multiplication on a sphere was derived from earlier work researching the imagery necessary to visualize the operation. Movements initially portrayed in Euclidean space were eventually seen to migrate naturally to a sphere. That led to the realization that the paradigm for bivectors which had been developed to portray rotors in two-dimensions was also meaningful for a formal definition of bivector operations in three dimensions. The three planes defined by the e_i which anchor a spherical manifold, host a canonical unit rotor of length $\pi/2$ on a sphere of radius 1. Here is the definition given by Hestenes in [2] equation (11) for rotational dynamics in the plane[4]:

> *A unit bivector i for the plane containing vectors σ_1 and σ_2 is determined by the product*
>
> $$i = \sigma_1 \sigma_2 = \sigma_1 \wedge \sigma_2 = -\sigma_2 \sigma_1$$

This informs the development in three dimensions as well by adopting basis elements of length $\pi/2$. Proceeding to bivector multiplication, the basis element $\boldsymbol{i_1} = \boldsymbol{e_2 e_3}$ is shown in (Figure 8).

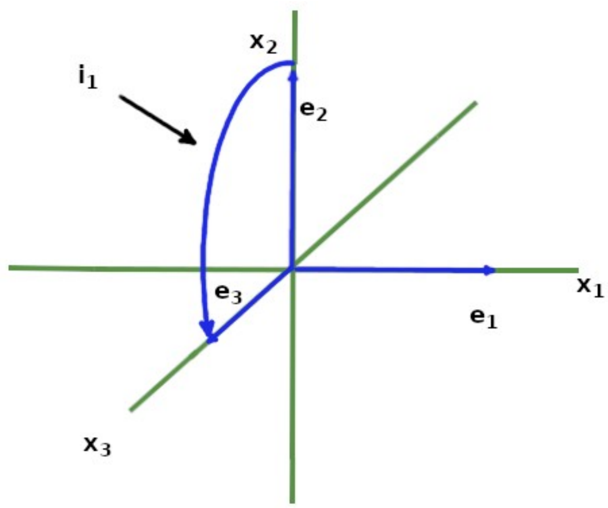

Figure 8: Basis $\boldsymbol{i_1}$ derived from the vector multiplication $\boldsymbol{e_2 e_3}$

The basis bivector $\boldsymbol{i_2} = \boldsymbol{e_3 e_1}$, can be synthesized in the same fashion (Figure 9).

4 In that work σ_1 and σ_2 are the standard unit basis elements of two-dimensional Euclidean space. The planar elements defined on the sphere as developed here are symbolized by e_1 and e_2.

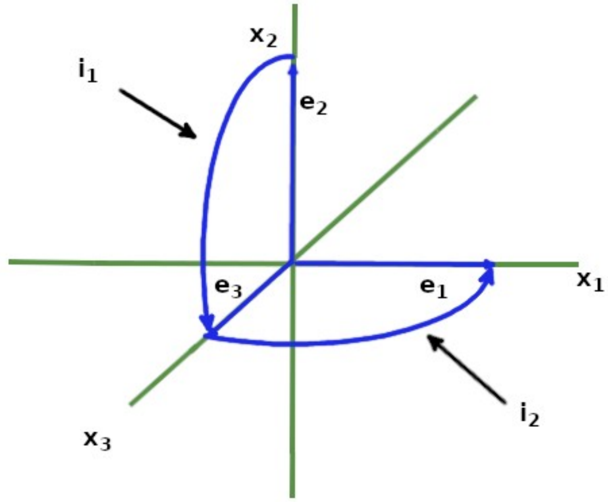

Figure 9: Unit bivectors i_1 *and* $\boldsymbol{i}_2$

Bivectors $\boldsymbol{i}_1$ and $\boldsymbol{i}_2$ are next shown on a unit sphere with the multiplication $\boldsymbol{i}_1\boldsymbol{i}_2$ generating $\boldsymbol{i}_3$. All the bivectors have the properties of rotors which Hestenes defined in two dimensions: they have basis length and their placement on a great circle is immaterial (Figure 10).

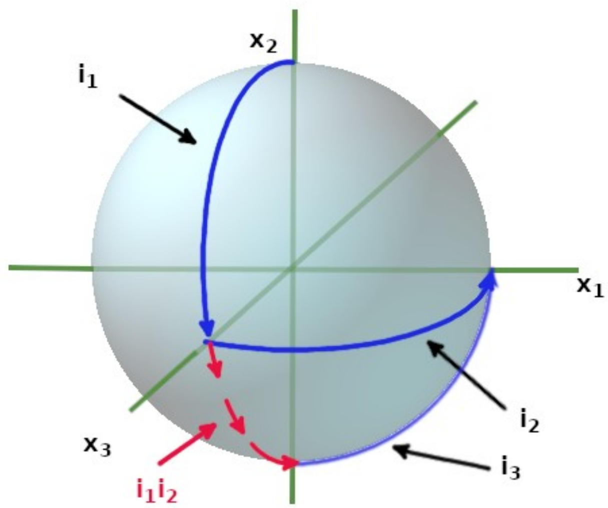

Figure 10: Bivector multiplication with $\boldsymbol{i}_1\boldsymbol{i}_2=\boldsymbol{i}_3$

The anti-commutative property of bivectors naturally derives from the definition of basis rotors on the sphere. Rotations reverse direction when the order changes. That is in line with the algebraic formulation since $\boldsymbol{i}_2\boldsymbol{i}_1=-\boldsymbol{i}_3$ in $\boldsymbol{G}^3$, so $\boldsymbol{i}_1$ is rotated to $-\boldsymbol{i}_3$ when operated on by $\boldsymbol{i}_2$ (Figure 11).

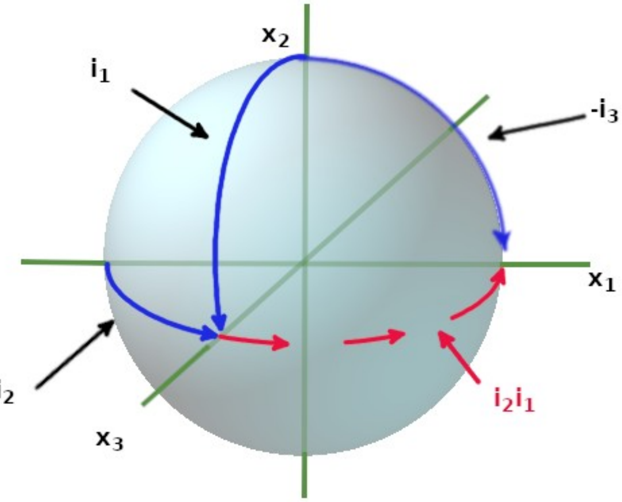

Figure 11: Bivector multiplication with $\boldsymbol{i}_2\boldsymbol{i}_1 = -\boldsymbol{i}_3$

The symmetry of the sphere insures the proof of anti-commutativity for each of the other basis bivector multiplications.

Geometric algebra on the sphere

The $\boldsymbol{i}_k, k=1,2,3$ are an orthonormal set of bivectors on the sphere, basis elements of length $\pi/2$. The rules for the inner product in the plane are defined in equation (10) of reference [2] . The result is similar here, but with a sign reversal:

$$\boldsymbol{i}_j \cdot \boldsymbol{i}_k = \frac{1}{2}(\boldsymbol{i}_j\boldsymbol{i}_k + \boldsymbol{i}_k\boldsymbol{i}_j) = -\delta_{jk} \tag{5}$$

where δ_{jk} is the Kronecker delta.

For $\boldsymbol{i}_1\boldsymbol{i}_2$ the inner product is:

$$\boldsymbol{i}_1 \cdot \boldsymbol{i}_2 = \frac{1}{2}(\boldsymbol{i}_1\boldsymbol{i}_2 + \boldsymbol{i}_2\boldsymbol{i}_1) = \frac{1}{2}(\boldsymbol{i}_1\boldsymbol{i}_2 - \boldsymbol{i}_1\boldsymbol{i}_2) = 0 \tag{6}$$

Figure 14 is taken from Figure 10 and Figure 11 for $\boldsymbol{i} \neq \boldsymbol{j}$. Visually, the contra-positioning annihiles rotational spin. Again, symmetry insures this result for all of the binary products of bivectors.

The inner product for the square of an element is:

$$\boldsymbol{i}_1 \cdot \boldsymbol{i}_1 = \frac{1}{2}(\boldsymbol{i}_1{}^2 + \boldsymbol{i}_1{}^2) = \frac{1}{2}2\boldsymbol{i}_1{}^2 = \boldsymbol{i}_1{}^2 = e_2e_3e_2e_3 = -e_2e_3e_3e_2 = e_2e_2 = -1 \tag{7}$$

This is analogous to the result in the plane where $\boldsymbol{i}^2 = -1$ (see equation (12) of reference [2]) . On a spherical manifold it is dimensional: it is also a rotation of length π but applied to the target bivector:

$$\boldsymbol{i}_1^2\boldsymbol{i}_2 = -\boldsymbol{i}_2 \tag{8}$$

This transition is shown in Figure 12 and Figure 13 with the squaring operation acting twice on a bivector.

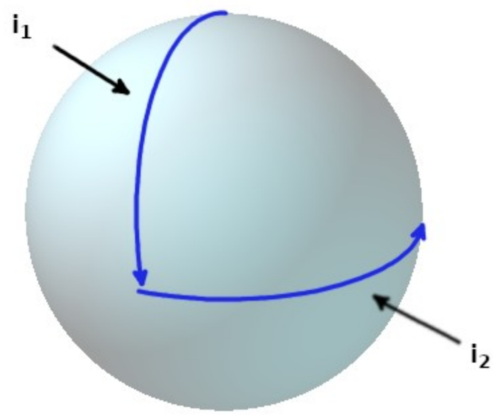

Figure 12: $\boldsymbol{i}_1$ *and* $\boldsymbol{i}_2$ *on the sphere*

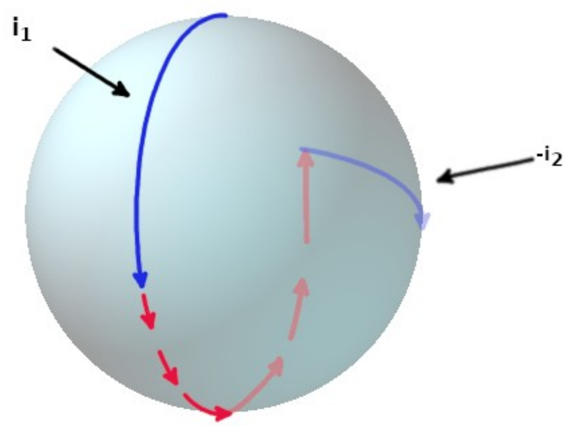

Figure 13: $\boldsymbol{i}_1^2$ *rotates* $\boldsymbol{i}_2$ *to* $-\boldsymbol{i}_2$ *on the sphere*

For the outer product, the calculation gives:

$$\boldsymbol{i}_1 \wedge \boldsymbol{i}_2 = \frac{1}{2}(\boldsymbol{i}_1 \boldsymbol{i}_2 - \boldsymbol{i}_2 \boldsymbol{i}_1) = \frac{1}{2}(\boldsymbol{i}_1 \boldsymbol{i}_2 + \boldsymbol{i}_1 \boldsymbol{i}_2) = \frac{1}{2} 2 \boldsymbol{i}_1 \boldsymbol{i}_2 = \boldsymbol{i}_3 \quad (9)$$

The full geometric product develops as expected:

$$\boldsymbol{i}_1 \boldsymbol{i}_2 = \boldsymbol{i}_1 \cdot \boldsymbol{i}_2 + \boldsymbol{i}_1 \wedge \boldsymbol{i}_2 = \frac{1}{2}(\boldsymbol{i}_1 \boldsymbol{i}_2 + \boldsymbol{i}_2 \boldsymbol{i}_1) + \frac{1}{2}(\boldsymbol{i}_1 \boldsymbol{i}_2 - \boldsymbol{i}_2 \boldsymbol{i}_1) = \frac{1}{2}(\boldsymbol{i}_1 \boldsymbol{i}_2 - \boldsymbol{i}_1 \boldsymbol{i}_2) + \frac{1}{2}(\boldsymbol{i}_1 \boldsymbol{i}_2 + \boldsymbol{i}_1 \boldsymbol{i}_2) = 0 + (\boldsymbol{i}_1 \boldsymbol{i}_2) = \boldsymbol{i}_3 \quad (10)$$

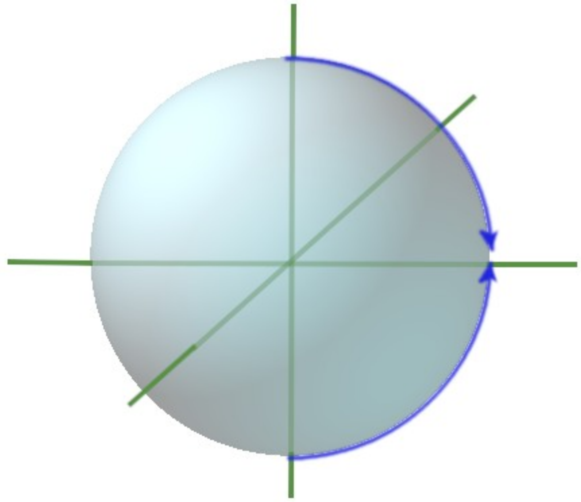

Figure 14: Sum of anti-commutative products $(\boldsymbol{i}_1\boldsymbol{i}_2+\boldsymbol{i}_2\boldsymbol{i}_1)$

This shows how multiplication of bivectors proceeds on a spherical manifold. When $\boldsymbol{i}_j=\boldsymbol{i}_k$ the squaring of the bivector is a reversing factor when applied to a second bivector for $\boldsymbol{i}_j^2=-1\,,\,j=1{,}2{,}3$. (as shown in equation (12) of reference [2]).

Euclidean evolution onto a spherical manifold

In a previous attempt at directed rotations – and how they informed Grassmann's outer product – bivectors were embedded in Euclidean three-space, the standard definition found in the literature[7]. Working in three dimensions using those elements did result in the emergence of a pure quaternion algebra. Rotational elements would transition from one plane to another under the influence of the third (Figure 15).

Exchanging the sign of the operator reversed the spin rotation. That mirrored the characteristic change in sign fundamental to Grassmann’s outer product, and to the group properties of rotations in three dimensions (Figure 16).

These results were intriguing but further investigation was put aside in favor of research into the historical development of geometric algebra, and of Clifford algebras in general. There was, nonetheless, an eventual realization that the operation of multiplication should be projected onto, and staged on a sphere.

In that domain, it was found that the operation also worked to transmute the rotations properly. This is in line with suggestions in the literature. Quaternions are exceedingly useful in the field of computer animation[8], and their deployment on a sphere one of the easiest ways to visualize the rotational transformations they enable[9]. With that re-orientation there came a growing understanding that coordinate free rotation on a sphere was an emergent property informing the orbital trajectories of bivector multiplication. That new stage is also where it became clear that rotations of $\pi/2$ took on a role analogous to the unit basis elements in Euclidean space. This is also in keeping with the definition of the standard basis for a geometric algebra. There will be more to say about that in what follows.

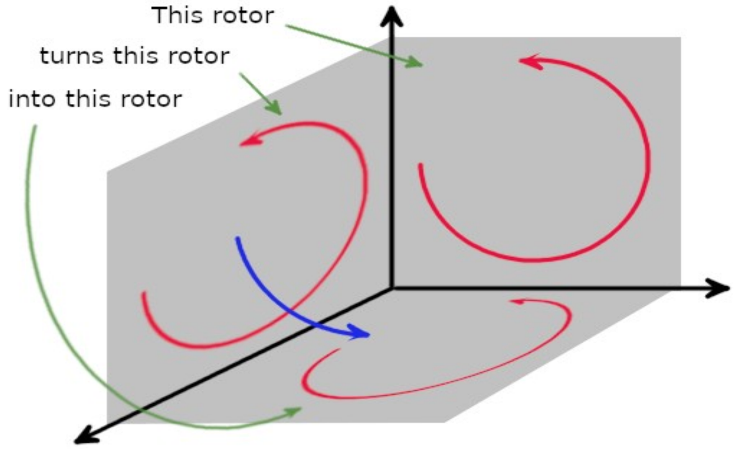

Figure 15: A rotor operating on Euclidean planar rotations

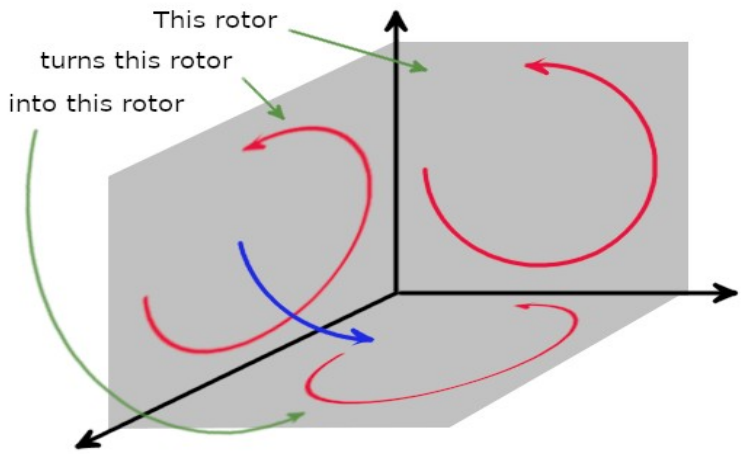

Figure 16: Opposite spin reverses spin on resulting rotor

There was ever-increasing interest in these results. Shortly afterwards, one of the trails followed in researching the scope of geometric algebra surfaced the papers of Hiley and his co-workers[1] [3] [10]. The statement about Gel'fand's postulate – that only a commutative algebra can be staged on a single manifold – led to a greatly renewed interest in generating a bivector algebra on a sphere. A challenge offered up to attempt the visualization of bivector addition in geometric algebra provided the opportunity to explore that possibility[6].

Mirroring the geometric algebra of 2-D in 3-D

The key to developing a rational addition on the sphere was the normal produced at the end of the algebraic calculation in that reference. When placed on a sphere, the protrusion of that normal into a signed portion of the manifold informed the placement of the plane orthogonal to it. Upon displaying the bivector sum in that space, one equivalent to vector addition in the two dimensions, it was clear that the plane would result in a placement halfway between the addends (Figure 6). A prospective understanding of bivector addition quickly emerged.

The positioning of the bivector sum halfway between the two bivectors being added (Figure 4) is nothing more than a spherical reflection of the position-less nature of vectors in the plane. Placement on the defining great circle is arbitrary.

Addition is commutative with this definition once a sense of direction is established: halfway between two great circles does not change the result, no matter which one is taken first. Moreover, multiplication by pseudoscalar reversion works exactly as it should: the extrusion of the normal from the disk (Figure 7) providing scaffolding for the bivector reflects the self-contained nature of the algebra on a sphere.

The defining insight was provided by Hestenes vision of a rotor operating on a circle irrespective of it's absolute location[2]. That idea brought its own questions discussed below, but it provided the intuitive leap needed to inform the transition to three dimensions. The placement of rotors on a sphere – the dynamic of rotation built-in without the need for pinning bivectors to a starting point – immediately clarified how this would all work. With the adoption of this paradigm, geometric algebra on a sphere can take on a canonical role as the algebra of motions on that manifold. This instantiation of G^3 expands on the 2-D version. Hestenes' statement that the vector product in the plane results in a "*directed arc of fixed length that can be rotated at will on the unit circle*" extends to the sphere, with convolutions of the orthogonal elements subsumed by the algebraic operations.

In the complex plane, multiplication by the "imaginary" $\boldsymbol{i}$ can also be understood as a rotation of $\pi/2$ radians with $\boldsymbol{i}^2$ doubling the rotation. Coordinate-free, that plane becomes the backdrop for those movements no matter where they are staged. This redefinition has implications. By embedding rotations into vector algebra, bivectors take their place as computational elements. The radical significance of duality in the plane is discussed in the next section.

Now, on the sphere rotation engages bivectors in analogy to the process in the complex plane, but with a 3-D twist. Any great circle on which rotations are being effectuated generalizes bivector multiplication to admit the additional dimensional paradigm. Bivector multiplication as defined for the orthogonal elements $\boldsymbol{i}_1 \boldsymbol{i}_2$ gives the proper bivector product when expressed in the underlying Euclidean basis through outer product multiplication of the $\boldsymbol{e}_i$. One more example:

$$\boldsymbol{i}_1 \boldsymbol{i}_2 = \boldsymbol{e}_2 \boldsymbol{e}_3 \boldsymbol{e}_3 \boldsymbol{e}_1 = \boldsymbol{e}_2 (-1) \boldsymbol{e}_1 = \boldsymbol{e}_1 \boldsymbol{e}_2 = \boldsymbol{i}_3 \tag{11}$$

But the rotation is itself applied orthogonally. Doubling of one of the $\boldsymbol{i}_k$ has the effect of reversing the direction of the multiplicand (Figure 13).

Two places at once

There is a distinction that arises naturally from the placement of rotors on a plane. Defining vector multiplication in two dimensions takes up all of that conceptual frame. In that space, the re-definition of the imaginary $\boldsymbol{i}$ to its bivector doppelgänger $\boldsymbol{e}_1 \boldsymbol{e}_2$ is a call to rotate. Rotations come to life in two dimensions, not just as one linear complex vector. That bivector implicitly commands all of that 2-D space. The placement of the scalar element associated with a complex number is thus occluded by that bivector.

Turning to the standard definition for a complex number $(x + \boldsymbol{i}\, y)$ – a vector-like directed line projected onto a plane from a fixed origin – we have a system for conceptualizing such numbers, an algebraic field which gives complex analysis its enormous power. But there is another way to come at this.

After defining multiplication and addition on a sphere for the pure quaternions[5], the next step was understanding what that meant for the scalar portion of the full quaternion algebra. It was logical to assume that the concept of a manifold informed that element also, that it's solitary placement on a separate real line was perfectly acceptable. With that understanding, the geometric algebra of three dimensions was then conceived to be dynamically transitional between the different manifolds of the three-dimensional geometric algebra. There will be one for each component of the graded algebra:

- the real line
- 3D Euclidean space for posting vectors
- the sphere – also 3D – where bivectors come to life
- the pseudoscalar encompassing the volumetric component for eliciting the dual space

But this conceptual leap can also reconfigure the framework for complex numbers. Gel'fand's postulate and what it means for a commutative algebra provides insight about the possibility of a dual representation for complex numbers. The key point is this: in two-dimensions the extended algebra with its rules for addition and multiplication and with its definition as a field *is commutative*, and:

> *"we can … abstract a unique manifold if our algebra is commutative"[3].*

That manifold is the complex plane. Wessel's creation brought it into play as the home for a vastly expanded numerical realm[11].

However, we can also embed the scalar and bivector portions of complex numbers as components in a geometric algebra of three grades: scalars on a real line; vectors on the plane; bivectors on a circle. So complex numbers can have a dual representation in a topological realm populated by those manifolds. Moreover, that representation as a non-commutative algebra has invariance properties not available to complex numbers on the complex plane[6].

This "duality" corresponds to the fact that there have always been two ways to model complex numbers: as two-dimensional vectors and as the rotational components informed by Euler's formula: $e^{i\theta}=\sin(\theta)+i\cos(\theta)$ [13]. The rules for operating on those vectors in the complex plane define a commutative algebra. The two-dimensional version of geometric algebra is non-commutative and, as a graded algebra, can be represented on three manifolds, one for each of the grades.

Making the grade

The development of rotors on a spherical manifold takes as its starting point the familiar planes defined by combinations of the unit vectors which produce those bivectors ($\boldsymbol{e_1 e_2}, \boldsymbol{e_2 e_3}, \boldsymbol{e_3 e_1}$). They in turn frame the space we work with (Figure 1 and Figure 16). These are elements of a standard basis in G^3 . The difference here is that they are no longer assumed to inhabit a purely Euclidean space. Instead they provide a platform for developing rotational dynamics on a sphere.

In order to define a multiplication that gives proper meaning to bivector algebra as the algebra of rotors on a spherical manifold, the way chosen is to let the product of those unit combinations sweep out an arc from the

5 See Figure 4, Figure 10, and Figure 11.

6 "...the complex numbers product is geometrically meaningless, because its geometric interpretation is always related to the real axis of the specific complex numbers coordinate system of the 2D plane."[12]

first to the second, say $\boldsymbol{e}_1$ to $\boldsymbol{e}_2$. That path along a great circle has length $\pi/2$. This leads to the following conjecture:

- The unit basis vectors associated with Euclidean space and their use in defining basis elements on a spherical manifold model the process in higher dimensions.

Released from the constraint of working on a single manifold, we are free to consider extensions of the idea of a physical space where basis elements are no longer restricted to the Euclidean realm. In placing the individual graded components of geometric algebra on separate manifolds, the natural assumption is that the binomial theorem informs the development of basis elements for each grade, providing a hint of what's to come in higher dimensions.

For the grades of a two dimensional algebra we can enumerate the basis elements as follows:

Table 1: Basis elements for three grades

Binomial Coefficient	Basis	Manifold
$\binom{2}{0}$	1	Real Line
$\binom{2}{1}$	$(1,1)$	Euclidean Plane
$\binom{2}{2}$	$\pi/2$	Circle

Here the assumption is that by introducing rotational dynamics on the circle we have swept the unit basis element $\boldsymbol{e}_1$ a length $\pi/2$ to $\boldsymbol{e}_2$. That we take as the basis element of the pseudoscalar on the circle. What emerges is a harbinger of what we see in higher dimensions as shown previously (Figure 8 and Figure 9) and what is suggested below.

Working with manifolds in three dimensional geometric algebra which has four grades, we have the basis elements we know and one which is unknown – the pseudoscalar basis:

Table 2: Basis elements for four grades

Binomial Coefficient	Basis	Manifold
$\binom{3}{0}$	1	Real Line
$\binom{3}{1}$	$(1,1,1)$	Euclidean Space
$\binom{3}{2}$	$(\pi/2,\pi/2,\pi/2)$	Sphere
$\binom{3}{3}$	?	Volume

Presumably the unknown basis, whatever it might entail, will be built into higher-dimensional Clifford algebras. Even and odd grading may also have a role to play in determining the form of the pseudoscalar basis in 3D, and those of higher dimensionality. With this model for each set of graded bases, the presumption is therefore that the mystery pseudoscalar basis will be incorporated into the standard basis in 4D as $\binom{4}{3}=6$ elements.

The adoption of $\pi/2$ as the basis element on a spherical manifold also reveals the algebra of bivectors as nothing more than a spin around the sphere. For example, working with the bivectors shown in Figure 9:

$$\boldsymbol{i}_1{}^4\boldsymbol{i}_2=(\boldsymbol{e}_2\boldsymbol{e}_3)^4\boldsymbol{i}_2=(\boldsymbol{e}_2\boldsymbol{e}_3)^2(\boldsymbol{e}_2\boldsymbol{e}_3)^2\boldsymbol{i}_2=(-1)(-1)i_2=\boldsymbol{i}_2 \tag{12}$$

Reversing the spin of $\boldsymbol{e}_2\boldsymbol{e}_3$ and changing the sign of the bivector multiplier results in the same configuration for $\boldsymbol{i}_2$ which is as it should be. It does not matter in which direction the great circle the bivector resides on is rotated in its full trip around the sphere:

$$(-\boldsymbol{i}_1)^4\boldsymbol{i}_2=(-\boldsymbol{e}_2\boldsymbol{e}_3)^4\boldsymbol{i}_2=(-\boldsymbol{e}_2\boldsymbol{e}_3)^2(-\boldsymbol{e}_2\boldsymbol{e}_3)^2\boldsymbol{i}_2=(-1)(-1)i_2=\boldsymbol{i}_2 \tag{13}$$

Two multiplications instead of four also result in the proper transition: the product has reversed the rotation of the bivector on the sphere:

$$(-\boldsymbol{i}_1)^2\boldsymbol{i}_2=(-\boldsymbol{e}_2\boldsymbol{e}_3)^2\boldsymbol{i}_2=(-\boldsymbol{e}_2\boldsymbol{e}_3)^2\boldsymbol{i}_2=(-1)\boldsymbol{i}_2=-\boldsymbol{i}_2 \tag{14}$$

On a sphere of radius 1 it also happens that the surface area bounded by three orthogonal unit bivectors of length $\pi/2$ is also $\pi/2$ (Figure 17). This is suggestive of a homology involving Euclidean space[14] where the unit basis defines an area of measure 1 .

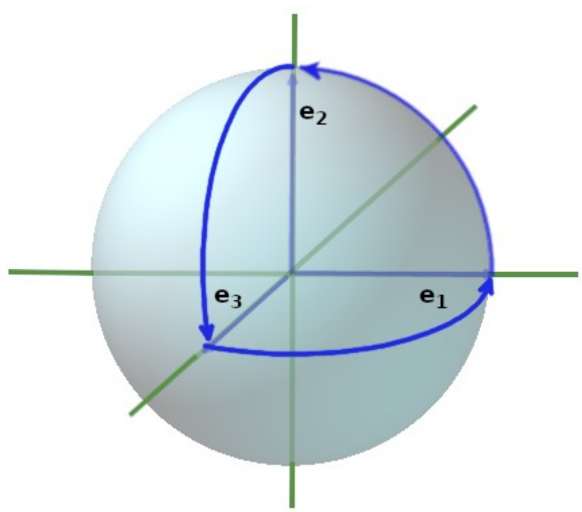

Figure 17: Bivectors of length $\pi/2$ enclose surface area of $\pi/2$

Spin cycle

An algebra with both an addition and a multiplication defined on a sphere – one with basis vectors residing on that manifold – opens a range of dynamical possibilities. As one example a sequence of multiplications involving six consecutive transformations by basis elements – a total of 3π – returns the bivector it operates on to its original configuration. Arbitrarily choosing $\boldsymbol{i}_3$ as the initial rotational bivector, a successive set of orthogonal multiplications shows that:

$$\begin{gathered} \boldsymbol{i}_2\boldsymbol{i}_3=\boldsymbol{i}_1 \\ \boldsymbol{i}_3\boldsymbol{i}_1=\boldsymbol{i}_2 \\ (-\boldsymbol{i}_1)\boldsymbol{i}_2=-\boldsymbol{i}_3 \\ \boldsymbol{i}_2(-\boldsymbol{i}_3)=-\boldsymbol{i}_1 \\ \boldsymbol{i}_3-\boldsymbol{i}_1=-\boldsymbol{i}_2 \\ (-\boldsymbol{i}_1)(-\boldsymbol{i}_2)=\boldsymbol{i}_3 \end{gathered} \tag{15}$$

Imagining these movements as continuous forces acting simultaneously leads to the idea of a swirling motion systematically covering specific portions of the sphere. The obvious tool to develop this idea is video animation delivering a visual representation of this motion.

Work on quaternions[9] [15] [16] [17] has used the concept of frames as a way to understand transitions to higher dimensions up to and including the 4π rotation of the belt-trick in 4D[18]. The algebra of basis elements on the sphere shown in (15) suggests that another possible way to approach this is by rotational extension into that additional dimension, with eight movements resulting in a 4π return to position.

The relocation of graded elements onto manifolds also offers a range of possibilities for conceptualizing models of the physical world. The center of a sphere can be driven by a force vector propelling it through space, while on that sphere rotational dynamics involving bivectors are in play. Additive and multiplicative forces on that driven sphere could result in pressure wave pulsations as well as those swirling motions. The possibilities would seem to be endless.

In imagining a linearly driven sphere hosting a spinning bivector, it is clear that three-dimensional basis elements in Euclidean space could also situate that sphere. The linear vector would transform its basis forward to the center of the sphere where a rotor is working. The open question is whether applying the operations of the geometric algebra could identify the relationship between the linear vector and the rotor it is propelling. Preliminary calculations bring up the same ideas developed in a previous section (Mirroring the geometric algebra of 2-D in 3-D) subject to further examination.

The ecology of multi-dimensional space

As has been stated often by David Hestenes, the elements of a geometric algebra are extensions of what we have long thought of as “numbers”:

> *“I submit that Clifford Algebra is as universal and basic as the real number system. Indeed, it is no more and no less than an extension of the real number system to incorporate the geometric concept of direction. Thus, a Clifford Algebra is a system of directed numbers.”[5]*

With even greater clarity, he observed that he understood the grammar of geometric algebra to be:

> *"An arithmetic of directed numbers encoding the geometric concepts of magnitude, direction, sense and dimension"[19]*

It was this very idea of the directionality of geometric elements – inherent in Grassmann's definition of the outer product[20] – that Felix Klein adopted in his writings about geometry[21]. That directionality, successfully deployed on multiple manifolds, brings a broader vision of how the disposition of forces can play out in those geometric spaces. Implicit in this simple but profound idea is that of dynamism. The linear vectors of physics are conceived in exactly that way. Weighting those vectorial components incorporates that dynamism, the drive of a directed force or flow irrespective of any specific starting point. The development of rotors in two and three dimensions is a direct analog to that linear drive. It does nothing more than bring spinning vortices into play in the same way. No longer dependent on a fixed starting point, they exist in and of themselves as rotational analogs of linear forces.

The profoundly visionary work of Hermann Grassmann [22] which he pursued for decades and which eventually led Clifford to his epiphany [23], reveals itself to transcend Euclidean space, providing the conceptual tools to suggest movement between multiple manifolds. Singular points on a real line, rotations on circles, directed projections in Euclidean space, twisting spherical spirals ... these are the two- and three-dimensional abstract dynamisms available to geometric algebra. They only hint at the extra-dimensional extensions that may find their proper home in this all-encompassing mathematical framework.

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