HASTINGS-CODY APPROXIMATIONS OF THE INTEGRAL OF A POWER TIMES THE COMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND

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Abstract. Hastings and later Cody tabulated minimax polynomial approximations for the Complete Elliptic Integral of the First Kind. The simplicity of this representation by polynomials and polynomials times a logarithm allows to integrate their terms analytically. We demonstrate how integrals of the Complete Elliptic Integral times a power of its argument achieve double precision accuracy for powers from 0 to 2 based on Cody's polynomials up to 9th order.

1. Introduction

The series representation of the complete Elliptic Integral $K(k)$ of the First Kind is the Gaussian hypergeometric function [3, p. 15][1, 17.2.6]

(1)
$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\theta}}d\theta = \frac{\pi}{2}{}_2F_1\left(\begin{array}{c}1/2,1/2\\ 1\end{array} \mid k^2\right) = \frac{\pi}{2}\sum_{n\ge 0}\frac{(1/2)_n(1/2)_n}{(1)_n}\frac{k^{2n}}{n!}.$$

For the integrated Elliptic Integral, the Byrd-Friedman book gives a power series [3, 610.00]

$$\int K dk = \frac{\pi k}{2}\left\{1+\sum_{n\ge 1}\frac{[(2n)!]^2k^{2n}}{(2n+1)2^{4n}(n!)^4}\right\}, \quad 0<k<1, \tag{2}$$

obtained by swapping the order of summation and integration:

$$\int K(k)dk = \frac{\pi}{2}\sum_{n\ge 0}\frac{(1/2)_n(1/2)_n}{(1)_n(2n+1)}\frac{k^{2n+1}}{n!} = \frac{\pi}{4}k\sum_{n\ge 0}\frac{(1/2)_n(1/2)_n}{(1)_n(n+1/2)}\frac{k^{2n}}{n!} \tag{3}$$
$$= \frac{\pi}{4}k\sum_{n\ge 0}\frac{(1/2)_n(1/2)_n(1/2)_n}{(1)_n\frac{1}{2}(3/2)_n}\frac{k^{2n}}{n!} = \frac{\pi}{2}k\,{}_3F_2\left(\begin{array}{c}1/2,1/2,1/2\\ 1,3/2\end{array} \mid k^2\right).$$

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In the same spirit, i-th moments of the integral are Saalschützian generalized hypergeometric series [21]:

$$\int k^i K(k)dk = \frac{\pi}{2}\sum_{n\ge 0}\frac{(1/2)_n(1/2)_n}{(1)_n(2n+1+i)}\frac{k^{2n+1+i}}{n!} \tag{4}$$

$$= \frac{\pi}{4}k^{1+i}\sum_{n\ge 0}\frac{(1/2)_n(1/2)_n}{(1)_n[n+(i+1)/2]}\frac{k^{2n}}{n!} = \frac{\pi}{4}k^{1+i}\sum_{n\ge 0}\frac{(1/2)_n(1/2)_n((1+i)/2)_n}{(1)_n\frac{1+i}{2}((3+i)/2)_n}\frac{k^{2n}}{n!}$$

$$= \frac{\pi}{2(1+i)}k^{1+i}{}_3F_2\left(\begin{array}{c}1/2,1/2,(1+i)/2\\ 1,(3+i)/2\end{array}\mid k^2\right).$$

The series is known to converge for $k^2 < 1$ [21], and provides reference values for other methods to compute these integrals.

Remark 1. *Special values for unit argument $k^2 = 1$ for $i = 0$ to 3 are* [19, 7.4.4.165,7.4.4.172]

$$ {}_3F_2\left(\begin{array}{c}1/2,1/2,(i+1)/2\\ 1,(i+3)/2\end{array}\mid 1\right) = \begin{cases} 4G/\pi, & i=0;\\ {}_2F_1\left(\begin{array}{c}1/2,1/2\\ 2\end{array}\mid 1\right) = 4/\pi, & i=1;\\ 3(2G+1)/(2\pi), & i=2;\\ 40/(9\pi), & i=3;\end{cases} \tag{5}$$

where $G \approx 0.915965594$ is Catalan's constant [10, A006752]*. Bailey's contiguous relations for unit argument read* [2, (4.5)]

$$(3+i)(1+i)(i-1){}_3F_2\left(\begin{array}{c}1/2,1/2,(i-1)/2\\ 1,(i+1)/2\end{array}\mid 1\right) \tag{6}$$

$$+(1+i)(2+i)^2{}_3F_2\left(\begin{array}{c}1/2,1/2,(i+3)/2\\ 1,(i+5)/2\end{array}\right)$$

$$=(3+i)(1+2i^2+2i){}_3F_2\left(\begin{array}{c}1/2,1/2,(i+1)/2\\ 1,(i+3)/2\end{array}\mid 1\right).$$

This means the ${}_3F_2$ for unit argument at $i\ge 4$ can be generated recursively from the four special values above, and in consequence the definite integrals $\int_0^1 k^iK(k)dk$ are all linear combinations of fractions and Catalan's constant.

2. Hastings-Cody Representations

Hastings' approximation of the complete elliptic integral of the first kind is [12][1, 17.3.34]

$$K \approx a_0 + a_1m_1 + \cdots + a_4m_1^4 - [b_0 + b_1m_1 + \cdots b_4m_1^4]\log m_1 \tag{7}$$

with $m_1 = 1-k^2$, which is a typical near-minimax approximation as Fig. 1 shows.

2.1. Moment Order zero.

An immediate benefit is that each term allows closed-form integration,

$$\int Kdk \approx \int dk\left[\sum_{s\ge 0}a_sm_1^s - \log m_1\sum_{s\ge 0}b_sm_1^s\right]. \tag{8}$$

s	a_s	b_s
0	1.38629436112	0.5
1	0.09666344259	0.12498593597
2	0.03590092383	0.06880248576
3	0.03742563713	0.03328355346
4	0.01451196212	0.00441787012

TABLE 1. Coefficients in Hastings' 4-th polynomial order approximation (7).

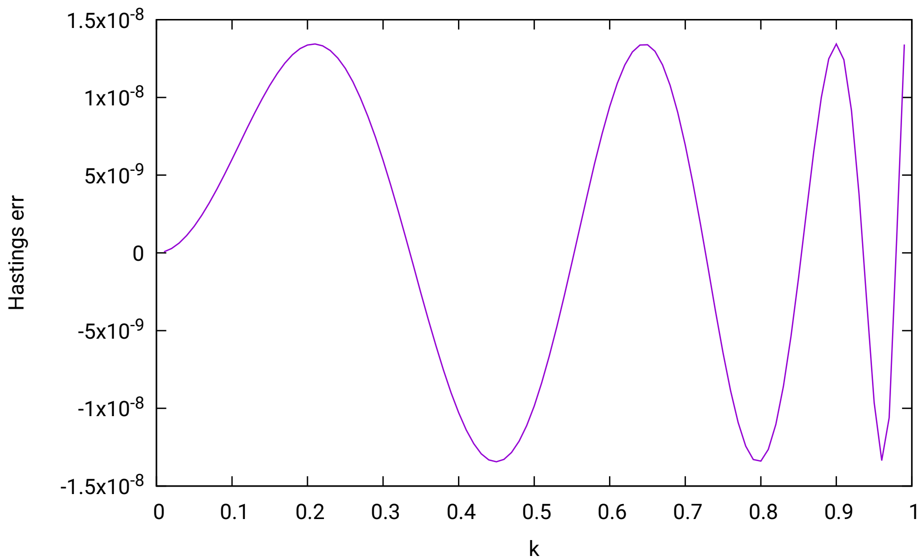

FIGURE 1. Absolute error: Hastings' approximation (7) minus $K(k)$.

For the terms with the a_s coefficients we define the elementary integrals

(9)

$$\bar{k}_s^{(0)} \equiv \int dk m_1^s = \int dk(1-k^2)^s = \sum_{l=0}^{s} \binom{i}{l} (-)^l \int dk k^{2l} = k \sum_{l=0}^{s} \binom{s}{l} \frac{1}{2l+1}(-k^2)^l.$$

Remark 2. *A recurrence is derived via partial integration*

$$(1+2s)\bar{k}_s^{(0)} = k(3-k^2)(1-k^2)^{s-1} + 2(s-1)[2\bar{k}_{s-2}^{(0)} - \bar{k}_{s-1}^{(0)}], s \geq 1. \quad (10)$$

This is anchored at $\bar{k}_0^{(0)} = k$, $\bar{k}_1^{(0)} = k - k^3/3 = k(3-k^2)/3$.

For the terms with the b_s coefficients the associated integrals involve an additional logarithmic factor in the integrand. By partial integration

(11)
$$\bar{l}_s^{(0)} \equiv \int dk m_1^s \log m_1 = \int dk(1-k^2)^s \log(1-k^2) = \bar{k}_s^{(0)} \log(1-k^2) - \int \bar{k}_s^{(0)}(-\frac{2k}{1-k^2})dk$$
$$= \bar{k}_s^{(0)} \log(1-k^2) + 2\int \bar{k}_s^{(0)} \frac{k}{1-k^2} dk = \bar{k}_s^{(0)} \log(1-k^2) + 2\sum_{l=0}^{s} \binom{s}{l} \frac{(-)^l}{2l+1} \int \frac{k^{2l+2}}{1-k^2} dk$$
$$= \bar{k}_s^{(0)} \log(1-k^2) - 2\sum_{l=0}^{s} \binom{s}{l} \frac{1}{2l+1} \sum_{l'=0}^{l+1} \binom{l+1}{l'} (-)^{l+1-l'} \int (1-k^2)^{l'-1} dk$$
$$= \bar{k}_s^{(0)} \log(1-k^2) - 2\sum_{l=0}^{s} \binom{s}{l} \frac{1}{2l+1} \left[(-)^{l+1} \operatorname{arctanh} k + \sum_{l'=1}^{l+1} (-)^{l+1-l'} \binom{l+1}{l'} \int (1-k^2)^{l'-1} dk \right]$$
$$= \bar{k}_s^{(0)} \log(1-k^2) + 2 \operatorname{arctanh} k \sum_{l=0}^{s} \binom{s}{l} \frac{1}{2l+1} (-)^l - \sum_{l=0}^{s} \binom{s}{l} \frac{2}{2l+1} \sum_{l'=0}^{l} (-)^{l-l'} \binom{l+1}{l'+1} \bar{k}_{l'}^{(0)}$$
$$= \bar{k}_s^{(0)} \log(1-k^2) + \frac{2}{\binom{s+1/2}{s}} \operatorname{arctanh}(k) - \sum_{l'=0}^{s} \bar{k}_{l'}^{(0)} \sum_{l=l'}^{s} \binom{s}{l} \frac{2}{2l+1} (-)^{l-l'} \binom{l+1}{l'+1}.$$

At $s=0$ this is $\bar{l}_0^{(0)} = k\log(1-k^2) - 2k + 2\operatorname{arctanh} k$, for example.

With polynomial orders from 4 (which is Hastings') up to 10 (tabulated by Cody [5]) the relative errors in the representation

$$\int dk K(k) \approx \sum_{s\ge 0} a_s \bar{k}_s^{(0)} - \sum_{s\ge 0} b_s \bar{l}_s^{(0)} \tag{12}$$

are illustrated in Figure 2.

2.2. Moment Order One.

The Hastings-Cody approximations for the moments of first order are

$$\int kK dk \approx \int dk \left[\sum_{s\ge 0} a_s k m_1^s - \log m_1 \sum_{s\ge 0} b_s k m_1^2 \right] \tag{13}$$
$$= \int dk \left[\sum_{s\ge 0} a_s k(1-k^2)^s - \log(1-k^2) \sum_{s\ge 0} b_s k(1-k^2)^2 \right].$$

These integrands are now odd functions of k:

$$\bar{k}_s^{(1)} \equiv \int k(1-k^2)^s dk = -\frac{1}{2}\int y^s dy = -\frac{1}{2}\frac{y^{s+1}}{s+1} + c = -\frac{1}{2}\frac{(1-k^2)^{s+1}}{s+1} + c. \tag{14}$$

We select the integration constant c to have this evaluate to 0 at the lower limit $k=0$, i.e., $\int K dk$ is interpreted as $\int_0^k K(k')dk'$:

$$\bar{k}_s^{(1)} = -\frac{1}{2}\frac{(1-k^2)^{s+1} - 1}{s+1}. \tag{15}$$

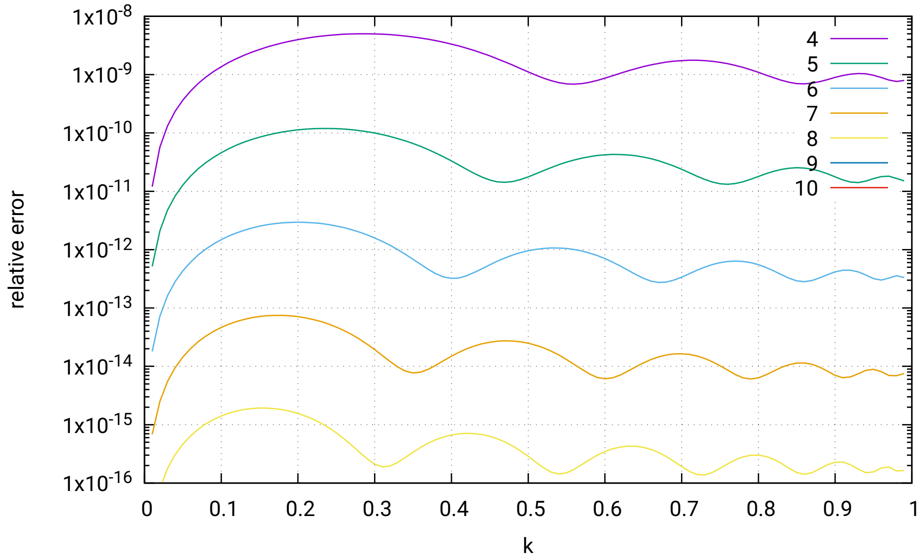

FIGURE 2. Relative error in the approximation (12) of $\int K(k)dk$ for upper limit k with Cody-Hastings polynomial orders from 4 to 10. Curves for orders 9 and 10 do not appear because the relative errors stay below the 10^{-16} limit chosen on the error scale.

The auxiliary integrals for the b_s terms are calculated with the substitution $y = 1-k^2$ [11, 2.723.1][18, 1.6.1.18]

(16)
$$\bar{l}_s^{(1)} \equiv \int k(1-k^2)^s \log(1-k^2)dk = -\frac{1}{2}\int y^s \log y dy = -\frac{1}{2}y^{s+1}\left[\frac{\log y}{s+1} - \frac{1}{(s+1)^2}\right] + c$$
$$= -\frac{1}{2}(1-k^2)^{s+1}\left[\frac{\log(1-k^2)}{s+1} - \frac{1}{(s+1)^2}\right] + c.$$

The integration constant c is selected to reach a limit of zero for $k = 0$:

$$\bar{l}_s^{(1)} = -\frac{1}{2}(1-k^2)^{s+1}\left[\frac{\log(1-k^2)}{s+1} - \frac{1}{(s+1)^2}\right] - \frac{1}{2(s+1)^2}. \tag{17}$$

The performance of the approximation

$$\int kK dk \approx \sum_{s\ge 0} a_s \bar{k}_s^{(1)} - \sum_{s\ge 0} b_s \bar{l}_s^{(1)} \tag{18}$$

is illustrated in Figure 3 for various polynomial orders. As for the moments of order zero, polynomial orders 9 suffice to generate results with double precision accuracy for all upper limits k.

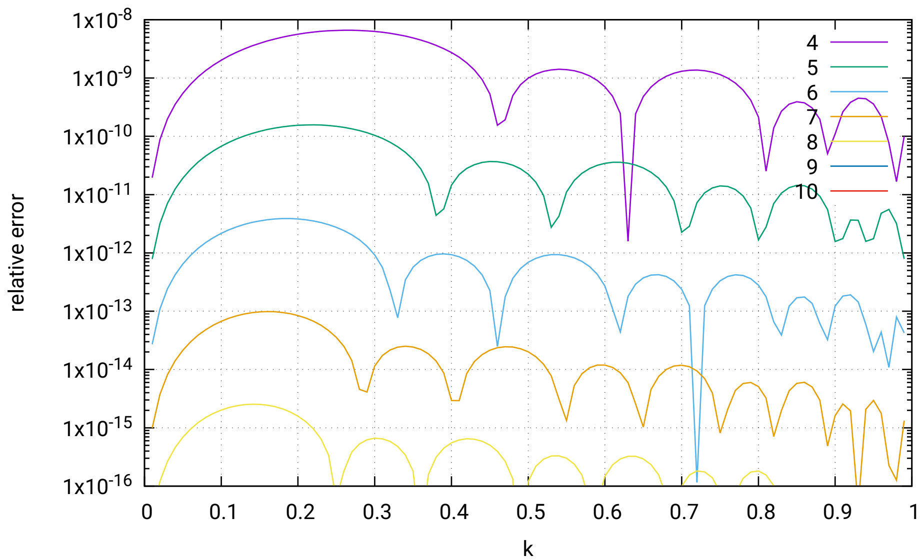

FIGURE 3. Relative error in the approximation (18) of $\int kK(k)dk$ for upper limit k with Cody-Hastings polynomial orders from 4 to 10.

2.3. **Moments of Higher Order.** Higher even powers of k in the integrand are recursively wound back to order zero by binomial expansion of the factor:

(19)
$$\bar{k}_s^{(2i)} \equiv \int k^{2i}(1-k^2)^s dk = (-)^i \int (-k^2)^i (1-k^2)^s dk = (-)^i \int (1-k^2-1)^i (1-k^2)^s dk$$
$$= (-)^i \sum_{l=0}^{i} \binom{i}{l} \int (1-k^2)^l (-)^{i-l} (1-k^2)^s dk = \sum_{l=0}^{i} (-)^l \binom{i}{l} \int (1-k^2)^l (1-k^2)^s dk$$
$$= \sum_{l=0}^{i} (-)^l \binom{i}{l} \int (1-k^2)^{s+l} dk = \sum_{l=0}^{i} (-)^l \binom{i}{l} \bar{k}_{s+l}^{(0)}, \quad i \ge 0.$$

Higher odd powers of k are wound back to order one:

(20)
$$\bar{k}_s^{(2i+1)} \equiv \int k^{2i+1}(1-k^2)^s dk = -\frac{1}{2}\int (k^2)^i y^s dy = -(-)^i \frac{1}{2} \int (-k^2)^i y^s dy$$
$$= -(-)^i \frac{1}{2} \int (1-k^2-1)^i y^s dy = -(-)^i \frac{1}{2} \int (y-1)^i y^s dy = -(-)^i \frac{1}{2} \sum_{l=0}^{i} \binom{i}{l} \int y^l (-)^{i-l} y^s dy$$
$$= -\frac{1}{2} \sum_{l=0}^{i} (-)^l \binom{i}{l} \frac{1}{l+s+1} y^{l+s+1} + c = -\frac{1}{2} \sum_{l=0}^{i} (-)^l \binom{i}{l} \frac{1}{l+s+1} (1-k^2)^{l+s+1} + c$$
$$= -\frac{1}{2} \sum_{l=0}^{i} (-)^l \binom{i}{l} \frac{(1-k^2)^{l+s+1} - 1}{l+s+1} = \sum_{l=0}^{i} (-)^l \binom{i}{l} \bar{k}_{l+s}^{(1)}, \quad i \ge 0.$$

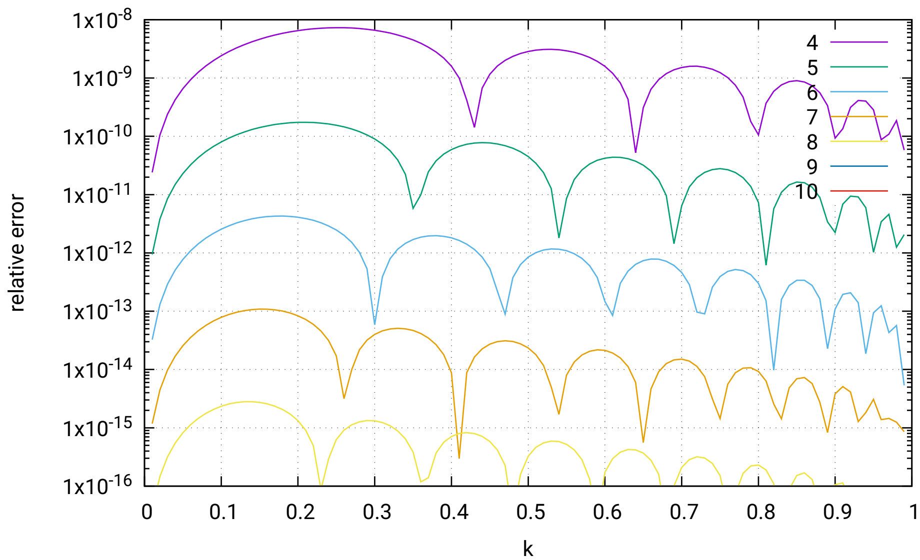

FIGURE 4. Relative error of $\int k^2 K dk$ according to (24) for $i = 2$ and polynomial orders 4 to 10.

The integrals over the terms for the b-coefficients are for even powers

$$\bar{l}_s^{(2i)} \equiv \int k^{2i}(1-k^2)^s \log(1-k^2) = (-)^i \int (1-k^2-1)^i (1-k^2)^s \log(1-k^2) \tag{21}$$
$$= \sum_{l=0}^{i} (-)^l \binom{i}{l} \bar{l}_{s+l}^{(0)}, \quad i \geq 0.$$

For odd powers $1 - k^2 = y$, $dy = -2kdk$, is substituted as above:

$$\bar{l}_s^{(2i+1)} \equiv \int k^{2i+1}(1-k^2)^s \log(1-k^2) dk = -(-)^i \frac{1}{2} \int (1-k^2-1)^i (1-k^2)^s \log(1-k^2) dy \tag{22}$$
$$= -\frac{1}{2} \sum_{l=0}^{i} \binom{i}{l} (-)^l \int (1-k^2)^l (1-k^2)^s \log(1-k^2) dy$$
$$= -\frac{1}{2} \sum_{l=0}^{i} \binom{i}{l} (-)^l \int y^{l+s} \log y dy = \sum_{l=0}^{i} \binom{i}{l} (-)^l \bar{l}_{l+s}^{(1)}, \quad i \geq 0.$$

Remark 3. *The Pascal-type recurrences are*

$$\bar{k}_s^{(i)} = \bar{k}_{s-1}^{i} - \bar{k}_{s-1}^{i+2}; \quad \bar{l}_s^{(i)} = \bar{l}_{s-1}^{i} - \bar{l}_{s-1}^{i+2}. \tag{23}$$

This defines the Hastings-Cody approximation for moments of order i:

$$\int k^i K dk \approx \sum_{s\geq 0} a_s \bar{k}_s^{(i)} - \sum_{s\geq 0} b_s \bar{l}_s^{(i)} \tag{24}$$

with the performance of the second moments $i = 2$ illustrated in Figure 4.

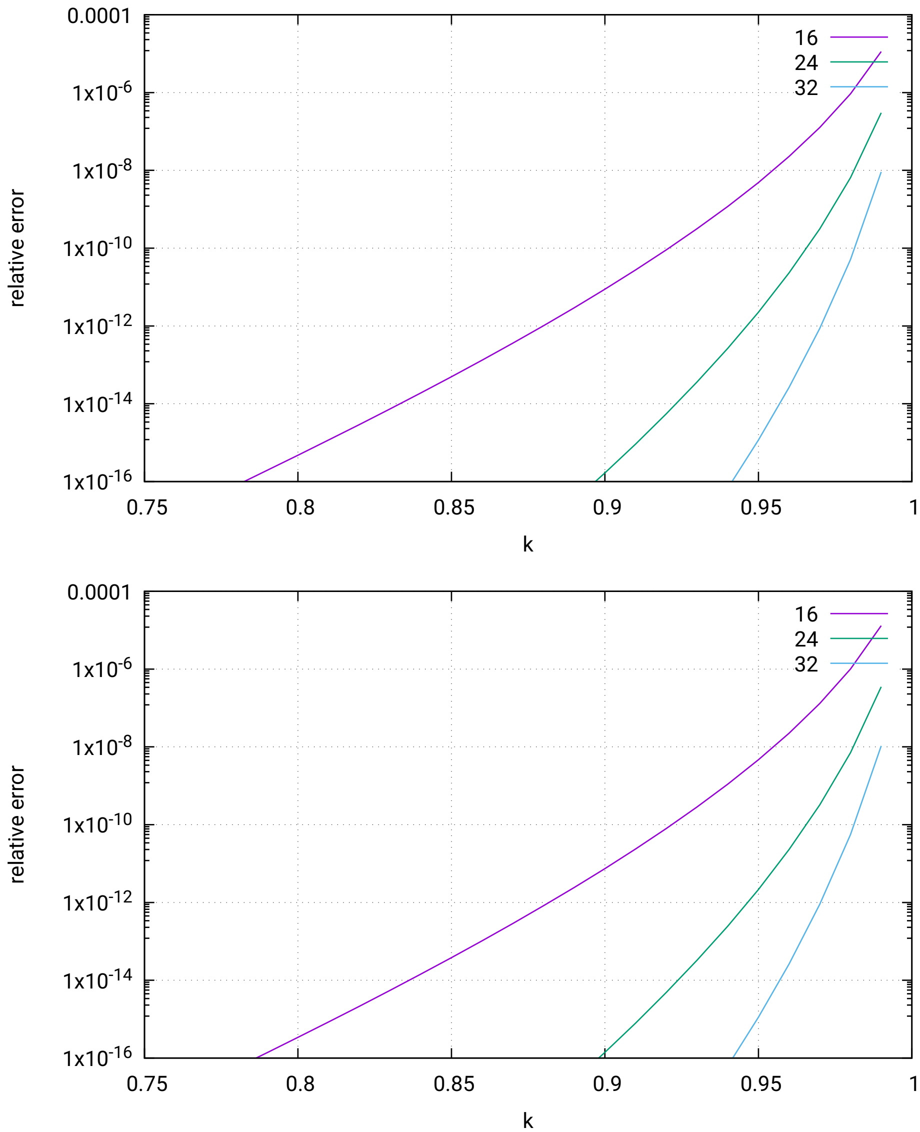

Figure 5. Relative errors of Gauss-Legendre quadratures with 16, 24 or 32 abscissae points for $\int k^i K(k) dk$ at $i = 1$ (top) and $i = 2$ (bottom).

3. Gauss-Legendre Quadrature

Examples of a Gauss-Legendre quadrature of $\int k^i K(k) dk$ for upper limits between 0.7 and 1.0 are shown in Figure 5; weights and abscissae have been copied from [16]. As expected, this approach cannot handle the curvature of $K(k)$ at large upper limits; 32 abscissae points are sufficient for double precision accuracy as long as the upper limits are smaller than ≈ 0.94. For upper limits smaller than 0.8, 16 abscissae points are sufficient.

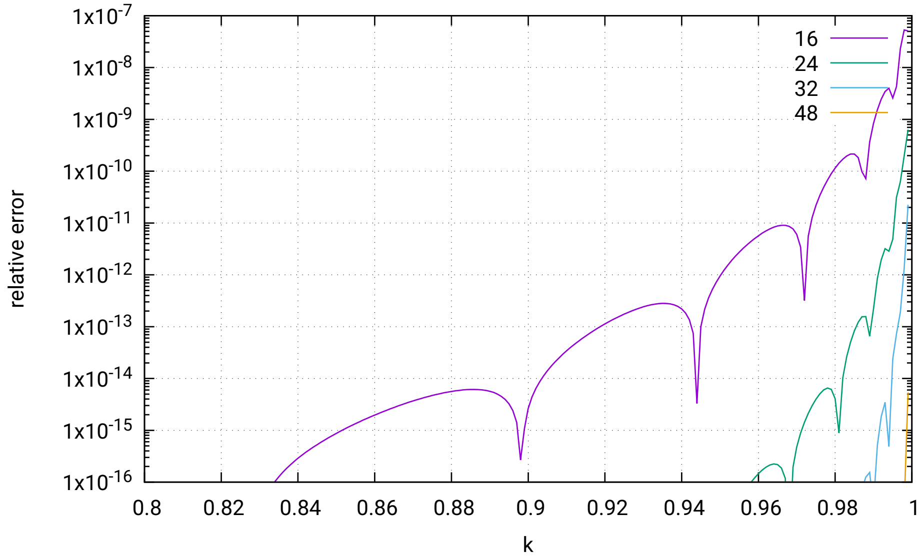

FIGURE 6. Relative errors of Gauss-Legendre quadratures for $\int K(k)dk$ with 16, 24, 32 or 48 abscissae points over the θ interval computed with (25).

4. Reversed Order of Integration

Integrals over Elliptic integrals are double integrals. Swapping the order of the integrations proposes [11, 2.261]

$$\int K(k)dk = \int dk \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\pi/2} d\theta \frac{1}{\sin\theta} \arcsin(k\sin\theta), \tag{25}$$

which is a rather smooth function of k and θ. This can be integrated with a Gauss-Legendre procedure over the (fixed) θ-interval. Figure 6 shows that this achieves double precision accuracy for $k \le 0.98$ if 32 abscissa points are used, or for $k \le 0.95$ if 24 points are used. The same strategy for first order moments reads

$$\int kK(k)dk = \int dkk \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\pi/2} d\theta \frac{1-\sqrt{1-k^2\sin^2\theta}}{\sin^2\theta}. \tag{26}$$

Remark 4. *The constant of integration over dk has again been chosen to tack the integral to zero at $k = 0$.*

The second moments are [11, 2.264.3]

$$\begin{aligned} \int k^2K(k)dk &= \int dkk^2 \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \\ &= \int_0^{\pi/2} d\theta \frac{1}{2\sin^2\theta}\left[-k\sqrt{1-k^2\sin^2\theta} + \int \frac{dk}{\sqrt{1-k^2\sin^2\theta}}\right] \\ &= \int_0^{\pi/2} d\theta \frac{1}{2\sin^2\theta}\left[-k\sqrt{1-k^2\sin^2\theta} + \frac{\arcsin(k\sin\theta)}{\sin\theta}\right]. \end{aligned} \tag{27}$$

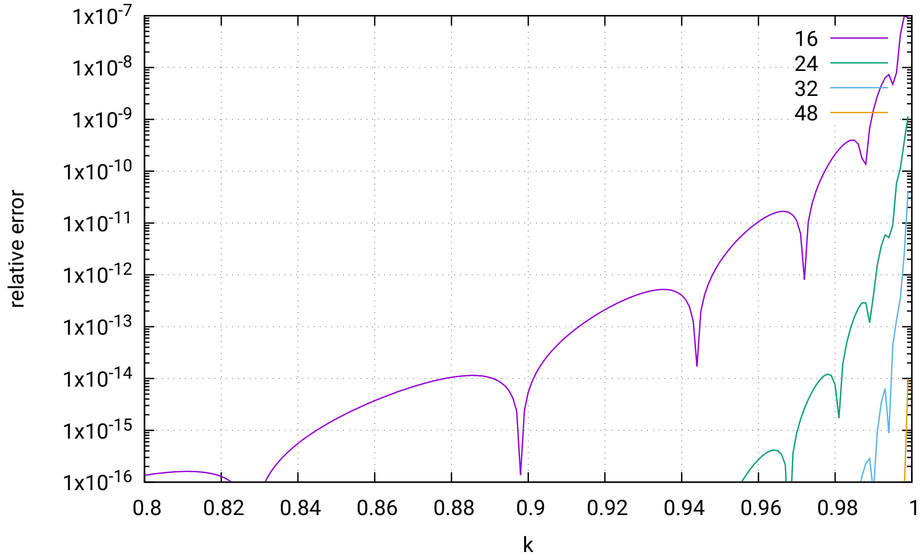

FIGURE 7. Relative errors of Gauss-Legendre quadratures for $\int kK(k)dk$ with 16, 24, 32 or 48 abscissae points over the θ interval computed with (26).

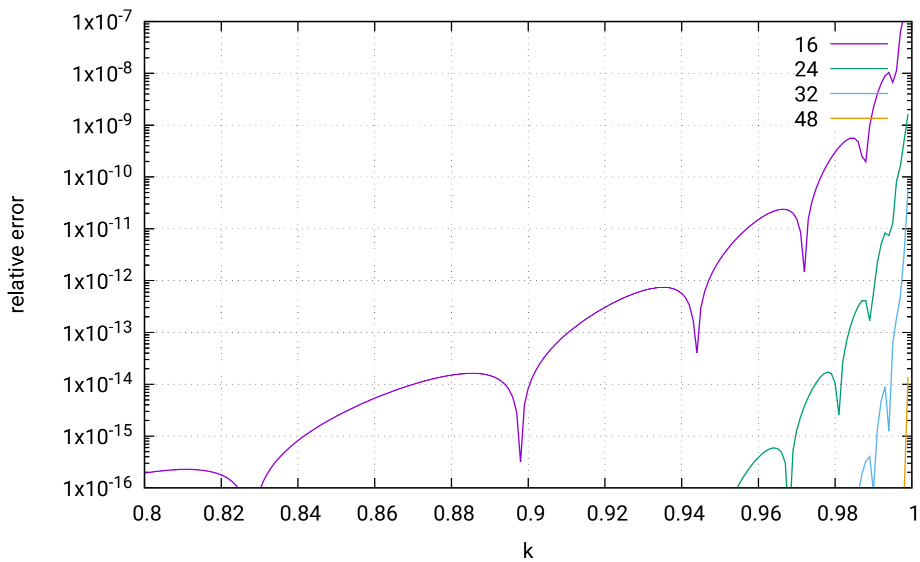

FIGURE 8. Relative errors of Gauss-Legendre quadratures for $\int k^2K(k)dk$ with 16, 24, 32 or 48 abscissae points over the θ interval computed with (27).

In summary, this methodology is an accurate and fast evaluation—compared with

Section 3 where each point requires evaluation of a $K(k)$—if the upper limit of the k-integral is not too close to 1.

5. Integrand with Squared Elliptic Integral

For integrated powers of K like $\int dk K^2(k)$, a reference evaluation emerges if the square of (1) is evaluated as a power series based on the product of two Gaussian Series [4]:

$$(28)\quad {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \mid px\right) {}_2F_1\left(\begin{matrix} a', b' \\ c' \end{matrix} \mid qx\right) = \sum_n \frac{(a)_n (b)_n}{(c)_n n!} {}_4F_3\left(\begin{matrix} a', b', 1-c-n, -n \\ 1-a-n, 1-b-n, c' \end{matrix} \mid q/p\right) (px)^n.$$

$$(29)\quad K^2 = \frac{\pi^2}{4} \sum_{n\ge 0} \frac{(1/2)_n (1/2)_n}{n!^2} {}_4F_3\left(\begin{matrix} 1/2, 1/2, -n, -n \\ 1/2-n, 1/2-n, 1 \end{matrix} \mid 1\right) k^{2n}.$$

Interchange of integration and summation yields series representations of the moments:

$$(30)\quad \int K^2 dk = \frac{\pi^2}{4} \sum_{n\ge 0} \frac{(1/2)_n (1/2)_n}{(n!)^2 (2n+1)} {}_4F_3\left(\begin{matrix} 1/2, 1/2, -n, -n \\ 1/2-n, 1/2-n, 1 \end{matrix} \mid 1\right) k^{1+2n}$$
$$= \frac{\pi^2}{4} k \big[1 + \frac{1}{6} k^2 + \frac{11}{160} k^4 + \frac{17}{64} k^6 + \frac{1787}{8192} k^8 + \frac{277}{16384} k^{10} + \frac{42631}{3407872} k^{12} + \frac{75937}{7864320} k^{14} + \frac{70223483}{9126805504} k^{16} + \frac{128223827}{20401094656} k^{18} + \cdots\big].$$

$$(31)\quad \int k K^2 dk = \frac{\pi^2}{4} \sum_{n\ge 0} \frac{(1/2)_n (1/2)_n}{(n!)^2 (2n+2)} {}_4F_3\left(\begin{matrix} 1/2, 1/2, -n, -n \\ 1/2-n, 1/2-n, 1 \end{matrix} \mid 1\right) k^{2+2n}$$
$$= \frac{\pi^2}{4} k^2 \big[\frac{1}{2} + \frac{1}{8} k^2 + \frac{11}{192} k^4 + \frac{17}{512} k^6 + \frac{1787}{81920} k^8 + \frac{3047}{196608} k^{10} + \frac{42631}{3670016} k^{12} + \frac{75937}{8388608} k^{14} + \frac{70223483}{9663676416} k^{16} + \frac{128223827}{21474836480} k^{18} + \cdots\big].$$

These do not converge well, as illustrated in Figure 9, although Wynn's series transformation [23] may gain some orders of magnitude if we compare Fig. 9 with Fig. 10.

Remark 5. *The poor convergence of this reference implementation is the reason not to plot some results in this section at $k > 0.85$.*

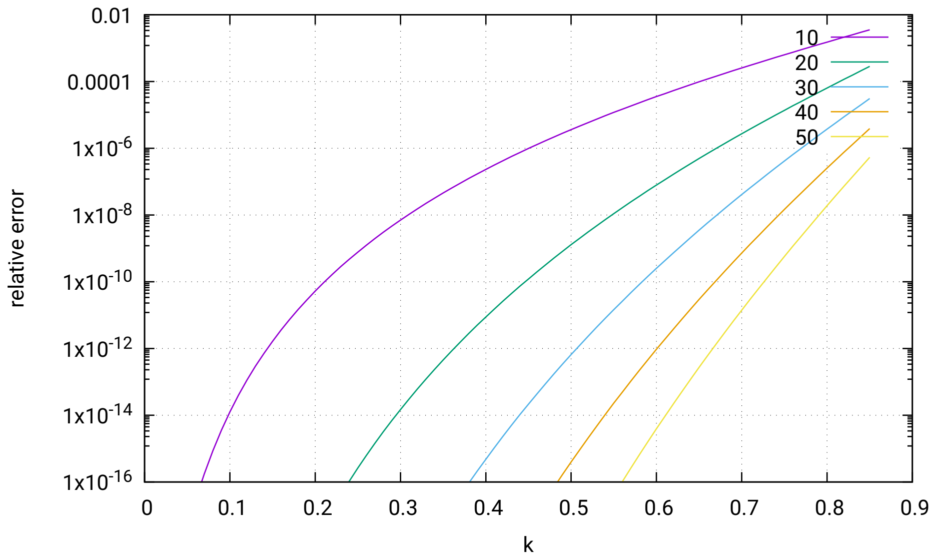

FIGURE 9. Relative errors of power series representations for $\int K(k)^2 dk$ truncated at the orders $O(k^o)$ for $o = 10, 20, \ldots$, which means essentially keeping the first 5, 10,... non-vanishing terms of the power series.

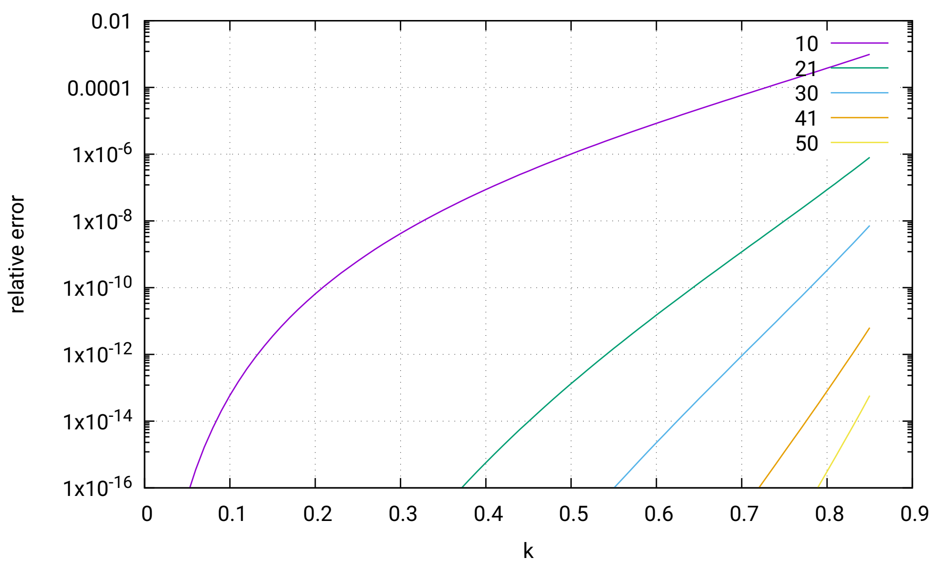

FIGURE 10. Relative errors of power series representations for $\int K(k)^2 dk$ truncated at the orders $O(k^o)$ for $o = 10, 21, \ldots$, and postprocessed with Wynn's convergence acceleration. (The truncation orders are slightly adjusted because the acceleration needs two more series terms to produce one more estimator...)

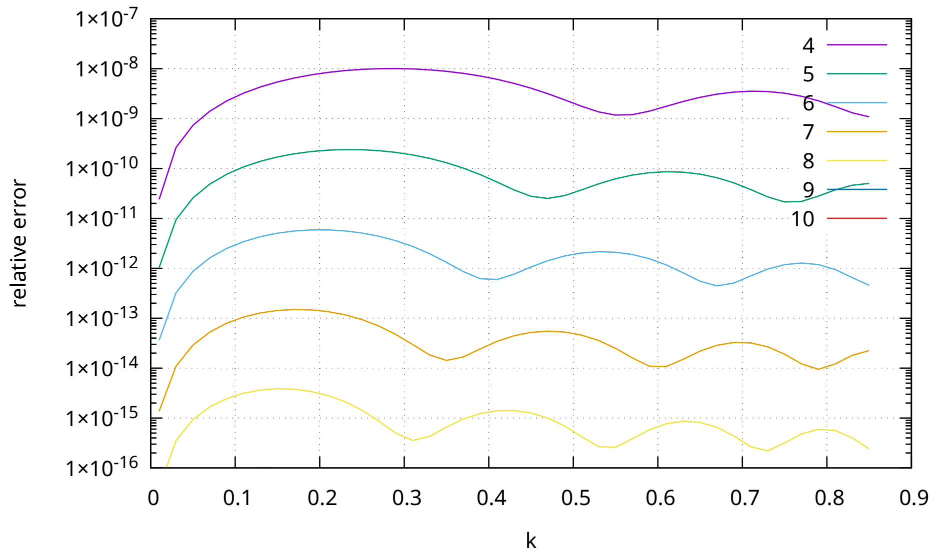

FIGURE 11. Relative errors of the Hastings-Cody representation (32) for $\int K(k)^2 dk$ for polynomial orders from 4 to 10.

The Hastings-Cody approach of the square proceeds with binomial expansion which convolves the a_s and b_s sums:

$$\int dk K^2(k) \approx \int dk \left[\sum_{s\ge 0} a_s(1-k^2)^s - \log(1-k^2)\sum_{s\ge 0} b_s(1-k^2)^s\right]^2 \tag{32}$$

$$= \int dk \left[\sum_{s\ge 0} a_s(1-k^2)^s\right]^2 + \int dk \log^2(1-k^2)\left[\sum_{s\ge 0} b_s(1-k^2)^s\right]^2$$

$$-2\int dk \log(1-k^2)\sum_{s\ge 0} a_s(1-k^2)^s \sum_{s'\ge 0} b_{s'}(1-k^2)^{s'}$$

$$= \sum_{s\ge 0}\bar{k}_s^{(0)}\sum_{s'\ge 0} a_{s-s'}a_{s'} + \int dk \log^2(1-k^2)\left[\sum_{s\ge 0} b_s(1-k^2)^s\right]^2 - 2\sum_{s\ge 0}\bar{l}_s^{(0)}\sum_{s'\ge 0} a_{s-s'}b_{s'}$$

$$= \sum_{s\ge 0}\bar{k}_s^{(0)}\sum_{s'\ge 0} a_{s-s'}a_{s'} + \sum_{s\ge 0}\bar{L}_s\sum_{s'\ge 0} b_{s-s'}b_{s'} - 2\sum_{s\ge 0}\bar{l}_s^{(0)}\sum_{s'\ge 0} a_{s-s'}b_{s'}.$$

In the middle term integrals with a squared logarithm are required,

$$\bar{L}_s \equiv \int dk \log^2(1-k^2)(1-k^2)^s, \tag{33}$$

which are detailed in App. B. a_s, b_s with negative indices or indices larger than the polynomial order are interpreted as zero. Figure 11 demonstrates that polynomial orders 9 achieve double precision accuracy for $\int dk K^2(k)$.

6. Summary

The approximation (24) is the Hastings-Cody equivalent for ith- moments of the Complete Elliptic Integral of the First Kind, where the auxiliary integrals $\bar{k}_s^{(i)}$ and $\bar{l}_s^{(i)}$ might be recursively generated from (9), (11), (15) and (17).

Appendix A. Auxiliary Integral denominator m_1

In preparation of the last section in this manuscript, partial fraction decomposition and [11, 2.727.2,2.728.2] yield for the negative index in (11)

(34)
$$\bar{l}_{-1}^{(0)} = \int \frac{\log(1-k^2)}{1-k^2}dk = \frac{1}{2}\int \left[\frac{\log(1+k)}{1+k}+\frac{\log(1-k)}{1+k}+\frac{\log(1+k)}{1-k}+\frac{\log(1-k)}{1-k}\right]dk$$
$$= \frac{1}{4}\log^2(1+k)+\frac{1}{2}\int\frac{\log(1-k)}{1+k}dk+\frac{1}{2}\int\frac{\log(1+k)}{1-k}-\frac{1}{4}\log^2(1-k)$$
$$= [\log 2+\frac{1}{2}\log(1-k^2)]\,\mathrm{arctanh}\,k+\frac{1}{4}(1-k)\Phi\left(\frac{1-k}{2},2,1\right)-\frac{1}{4}(1+k)\Phi\left(\frac{1+k}{2},2,1\right)$$

where Φ is the Lerch function [11, 9.550][7, §1.11]

$$\Phi(z,s,v) = \sum_{n\ge 0}\frac{z^n}{(v+n)^s}. \qquad (35)$$

Appendix B. Auxiliary Integral with Squared Logarithm

By partial integration of (33)

$$
\begin{aligned}
\bar L_s &= \bar k_s^{(0)} \log^2(1-k^2) - 2\int dk \bar k_s^{(0)} \log(1-k^2)\frac{-2k}{1-k^2} \\
&= \bar k_s^{(0)} \log^2(1-k^2) + 4\int dk \bar k_s^{(0)} \log(1-k^2)\frac{k}{1-k^2} \\
&= \bar k_s^{(0)} \log^2(1-k^2) - 4\sum_{l=0}^{s}\binom{s}{l}\frac{1}{2l+1}\int dk \log(1-k^2)\frac{(-k^2)^{l+1}}{1-k^2} \\
&= \bar k_s^{(0)} \log^2(1-k^2) - 4\sum_{l=0}^{s}\binom{s}{l}\frac{1}{2l+1}\int dk \log(1-k^2)\frac{(1-k^2-1)^{l+1}}{1-k^2} \\
&= \bar k_s^{(0)} \log^2(1-k^2) - 4\sum_{l=0}^{s}\binom{s}{l}\frac{(-)^l}{2l+1}\sum_{l'=0}^{l+1}\binom{l+1}{l'}(-)^{1-l'}\int dk \log(1-k^2)\frac{(1-k^2)^{l'}}{1-k^2} \\
&= \bar k_s^{(0)} \log^2(1-k^2) - 4\sum_{l=0}^{s}\binom{s}{l}\frac{(-)^l}{2l+1}\Big[(-)\int dk \log(1-k^2)\frac{1}{1-k^2} \\
&\quad + \sum_{l'=1}^{l+1}\binom{l+1}{l'}(-)^{1-l'}\int dk \log(1-k^2)\frac{(1-k^2)^{l'}}{1-k^2}\Big] \\
&= \bar k_s^{(0)} \log^2(1-k^2) + 4\bar l_{-1}^{(0)}\sum_{l=0}^{s}\binom{s}{l}\frac{(-)^l}{2l+1} - 4\sum_{l=0}^{s}\binom{s}{l}\frac{(-)^l}{2l+1}\sum_{l'=0}^{l}\binom{l+1}{l'+1}(-)^{l'}\int dk \log(1-k^2)(1-k^2)^{l'} \\
&= \bar k_s^{(0)} \log^2(1-k^2) + 4\bar l_{-1}^{(0)}\frac{1}{\binom{s+1/2}{s}} - 4\sum_{l'=0}^{s}(-)^{l'}\bar l_{l'}^{(0)}\sum_{l=l'}^{s}\binom{s}{l}\frac{(-)^l}{2l+1}\binom{l+1}{l'+1}.
\end{aligned} \tag{36}
$$

The $\bar k_s^{(0)}$ are computed from (9), the $\bar l_s^{(0)}$ from (11), and $\bar l_{-1}^{(0)}$ from App. A.

References

1. Milton Abramowitz and Irene A. Stegun (eds.), *Handbook of mathematical functions*, 9th ed., Dover Publications, New York, 1972. MR 0167642
2. W. N. Bailey, *Contiguous hypergeometric functions of the type $_3f_2$*, Proc. Glasg. Math. Ass. **2** (1954), no. 2, 62–65. MR 0064918
3. Paul F. Byrd and Morris D. Friedman, *Handbook of elliptical integrals for engineers and physicists*, 2nd ed., Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 67, Springer, Berlin, Göttingen, 1971, E: [9, 8, 14]. MR 0277773
4. T. W. Chaundy, *An extension of hypergeometric functions (i)*, Q. J. Math. **14** (1943), 55–78. MR 0010749
5. W. J. Cody, *Chebyshev polynomial expansions of complete elliptic integrals*, Math. Comp. **19** (1965), no. 90, 249–259. MR 0178563
6. G. Dôme and K. S. Kölbig, *Table errata 621*, Math. Comp. **65** (1996), no. 215, 1379–1386.
7. Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, and Francesco G. Tricomi (eds.), *Higher transcendental functions*, vol. 1, McGraw-Hill, New York,London, 1953, E:[22]. MR 0058756
8. Henry E. Fettis, *Table errata 576*, Math. Comp. **36** (1981), no. 153, 317.
9. ______, *Table errata 580*, Math. Comp. **36** (1981), no. 153, 319.
10. O. E. I. S. Foundation Inc., *The On-Line Encyclopedia Of Integer Sequences*, (2026), https://oeis.org/. MR 3822822
11. I. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, 8 ed., Elsevier, Amsterdam, 2015, E: https://www.mathtable.com/errata/ [20]. MR 3307944

12. Cecil Hastings (ed.), *Approximations for digital computers*, Princeton University Press, Princeton, 1955. MR 0068915
13. K. S. Kölbig, *Table errata 629*, Math. Comp. **66** (1997), no. 220, 1765–1767.
14. ______, *Table errata 632*, Math. Comp. **66** (1997), no. 220, 1765–1767.
15. Ernst D. Krupnikov, *Table errata 601*, Math. Comp. **41** (1983), no. 164, 782–783. MR 0717727
16. Richard J. Mathar, *Gauss-Laguerre and Gauss-Hermite quadrature on 64, 96 and 128 nodes*, vixra:1303.0013 (2013).
17. Richard J. Mathar and Artur Jasinski, *Errata to chapter 7 (hypergeometric constants) in vol. 3 "integrals and series — more special functions" (1990) by prudnikov et al.*, viXra:2510.0141 (2025).
18. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and series*, vol. 1, Taylor and Francis, 1986, E: [13].
19. ______, *Integrals and series*, vol. 3, Gordon and Breach, 1990, E: [6, 15, 17]. MR 1054647
20. Robert Reynolds, *Derivation of some definite integrals*, Scientia A (2026), 37–137.
21. Lucy Joan Slater, *Generalized hypergeometric functions*, Cambridge University Press, 1966. MR 0201688
22. H. van Haeringen and L. P. Kok, *Table errata 594*, Math. Comput. **41** (1983), no. 164, 775–783.
23. P. Wynn, *On a device for computing the $e_m(s_n)$ transformation*, Math. Tabl. Aids Comput. **10** (1956), no. 54, 91–96. MR 0084056

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