# A proof for the Collatz conjecture

J. Chauvet, jerome.chauvet.msc@gmail.com, Nîmes (France)

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## Abstract

This proof addresses the Collatz conjecture and relies on the quantitative characterization undergone by the evolving densities of even versus odd natural numbers within the limit set produced by infinite iterations of the transformation law $T$ from any initial value taken in $\mathbb{N}$. This couple of limit densities at infinity for even and odd numbers naturally comes along with respective frequencies of occurrence for the two competing transformations of the Collatz system, from which we deduce the final cyclic attractor, hence proving the conjecture.

## Introduction

Although being intensively studied since 1937 [1], the Collatz conjecture has remained unproven until now. It states that the following dynamical system :

$$T(X_n) := \left\{ \begin{array}{ll} \text{if } X_n \text{ is odd, then } X_{n+1} = 3X_n + 1 & (*) \\ \text{if } X_n \text{ is even, then } X_{n+1} = \dfrac{X_n}{2} & (**) \end{array} \right\}$$

must evolve in the limit of the greatest values for $n$ so that :

$$n \to +\infty \qquad X_n = 1,\ X_{n+1} = 4,\ X_{n+2} = 2,\ X_{n+3} = 1,\ \ldots$$

Despite numerous verifications through direct computation of $T$ that this cyclic attractor must be the only possible end state, no general proof could be established for any initial value $X_0 \in \mathbb{N}$ .

## Our method

Let us remark before all the following :

If $X_n$ is odd, then it automatically satisfies the equation $X_n = 2k+1 \ (k \in \mathbb{N})$ . In accordance with the definition of $T$, then we will have the following holding as well :

$$\begin{aligned} X_{n+1} &= 3(2k+1)+1 \\ &= 6k+4 \\ &= 2(3k+2) \end{aligned}$$

which means that $T$ always produces an even number as an output whenever an odd number is fed to it.

However, in case of $X_n$ being even, then $X_n = 2k \ (k \in \mathbb{N})$ must hold, so that, in accordance with the definition of $T$, we will also have the following :

$$X_{n+1} = \frac{2k}{2} = k$$

which means that whichever $X_n$ we take at random amongst the even naturals as an input to $T$, the output will have half chance to be odd, half chance to be even, only depending on $k$.

Writing $T_\omega$ for (*) and $T_\epsilon$ for (**), therefore we obtain :

| Iteration | Input density | Function | Redistribution | Output density |
|---|---|---|---|---|
| n=0 | 1/2 odd | $T_\omega$ | 1/2 even | 3/4 even |
| | 1/2 even | $T_\epsilon$ | (1/2).(1/2) even | |
| | | | (1/2).(1/2) odd | 1/4 odd |

| | | | | |
|---|---|---|---|---|
| n=1 | 1/4 odd | $T_\omega$ | 1/4 even | 5/8 even |
| | 3/4 even | $T_\epsilon$ | (1/2).(3/4) even | |
| | | | (1/2).(3/4) odd | 3/8 odd |

| | | | | |
|---|---|---|---|---|
| n=2 | 3/8 odd | $T_\omega$ | 3/8 even | 11/16 even |
| | 5/8 even | $T_\epsilon$ | (1/2).(5/8) even | |
| | | | (1/2).(5/8) odd | 5/16 odd |

...

| | | | | |
|---|---|---|---|---|
| $n \to +\infty$ | ?/? odd | $T_\omega$ | ?/? even | ?/? even |
| | ?/? even | $T_\epsilon$ | (1/2).(?/?) even | |
| | | | (1/2).(?/?) odd | ?/? odd |

The latter can be expressed in form of a Markov process [2] like the following :

$$\begin{pmatrix} \omega \\ \epsilon \end{pmatrix}_n = \begin{pmatrix} 0 & 1/2 \\ 1 & 1/2 \end{pmatrix} \begin{pmatrix} \omega \\ \epsilon \end{pmatrix}_n = M \begin{pmatrix} \omega \\ \epsilon \end{pmatrix}_n$$

To be relevant, this Markov process must be iterated from zero up to $n$ in parallel with the transformation $T$, yielding each time the corresponding densities for odd ($\omega$), respectively even ($\varepsilon$), numbers within the $n^{\text{th}}$ image set of $T$, represented by the vector above.

Using the standard method for determining the eigenvalues and eigenvectors of $M$ through calculation of the characteristic polynomial of the endomorphism [3], we find that :

$$\begin{aligned} det\, M = 0 &\Leftrightarrow (-\lambda)(1/2 - \lambda) - (1)(1/2) \\ &\Leftrightarrow \lambda^2 - \frac{1}{2}(\lambda + 1) = 0 \end{aligned}$$

which has two distinct solutions $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2}$. Thus, *M* can be reduced and turned into a diagonal matrix *D* like the following :

$$\begin{aligned} M^n &= P\, D^n\, P^{-1} \\ &= P \begin{pmatrix} 1^n & 0 \\ 0 & (-1)^n \frac{1}{2^n} \end{pmatrix} P^{-1} \end{aligned}$$

where *P* is the transition matrix and $P^{-1}$ its inverse such that $PP^{-1} = I$ (i.e. the identity matrix).

From there on, we have to determine the following limit :

$$\lim_{n \to +\infty} P\, D^n\, P^{-1} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

by simply finding the two eigenvectors of *M* such that :

$$\begin{pmatrix} 0 & 1/2 \\ 1 & 1/2 \end{pmatrix} \begin{pmatrix} u_i \\ v_i \end{pmatrix} = \lambda_i \begin{pmatrix} u_i \\ v_i \end{pmatrix}$$

which turn out to be :

$$\lambda_i = \lambda_1 \Rightarrow \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \qquad \lambda_i = \lambda_2 \Rightarrow \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

From the latter, we derive the transition matrix :

$$P = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1 \\ 1 & -1 \end{pmatrix}$$

and its inverse using the standard formula for 2x2 matrices :

$$\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}^{-1} = \frac{1}{u_1 v_2 - v_1 u_2} \begin{pmatrix} v_2 & -u_2 \\ -v_1 & u_1 \end{pmatrix}$$

Hence :

$$P^{-1} = \frac{1}{-3/2} \begin{pmatrix} -1 & -1 \\ -1 & 1/2 \end{pmatrix} = \begin{pmatrix} 2/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix}$$

Since $M^n \begin{pmatrix} \omega \\ \epsilon \end{pmatrix}_0 = P\, D^n\, P^{-1} \begin{pmatrix} \omega \\ \epsilon \end{pmatrix}_0$ for all *n*, we can now determine the limit state of the Markov process :

$$\lim_{n \to +\infty} \begin{pmatrix} \omega \\ \epsilon \end{pmatrix}_n = \lim_{n \to +\infty} P\, D^n\, P^{-1} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \lim_{n \to +\infty} \begin{pmatrix} 1/2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & (-1/2)^n \end{pmatrix} \begin{pmatrix} 2/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

Since $\lim_{n \to +\infty} 1^n = 1$ and $\lim_{n \to +\infty} (-1/2)^n = 0$ , then the matrix product $P\, D^\infty\, P^{-1}$

becomes :

$$\lim_{n\to+\infty}\begin{pmatrix}\omega\\ \epsilon\end{pmatrix}_n=\begin{pmatrix}2/6 & 2/6\\ 2/3 & 2/3\end{pmatrix}\begin{pmatrix}1/2\\ 1/2\end{pmatrix}=\begin{pmatrix}\omega_\infty\\ \epsilon_\infty\end{pmatrix}=\begin{pmatrix}1/3\\ 2/3\end{pmatrix}$$

This result means that the limit set obtained by iterating *T*([*a*,*b*]) infinitely many times out of an arbitrary initial interval [*a*,*b*] of $\mathbb{N}$ starts with both densities being 1/2 with respect to even and odd numbers, and ends up with one third for odd numbers and two thirds for even numbers at infinity.

As a consequence, for *n* aiming towards infinity, the transformation law *T*, which comes in 2 different forms $T\omega$ (*) and $T\varepsilon$ (**), will be affected with a frequency of occurrence defined by the density of each type of input, odd or even, given by the limit vector on which the Markov process operates. In this limit, $T\omega$ will happen for one third of the total application of *T* whilst $T\varepsilon$ will happen for two thirds as well.

Thus, we are led to translate this limit regime of *T* into a new function *H*, defined in 3 equivalent forms through permutation :

$$\lim_{n\to+\infty}X_{n+3}=\begin{Bmatrix}H_1(X_n)=T_\omega(T_\epsilon(T_\epsilon(X_n)))\\ H_2(X_n)=T_\epsilon(T_\omega(T_\epsilon(X_n)))\\ H_3(X_n)=T_\epsilon(T_\epsilon(T_\omega(X_n)))\end{Bmatrix}$$

Direct calculation of those 3 functions $H_1$, $H_2$ and $H_3$ yields three possible limit transformation laws defined like the following :

$$H_1(X_n)=3\left(\frac{1}{2}\left(\frac{1}{2}X_n\right)\right)+1=\frac{3}{4}X_n+1\qquad H_2(X_n)=\frac{1}{2}\left(3\left(\frac{X_n}{2}\right)+1\right)=\frac{3}{4}X_n+\frac{1}{2}\qquad H_3(X_n)=\frac{1}{2}\left(\frac{1}{2}(3X_n+1)\right)=\frac{3}{4}X_n+\frac{1}{4}$$

Since any of these *H* functions will be iterated infinitely many times, i.e. in the limit $n\to+\infty$ , the following must hold as well :

$$H_i^n(X_n)=(3/4)^n X_n+\beta((3/4)^0+(3/4)^1+\ \ldots+(3/4)^n)$$

where β = 1, 1/2 or 1/4. Moreover, since $\sum_{i=0}^{+\infty}(3/4)^i=\frac{1}{1-3/4}=\frac{1}{1/4}=4$ [4], we consequently get the expression for the limit behavior of the transformation *T* :

$$\lim_{n\to+\infty}H_i^n(X_n)=(3/4)^n X_n+4\beta=4\beta$$

for which $\lim_{n\to+\infty}(3/4)^n=0$ extinguishes all finite value $X_n$ . More precisely, replacing β with one of the three possible values as above, we will have :

$$\lim_{n\to+\infty} H_1^n(X_n) = 4$$
$$\lim_{n\to+\infty} H_2^n(X_n) = 4/2 = 2$$
$$\lim_{n\to+\infty} H_3^n(X_n) = 4/4 = 1$$

## Conclusion

Given the primary definition of $T$ (cf. introduction), this triplet of values defines the attractor state (i.e. final state) of the transformation as $T^n \to H^n$ in the limit of $n$ tends to infinity in form of a cycle $4 \to 2 \to 1 \to 4 \to \ldots$ , which proves the Collatz conjecture.

## References

[1] https://en.wikipedia.org/wiki/Collatz_conjecture
[2] https://en.wikipedia.org/wiki/Markov_chain
[3] https://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors
[4] https://en.wikipedia.org/wiki/Geometric_series