Regularized Gamma–Bernoulli Identities for the Riemann Zeta Function and a Conditional Critical-Line Criterion

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Abstract

A Gamma–Bernoulli approach to the critical-line problem for the Riemann zeta function is developed. Starting from the Mellin–theta representation and the functional equation, one obtains explicit identities for the reflected Gamma quotient and for the regularization built into the Weierstrass product for Γ. On the Bernoulli side, the kernel $(\mathrm{e}^u - 1)^{-1}$ is decomposed into its singular part and an analytic remainder, which yields a concrete zero-conditioned identity after continuation. The analysis shows that the harmonic divergence visible in raw finite Gamma products is a truncation phenomenon and therefore cannot by itself force $\mathrm{Re}(\rho) = \frac{1}{2}$. What remains is a coercive estimate which, if established, would convert the same mechanism into a critical-line theorem.

Contents

Keywords. Riemann zeta function; Gamma function; Bernoulli numbers; functional equation; critical strip; regularized products.

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1 Introduction

The Riemann zeta function entered analytic number theory through Riemann's 1859 memoir, where the relation between the zeros of $\zeta(s)$ and the distribution of prime numbers was first placed in a systematic analytic form [1]. The conjecture now called the Riemann Hypothesis (RH) asserts that every nontrivial zero ρ of $\zeta(s)$ satisfies

$$\mathrm{Re}(\rho) = \frac{1}{2}. \tag{1.1}$$

The significance of RH extends far beyond the location of the zeros themselves. It controls the size of the oscillatory terms in prime-counting formulae and influences the sharpness of estimates throughout multiplicative number theory [2]. Classical analytic accounts of the continuation of $\zeta(s)$, the functional equation, and the geometry of the critical strip are given by Titchmarsh [3], Edwards [4], Patterson [5], and Ivić [6].

The problem remains open. Bombieri's survey places RH among the Millennium Prize Problems and explains its arithmetic scope [7], while Conrey's survey describes the analytic, spectral, and probabilistic viewpoints that have shaped modern work on the subject [8]. A useful way to measure the present difficulty is to compare the distinct criteria known to be equivalent to RH. Li's criterion recasts RH as the positivity of a sequence attached to the completed zeta function [9]. Robin's criterion translates RH into an inequality for the divisor-sum function [10]. Lagarias gave an elementary reformulation in terms of harmonic numbers and the divisor function [11]. These equivalent criteria show that the same obstruction can be expressed through positivity, divisor growth, or harmonic averages, and they illustrate why no single point of view has yet resolved the problem.

The argument developed here belongs to the special-function side of the subject. Its ingredients are classical: Mellin transforms, the Jacobi theta transformation, Bernoulli expansions of the kernel $(\mathrm{e}^u - 1)^{-1}$, and the Weierstrass product for Γ. A contemporary review of the fourth edition of Whittaker and Watson records the long-standing role of such methods in complex analysis [12], while the *NIST Handbook of Mathematical Functions* provides a modern reference for the same analytic infrastructure [13]. The issue is therefore not the validity of the underlying identities, but the extent to which they constrain the real part of a zero once the functional equation is combined with a zero-conditioned continuation.

The aim is to determine precisely what this Gamma–Bernoulli mechanism yields after the necessary regularization has been carried out. Throughout,

$$s = \sigma + \mathrm{i}t, \qquad \sigma, t \in \mathbb{R}, \tag{1.2}$$

and

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{1.3}$$

denotes the completed zeta function. The symbol ρ is reserved for a nontrivial zero of ζ.

The analysis leads to three concrete conclusions. First, the reflected Gamma quotient admits an exact regularized product representation, and the logarithmic drift visible in finite unregularized products is therefore a truncation effect rather than an invariant obstruction. Second, the Bernoulli decomposition of $(\mathrm{e}^u - 1)^{-1}$ yields, after continuation, an exact zero-conditioned identity in the critical strip. Third, once these two components are placed in their correct regularized form, the unresolved point is coercive: one needs a sign estimate for the combined Gamma–Bernoulli balance in order to deduce $\sigma = \frac{1}{2}$ for every nontrivial zero. The result is thus a conditional critical-line criterion rather than a proof of RH.

Section 2 records the standard Mellin–theta representation and the functional equation. Section 3 derives the continued Bernoulli identity and its zero-conditioned form. Sections 4–7 analyze the reflected Gamma quotient, its regularization, the local Gamma balance, and the Bernoulli remainder. Section 8 collects the numerical interpretation, the structural role of the critical line, and the concluding analytic consequence. Technical complements are gathered at the end.

2 Standard analytic setup

This section records the analytic identities on which the later argument rests. Each formula is standard, but the argument depends on keeping their domains of validity explicit before any continuation is used at a zeta zero.

For $\mathrm{Re}(s) > 1$, the Riemann zeta function is defined by the absolutely convergent Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \tag{2.1}$$

and Euler's product gives

$$\zeta(s) = \prod_{p}(1 - p^{-s})^{-1} \qquad (\mathrm{Re}(s) > 1), \tag{2.2}$$

where the product runs over primes. This Euler product is the analytic link between $\zeta(s)$ and the primes and is the natural starting point for the discussion below.

The Mellin-transform route begins with the Gamma integral

$$\Gamma(z) = \int_0^{\infty} \mathrm{e}^{-u}u^{z-1}\,\mathrm{d}u, \qquad \mathrm{Re}(z) > 0. \tag{2.3}$$

Replacing u with $\pi n^2 x$ and summing over $n \geq 1$ yields

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty \left(\sum_{n=1}^\infty \mathrm{e}^{-\pi n^2 x}\right) x^{s/2-1}\,\mathrm{d}x, \qquad \mathrm{Re}(s) > 1. \tag{2.4}$$

If we define

$$S(x) := \sum_{n=1}^\infty \mathrm{e}^{-\pi n^2 x}, \tag{2.5}$$

then

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty S(x) x^{s/2-1}\,\mathrm{d}x. \tag{2.6}$$

Now introduce the Jacobi theta function

$$\theta(x) := \sum_{n=-\infty}^\infty \mathrm{e}^{-\pi n^2 x} = 1 + 2S(x), \tag{2.7}$$

which satisfies the classical modular relation

$$\theta(x) = x^{-1/2}\theta(x^{-1}). \tag{2.8}$$

Equivalently,

$$S(x^{-1}) = -\frac{1}{2} + \frac{1}{2}x^{1/2} + x^{1/2}S(x). \tag{2.9}$$

Splitting the Mellin integral at $x = 1$ and using the previous identity yields the classical completed-zeta representation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty S(x)\left(x^{s/2-1} + x^{-(s+1)/2}\right)\mathrm{d}x. \tag{2.10}$$

The right-hand side is invariant under $s \mapsto 1 - s$. Hence one recovers the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{2.11}$$

Remark 2.1. Equation (2.11) immediately implies that if ρ is a nontrivial zero, then so are $1-\rho$, $\overline{\rho}$, and $1-\overline{\rho}$. This gives the familiar quartet symmetry. It does *not* by itself imply $\mathrm{Re}(\rho) = \frac{1}{2}$. Any method that starts from the functional equation must therefore produce at least one further coercive input.

Figure 1 displays the profile of $\log|\xi(\sigma + it)|$ as a function of σ for representative fixed heights t. The graph makes the symmetry

$$\xi(s) = \xi(1-s)$$

visible at the level of the modulus: reflection about $\sigma = \frac{1}{2}$ leaves the completed zeta function unchanged. This is the geometric starting point of the entire argument. Every identity derived from the functional equation is centered on the same vertical line, but the figure also makes clear that symmetry alone does not force a zero onto that line.

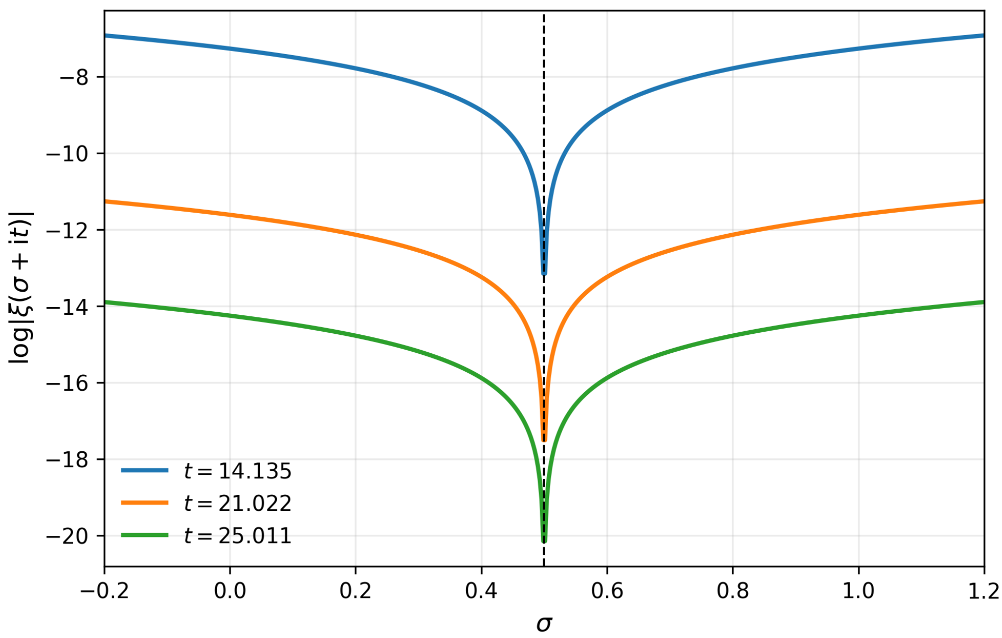

Figure 1: Symmetry profile of $\log|\xi(\sigma+it)|$ about $\sigma=\frac{1}{2}$.

3 Bernoulli expansions and zero-conditioned identities

The Bernoulli side of the method begins with a local decomposition of the kernel $(\mathrm{e}^u-1)^{-1}$. The purpose of this section is to separate the singular contribution, which can be integrated explicitly, from the analytic remainder, which survives after continuation into the critical strip. This separation leads to an exact identity at a nontrivial zero and makes clear which part of the argument is genuinely arithmetic and which part is only a consequence of analytic regularization.

The Bernoulli numbers B_n are defined by the generating function

$$\frac{u}{\mathrm{e}^u-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}u^n,\qquad |u|<2\pi. \tag{3.1}$$

Consequently,

$$\frac{1}{\mathrm{e}^u-1}=\frac{1}{u}-\frac{1}{2}+\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}u^{2n-1},\qquad |u|<2\pi. \tag{3.2}$$

The corresponding local and asymptotic expansions for the Gamma and digamma functions are standard. Here they are used only to identify the singular contribution explicitly and to isolate the analytic remainder.

The Mellin transform of the kernel $(\mathrm{e}^u-1)^{-1}$ is

$$\Gamma(s)\zeta(s)=\int_0^{\infty}\frac{u^{s-1}}{\mathrm{e}^u-1}\,\mathrm{d}u,\qquad \mathrm{Re}(s)>1. \tag{3.3}$$

Replacing s by $1-s$ gives

$$\Gamma(1-s)\zeta(1-s)=\int_0^{\infty}\frac{u^{-s}}{\mathrm{e}^u-1}\,\mathrm{d}u,\qquad \mathrm{Re}(s)<0. \tag{3.4}$$

Formula (3.4) does not directly apply in the critical strip, so the singular part of the kernel must first be isolated.

Define

$$h(u) := \frac{1}{e^u - 1} - \frac{1}{u} + \frac{1}{2}. \tag{3.5}$$

Since $h(u) = O(u)$ as $u \to 0^+$, the integrals

$$I_0(s) := \int_0^1 h(u) u^{-s} \, du, \qquad I_1(s) := \int_1^\infty \frac{u^{-s}}{e^u - 1} \, du \tag{3.6}$$

converge absolutely for $0 < \mathrm{Re}(s) < 2$ and define analytic functions there.

Proposition 3.1. *For* $0 < \mathrm{Re}(s) < 1$ *one has*

$$\Gamma(1-s)\zeta(1-s) = -\frac{1}{s} - \frac{1}{2(1-s)} + I_0(s) + I_1(s). \tag{3.7}$$

Proof. For $\mathrm{Re}(s) < 0$, split (3.4) at $u = 1$ and insert (3.5). This gives

$$\begin{aligned} \Gamma(1-s)\zeta(1-s) &= \int_0^1 \left(\frac{1}{u} - \frac{1}{2} + h(u) \right) u^{-s} \, du + \int_1^\infty \frac{u^{-s}}{e^u - 1} \, du \\ &= \int_0^1 u^{-s-1} \, du - \frac{1}{2} \int_0^1 u^{-s} \, du + I_0(s) + I_1(s). \end{aligned} \tag{3.8}$$

The elementary integrals are

$$\int_0^1 u^{-s-1} \, du = -\frac{1}{s}, \qquad \int_0^1 u^{-s} \, du = \frac{1}{1-s}. \tag{3.9}$$

Substituting into (3.8) yields (3.7) for $\mathrm{Re}(s) < 0$. Since the right-hand side of (3.7) is analytic on $0 < \mathrm{Re}(s) < 1$, the identity extends to that strip by analytic continuation. □

Proposition 3.1 gives a concrete identity at the reflected zeros. If ρ is a nontrivial zero of ζ, then $1 - \rho$ is also a zero by Remark 2.1, and therefore

$$I_0(\rho) + I_1(\rho) = \frac{1}{\rho} + \frac{1}{2(1-\rho)}. \tag{3.10}$$

The right-hand side is explicit, while the left-hand side contains the regularized Bernoulli kernel. This identity is the correct starting point for any attempt to extract information on $\sigma = \mathrm{Re}(\rho)$ from the Bernoulli side.

What fails in the unregularized argument is the next inference: one cannot conclude $\sigma = \frac{1}{2}$ from a harmonic divergence until the Weierstrass normalization of the Gamma product has been taken into account. The next section makes that normalization explicit.

4 The Weierstrass product and the key regularized identity

The next step is to write the reflected Gamma quotient in a form in which the truncation error and the canonical regularization are completely separated. That distinction is decisive: the raw finite products carry a harmonic drift, whereas the exact Weierstrass product already contains the exponential counterterm that cancels it.

We start from the Weierstrass product for the Gamma function

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \frac{e^{z/k}}{1 + z/k}, \qquad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}, \tag{4.1}$$

where γ is Euler's constant.

A natural comparison in this method is between values of Γ at points related by $x \mapsto 1 - x$. The following lemma gives the exact quotient identity in that setting.

Lemma 4.1 (Gamma quotient identity)**.** *For $x \in \mathbb{C} \setminus \mathbb{Z}$ one has*

$$\frac{\Gamma(2-x)}{\Gamma(x+1)} = e^{-\gamma(1-2x)} \prod_{k=1}^{\infty} \left(e^{(1-2x)/k} \frac{k+x}{k+1-x} \right). \tag{4.2}$$

Proof. Apply (4.1) to $\Gamma(1-x)$ and $\Gamma(x)$ and divide. Then use the functional relation $\Gamma(z+1) = z\Gamma(z)$ to replace $\Gamma(1-x)/\Gamma(x)$ by $\Gamma(2-x)/\Gamma(x+1)$. □

Equation (4.2) may be rearranged as

$$1 = e^{\gamma(1-2x)} \frac{\Gamma(2-x)}{\Gamma(x+1)} \prod_{k=1}^{\infty} \left(e^{-(1-2x)/k} \frac{k+1-x}{k+x} \right). \tag{4.3}$$

This is exact. However, if one removes the exponential correction $e^{-(1-2x)/k}$ and keeps only the raw quotient $(k+1-x)/(k+x)$, then one obtains a divergent product whenever $x \neq \frac{1}{2}$. This is the source of the tempting but incorrect "harmonic-series forces the half-line" inference.

The correct statement is the following exact limit.

Theorem 4.2 (Regularized product)**.** *Let $x \in \mathbb{C} \setminus \mathbb{Z}$ and define*

$$R_N(x) := e^{\gamma(1-2x)-(1-2x)H_N} \frac{\Gamma(2-x)}{\Gamma(x+1)} \prod_{k=1}^{N} \frac{k+1-x}{k+x}, \tag{4.4}$$

where $H_N = \sum_{k=1}^{N} 1/k$ is the Nth harmonic number. Then

$$\lim_{N \to \infty} R_N(x) = 1. \tag{4.5}$$

Proof. The finite product defining $R_N(x)$ is exactly the Nth truncation of (4.3). Since the infinite product in (4.3) converges to 1, the conclusion follows. □

Corollary 4.3. *For $x \neq \frac{1}{2}$, the divergence of the raw factor*

$$\frac{\Gamma(2-x)}{\Gamma(x+1)} \prod_{k=1}^{N} \frac{k+1-x}{k+x} \tag{4.6}$$

as $N \to \infty$ is exactly canceled by the harmonic renormalization $e^{\gamma(1-2x)-(1-2x)H_N}$. Therefore the raw appearance of $\sum_{k \geq 1} 1/k$ does not by itself imply $x = \frac{1}{2}$.

Remark 4.4. The regularized identity shows exactly where the finite-product argument breaks down. The harmonic divergence belongs to the unregularized truncation and is removed by the normalization already implicit in the Weierstrass product.

Figure 2 compares the raw finite product attached to the reflected Gamma quotient with its canonically regularized form. Theorem 4.2 shows that the factor involving H_N is not optional: it is exactly the counterterm that removes the harmonic drift from the truncation. The figure is therefore an analytic diagnostic. It separates the genuine Gamma ratio, which stabilizes after renormalization, from the artificial growth created by the unregularized partial products.

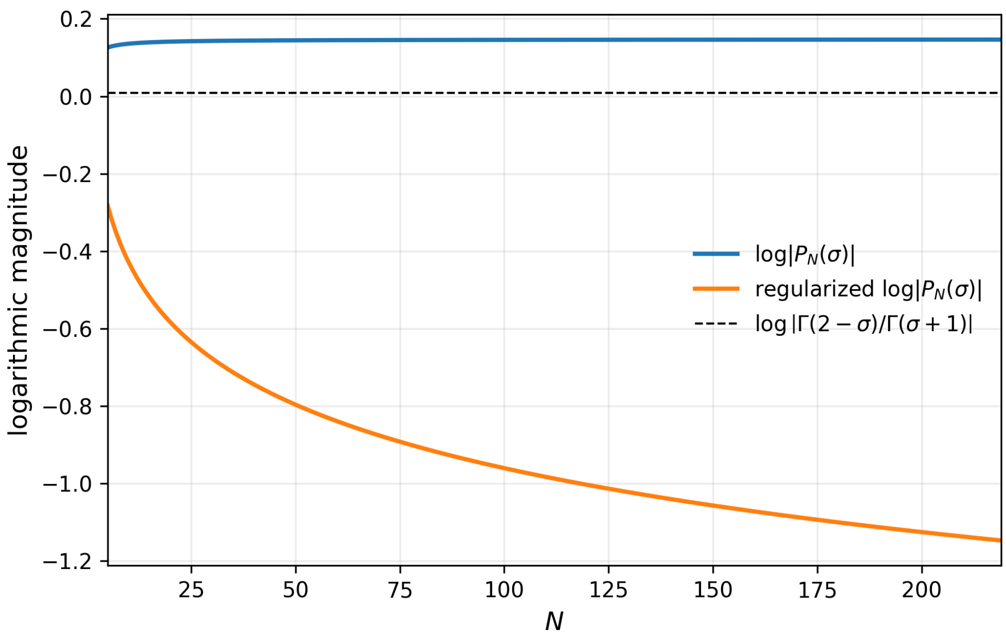

Figure 2: Raw and regularized reflected Gamma quotients as the truncation length increases.

5 Local structure of the Gamma balance

A recurring quantity in this method is the imbalance between the two Gamma factors naturally attached to the reflected pair $\sigma + it$ and $1 - \sigma + it$. For fixed $t \in \mathbb{R}$ define

$$G_t(\sigma) := \log\left|\Gamma\left(\frac{\sigma + it}{2}\right)\right| - \log\left|\Gamma\left(\frac{1 - \sigma + it}{2}\right)\right|, \qquad 0 < \sigma < 1. \tag{5.1}$$

This quantity measures the local imbalance between the reflected Gamma factors.

Proposition 5.1. *For each fixed $t \in \mathbb{R}$ and $\sigma \in (0, 1)$ one has*

$$G_t(1 - \sigma) = -G_t(\sigma), \tag{5.2}$$

$$G_t\left(\frac{1}{2}\right) = 0, \tag{5.3}$$

$$G_t'(\sigma) = \frac{1}{2}\operatorname{Re}\psi\left(\frac{\sigma + it}{2}\right) + \frac{1}{2}\operatorname{Re}\psi\left(\frac{1 - \sigma + it}{2}\right), \tag{5.4}$$

where $\psi = \Gamma'/\Gamma$ is the digamma function.

Proof. Equation (5.2) follows immediately from the definition (5.1). Evaluating at $\sigma = \frac{1}{2}$ gives (5.3). For the derivative, note that if $z(\sigma)$ is differentiable and avoids the poles of Γ, then

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\log|\Gamma(z(\sigma))| = \mathrm{Re}\big(\psi(z(\sigma))z'(\sigma)\big)\,. \tag{5.5}$$

Apply this to $z_1(\sigma) = \frac{\sigma+\mathrm{i}t}{2}$ and $z_2(\sigma) = \frac{1-\sigma+\mathrm{i}t}{2}$. ☐

The antisymmetry (5.2) shows that the critical line is the natural center of the Gamma comparison. The next statement makes the local structure precise.

Lemma 5.2. *For each fixed $t \in \mathbb{R}$, the function $\delta \mapsto G_t(\frac{1}{2}+\delta)$ is odd in a neighborhood of $\delta = 0$. Consequently,*

$$G_t\left(\frac{1}{2}+\delta\right) = \sum_{m=0}^{M} \frac{G_t^{(2m+1)}(\frac{1}{2})}{(2m+1)!}\,\delta^{2m+1} + O_{M,t}(\delta^{2M+3}) \qquad (\delta \to 0). \tag{5.6}$$

In particular,

$$G_t\left(\frac{1}{2}+\delta\right) = \mathrm{Re}\,\psi\left(\frac{1}{4}+\frac{\mathrm{i}t}{2}\right)\delta + O_t(\delta^3). \tag{5.7}$$

Proof. By (5.2), $G_t(\frac{1}{2}-\delta) = -G_t(\frac{1}{2}+\delta)$, so the translated function is odd. The Taylor expansion of an odd analytic function contains only odd powers. The coefficient of δ is $G_t'(\frac{1}{2})$, and (5.4) gives

$$G_t'\left(\frac{1}{2}\right) = \mathrm{Re}\,\psi\left(\frac{1}{4}+\frac{\mathrm{i}t}{2}\right). \tag{5.8}$$

☐

The previous lemma quantifies the first-order response of the Gamma side to a displacement away from the critical line.

Proposition 5.3 (Large-t local coercivity on the Gamma side)**.** *There exist constants $T > 0$, $\delta_0 > 0$, and $c > 0$ such that for $|t| \geq T$ and $|\delta| \leq \delta_0$ one has*

$$\mathrm{sgn}(\delta)\,G_t\left(\frac{1}{2}+\delta\right) \geq c|\delta|. \tag{5.9}$$

Proof. The digamma asymptotic formula in any fixed sector away from the negative real axis gives

$$\psi(z) = \log z - \frac{1}{2z} + O(|z|^{-2}), \qquad |z| \to \infty. \tag{5.10}$$

Substituting $z = \frac{1}{4} + \frac{\mathrm{i}t}{2}$ yields

$$G_t'\left(\frac{1}{2}\right) = \mathrm{Re}\,\psi\left(\frac{1}{4}+\frac{\mathrm{i}t}{2}\right) = \log\left(\frac{|t|}{2}\right) + O(|t|^{-2}), \tag{5.11}$$

hence $G_t'(\frac{1}{2}) \geq 2c > 0$ for all $|t| \geq T$ if T is large enough. Now use (5.7) and continuity of the cubic remainder: after shrinking δ_0 if necessary,

$$\left|G_t\left(\frac{1}{2}+\delta\right) - G_t'\left(\frac{1}{2}\right)\delta\right| \leq c|\delta| \tag{5.12}$$

for all $|t| \geq T$ and $|\delta| \leq \delta_0$. This implies (5.9). ☐

The limitation is equally clear. The quantity $G_t(\sigma)$ depends only on the Gamma factors. To reach RH, one would still need a zero-conditioned theorem showing that the Bernoulli side preserves a compatible sign or monotonicity.

6 Localized Bernoulli control

The Bernoulli expansion enters the method through kernels such as $(\mathrm{e}^u - 1)^{-1}$. The essential point is the existence of a controlled remainder after finitely many Bernoulli terms are removed.

Proposition 6.1 (Localized Bernoulli expansion)**.** *Let $M \geq 1$. Then there exists a bounded analytic function $R_M(u)$ on a neighborhood of $[0,1]$ such that*

$$\frac{1}{\mathrm{e}^u - 1} = \frac{1}{u} - \frac{1}{2} + \sum_{n=1}^{M-1} \frac{B_{2n}}{(2n)!} u^{2n-1} + u^{2M-1} R_M(u) \qquad (0 < u \leq 1). \tag{6.1}$$

Consequently, for $0 < u \leq 1$,

$$\left| \frac{1}{\mathrm{e}^u - 1} - \left(\frac{1}{u} - \frac{1}{2} + \sum_{n=1}^{M-1} \frac{B_{2n}}{(2n)!} u^{2n-1} \right) \right| \leq C_M u^{2M-1} \tag{6.2}$$

for some constant $C_M > 0$ depending only on M.

Proof. The function

$$F(u) := \frac{u}{\mathrm{e}^u - 1} \tag{6.3}$$

is analytic at $u = 0$, with Taylor series

$$F(u) = 1 - \frac{u}{2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} u^{2n}. \tag{6.4}$$

Divide by u and group the first M nonzero terms. The remainder is analytic and therefore bounded on a small complex neighborhood of $[0,1]$, which yields (6.2). □

Proposition 6.1 explains what the Bernoulli part of the method can legitimately provide: a finite explicit main term and a controlled error. Such a decomposition is useful, but by itself it does not force the real part of a zero. The missing step remains a zero-conditioned sign mechanism.

7 Numerical and analytic consequences of the method

This section collects the consequences of the preceding analytic identities that can be displayed explicitly. The numerical material is not used as evidence for RH. Its role is more limited and more precise: each plot is attached to a quantity already introduced in the proofs and is included only to display the behavior predicted by the corresponding proposition or theorem.

7.1 Numerical illustrations aligned with the method

Figure 3 plots the function $G_t(\sigma)$ for several fixed heights t. Proposition 5.1 proves that $G_t(\sigma)$ is odd about $\sigma = \frac{1}{2}$, while Proposition 5.3 identifies the derivative at the critical line as the local quantity governing the Gamma contribution. The graph shows both effects simultaneously: the sign change occurs exactly at $\sigma = \frac{1}{2}$, and the transition becomes sharper as the height increases.

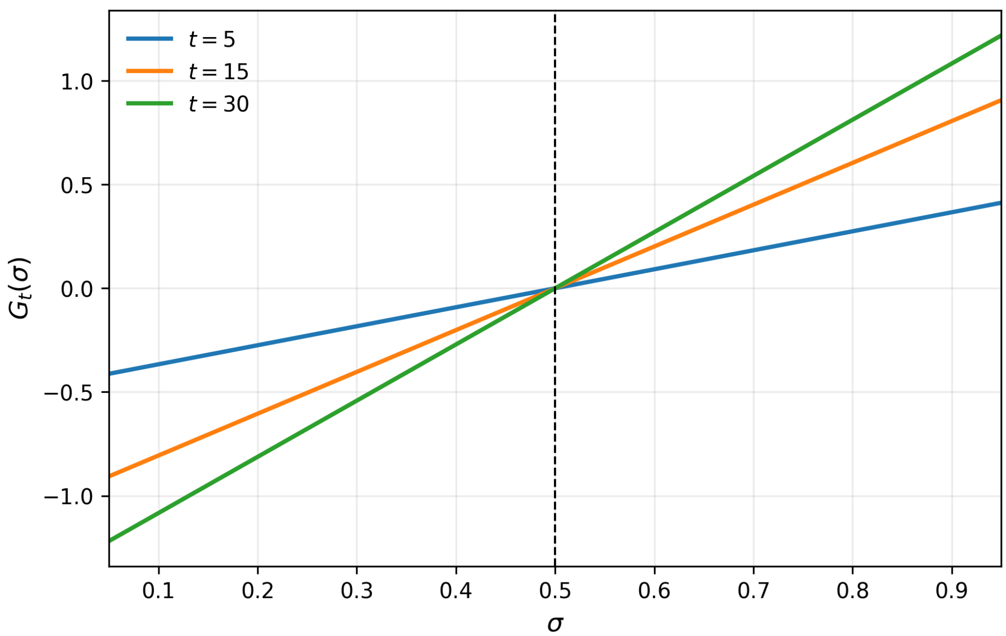

Figure 3: The odd Gamma-balance profile $G_t(\sigma)$ across the critical line.

Figure 4 records the remainder produced by truncating the Bernoulli expansion of $(e^u - 1)^{-1}$ on a fixed compact interval. The point of the estimate in Proposition 6.1 is not merely that the remainder is small, but that it is analytic and uniformly controlled once the singular part has been extracted. The figure displays that mechanism directly: successive truncations reduce the local defect by the order predicted by the Bernoulli expansion.

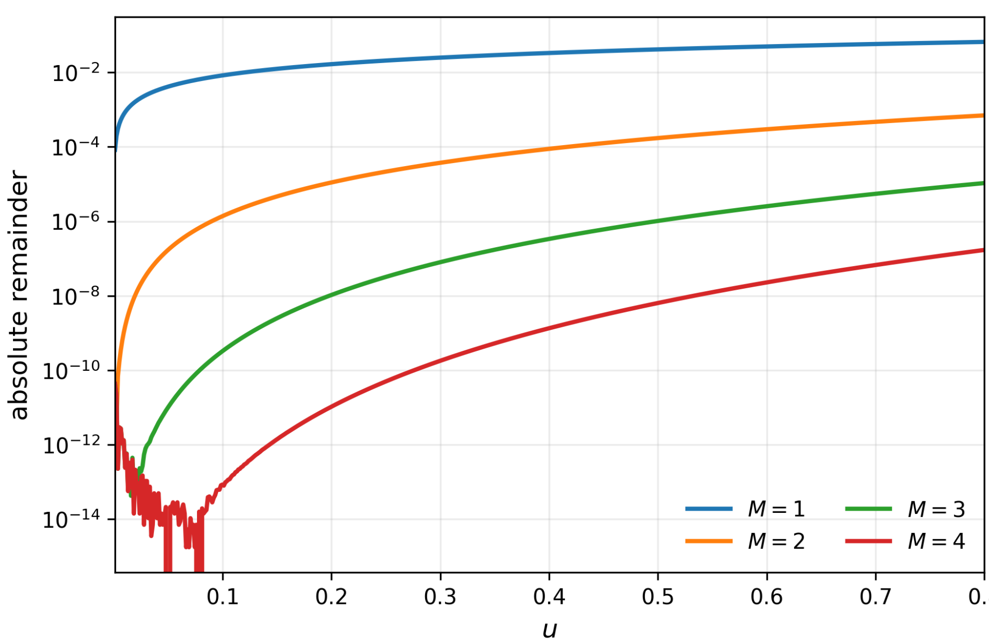

Figure 4: Decay of the localized Bernoulli remainder after successive truncations.

7.2 Finite truncations and the source of the raw drift

It is useful to isolate the exact mechanism that creates the apparent divergence in the unregularized product. Define the finite truncation

$$P_N(\sigma) := \prod_{k=1}^{N} \left(\mathrm{e}^{(1-2\sigma)/k} \frac{k+\sigma}{k+1-\sigma} \right), \qquad 0 < \sigma < 1. \tag{7.1}$$

This is the object that informally suggests that the factor $1-2\sigma$ should be special. The next proposition shows exactly what happens.

Proposition 7.1. *For fixed $\sigma \in (0,1)$ one has*

$$\log |P_N(\sigma)| = (1-2\sigma)H_N - \log \left| \frac{\Gamma(2-\sigma)}{\Gamma(\sigma+1)} \right| - \gamma(1-2\sigma) + o(1) \qquad (N \to \infty). \tag{7.2}$$

Equivalently,

$$\log |P_N(\sigma)| = (1-2\sigma)\log N + O_\sigma(1) \qquad (N \to \infty). \tag{7.3}$$

Proof. Take logarithms in the finite-N identity from Theorem 4.2. Since $H_N = \log N + \gamma + o(1)$, the first formula is immediate, and the second follows by absorbing the bounded terms into $O_\sigma(1)$. □

Proposition 7.1 gives the exact local explanation of the drift visible in the raw product. The coefficient of $\log N$ is indeed $1-2\sigma$, but the divergent term is precisely the one removed by the canonical Weierstrass renormalization. This identifies the source of the apparent divergence and shows why the raw product, taken by itself, cannot determine the location of a zero.

The regularized identity of Theorem 4.2 shows that the raw drift is an artifact of truncation, not a zero-forcing mechanism by itself.

7.3 How the method interacts with a zeta zero

Let

$$\rho = \sigma + \mathrm{i}t \tag{7.4}$$

be a nontrivial zero of ζ. Section 3 already gives one exact consequence of this assumption, namely (3.10). Here the exact identity from Section 3 is compared with the reflected Gamma algebra from Sections 4 and 5.

The estimates used in this section are purely local in the variables (σ, t) and rely only on the reflected Gamma identities already established above.

A basic model is the Gamma reflection formula

$$\Gamma(\rho)\Gamma(1-\rho) = \frac{\pi}{\sin(\pi\rho)}. \tag{7.5}$$

Taking absolute values gives

$$|\Gamma(\rho)|\,|\Gamma(1-\rho)| = \frac{\pi}{|\sin(\pi\rho)|}. \tag{7.6}$$

This identity is invariant under $\rho \mapsto 1-\rho$, so it records the symmetry of the reflected pair but does not isolate σ.

To obtain a quantity that changes under $\sigma \mapsto 1-\sigma$, one passes to a quotient such as

$$Q(\sigma,t) := \frac{\Gamma(\sigma+\mathrm{i}t)}{\Gamma(1-\sigma-\mathrm{i}t)}. \tag{7.7}$$

Its logarithm decomposes into a symmetric part and an antisymmetric part, and the antisymmetric component is measured locally by the Gamma balance $G_t(\sigma)$ from Section 5. The zero condition therefore enters the method in two distinct ways: through the explicit Bernoulli identity (3.10) and through the reflected Gamma quotient.

The central difficulty is now explicit. The Bernoulli identity (3.10) is exact, and the Gamma quotient identities in Sections 4 and 5 are exact, but neither family of identities by itself yields a sign condition forcing $\sigma = \frac{1}{2}$. A proof in this setting would require a single zero-conditioned expression whose real part cannot vanish away from the critical line.

7.4 A conditional critical-line criterion

Section 3 gives the exact identity (3.10) at a nontrivial zero $\rho = \sigma + \mathrm{i}t$, whereas Sections 4 and 5 describe the reflected Gamma contribution. To isolate the remaining burden of the method it is convenient to package the Bernoulli side into a single functional.

Definition 7.2. For $0 < \sigma < 1$ and $t \in \mathbb{R}$ define

$$\mathcal{B}(\sigma,t) := I_0(\sigma+\mathrm{i}t) + I_1(\sigma+\mathrm{i}t) - \frac{1}{\sigma+\mathrm{i}t} - \frac{1}{2(1-\sigma-\mathrm{i}t)}, \tag{7.8}$$

where I_0 and I_1 are given in Section 3.

By (3.10), every nontrivial zero $\rho = \sigma + \mathrm{i}t$ satisfies

$$\mathcal{B}(\sigma,t) = 0. \tag{7.9}$$

What is still missing is a theorem showing that the vanishing of $\mathcal{B}(\sigma,t)$ is impossible unless $\sigma = \frac{1}{2}$.

Theorem 7.3 (Conditional critical-line criterion)**.** *Assume that there exists a continuous function* $W : (0,1) \to [0,\infty)$ *such that*

$$W(\sigma) = 0 \iff \sigma = \frac{1}{2} \tag{7.10}$$

and

$$\operatorname{Re}\mathcal{B}(\sigma,t) \geq W(\sigma) \qquad \textit{for every pair } (\sigma,t) \textit{ arising from a nontrivial zero } \rho = \sigma+\mathrm{i}t. \tag{7.11}$$

Then RH holds.

Proof. Let $\rho = \sigma + \mathrm{i}t$ be a nontrivial zero. Equation (7.9) gives $\mathcal{B}(\sigma,t) = 0$, so (7.11) yields

$$0 = \operatorname{Re}\mathcal{B}(\sigma,t) \geq W(\sigma). \tag{7.12}$$

Hence $W(\sigma) = 0$, and therefore $\sigma = \frac{1}{2}$. □

Theorem 7.3 isolates the remaining analytic burden of the method. Once the zero condition has been written in the explicit form (7.9), the remaining task is to prove a sign estimate for the concrete continued Bernoulli functional $\mathcal{B}(\sigma,t)$.

7.5 A more explicit route to coercivity

The conditional theorem raises a natural question: what form of Bernoulli analysis could produce a function $W(\sigma)$ with the required sign property?

One plausible route is to work with the completed function $\xi(s)$ rather than with ζ directly. Since ξ is entire and satisfies $\xi(s) = \xi(1-s)$, one can hope to compare the logarithmic derivatives

$$\frac{\xi'(\sigma + \mathrm{i}t)}{\xi(\sigma + \mathrm{i}t)} \quad \text{and} \quad \frac{\xi'(1-\sigma + \mathrm{i}t)}{\xi(1-\sigma + \mathrm{i}t)} \tag{7.13}$$

after subtracting the principal singular terms. The Gamma factor contributes digamma functions,

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{7.14}$$

and the Bernoulli expansion of ψ is classical:

$$\psi(z) = \log z - \frac{1}{2z} - \sum_{n=1}^{m-1} \frac{B_{2n}}{2n\, z^{2n}} + O_m(|z|^{-2m}), \qquad |\arg z| < \pi. \tag{7.15}$$

This expansion is especially useful because its real part can often be estimated uniformly in sectors.

Thus a sharpened version of the method may proceed as follows:

1. pass from the zero condition $\xi(\rho) = 0$ to an integral or logarithmic identity involving ψ and Bernoulli coefficients;
2. separate the identity into a symmetry-neutral part and a defect term that vanishes on the critical line;
3. show that the real part of this defect term has a fixed sign when $\sigma \neq \frac{1}{2}$.

The third step remains the unresolved point of the method.

8 Interpretation of the critical line and concluding remarks

8.1 Why the critical line remains central

The line $\sigma = \frac{1}{2}$ appears repeatedly for structural reasons.

First, the completed zeta function is symmetric about that line:

$$\xi\left(\frac{1}{2} + z\right) = \xi\left(\frac{1}{2} - z\right). \tag{8.1}$$

Second, the reflection formulas for Γ pair z with $1 - z$, so the Gamma algebra is centered on the same vertical axis. Third, the harmonic factor in Theorem 4.2 vanishes exactly when $1 - 2x = 0$, which is the unique point at which the raw finite quotient and the regularized quotient agree without correction.

This repeated appearance of the critical line is not a notational accident. It expresses the fact that the functional equation, the Gamma reflection formulas, and the Weierstrass normalization all use the same center of symmetry. What is missing is a zero-conditioned inequality that turns this symmetry into rigidity.

Comparison with other criteria for RH makes the obstruction transparent. Robin's criterion translates RH into an inequality for the divisor-sum function, while Lagarias gave an elementary reformulation in terms of harmonic numbers and $\sigma(n)$. In each case the decisive ingredient is a sign or positivity statement valid on the full domain of the criterion. The same obstruction appears here: the missing step is the coercive estimate from Theorem 7.3.

8.2 Conclusion

The analysis above identifies, in explicit form, the point at which the Gamma–Bernoulli method becomes conditional. The Mellin–theta representation and the functional equation fix the line $\sigma = \frac{1}{2}$ as the symmetry axis of the completed zeta function. On the Bernoulli side, the kernel $(\mathrm{e}^u - 1)^{-1}$ is decomposed into a singular part and an analytic remainder, leading to the continued identity (3.7) and the zero-conditioned relation (3.10). On the Gamma side, the Weierstrass product yields the exact regularized identity of Theorem 4.2, which separates the intrinsic reflected Gamma quotient from the harmonic drift of finite truncations.

Two consequences follow. First, the logarithmic term visible in raw finite products is not a rigidity mechanism for the real part of a zero; it is a truncation artefact removed by canonical regularization. Second, after that regularization has been carried out, the remaining obstruction is coercive. The Gamma contribution is odd about the critical line and its local slope can be described explicitly, while the Bernoulli side contributes an analytic remainder that is locally controlled. What is still missing is a sign estimate strong enough to combine these two pieces into a zero-forcing inequality.

The mathematical content of the method is therefore clear. It provides a regularized Gamma–Bernoulli formulation of the critical-line problem, isolates the exact identity available at a nontrivial zero, and shows precisely why the naive divergence argument from finite products does not prove RH. Any completion of this approach would require the coercive estimate stated in Theorem 7.3; without that estimate, the argument remains a conditional critical-line criterion rather than a proof of the Riemann Hypothesis.

A Technical complements

A.1 Derivation of the regularized product

For completeness we record the finite-N derivation behind Theorem 4.2. Truncating (4.1) gives

$$\Gamma(z) = \frac{\mathrm{e}^{-\gamma z}}{z} \lim_{N\to\infty} \prod_{k=1}^{N} \frac{\mathrm{e}^{z/k}}{1 + z/k}. \tag{A.1}$$

Apply this to $z = 1 - x$ and $z = x$. Dividing yields

$$\frac{\Gamma(1-x)}{\Gamma(x)} = \frac{x}{1-x}\,\mathrm{e}^{-\gamma(1-2x)} \lim_{N\to\infty} \prod_{k=1}^{N} \left(\mathrm{e}^{(1-2x)/k} \frac{k+x}{k+1-x} \right). \tag{A.2}$$

Now use

$$\frac{\Gamma(2-x)}{\Gamma(x+1)} = \frac{1-x}{x} \frac{\Gamma(1-x)}{\Gamma(x)}, \tag{A.3}$$

and rearrange to obtain

$$1 = \lim_{N\to\infty} \left[\mathrm{e}^{\gamma(1-2x)-(1-2x)H_N} \frac{\Gamma(2-x)}{\Gamma(x+1)} \prod_{k=1}^{N} \frac{k+1-x}{k+x} \right]. \tag{A.4}$$

This is exactly the statement of Theorem 4.2.

A.2 A short note on the role of regularization

In analytic number theory, divergent expressions often become meaningful only after a canonical subtraction has been performed. The passage from

$$\sum_{k=1}^{N} \frac{1}{k} \tag{A.5}$$

to

$$H_N - \log N - \gamma \tag{A.6}$$

is the simplest example. The Gamma function itself is a monument to this principle: its Weierstrass product packages a divergent formal product into a renormalized convergent one.

The same principle applies here. A formal divergence becomes relevant only when one proves that the necessary renormalization cannot be absorbed or canceled. For the Gamma quotient studied here, that cancellation occurs explicitly.

A.3 A computation-friendly reformulation

For numerical work it is often preferable to differentiate the regularized Gamma quotient rather than the quotient itself. Define

$$\Phi_t(\sigma) := \frac{\partial}{\partial\sigma} \left[\log\Gamma\left(2 - \frac{\sigma + \mathrm{i}t}{2}\right) - \log\Gamma\left(1 + \frac{\sigma + \mathrm{i}t}{2}\right) \right]. \tag{A.7}$$

A direct differentiation gives

$$\Phi_t(\sigma) = -\frac{1}{2}\psi\left(2 - \frac{\sigma + \mathrm{i}t}{2}\right) - \frac{1}{2}\psi\left(1 + \frac{\sigma + \mathrm{i}t}{2}\right). \tag{A.8}$$

Taking real parts and reflecting $\sigma \mapsto 1 - \sigma$ shows that the same odd/even structure from Section 4 persists at the logarithmic-derivative level. This reformulation is useful because the digamma function is numerically stable and asymptotically transparent.

In particular, if a future refinement of the Bernoulli part of the method produces a zero-conditioned identity of the form

$$\operatorname{Re}\Phi_t(\sigma) + \mathcal{E}(\sigma, t) = 0 \qquad \text{whenever } \zeta(\sigma + \mathrm{i}t) = 0, \tag{A.9}$$

with an explicit error term $\mathcal{E}$, then one can try to deduce coercivity directly from asymptotics of ψ. This explains the preference for the regularized formulation.

A.4 Stirling analysis at the critical line

For completeness we sketch the derivation of (5.11). Let

$$z_t := \frac{1}{4} + \frac{\mathrm{i}t}{2}. \tag{A.10}$$

Then

$$G_t'\left(\frac{1}{2}\right) = \operatorname{Re}\psi(z_t). \tag{A.11}$$

Stirling's formula for the logarithmic derivative of Γ gives

$$\psi(z_t) = \log z_t - \frac{1}{2z_t} + O(|z_t|^{-2}), \qquad |t| \to \infty. \tag{A.12}$$

Taking real parts yields

$$\operatorname{Re}\psi(z_t) = \log|z_t| - \operatorname{Re}\left(\frac{1}{2z_t}\right) + O(|t|^{-2}). \tag{A.13}$$

Since

$$|z_t| = \left(\frac{1}{16} + \frac{t^2}{4}\right)^{1/2} = \frac{|t|}{2}\left(1 + O(t^{-2})\right), \tag{A.14}$$

we obtain

$$\log|z_t| = \log\left(\frac{|t|}{2}\right) + O(t^{-2}). \tag{A.15}$$

Moreover,

$$\operatorname{Re}\left(\frac{1}{2z_t}\right) = O(t^{-2}), \tag{A.16}$$

and hence

$$G_t'\left(\frac{1}{2}\right) = \log\left(\frac{|t|}{2}\right) + O(t^{-2}), \tag{A.17}$$

which is the asymptotic used in Proposition 5.3.

The Gamma-side slope at the critical line is governed by the leading logarithm in Stirling's formula. Any refinement of the method must therefore show that the zeta-dependent correction term cannot overturn this dominant Gamma contribution.

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