Definitive Proof of Brocard's Conjecture

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1 Abstract

This paper presents a formal proof of Brocard's Conjecture, which posits that there are at least four prime numbers between the squares of any two consecutive primes p_i^2 and p_{i+1}^2 for $i > 1$. By defining the function $\pi^*(n)$ that approximates the prime counting function $\pi(n)$, we establish a lower bound for the number of primes in these intervals. Using mathematical induction, we demonstrate that the minimum number of primes in the interval, $\Delta\pi^*(p_i)$, is consistently greater than or equal to 4 for all $p_i \geq 3$. The proof is further supported by a rigorous error analysis, bounding the maximum possible deviation between the estimated prime count $\pi^*(n)$ and the actual prime counting function $\pi(n)$.

2 Introduction

Brocard's Conjecture (not to be confused with Brocard's problem) posits that there are at least four prime numbers between the squares of any two consecutive primes p_i^2 and p_{i+1}^2 for $i > 1$. This conjecture was proposed by French mathematician Henri Brocard in 1904 [1]. Though it has been computationally confirmed for prime numbers up to p_n where $n = 4 \times 10^5$ [2], it remains one of the significant unsolved problems in number theory.

This paper introduces a methodology that approximates the number of composite integers within a set $P_n = \{2, 3, 4, ...n\}$ by iteratively identifying integers divisible by prime numbers $p \leq \sqrt{n}$. Through this recursive sieving process, we derive the function:

$$\pi^*(n) = n \prod_{\substack{q \leq \lambda(\sqrt{n}) \\ q \text{ is prime}}} \frac{q-1}{q} \tag{1}$$

where $\lambda(\sqrt{n})$ is the largest prime number less than or equal to $\sqrt{n}$.

This allows for the definition of $W(n)$, the fraction of integers less than or equal to n that are prime, which remains constant between prime-squared boundaries. By analyzing the growth of these intervals relative to the density of primes, we identify a lower bound, $\Delta\pi^{LB}(p_i)$, which occurs at twin primes, as also observed by Feliksiak in 2022 [3]. We utilize mathematical induction to show that the lower bound strictly satisfies the conditions of Brocard's Conjecture and becomes stronger with increasing primes. Finally, we address the precision of this inductive step through an error term $E(n) = 2\pi(\sqrt{n})$, ensuring that fluctuations in prime distribution do not invalidate the established lower bound.

3 Functions

We define the following functions:

Let $\lambda(x)$ denote the largest prime number less than or equal to x. For example, $\lambda(10.5) = 7$, $\lambda(20) = 19$ and $\lambda(19) = 19$.

Let $k_p(n)$ be the number of composite integers less than or equal to n that are divisible by prime number p.

Let $k_p^*(n)$ be the derived approximation to $k_p(n)$. The asterisk indicates approximation.

Let $K_p(n)$ be the number of composite integers less than or equal to n that are divisible by prime number p and not divisible by a prime number less than p.

Let $K_p^*(n)$ be the derived approximation to $K_p(n)$.

Let $K(n) = \sum_{\substack{p<\lambda(\sqrt{n}) \\ p \text{ is prime}}} K_p(n)$ and is equal to the number of composite integers less than n.

Let $K^*(n) = \sum_{\substack{p<\lambda(\sqrt{n}) \\ p \text{ is prime}}} K_p^*(n)$ and approximates $K(n)$.

Let $\pi_p(n)$ be the number of integers less than or equal to n that are not divisible by prime number p and any prime number less than p.

Let $\pi_p^*(n)$ be the derived approximation to $\pi_p(n)$.

Let $W(n)$ be the derived approximation to the density of prime numbers in P_n.

Let $\pi(n)$ denote the prime counting function. $\pi(n)$ is equal to the number of prime numbers less than or equal to n.

Let $\pi^*(n) = nW(n)$ which approximates $\pi(n)$.

Let $\Delta\pi(p_i) = \pi(p_{i+1}^2) - \pi(p_i^2)$ which is the number of primes between p_i^2 and p_{i+1}^2

Let $\Delta\pi^*(p_i) = \pi^*(p_{i+1}^2) - \pi^*(p_i^2)$ which approximates $\Delta\pi(p_i)$.

Let $\Delta\pi^{LB}(p_i)$ represent the lower bound on $\Delta\pi^*(p_i)$.

Let e_p be the maximum error contribution to $\pi^*(n)$ for prime number p.

Let $E(n) = 2\pi(\sqrt{n})$ and is the maximimum error deviation between $\pi(n)$ and $\pi^*(n)$.

Let $LB(p_i)$ represent the lower bound on $\Delta\pi^{LB}(p_i)$ due to $E(n)$.

4 Methodology

We start by deriving a function that approximates the number of prime numbers less than or equal to n. We derive equations for the number of composite integers divisible by 2, 3, 5 ... $\lambda(\sqrt{n})$ and subtract them from the number of integers less than or equal to n.

Consider the set $P_n = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, \ldots n\}$. The integer 1 is not in P_n since it is not considered a prime number. Thus $|P_n| = n - 1$.

Define $k_2(n)$ as the number of composite integers in the set P_n that are divisible by 2 as follows:

$$k_2(n) = \lfloor n/2 \rfloor - 1 \tag{2}$$

Note that we have to subtract 1 since we do not want to count 2 as a composite integer.

The number of integers in P_n that are not divisible by 2 can be defined as follows:

$$\pi_2(n) = |P_n| - k_2(n) \tag{3}$$

$$\pi_2(n) = (n - 1) - (\lfloor n/2 \rfloor - 1) \tag{4}$$

$$\pi_2(n) = n - \lfloor n/2 \rfloor \tag{5}$$

Define $k_2^*(n)$ as $n/2$ which approximates $\lfloor n/2 \rfloor$ and $k_2(n)$ for large values of n.

$$k_2^*(n) = n/2 \tag{6}$$

Define $\pi_2^*(n)$ as the approximate number of integers that are in P_n that are not divisible by 2 as follows:

$$\pi_2^*(n) = n - k_2^*(n) \tag{7}$$
$$\pi_2^*(n) = n - n/2 \tag{8}$$
$$\pi_2^*(n) = n/2 \tag{9}$$

The same process can be applied to approximate the number of integers in P_n that are divisible by 3. Define $k_3(n)$ as the number of composite integers in the set P_n that are divisible by 3 as follows:

$$k_3(n) = \lfloor n/3 \rfloor - 1 \tag{10}$$

Note that we have to subtract 1 since we do not want to count 3 as a composite integer.
As n becomes large, the -1 in equation 10 becomes negligible, and $\lfloor n/3 \rfloor$ is approximated as $n/3$. Define $k_3^*(n)$ as the approximate number of integers in P_n that are divisible by 3 as follows:

$$k_3^*(n) = n/3 \tag{11}$$

To find the number of integers in P_n that are not divisible by 2 or 3, we cannot subtract $k_3(n)$ from $\pi_2(n)$ since approximately half of the number of integers that are divisible by 3 are also divisible by 2. Define $K_3^*(n)$ as the number of integers in $k_3^*(n)$ that are not divisible by 2 as follows:

$$K_3^*(n) = (1/2)k_3^*(n) \tag{12}$$
$$K_3^*(n) = (1/2)(1/3)n \tag{13}$$

Define $\pi_3^*(n)$ as the approximate number of integers in P_n that are not divisible by 2 or 3 as follows:

$$\pi_3^*(n) = \pi_2^*(n) - K_3^*(n) \tag{14}$$
$$\pi_3^*(n) = (1/2)n - (1/2)(n/3) \tag{15}$$
$$\pi_3^*(n) = (1/2)(n - n/3) \tag{16}$$
$$\pi_3^*(n) = (1/2)(2/3)n \tag{17}$$

The same process can be applied to approximate the number of integers in P_n that are divisible by 5. Define $k_5(n)$ as the number of composite integers in the set P_n that are divisible by 5 as follows:

$$k_5(n) = \lfloor n/5 \rfloor - 1 \tag{18}$$

Note that we have to subtract 1 since we do not want to count 5 as a composite integer.
As n becomes large, the -1 in equation 18 becomes negligible, and $\lfloor n/5 \rfloor$ is approximated as $n/5$. Define $k_5^*(n)$ as the approximate number of integers in P_n that are divisible by 5 as follows:

$$k_5^*(n) = n/5 \tag{19}$$

Approximately half of the number of integers in P_n that are divisible by 5 are also divisible by 2, and approximately one third of the numbers are divisible by 3. Define $K_5^*(n)$ as the number of integers in $k_5^*(n)$ that are not divisible by 2 or 3 as follows:

$$K_5^*(n) = (1/2)(2/3)k_5^*(n) \tag{20}$$

$$K_5^*(n) = (1/2)(2/3)(1/5)n \tag{21}$$

Define $\pi_5^*(n)$ as the approximate number of integers in P_n that are not divisible by 2, 3 or 5 as follows:

$$\pi_5^*(n) = \pi_3^*(n) - K_5^*(n) \tag{22}$$

$$\pi_5^*(n) = (1/2)(2/3)n - (1/2)(2/3)(n/5) \tag{23}$$

$$\pi_5^*(n) = (1/2)(2/3)(n - n/5) \tag{24}$$

$$\pi_5^*(n) = (1/2)(2/3)(4/5)n \tag{25}$$

The number of composite integers in P_n that are divisible by prime number p is

$$k_p(n) = \lfloor n/p \rfloor - 1 \tag{26}$$

As n becomes large, the term -1 becomes negligible, and $\lfloor n/p \rfloor$ is approximated as n/p. Thus the number of integers not divisible by p can be approximated by the following equation:

$$k_p^*(n) = n/p \tag{27}$$

Define $K_p^*(n)$ as the approximate number of integers in P_n that are divisible by p but not by any prime number less than p as follows:

$$K_p^*(n) = (1/2)(2/3)(4/5)...(p_{i-1} - 1/p_{i-1})k_p^*(n) \tag{28}$$
$$K_p^*(n) = (1/2)(2/3)(4/5)...(p_{i-1} - 1/p_{i-1})(n/p) \tag{29}$$
$$K_p^*(n) = (n/p) \prod_{\substack{q<p \\ q \text{ is prime}}} (q-1)/q \tag{30}$$

Note that q is strictly less than p.

Thus, for any prime number p_i, the number of integers less than or equal to n that are not divisible by p_i or any prime less than p_i is as follows:

$$\pi_{p_i}^*(n) = \pi_{p_{i-1}}^*(n) - K_{p_i}^*(n) \tag{31}$$
$$\pi_{p_i}^*(n) = (1/2)(2/3)...((p_{i-1} - 1)/p_{i-1})n - (1/2)(2/3)...((p_{i-1} - 1)/p_{i-1})(1/p_i)n \tag{32}$$
$$\pi_{p_i}^*(n) = (1/2)(2/3)...((p_i - 1)/p_i)n \tag{33}$$
$$\pi_p^*(n) = n \prod_{\substack{q \le p \\ q \text{ is prime}}} \frac{q-1}{q} \tag{34}$$

For a given n, the number of prime numbers less than or equal to n can be approximated by the following equation for large n:

$$\pi^*(n) = n \prod_{\substack{q \le \lambda(\sqrt{n}) \\ q \text{ is prime}}} \frac{q-1}{q} \tag{35}$$

Define $W(n)$, the fraction of integers less than or equal to n that are prime, as follows:

$$W(n) = \prod_{\substack{q \le \lambda(\sqrt{n}) \\ q \text{ is prime}}} \frac{q-1}{q} \tag{36}$$

Then the number of primes less than or equal to n simplifies to

$$\pi^*(n) = nW(n) \tag{37}$$

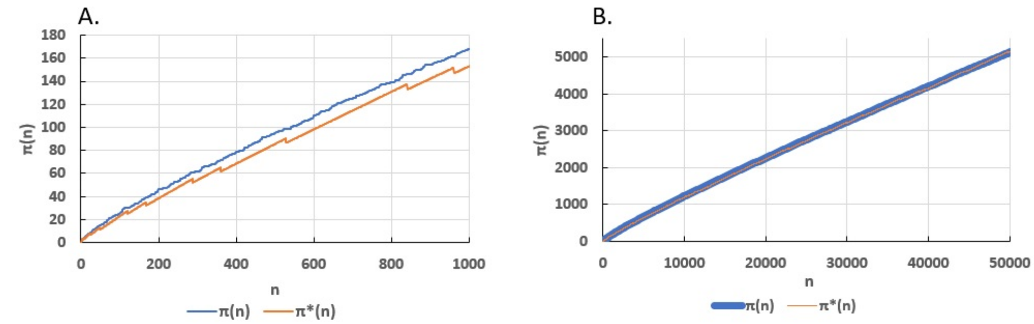

Figure 1: A.) The prime counting function $\pi(n)$ (blue line) is underestimated by $\pi^*(n)$ (orange line) for values of $n <$ 1,000. B.) But as n gets larger, the value of $\pi^*(n)$ approaches $\pi(n)$ and the curves are virtually identical. The line for $\pi(n)$ is made thicker so it can be seen, otherwise it would be obscured by the line for $\pi^*(n)$.

To verify that the derivation of the equation for $\pi^*(n)$ is correct and to determine at what point $\pi^*(n)$ begins to closely coincide with the prime counting function, $\pi(n)$, figure 1 compares $\pi^*(n)$ (orange line) to $\pi(n)$ (blue line). Figure 1A shows that $\pi^*(n)$ underestimates $\pi(n)$ for values of $n < 1,000$. Figure 1B shows that as n increases to 50,000, the curves for $\pi(n)$ and $\pi^*(n)$ lie nearly on top of each other. These graphs support that the number of primes less than or equal to n can be accurately predicted by equation for $\pi^*(n)$ for large values of n.

5 Proof of Brocards Conjecture

To prove Brocard's conjecture, we use mathematical induction. W(n) only changes when n becomes a prime number squared. Thus, we consider only values of $n = p^2$. From the behavior of $W(p_i)$, we conclude the following:

$$W(p_{i+1}^2) = \left(\frac{p_{i+1} - 1}{p_{i+1}}\right) W(p_i^2) \tag{38}$$

Define $\Delta\pi^*(p_i)$ as the number of prime numbers between p_i^2 and p_{i+1}^2. Using our approximation for the prime counting function, we have the following:

$$\Delta\pi^*(p_i) = \pi^*(p_{i+1}^2) - \pi^*(p_i^2) \tag{39}$$
$$\Delta\pi^*(p_i) = p_{i+1}^2 W(p_{i+1}^2) - p_i^2 W(p_i^2) \tag{40}$$
$$\Delta\pi^*(p_i) = p_{i+1}^2 \left(\frac{p_{i+1}-1}{p_{i+1}}\right) W(p_i^2) - p_i^2 W(p_i^2) \tag{41}$$
$$\Delta\pi^*(p_i) = p_{i+1}(p_{i+1}-1) W(p_i^2) - p_i^2 W(p_i^2) \tag{42}$$
$$\Delta\pi^*(p_i) = (p_{i+1}^2 - p_{i+1}) W(p_i^2) - p_i^2 W(p_i^2) \tag{43}$$
$$\Delta\pi^*(p_i) = W(p_i^2)(p_{i+1}^2 - p_{i+1} - p_i^2) \tag{44}$$

Define $p_{i+1} = p_i + g$ where g is the gap between p_i and p_{i+1}. Substituting $p_i + g$ for p_{i+1} in the term $p_{i+1}^2 - p_{i+1} - p_i^2$ in equation 44 yields the following:

$$p_{i+1}^2 - p_{i+1} - p_i^2 = (p_i + g)^2 - (p_i + g) - p_i^2 \tag{45}$$
$$p_{i+1}^2 - p_{i+1} - p_i^2 = (p_i + g)^2 - p_i - g - p_i^2 \tag{46}$$
$$p_{i+1}^2 - p_{i+1} - p_i^2 = p_i^2 + 2gp_i + g^2 - p_i - g - p_i^2 \tag{47}$$
$$p_{i+1}^2 - p_{i+1} - p_i^2 = 2gp_i + g^2 - p_i - g \tag{48}$$
$$p_{i+1}^2 - p_{i+1} - p_i^2 = 2gp_i - p_i + g^2 - g \tag{49}$$
$$p_{i+1}^2 - p_{i+1} - p_i^2 = p_i(2g - 1) + g^2 - g \tag{50}$$

Substituting into equation 44 yields the following:

$$\Delta\pi^*(p_i) = W(p_i^2)(p_i(2g-1) + g^2 - g) \tag{51}$$

The minimum value of $\Delta\pi^*(p_i)$ occurs when $g = 2$. In other words, $\Delta\pi^*(p_i)$ is minimized when p_i and p_{i+1} are twin primes. The observation that $\Delta\pi(p_i)$ was minimized at the twin primes was also made by Feliksiak in 2022 [3]. This follows because the smaller the difference between p_i and p_{i+1}, the smaller the difference between p_i^2 and p_{i+1}^2, and thus the fewer primes within the interval. Substituting 2 for g yields the following:

$$\Delta\pi^*(p_i) \geq W(p_i^2)(p_i(2*2-1) + 2^2 - 2) \tag{52}$$
$$\Delta\pi^*(p_i) \geq W(p_i^2)(3p_i + 2) \tag{53}$$
$$\Delta\pi^{LB}(p_i) = W(p_i^2)(3p_i + 2) \tag{54}$$

where $\Delta\pi^{LB}(p_i)$ is the lower bound on $\Delta\pi^*(p_i)$.

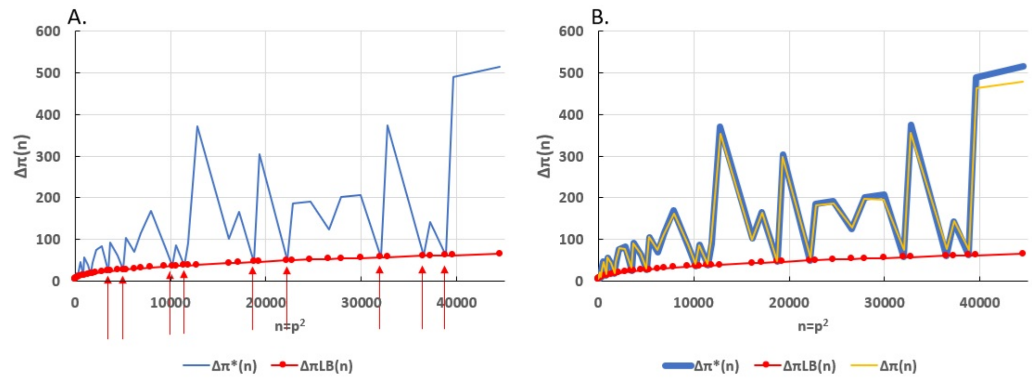

Figure 2: A.) The function $\Delta\pi^{LB}(n)$ (red line) is the lower bound of function $\Delta\pi^{*}(n)$ (blue line) and they coincide at values where p_i and p_{i+1} are twin primes indicated by the red arrows. B.) When graphing both $\Delta\pi^{*}(n)$ (blue line) and $\Delta\pi(n)$ (yellow line), the curves coincide very closely indicating that $\Delta\pi^{*}(n)$ is a good approximation to $\Delta\pi(n)$. The line for $\Delta\pi^{*}(n)$ is made thicker so it can be seen, otherwise it would be obscured by the line for $\Delta\pi(n)$.

To support that the lower bound $\Delta\pi^{LB}(n)$ is properly derived, figure 2 compares $\Delta\pi^{LB}(n)$ and $\Delta\pi^{*}(n)$. In figure 2A, $\Delta\pi^{*}(n)$ (blue line) is greater than or equal to $\Delta\pi^{LB}(n)$ (red line) for values of $n = p^2$ up to 211^2. The points where $\Delta\pi^{*}(n) = \Delta\pi^{LB}(n)$, indicated by red arrows, occurs when $n = p_i^2$ where p_i and p_{i+1} are twin primes. When graphing both $\Delta\pi^{*}(n)$ (blue line) and $\Delta\pi(n)$ (yellow line), figure 2B, the curves coincide very closely indicating that $\Delta\pi^{*}(n)$ is a good approximation to $\Delta\pi(n)$.

If $\Delta\pi^{LB}(p_i) \geq 4$ for all p_i and $\Delta\pi(p_i) \geq \Delta\pi^{LB}(p_i)$, then $\Delta\pi(p_i) \geq 4$, thus proving Brocard's conjecture. We prove this by mathematical induction.

Base case $p_i = 5$.
Substituting $p_i = 5$ into equation 54 yields:

$$\Delta\pi^{LB}(5) = W(5^2)(3 * 5 + 2) \tag{55}$$

$$\Delta\pi^{LB}(5) = (1/2)(2/3)(4/5)(17) \tag{56}$$

$$\Delta\pi^{LB}(5) = 68/15 \approx 4.53333 > 4 \tag{57}$$

Inductive step.

Assuming that $\Delta\pi^{LB}(p_i) \geq 4$, we prove that $\Delta\pi^{LB}(p_{i+1}) \geq 4$.

$$\Delta\pi^{LB}(p_{i+1}) = W(p_{i+1}^2)(3p_{i+1} + 2) \tag{58}$$

Using equation 38 we obtain the following:

$$\Delta\pi^{LB}(p_{i+1}) = \left(\frac{p_{i+1} - 1}{p_{i+1}}\right) W(p_i^2)(3p_{i+1} + 2) \tag{59}$$

$$\Delta\pi^{LB}(p_{i+1}) = \left(\frac{p_{i+1} - 1}{p_{i+1}}\right) \left(\frac{3p_{i+1} + 2}{3p_i + 2}\right) W(p_i^2)(3p_i + 2) \tag{60}$$

$$\Delta\pi^{LB}(p_{i+1}) = \left(\frac{p_{i+1} - 1}{p_{i+1}}\right) \left(\frac{3p_{i+1} + 2}{3p_i + 2}\right) \Delta\pi^{LB}(p_i) \tag{61}$$

Since we assume $\Delta\pi^{LB}(p_i) > 4$, if we prove that

$$\left(\frac{p_{i+1} - 1}{p_{i+1}}\right) \left(\frac{3p_{i+1} + 2}{3p_i + 2}\right) \geq 1 \tag{62}$$

then the product of the two terms is greater than 4. Equation 62 is equivalent to the following:

$$(p_{i+1} - 1)(3p_{i+1} + 2) \geq p_{i+1}(3p_i + 2) \tag{63}$$

$$3p_{i+1}^2 + 2p_{i+1} - 3p_{i+1} - 2 \geq 3p_{i+1}p_i + 2p_{i+1} \tag{64}$$

$$3p_{i+1}^2 - p_{i+1} - 2 \geq 3p_{i+1}p_i + 2p_{i+1} \tag{65}$$

$$3p_{i+1}^2 - 3p_{i+1} - 3p_{i+1}p_i - 2 \geq 0 \tag{66}$$

Define $p_{i+1} = p_i + g$ where g is the gap between p_i and p_{i+1}. Substituting $p_i + g$ for p_{i+1} gives us the following:

$$3(p_i + g)^2 - 3(p_i + g) - 3(p_i + g)p_i - 2 \geq 0 \tag{67}$$

$$3(p_i^2 + 2p_ig + g^2) - 3p_i - 3g - 3(p_i^2 + p_ig) - 2 \geq 0 \tag{68}$$

$$3p_i^2 + 6p_ig + 3g^2 - 3p_i - 3g - 3p_i^2 - 3p_ig - 2 \geq 0 \tag{69}$$

$$6p_ig + 3g^2 - 3p_i - 3g - 3p_ig - 2 \geq 0 \tag{70}$$

$$3p_ig + 3g^2 - 3p_i - 3g - 2 \geq 0 \tag{71}$$

$$3p_ig - 3p_i + 3g^2 - 3g - 2 \geq 0 \tag{72}$$

The LHS of equation 72 can be shown to be always increasing with respect to g by taking the derivative. Define $f(g)$ as the LHS of equation 72 and take the derivative wrt g as follows:

$$f(g) = 3p_i g - 3p_i + 3g^2 - 3g - 2 \tag{73}$$
$$f'(g) = 3p_i + 6g - 3 \tag{74}$$

The function $f'(g)$ in equation 74 increases linearly with g. Therefore, the lowest value of g will minimize the LHS of equation 72 and since prime gaps cannot be less than 2 for primes greater than 3, the lowest allowable value for g is 2. If equation 72 holds true for $g = 2$, then it holds for all values of g. Substituting 2 for g in equation 72 yields:

$$3p_i * 2 - 3p_i + 3 * 2^2 - 3 * 2 - 2 \geq 0 \tag{75}$$
$$6p_i - 3p_i + 12 - 6 - 2 \geq 0 \tag{76}$$
$$3p_i + 4 \geq 0 \tag{77}$$

Not only does the inequality hold true for all g, but the larger p_i becomes, the inequality becomes stronger. Since we assumed $\Delta\pi^{LB}(p_i) \geq 4$ and $\left(\frac{p_{i+1}-1}{p_{i+1}}\right)\left(\frac{3p_{i+1}+2}{3p_i+2}\right) > 1$, then the product of these two terms, $\Delta\pi^{LB}(p_{i+1})$, must be greater than 4. Therefore, for any consecutive pair of prime numbers p_i and p_{i+1}, the difference between $\pi^*(p_i^2)$ and $\pi^*(p_{i+1}^2)$ is greater than or equal to 4.

6 Error Analysis

We show that $\Delta\pi^{LB}(p_i)$ is the lower bound on $\Delta\pi^*(p_i)$ where $\Delta\pi^*(p_i)$ is the approximate number of primes between p_i^2 and p_{i+1}^2. We also show that $\Delta\pi^{LB}(p_i) \geq 4$ for all primes $p_i > 2$. However, $\Delta\pi^{LB}(p_i)$ relies on the approximation $\pi^*(n)$ which does not equal $\pi(n)$.

In order for the proof to be rigorous, we must show that the difference between $\pi^*(p_i)$ and $\pi(p_i)$ is small enough that it does not invalidate the proof at any point. To do this, we must find the maximum error contribution from each $p_i < \lambda(\sqrt{n})$. The error in the number of composite integers less than or equal to n, $K^*(n)$, is the same as the error in the number of prime numbers less than or equal to n but with the opposite sign. To find the error in $K^*(n)$, the maximum error contribution from each prime number less than $\sqrt{n}$ is calculated. Then by the triangle inequality, the sum of the maximum individual errors is greater than or equal to the maximum total error for $K^*(n)$. Let e_p be the maximum error contribution of prime number p.

The first approximation for deriving $\pi^*(n)$ is $k_2^*(n) = n/2$ for $k_2(n)$, the actual number of composite integers less than or equal to n that are divisible by 2. It follows that $n/2$ always overestimates $k_2(n)$. The difference between $k_2^*(n)$ and $k_2(n)$ is the error contribution for this approximation which we can call $e_2(n)$. Using equation 2, we have the following:

$$e_2(n) = k_2^*(n) - k_2(n) \tag{78}$$
$$e_2(n) = n/2 - (\lfloor n/2 \rfloor - 1) \tag{79}$$
$$e_2(n) = n/2 - \lfloor n/2 \rfloor + 1 \tag{80}$$

Define the maximal value of the function $e_2(n)$ as e_2 since it is no longer a function of n. The maximal value of $n/2 - \lfloor n/2 \rfloor$ is $1/2$ and occurs at values of $n = 3, 5, 7, ..., 2i - 1$. Therefore

$$e_2 = 1/2 + 1 \tag{81}$$
$$e_2 = 3/2 \tag{82}$$

The next approximation is $k_3^*(n) = n/3$ for $k_3(n)$. It follows that $n/3$ always overestimates $k_3(n)$. The difference between $k_3^*(n)$ and $k_3(n)$ is the error contribution for this approximation which we can call $e_3(n)$. Using equation 10, we have the following:

$$e_3(n) = k_3^*(n) - k_3(n) \tag{83}$$
$$e_3(n) = n/3 - (\lfloor n/3 \rfloor - 1) \tag{84}$$
$$e_3(n) = n/3 - \lfloor n/3 \rfloor + 1 \tag{85}$$

Define the maximal value of the function $e_3(n)$ as e_3 since it is no longer a function of n. The maximal value of $n/3 - \lfloor n/3 \rfloor$ is $2/3$ and occurs at values of $n = 5, 8, 11, ..., 3i - 1$. Therefore

$$e_3 = 2/3 + 1 \tag{86}$$
$$e_3 = 5/3 \tag{87}$$

The inclusion/exclusion principle states that given sets A and B, $|A \cup B| = |A| + |B| - |A \cap B|$. Note that $|A \cup B| \leq |A| + |B|$. Using the inclusion/exclusion principle, the number of integers less than or equal to n that are divisible by either 2 or 3 is as follows:

$$K_3(n) = k_2(n) + k_3(n) - k_{2,3}(n) \tag{88}$$

where $k_{2,3}(n)$ are the number of integers less than or equal to n that are divisible by both 2 and 3. In other words, $k_{2,3}(n)$ is the number of integers less than or equal to n that are divisible by 6 or $\lfloor n/6 \rfloor$.

$K_3(n)$ is approximated by $K_3^*(n)$ which is defined as follows:

$$K_3^*(n) = k_2^*(n) + k_3^*(n) - k_{2,3}^*(n) \tag{89}$$

where $k_{2,3}^*(n)$ is the approximation for $k_{2,3}(n)$ and is equal to $n/6$.

Subtracting the equation 88 from equation 89 gives:

$$K_3^*(n) - K_3(n) = (k_2^*(n) - k_2(n)) + (k_3^*(n) - k_3(n)) - (k_{2,3}^*(n) - k_{2,3}(n)) \tag{90}$$

$$K_3^*(n) - K_3(n) = e_2(n) + e_3(n) - (k_{2,3}^*(n) - k_{2,3}(n)) \tag{91}$$

$$K_3^*(n) - K_3(n) = e_2(n) + e_3(n) - (n/6 - \lfloor n/6 \rfloor) \tag{92}$$

For any composite integer c, it is true that $n/c - \lfloor n/c \rfloor \geq 0$. Since $n/6 - \lfloor n/6 \rfloor \geq 0$, the following inequality holds.

$$K_3^*(n) - K_3(n) \leq e_2(n) + e_3(n) \tag{93}$$

$$E_{2,3} = e_2 + e_3 \tag{94}$$

where $E_{2,3}$ is the maximum difference between $K_3^*(n)$ and $K_3(n)$.

The next approximation is $k_5^*(n) = n/5$ for $k_5(n)$. It follows that $n/5$ always overestimates $k_5(n)$. The difference between $k_5^*(n)$ and $k_5(n)$ is the error contribution $e_5(n)$. Using equation 18, we have the following:

$$e_5(n) = k_5^*(n) - k_5(n) \tag{95}$$

$$e_5(n) = n/5 - (\lfloor n/5 \rfloor - 1) \tag{96}$$

$$e_5(n) = n/5 - \lfloor n/5 \rfloor + 1 \tag{97}$$

Define the maximal value of the function $e_5(n)$ as e_5 since it is no longer a function of n. The maximal value of $n/5 - \lfloor n/5 \rfloor$ is $4/5$ and occurs at values of $n = 9, 14, 19, ..., 5i - 1$. Therefore

$$e_5 = 4/5 + 1 \tag{98}$$

$$e_5 = 9/5 \tag{99}$$

Note that for any prime number p, the maximal error contribution for p is as follows:

$$e_p = (2p - 1)/p \tag{100}$$

The inclusion/exclusion principle states that given sets A, B and C, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. Note that $|A \cup B \cup C| \leq |A| + |B| + |C|$. Using the inclusion/exclusion principle, the number of integers less than or equal to n that are divisible by either 2,3 or 5 is as follows:

$$K_5(n) = k_2(n) + k_3(n) + k_5(n) - c(n) \tag{101}$$

$$K_5^*(n) = k_2^*(n) + k_3^*(n) + k_5^*(n) - c^*(n) \tag{102}$$

where $c(n) = k_{2,3}(n) + k_{2,5}(n) + k_{3,5}(n) - k_{2,3,5}(n)$ and $c^*(n)$ is the approximation to $c(n)$. In this case $c^*(n) = n/6 + n/10 + n/15 - n/30$.

Subtracting the two equations gives:

$$K_5^*(n) - K_5(n) = (k_2^*(n) - k_2(n)) + (k_3^*(n) - k_3(n)) + (k_5^*(n) - k_5(n)) - (c^*(n) - c(n)) \tag{103}$$

$$K_5^*(n) - K_5(n) = e_2(n) + e_3(n) + e_5(n) - (c^*(n) - c(n)) \tag{104}$$

$$K_5^*(n) - K_5(n) < e_2(n) + e_3(n) + e_5(n) \tag{105}$$

$$E_{2,3,5} = e_2 + e_3 + e_5 \tag{106}$$

where $E_{2,3,5}$ is the maximum difference between $K_5^*(n)$ and $K_5(n)$.

By the inclusion/exclusion principle, for any number of sets A, B, C, ...,Z the union of the sets $|A \cup B \cup C \cup ... \cup Z| \leq |A| + |B| + |C| + ... + Z$.

Therefore, the maximum difference between $K_p^*(n)$ and $K_p(n)$ is defined as follows:

$$E_{2,3,5,...,p} = e_2 + e_3 + e_5 + ... + e_p(n) \tag{107}$$

Summing up these error contributions for all $p \leq \lambda(\sqrt{n})$ gives us the total maximal error, $E(n)$.

$$E_{2,3,5,...,\lambda(\sqrt{n})} = \sum_{p \leq \lambda(\sqrt{n})} \frac{2p-1}{p} \tag{108}$$

$$E_{2,3,5,...,\lambda(\sqrt{n})} < \sum_{p \leq \lambda(\sqrt{n})} 2 \tag{109}$$

$$E_{2,3,5,...,\lambda(\sqrt{n})} < 2\pi(\sqrt{n}) \tag{110}$$

$$E(n) = 2\pi(\sqrt{n}) \tag{111}$$

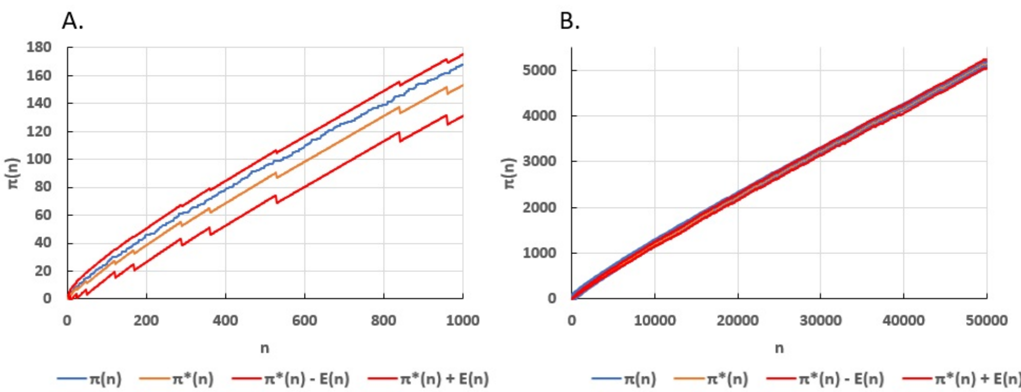

Figure 3: A.) The value of $\pi(n)$ lies comfortably within $\pi^*(n) + E(n)$ and $\pi^*(n) - E(n)$ (red lines) for $n \leq 1000$. B.) For large values of n up to 50,000, the error is small and it is hard to see graphically. However, it is computationally confirmed that $\pi(n)$ lies within $\pi^*(n) + E(n)$ and $\pi^*(n) - E(n)$ for $n \leq 50,000$.

where $E(n)$ is the maximal difference between $\pi^*(n)$ and $\pi(n)$. In other words the following holds true:

$$\pi^*(n) - E(n) \leq \pi(n) \leq \pi^*(n) + E(n) \tag{112}$$

To check the calculation for $E(n)$ is correct, the values of $\pi^*(n) + E(n)$ and $\pi^*(n) - E(n)$ are compared to $\pi(n)$ in figure 3. For values of $n \leq 1000$, figure 3A shows that $\pi(n)$ (blue line) lies comfortably within $\pi^*(n)+E(n)$ and $\pi^*(n) - E(n)$ (red lines). For larger values of n up to 50,000, figure 3B, the error is small and is hard to see graphically. However, it is computationally confirmed that $\pi(n)$ lies within $\pi^*(n)+E(n)$ and $\pi^*(n)-E(n)$ for $n \leq 50,000$.

From equation 54 we have the following:

$$\Delta\pi(p_i) = W(p_i^2)(3p_i + 2) \tag{113}$$

$$\Delta\pi(p_i) = p_i^2 W(p_i^2)\frac{(3p_i + 2)}{p_i^2} \tag{114}$$

$$\Delta\pi^{LB}(p_i) = \pi^*(p_i^2)\frac{(3p_i + 2)}{p_i^2} \tag{115}$$

From equation 112, we can now determine a new rigorous lower bound that takes into account the error in approximating $\pi(n)$ with $\pi^*(n)$. This gives us the following:

$$\Delta\pi(p_i) \geq (\pi^*(p_i^2) - E(p_i^2))\frac{(3p_i + 2)}{p_i^2} \tag{116}$$

$$\Delta\pi(p_i) \geq (\pi^*(p_i^2) - 2\pi(p_i))\frac{(3p_i + 2)}{p_i^2} \tag{117}$$

$$\Delta\pi(p_i) \geq \pi^*(p_i^2)\frac{(3p_i + 2)}{p_i^2} - 2\pi(p_i)\frac{(3p_i + 2)}{p_i^2} \tag{118}$$

$$\Delta\pi(p_i) \geq \Delta\pi^{LB}(p_i) - 2\pi(p_i)\frac{(3p_i + 2)}{p_i^2} \tag{119}$$

Let $LB(p_i)$ be the RHS of equation 119.

$$LB(p_i) = \Delta\pi^{LB}(p_i) - 2\pi(p_i)\frac{(3p_i + 2)}{p_i^2} \tag{120}$$

Since we prove that $\Delta\pi^{LB}(p_i)$ increases monotonically with increasing p_i, and the error term $2\pi(p_i)\frac{(3p_i+2)}{p_i^2}$ decreases monotonically with increasing p_i, then there exists a p such that $LB(p) > 4$ and for all primes $q > p$, $LB(q) > 4$. For $p = 11$, the value of $LB(11) \approx 4.38$ which is greater than 4. Therefore, $LB(q) > 4$ for all subsequent primes $q > p$. Given that $\Delta\pi(p) \geq LB(p)$, the actual number of primes between the squares of consecutive primes must be at least 4, satisfying Brocard's Conjecture.

As further confirmation that $\Delta\pi(n) \geq LB(n)$, the values of $\Delta\pi(n)$ and $LB(n)$ are calculated for values of $n = p^2$ up to $49,729$ and in all cases, $\Delta\pi(n) > LB(n)$. Figure 4A shows some of the values of $\Delta\pi(n)$ and $LB(n)$ for values of $n = p^2$ up to 5,329. The values highlighted in yellow are when p_i and p_{i+1} are twin primes and these are the values where $LB(n)$ and $\Delta\pi(n)$ are the closest. Figure 4B shows the data in graphical form.

7 Summary

This paper provides a formal proof of Brocard's Conjecture, demonstrating that for all $i > 1$, there are at least four prime numbers between the squares of consecutive primes p_i^2 and p_{i+1}^2 . The proof is built upon several critical methodological layers:

Recursive Prime Approximation: An approximation to the prime counting function is derived, $\pi^*(n) = nW(n)$, where $W(n)$ represents the density

A.

n=p^2	p	Δπ(n)	$\Delta\pi^{LB}(n)$	LB(n)
9	3	5	3.66666663	-1.222222
25	5	6	4.53333339	0.4533334
49	7	15	5.25714289	1.5020408
121	11	9	7.27272735	4.3801654
169	13	22	7.86413579	4.9528932
289	17	11	9.56784408	7.00037
361	19	27	10.09041718	7.4754587
529	23	47	11.6147622	9.1988832
841	29	16	14.05730258	11.940775
961	31	57	14.52095425	12.346136
1369	37	44	16.80547413	14.824466
1681	41	20	18.13670875	16.203336
1849	43	46	18.5652414	16.581466
2209	47	80	19.83468487	17.89263
2809	53	78	21.91001183	20.075907
3481	59	32	23.94670413	22.198356
3721	61	90	24.34365975	22.553818
4489	67	66	26.31354208	24.595119
5041	71	30	27.4765012	25.77049
5329	73	106	27.85639363	26.114603

B.

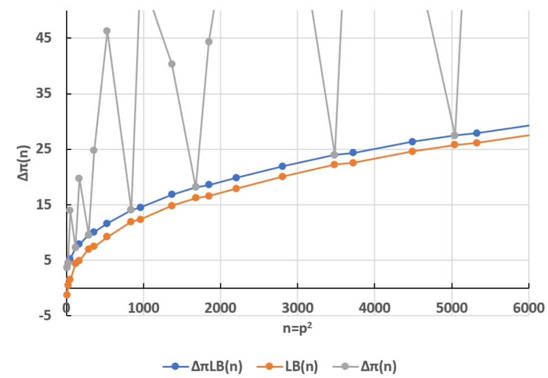

Figure 4: A.) The function $LB(n)$ is compared function $\Delta\pi(n)$ for various values of $n = p^2$ and $\Delta\pi(n)$ is greater than $LB(n)$ for values of n up to 5,359. The values highlighted in yellow are when p_i and p_{i+1} are twin primes and these are the values where $LB(n)$ and $\Delta\pi(n)$ are closest. B.) The graph comparing $LB(n)$ (orange line) and $\Delta\pi(n)$ (gray line) shows that $LB(n)$ is always less than $\Delta\pi(n)$.

fraction of primes based on a recursive sieving process that excludes composite integers divisible by primes less than or equal to $\sqrt{n}$.

Establishment of a Lower Bound: By analyzing the interval between p_i^2 and p_{i+1}^2, a lower bound function is derived, $\Delta\pi^{LB}(p_i) = W(p_i^2)(3p_i + 2)$, which represents the minimum expected number of primes when the prime gap is at its smallest (twin primes).

Inductive Proof: Using mathematical induction, it is proven that $\Delta\pi^{LB}(p_i) \geq 4$ for all $p_i \geq 5$. The inductive step confirms that the product of density changes and interval growth strictly increases or remains sufficient to satisfy the conjecture as p_i increases.

Rigorous Error Analysis: To bridge the gap between the approximation $\pi^*(n)$ and the actual prime counting function $\pi(n)$, a maximum error term $E(n) < 2\pi(\sqrt{n})$ is derived. This error is integrated into a final, rigorous lower bound $LB(p_i)$, which is computationally and analytically shown to stay above 4 for all $p_i \geq 11$.

8 Future Directions

The methodology established in this paper, specificlly the recursive prime counting approximation $\pi^*(n)$ and the rigorous error analysis of the lower bound $LB(p_i)$, provides a versatile framework for exploring several other unsolved problems in number theory. Potential areas for future research include the following:

Extensions of other prime gap conjectures: A primary objective is to apply these techniques to the Legendre Conjecture, which posits that at least one prime exists between n^2 and $(n+1)^2$ for all $n > 0$ [4]. Given that Brocard's conjecture is a stronger version of this problem, the established lower bound approach should be directly adaptable.

Generalization of Interval Densities: The density fraction $W(n)$ could be further refined to analyze the distribution of primes in even smaller intervals, such as those described by Oppermann's Conjecture which posits that there is at least one prime in the interval $(n^2 - n,\, n^2)$ and at least one prime in the interval $(n^2, n^2 + n)$ [5].

Optimization of the Error Term: While the current error analysis $E(n) = 2\pi(\sqrt{n})$ is sufficient to support this proof, future work will aim to tighten this bound further. Refining this term could provide more precise estimates for the minimum number of primes in tighter intervals and may offer insights into the Riemann Hypothesis [6].

Application to Twin Prime Distributions: The observation that the lower bound $\Delta\pi^{LB}$ coincides with the actual prime count precisely at twin prime intervals suggests that this sieve methodology could be utilized to further study the density and distribution of twin primes and offer insights into the Twin Prime conjecture [7]

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