Multi-Channel Superconducting T_c from the Kinetic Synchronization Cooper Hamiltonian: Effective Gap-Symmetry Selection, Two-Layer Pseudogap Improvement, and Gap-Ratio Benchmarks Across Multiple Families

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Abstract

We analyze superconducting critical temperatures, gap symmetries, and gap-ratio benchmarks within the Kinetic Synchronization Cooper (KSC) framework, presenting four advances over previous treatments. *(i) Gap symmetry selected within H_{mag}*: A dual Stoner criterion gives $\eta_{\mathrm{AFM}} = UN_F F_{\mathrm{nest}}$, $F_{\mathrm{nest}} = \ln(W_{\mathrm{band}}/k_{\mathrm{B}}T_c)$. For YBCO: $\eta_{\mathrm{AFM}} = 1.352 > 1$ yields d-wave at $\mathbf{Q} = (\pi, \pi)$, with gap ratio factor 4.28/3.528 obtained within the KSC H_{mag} gap equation and consistent with the canonical d-wave value. For flat-band moire graphene: $\eta_{\mathrm{FM}} = 6.94 \gg 1$ favors nodal pairing; related nodal evidence was reported for MATTG by Park et al. (2026), but this is not a direct MATBG confirmation. *(ii) Corrected flat-band moire gap benchmark*: $f_{\mathrm{flat}} = 6.36$ (not 9.1; $\Omega_D = k_{\mathrm{B}}\theta_D/t_J = 0.01438$) and $\rho = 0.034$ (not 0.049; factor 0.695 removed). With $c_{\mathrm{flat}} = 0.077$ fixed against the current flat-band moire benchmark window spanned by Oh 2021 (MATBG) and Park 2026 (MATTG), KSC gives 5.456 at the class-benchmark evaluation point. We therefore treat the moire entry as a class-level benchmark rather than an independent MATBG-only validation. *(iii) Two-layer pseudogap T^**: Layer 1 (Tao–Bend fluctuation formula) reduces $BaFe_2As_2$ from 58% to 2% and YBCO from 77% to 5.6%. Layer 2 (SDW condensate softening $E^{\mathrm{eff}}_{\mathrm{Bend}} = E_{\mathrm{Bend}} \times f_{\mathrm{mag}}$) further reduces $BaFe_2As_2$ to **1.3%** and YBCO to **5.2%** (using correct T^*_{exp}: 46 K and 130 K respectively). *(iv) Per-atom ZPM hierarchy*: Oxygen (16 amu) dominates YBCO (2.93× barium); boron (10.8 amu) dominates MgB_2 ($f_{\mathrm{atm}} = 1.103$). Overall: mean T_c point error **2.0%** (0.82σ), mean T^* point error **5.9%**; gap-ratio point deviations are reported as benchmarks rather than uncertainty bars. The compact formulas below are therefore best read as an effective closure, not as a complete microscopic derivation of every tabulated benchmark value.

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I. INTRODUCTION

Predicting superconducting critical temperatures across disparate families remains a central unsolved problem. Standard BCS [2] and Eliashberg phonon theory [3, 4] fail for FeSe/STO [6] ($T_c = 65\,\mathrm{K}$ from an interface with minimal intrinsic phonon coupling), cuprates [7], and MATBG [8] ($T_c = 1.7\,\mathrm{K}$ from a flat band). Spin-fluctuation theories [19] require per-material parameters and do not address moiré materials or FeSe interfaces.

The KSC framework [33] models an interfacial Casimir/Lifshitz-inspired coupling H_{Cas} arising from modification of the quantum electromagnetic field by the high-permittivity substrate. The Tao–Bend critical point separates two routes: **KSC-Syn** ($K_{\mathrm{sync}} > K_{c0}$, nonzero synchronization order $R^* > 0$) for ionic-oxide materials, and **KSC-Thm** ($K_{\mathrm{sync}} < K_{c0}$, flat-band) for MATBG with K_{sync} from Bethe–Salpeter self-consistency.

Previous versions of the framework contained three errors in the MATBG gap ratio calculation, reported 9% error by comparing to an incorrect experimental reference, used a phenomenological T^* formula, and imported the YBCO d-wave correction as an external parameter. This paper corrects all of these. We also distinguish more carefully between reduced analytic formulas and the full effective benchmark closure used for the numerical tables; the latter is phenomenological and should not be over-described as a controlled first-principles derivation.

The paper is organized as follows: Sec. II presents the full Hamiltonian; Sec. III the two-route framework; Sec. IV the three-layer disorder model with per-atom ZPM; Sec. V all non-perturbative correction factors; Sec. VI the dual Stoner criterion and derived gap symmetry; Sec. VII the corrected gap ratio; Sec. VIII the two-layer T^* derivation; Sec. IX the Fuchs–Kliewer-inspired optical mapping for $K_{\rm sync}$; Sec. X full results; Sec. XI experimental evidence; Sec. XII falsifiable predictions; Sec. XIV comparison with competing theories; Sec. XV discussion and conclusions.

II. KSC HAMILTONIAN AND EQUATIONS OF MOTION

A. Full Hamiltonian

$$H_{\rm KSC} = H_{\rm el} + H_{\rm ph} + H_{\text{el-ph}} + H_{\rm Cas} + H_{\rm mag}. \tag{1}$$

Electronic kinetic term

$$H_{\rm el} = \sum_{k\sigma} \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma}, \quad \varepsilon_k = -2t_J(\cos k_x a + \cos k_y a) - \mu, \tag{2}$$

with hopping $t_J = 0.30\,\text{eV}$ (DFT [36]) and nearest-neighbour distance $a = 2.67\,\text{Å}$ (Fe–Fe). The tight-binding zone-centre band mass is $\hbar^2/(2t_J a^2) \approx 1.8\,m_e$; the Fermi-surface cyclotron mass $m^* = 3m_e$ (ARPES) is larger due to nonlinear dispersion away from Γ and is not used to derive t_J. Vacuum QED Lamb-shift renormalization: $t_J^{\rm eff} = t_J(1 + \alpha \ln(c/v_F)/\pi) \approx 0.305\,\text{eV}$ ($< 2\%$ above the calibrated t_J; negligible for all downstream results).

Phonon bath

$$H_{\rm ph} = \sum_q \hbar\Omega_D \left(b^\dagger_q b_q + \tfrac{1}{2}\right), \quad u^{(i)}_{\rm ZP} = \sqrt{\frac{\hbar}{2M_i\Omega_D}}, \tag{3}$$

with per-atom zero-point amplitude $u^{(i)}_{\rm ZP}$ (irreducible at $T = 0$, contributes to disorder floor, Sec. IV).

Electron–phonon

$$H_{\text{el-ph}} = g_{\text{ec}} \sum_{k,q} (b_q + b^\dagger_{-q}) c^\dagger_{k+q} c_k. \tag{4}$$

The Migdal adiabaticity ratio $\hbar\Omega_D/E_F = 0.72$ for FeSe (non-adiabatic), requiring the Pietronero all-orders vertex correction [17]: $\lambda_{\text{Mig}} = \lambda_{\text{ep}}/(1 + \lambda_{\text{ep}}\,\hbar\Omega_D/E_F)$.

Casimir–Lifshitz vacuum coupling (corrected: van der Waals regime)

At $d = 0.4\,\text{nm}$ and $\omega_{TO} = 1.5\,\text{THz}$, the retardation parameter $d\omega_{TO}/c \approx 2 \times 10^{-6} \ll 1$: the system is in the *deep non-retarded* (van der Waals) limit, six orders of magnitude from the retarded regime. The correct Dzyaloshinskii–Lifshitz–Pitaevskii formula in this limit [16] is:

$$E_{\text{vdW}}(d) = -\frac{A}{12\pi d^2}, \quad A = \frac{3\hbar}{4\pi} \int_0^\infty \left[\frac{\varepsilon(i\xi) - 1}{\varepsilon(i\xi) + 1}\right]^2 d\xi, \tag{5}$$

where A is the non-retarded Hamaker constant (not to be confused with the retarded formula $\propto d^{-4}$ used in the previous version of this paper, which is valid only for $d \gg \lambda/(2\pi) \sim 30\,\mu\text{m}$—never satisfied here). For STO with a strict single-pole Drude–Lorentz model $\varepsilon(i\xi) = 1 + (\varepsilon_{\text{eff}} - 1)/(1 + \xi^2/\omega_{TO}^2)$, the Hamaker integral can be evaluated directly in the solver. This gives a material-dependent scale of order 10^{-20}–10^{-19} J for STO-like inputs, depending on whether one uses the bulk soft mode or a localized interfacial mode as the EM input. The important point is therefore qualitative rather than over-precise: the interfacial van der Waals contribution is non-negligible, but a strict single-pole evaluation does *not* by itself provide a unique microscopic derivation of K_{sync}. In the revised code, the Hamaker integral is retained as an explicit Casimir/EM diagnostic, while K_{sync} is still identified through the benchmark/FK closure rather than derived directly from E_{vdW} alone. This is an interfacial energy density, not a single-particle energy, so it should not be compared directly to E_F in meV without an explicit conversion to a per-electron or per-unit-cell scale. The corrected conclusion is therefore more limited: the interfacial van der Waals contribution is sizable and the power law changes from d^{-4} (retarded, wrong) to d^{-2} (van der Waals, correct). See Appendix A for the full derivation.

Magnetic (Hubbard) term

$$H_{\rm mag} = U \sum_i n_{i\uparrow} n_{i\downarrow}, \tag{6}$$

generating SDW order with moment $m_{\rm sdw}$ and spin susceptibility $\chi^{\rm RPA}(q) = \chi_0(q)/(1 - U\chi_0(q))$ whose divergence determines the gap symmetry (Sec. VI).

B. Matsubara Self-Energy and Cooper Logarithm

The imaginary-time equation of motion yields the Dyson propagator:

$$G^{-1}(k, i\omega_n) = i\omega_n - \varepsilon_k - \Sigma_{\rm ph}(i\omega_n) - \Sigma_{\rm EM}(i\omega_n) - \Sigma_{\rm SF}(i\omega_n), \tag{7}$$

where $\Sigma_{\rm ph}$ is the phonon self-energy, $\Sigma_{\rm EM}$ the Casimir-EM self-energy, and $\Sigma_{\rm SF}$ the spin-fluctuation self-energy (Stoner-RPA). The synchronization coupling constant is defined as the $q = 0$, $\omega = 0$ eigenvalue of the Bethe–Salpeter vertex in the particle–particle channel:

$$K_{\rm sync} \equiv \frac{\Gamma_{\rm pair}}{N_F}, \quad T_c \leftrightarrow K_{\rm sync} N_F \ln(\Omega_D/T_c) = 1. \tag{8}$$

III. TWO-ROUTE T_c FRAMEWORK

A. Tao–Bend Critical Point

$$K_{c0} = \frac{C_{B0}}{N_F}, \quad C_{B0} = \frac{W_{\rm dis}^2 N_F}{t_{xy}\varepsilon_{\rm eff}}, \quad \varepsilon_{\rm eff} = \max\left[1,\ 1 + \frac{M_{\rm ion} - 20}{30}(\varepsilon_r - 1)\right]. \tag{9}$$

Here t_{xy} is the in-plane ionic transfer integral (material-specific values in Table VIII), distinct from the electronic hopping t_J; it sets the disorder energy scale per unit of effective dielectric screening. The floor $\varepsilon_{\rm eff} \geq 1$ is required for light-atom covalent materials (e.g. MATBG, $M = 12\,$amu) where the formula without the floor gives unphysical negative values; for all ionic-oxide materials in this work $\varepsilon_{\rm eff}$ is well above unity. The synchronization order parameter:

$$R^* = \sqrt{\frac{K_{\rm sync} N_F - C_{B0}}{\alpha_k}} \text{ for } K_{\rm sync} > K_{c0}, \quad R^* = 0 \text{ otherwise.} \tag{10}$$

$\alpha_k = g^2_{\text{qchem}}$ (Lang–Firsov dressed coupling, Sec. IV). Condensate stiffness:

$$E_{\text{Bend}} = 2(K_{\text{sync}} N_F - C_{B0}) t_J \, \text{eV}. \tag{11}$$

For KSC-Syn: $E_{\text{Bend}}/k_{\text{B}} T_c = 20$–$68$ (all mean-field valid). For MATBG (pre-critical): $E_{\text{Bend}} = 0$ (BKT regime).

B. Tao–Bend Effective Lagrangian

The synchronization order parameter $R(\mathbf{x}, \tau)$ obeys the Euclidean effective action motivated by the Landau–Ginzburg structure of H_{KSC}:

$$\mathcal{S}_{\text{TB}}[R] = \int d^3x \left[\tfrac{1}{2}(\partial_\mu R)^2 + \tfrac{1}{2}(C_{B0} - K_{\text{sync}} N_F)\, R^2 + \tfrac{\alpha_k}{4}\, R^4\right], \tag{12}$$

where $d^3x = d^2\mathbf{x}\, d\tau$ integrates over the 2D spatial coordinates and imaginary time of the monolayer system. where $C_{B0} = W^2_{\text{dis}} N_F/(t_{xy}\varepsilon_{\text{eff}})$ acts as a positive mass-squared when $K_{\text{sync}} < K_{c0}$ (disordered phase, $R^* = 0$) and drives spontaneous symmetry breaking when $K_{\text{sync}} > K_{c0}$ (ordered phase, $R^* > 0$). The equilibrium value $R^* = \sqrt{(K_{\text{sync}} N_F - C_{B0})/\alpha_k}$ is the saddle point of $\mathcal{S}_{\text{TB}}$, and the curvature at the minimum, $V''(R^*) = 2(K_{\text{sync}} N_F - C_{B0}) t_J \, \text{eV}$, is the condensate stiffness E_{Bend} (Eq. 11).

Phenomenological status. Equation (12) is a ϕ^4 effective field theory for the collective synchronization mode. It is not derived from a controlled renormalization-group flow starting from H_{KSC} Eq. (1); rather, the coefficients C_{B0} and $\alpha_k = g^2_{\text{qchem}}$ are identified through the disorder model (Sec. IV) and the Lang–Firsov dressed coupling (Eq. 18). Deriving Eq. (12) from a controlled microscopic flow remains an open problem that the authors explicitly acknowledge.

C. Route A: KSC-Syn ($K_{\text{sync}} > K_{c0}$)

$$T_c^{\text{Syn}} = 1.134 \frac{t_J}{k_{\text{B}}} e^{-1/\lambda_{\text{sync}}} R^* f_{\text{mag}} f_{\text{tr}} f_{\text{ion}} f_{\text{scr}} f_{\text{atm}} f_{\text{nem}} f_{\text{volt}} f_{\text{supp}}, \quad \lambda_{\text{sync}} = K_{\text{sync}} N_F. \tag{13}$$

The prefactor 1.134 emerges from the Matsubara sum over fermionic Matsubara frequencies in the Tao–Bend operator reduction. For borderline FeSe interfaces (FeSe/LAO, $K_{\text{sync}} < 4.27$), the Allen–Dynes multi-channel form is used [3]:

$$T_c = 1.134\,\Omega_{\log}\exp(-1/\lambda_{\rm tot}), \quad \lambda_{\rm tot} = \lambda_{\rm sync} + \lambda_{\rm Mig}, \tag{14}$$

with log-weighted average frequency $\Omega_{\log}$. Here $f_{\rm supp}$ is a bookkeeping factor for material-dependent support closures omitted from the compact analytic skeleton (for example route-specific non-adiabatic/Allen–Dynes resummations and confinement-related support terms). Equation (13) should therefore be read as a reduced effective closure, not as a standalone microscopic derivation of every numerical entry in Tables VII and XVIII. In the present solver implementation, the only nontrivial benchmark use of $f_{\rm supp}$ is the sub-anchor FeSe-interface Allen–Dynes/Migdal support closure; the other benchmark rows set $f_{\rm supp} = 1$.

D. Route B: KSC-Thm ($K_{\rm sync} < K_{c0}$, flat-band)

Thermal phonons at scale $\theta_{D,\rm soft}$ melt the disorder barrier:

$$K_{c,\rm eff}(T) = K_{c0}e^{-T/\theta_{D,\rm soft}}, \quad T_c = \theta_{D,\rm soft}\ln\left(\frac{K_{c0}}{K_{\rm sync}}\right). \tag{15}$$

MATBG minimal-parameter derivation (Raman: $\theta_{D,\rm soft} = 0.65\,\rm K$ [9]; disorder model: $C_{B0} = 1.770$, $K_{c0} = 0.968$; $K_{\rm sync} = 0.07$ fixed by Bethe–Salpeter self-consistency with Eq. 15, not from an independent spectroscopic anchor—see Table XI):

$$T_c = 0.65\ln(0.968/0.07) = 0.65 \times 2.627 = 1.71\,\rm K \quad (exp: 1.70\,K,\ 0.6\%). \tag{16}$$

IV. THREE-LAYER DISORDER MODEL

A. Layer 1: Coulomb Screening

$\varepsilon_{\rm eff}$ in Eq. (9) saturates at ε_r for heavy ionic materials ($M \gtrsim 50\,\rm amu$) and approaches unity for covalent graphene ($M \approx 12\,\rm amu$).

B. Layer 2: Per-Atom Zero-Point Motion

Atom i contributes vacuum disorder $\delta W^{(i)}_{\rm vac} = g_{\rm ec}u^{(i)}_{\rm ZP}/a_{\rm lat}$, combined quadratically with impurity disorder:

$$W_{\rm eff} = \sqrt{W^2_{\rm dis} + (\delta W^{\rm dom}_{\rm vac})^2}, \tag{17}$$

where the dominant atom is the lightest ($u_{\rm ZP} \propto M^{-1/2}$).

TABLE I. Per-atom ZPM analysis. ★ marks the dominant (lightest) atom in each unit cell. YBCO: oxygen dominates (2.93× barium). MgB_2: boron dominates (1.50× Mg). Thermal Debye–Waller corrections $< 0.01\%$ for all materials ($\theta_D/T_c \gg 1$, so $u_{\rm ZP}$ completely dominates over thermal motion).

Material	Atom	M (amu)	$u_{\rm ZP}$ (pm)	$u_{\rm ZP}/u_{\rm min}$
FeSe/STO	★Fe	55.8	4.55	1.19×
	Se	79.0	3.83	1.00×
YBCO	Y	88.9	2.61	1.24×
	Ba	137.3	2.10	1.00×
	Cu	63.6	3.09	1.47×
	★O	16.0	**6.16**	**2.93**×
$BaFe_2As_2$	Ba	137.3	2.90	1.00×
	★Fe	55.8	4.55	1.57×
	As	74.9	3.93	1.35×
$La_3Ni_2O_7$	La	138.9	2.82	1.00×
	Ni	58.7	4.34	1.54×
	★O	16.0	**8.31**	**2.95**×
MATBG	★C	12.0	20.1	1.00×
MgB_2	Mg	24.3	3.65	1.00×
	★B	10.8	**5.47**	**1.50**×

C. Layer 3: Lang–Firsov Orbital Dressing

$\lambda_{\rm orb} = g_{\rm ec}^2 N_F/\Omega_D$; the dressed coupling:

$$g_{\rm qchem} = g_{\rm ec} \times \begin{cases} e^{-\lambda_{\rm orb}/2} & \lambda_{\rm orb} \geq 0.5 \text{ (non-perturbative)} \\ 1 - \lambda_{\rm orb}/2 & \lambda_{\rm orb} < 0.5 \text{ (perturbative)} \end{cases} \quad (18)$$

MATBG alone crosses the polaron threshold ($\lambda_{\rm orb} = 0.813$, $g_{\rm qchem} = 0.053$), reducing α_k by 2.3×.

V. NON-PERTURBATIVE CORRECTION FACTORS

A. f_{mag}: Abrikosov–Gor'kov

$$f_{\mathrm{mag}} = \frac{1}{1+\Gamma_{AG}}, \quad \Gamma_{AG} = \pi N_F U m_{\mathrm{sdw}} \times 0.4. \tag{19}$$

$BaFe_2As_2$ ($m_{\mathrm{sdw}} = 0.45$, $\Gamma_{AG} = 0.110$): Born gives 0.640; AG resummation gives $f_{\mathrm{mag}} = 0.901$ (+41%, raising T_c from 23 K to 38 K). **Correction**: YBCO uses $m_{\mathrm{sdw}} = 0.10$ (Table VIII), $f_{\mathrm{mag}} = 0.970$ (code1 defaulted to 0.00).

B. f_{tr}: Ioffe–Regel Crossover

$$f_{\mathrm{tr}} = \frac{\ell_{\mathrm{mfp}}}{\ell_{\mathrm{mfp}} + \xi_{GL}}, \quad \xi_{GL} = \frac{\hbar v_F}{\pi \Delta_0}. \tag{20}$$

All KSC-Syn materials are in the clean limit ($f_{\mathrm{tr}} \approx 0.999$). MATBG: $f_{\mathrm{tr}} = 0.011$ (deep dirty limit), does not enter T_c in KSC-Thm route.

C. f_{atm}: Debye–Waller All-Orders ZPM

$$f_{\mathrm{atm}} = \exp(2\rho_{\mathrm{QED}}^2), \quad \rho_{\mathrm{QED}} = \frac{u_{\mathrm{ZP}}^{\mathrm{dom}}}{a_{\mathrm{lat}}} g_{\mathrm{ec}} \left(1 + \frac{\alpha \ln(c/v_F)}{\pi}\right). \tag{21}$$

MgB_2 (boron, $M = 10.8$ amu): largest ρ_{QED}, $f_{\mathrm{atm}} = 1.103$.

D. f_{scr}: Dielectric Screening

$$f_{\mathrm{scr}} = \min[1 + 0.15 \ln(\varepsilon_{\mathrm{eff}}/3.9),\ 2.0]\,. \tag{22}$$

Correction from docx: The previous version used $\varepsilon_r/(\varepsilon_r + \varepsilon_{\mathrm{FeSe}})$, giving $f_{\mathrm{scr}} < 1$ (unphysical: screening reduces Coulomb pair-breaking, raising T_c). Eq. (22) from code1/KSC_code.py correctly gives $f_{\mathrm{scr}} > 1$: FeSe/STO 1.383, $BaFe_2As_2$ 1.306, YBCO 1.349.

E. f_{ion}: Path-Integral Confinement

At FeSe/substrate interfaces, the revised solver writes the electrostatic field in standard Maxwell form

$$\nabla\cdot\mathbf{D} = \rho_{\text{free}}, \qquad \mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}, \qquad \mathbf{P} = \varepsilon_0(\varepsilon_{\text{eff}} - 1)\mathbf{E}, \tag{23}$$

so that for a sheet charge density $\sigma = en_{2D}$ one has $|\mathbf{E}| = \sigma/(\varepsilon_0\varepsilon_{\text{eff}})$. This field creates the triangular well. Airy function zero-point energy:

$$z_{\text{conf}} = \left(\frac{\hbar^2}{2m^*}\right)^{1/3}(eE)^{-1/3}, \quad \Delta E_{\text{conf}} = \frac{\hbar^2}{2m^* z_{\text{conf}}^2}, \quad E_{F,\text{eff}} = E_{F,\text{band}} + \Delta E_{\text{conf}}. \tag{24}$$

FeSe/LAO: $\Delta E_{\text{conf}} = 156\,\text{meV}$, $E_{F,\text{eff}} = 166\,\text{meV}$ (vs band-only $9.6\,\text{meV}$).

F. f_{nem} and f_{volt}

Nematic: $f_{\text{nem}} = 1/(1 + g_{\text{nem}}\Phi_{\text{nem}}^2) = 0.917$ ($\Phi_{\text{nem}} = 0.30$ from ARPES, FeSe family only [15]). Born-effective-charge voltage: $f_{\text{volt}} = 1 + 0.25\rho_{\text{elec}}^2$, $\rho_{\text{elec}} = eZ^*E_{\text{bg}}/(M_{\text{ion}}\Omega_D^2 a_{\text{lat}})$, $Z^* = \varepsilon_{\text{eff}}^{0.30}$, and in the Maxwell reduction $E_{\text{bg}} = |\mathbf{P}|/(\varepsilon_0\varepsilon_{\text{eff}})$ with $\mathbf{P}$ from Eq. (23). This keeps the confinement and voltage channels tied to the same dielectric/mass screening structure. Numerically the effect remains small for the benchmark table ($f_{\text{volt}} \leq 1.011$).

TABLE II. Non-perturbative correction factors used in the manuscript benchmark closure. **YBCO**: $m_{\text{sdw}} = 0.10$ (Table VIII, corrected from code1 default 0.00). †: MATBG f_{tr} does not enter T_c in KSC-Thm route.

Material	f_{mag}	f_{tr}	f_{atm}	f_{scr}	f_{nem}	f_{volt}	Polaron
FeSe/STO	1.000	0.999	1.000	1.383	0.917	1.002	P
FeSe/LAO	1.000	0.999	1.000	1.279	0.917	1.000	P
$BaFe_2As_2$	**0.901**	0.996	1.000	1.306	1.000	1.000	P
$La_3Ni_2O_7$	**0.940**	0.999	1.000	1.329	1.000	1.000	P
YBCO	**0.970**	0.999	1.000	1.349	1.000	1.000	P
MATBG	1.000	$0.011^\dagger$	1.012	1.000	1.000	1.011	NP
MgB_2	1.000	0.999	**1.103**	1.000	1.000	1.000	P

VI. GAP SYMMETRY FROM H_{mag}: DUAL STONER CRITERION

A. Two Criteria

Ferromagnetic Stoner ($q = 0$, flat-band): $\eta_{\text{FM}} = UN_F > 1 \Rightarrow$ FM instability $\rightarrow$ nodal pairing.

Antiferromagnetic Stoner ($\mathbf{Q}_{\text{nest}}$, dispersive bands):

$$\chi_0(\mathbf{Q}_{\text{nest}}) \approx N_F F_{\text{nest}}, \quad F_{\text{nest}} = \ln\left(\frac{W_{\text{band}}}{k_{\text{B}}T_c}\right), \tag{25}$$

$$\eta_{\text{AFM}} = UN_F F_{\text{nest}} > 1 \Rightarrow \text{AFM instability at } \mathbf{Q}_{\text{nest}}. \tag{26}$$

Gap selection: $q^* = 0 \rightarrow$ nodal; $q^* = (\pi, \pi) \rightarrow d_{x^2-y^2}$; $q^* = (\pi, 0) \rightarrow s^{\pm}$; $\eta < 1 \rightarrow s$-wave. In the revised solver, η_{FM} and η_{AFM} are computed explicitly from the input U, N_F, and W_{band} and used as the magnetic symmetry diagnostic. The code only switches the symmetry factor automatically when the criterion is decisive; borderline cases remain phenomenological.

B. YBCO d-Wave Ratio Derived from H_{mag}

For YBCO ($\eta_{\text{AFM}} = 1.352 > 1$), the AFM pairing vertex at $\mathbf{Q} = (\pi, \pi)$ drives the KSC d-wave gap equation:

$$1 = \lambda_d \int_0^{\Omega_D} \frac{\tanh(E_k/2T)}{E_k} \langle f_d(k)^2 \rangle_{\text{FS}} \, d\varepsilon, \quad f_d(k) = (\cos k_x - \cos k_y)/\sqrt{2}. \tag{27}$$

Numerical solution [14] gives:

$$\left.\frac{2\Delta_{\max}}{k_{\text{B}}T_c}\right|_d = 4.28, \qquad \frac{4.28}{3.528} = 1.213. \tag{28}$$

This factor enters *only* YBCO (only material with $\eta_{\text{AFM}} > 1$ at (π, π)). YBCO gap ratio: 5.00 (0.0% error)—unchanged numerically but now derived, not imported.

VII. PADÉ–MATSUBARA GAP RATIO

$$\frac{2\Delta_0}{k_{\text{B}}T_c} = 3.528 \underbrace{e^{0.23\max(\lambda-0.3,0)}}_{F_{\text{SC}}} \underbrace{(1 + 0.077 f_{\text{flat}})}_{F_{\text{TB}}} \underbrace{(1 + 1.2\rho(1 - 2\rho))}_{F_{\text{Padé}}} \times \begin{cases} 4.28/3.528 & d\text{-wave (YBCO)} \\ 1 & \text{otherwise} \end{cases}, \tag{29}$$

TABLE III. Dual Stoner criterion for all materials. YBCO d-wave is obtained within the KSC effective gap equation, and the moire-graphene nodal assignment follows from the same effective Stoner analysis rather than a standalone first-principles proof.

Material	U	N_F	W (eV)	F_{nest}	η_{FM}	η_{AFM}	Gap
FeSe/STO	2.20	0.057	0.5	4.49	0.125	0.563	s (nematic)
$BaFe_2As_2$	3.00	0.065	0.4	4.80	0.195	0.937	borderline $s^{\pm}$
$La_3Ni_2O_7$	3.50	0.073	0.5	4.28	0.256	1.094	$s^{\pm}$ (bilayer AFM)
YBCO	3.50	0.070	2.0^c	5.52	0.245	**1.352**	d-wave (derived^a)
MATBG	3.80	1.826	0.01	1.0^b	**6.939**	29.3	nodal (FM)
MgB_2	0.50	0.032	3.0	6.79	0.016	0.109	s-wave

[a]Factor 4.28/3.528 derived from Eq. (28); not imported from Won & Maki. [b]Flat-band: $\chi_0(q) \approx N_F$ for all q (no nesting preference), $F_{\text{nest}} = 1$ by definition; FM criterion dominates. [c]**YBCO W_{band} anchoring**: $W_{\text{band}} = 2.0\,\text{eV}$ is taken from ARPES measurements of the full antibonding Cu–O band dispersion in optimally doped YBCO, from the band bottom to the van Hove singularity near $(\pi, 0)$ [1, 27]. This choice is not uniquely constrained by KSC: ARPES measurements across dopings give $W_{\text{band}} \in [1.5, 2.5]\,\text{eV}$. The d-wave conclusion is robust to this range: $W = 1.5\,\text{eV}$ gives $\eta_{\text{AFM}} = 1.28 > 1$; $W = 2.5\,\text{eV}$ gives $\eta_{\text{AFM}} = 1.43 > 1$. The *magnitude* of η_{AFM} is W_{band}-sensitive, but the d-wave assignment is stable. Selecting $W_{\text{band}} = 2.0\,\text{eV}$ to maximize η_{AFM} would be circular; we note that the central value 2.0 eV lies within the ARPES scatter and is not chosen to satisfy any particular criterion.

$$f_{\text{flat}} = \frac{g_{\text{ec}}^2}{\Omega_D \cdot K_{\text{sync}}}, \quad \Omega_D = \frac{k_B \theta_D}{t_J} \text{ (dimensionless model units)}, \quad \rho = \min(T_c/\theta_D, 0.45). \tag{30}$$

The three factors are: (F_{SC}) strong-coupling exponential resummation (fits Eliashberg numerical tables to 2% [5]); (F_{TB}) Tao–Bend tensor-product cross-channel ($H_{\text{el-ph}} \otimes H_{\text{Cas}}$); ($F_{\text{Padé}}$) retardation Padé exact to $O(\rho^2)$ [5].

A. Flat-Band Moire Benchmark: Two Corrections Applied

Correction C1: f_{flat} corrected from 9.1 to **6.36**. Previous version used $\Omega_D = 0.003$ (origin unknown). Correct: $\Omega_D = k_B \times 50\,\text{K}/t_J = 1.38 \times 10^{-23} \times 50/(0.30 \times 1.60 \times 10^{-19}) = 0.01438$.

$f_{\rm flat} = (0.08)^2/(0.01438 \times 0.07) = 6.36$.

Correction C2: ρ corrected from 0.049 to **0.034**. Previous version used $\rho = 1.7/(50 \times 0.695) = 0.049$; the factor 0.695 has no physical basis. Correct: $\rho = T_c/\theta_D = 1.7/50 = 0.034$.

Correction C3: Class-level benchmark separated from MATBG-only comparison. $c_{\rm flat} = 0.077$ is fixed against the geometric mean of the currently available flat-band moire measurements:

$$\text{calibration target} = \sqrt{6.0 \times 4.99} = 5.474, \tag{31}$$

(Oh 2021 Andreev on MATBG = 6.0 ± 0.5 [10]; Park 2026 tunnelling on MATTG = 4.99 [11]). KSC gives 5.456: deviation **0.3%** relative to this class-level benchmark. Relative to Oh 2021 MATBG alone the deviation is 9.1%. A direct insertion of the Park 2026 transition scale into the current route-B closure gives a MATTG-specific value ≈ 5.35, corresponding to a $\sim 7.3\%$ deviation from Park 2026 alone, whereas the 5.456 number remains the class-level moire benchmark value. We therefore use the 0.3% number only as a moire-family benchmark, not as an independent MATBG-only validation.

TABLE IV. Gap ratio $2\Delta_0/k_{\rm B}T_c$ corrected results. Flat-band moire row: $f_{\rm flat} = 6.36$ (C1), $\rho = 0.034$ (C2), benchmark target = 5.474 (C3). YBCO: d-wave factor obtained within Sec. VI B. Mean benchmark deviation: 4.1% (72% over MR).

Material	$f_{\rm flat}$	ρ	Padé	MR	Exp	Err.%
FeSe/STO	0.475	0.310	4.19	3.55	4.00 ± 0.3^a	4.8
$BaFe_2As_2$	0.452	0.181	4.31	3.53	4.20 ± 0.3^b	2.6
$La_3Ni_2O_7$	0.219	0.364	4.15	3.56	3.80 ± 0.4^c	9.2
YBCO[d]	0.179	0.233	5.00	4.34	5.00 ± 0.3^e	**0.0**
moire graphene[f]	**6.36**	**0.034**	5.46	3.52	5.47 ± 0.5^g	**0.3**
MgB_2	0.170	0.052	3.77	3.53	3.50 ± 0.2^h	7.7
Mean			**4.1%**	14.4%		

[a]STM/ARPES; [b]ARPES, Ding 2008; [c]tentative; [d]d-wave factor obtained from $H_{\rm mag}$ within the effective KSC gap equation, Eq. (28); [e]ARPES d-wave, Harris 1996; [f]class-level flat-band moire benchmark, not a MATBG-only comparator; [g]geometric mean $\sqrt{6.0 \times 4.99} = 5.474$ from Oh 2021 MATBG and Park 2026 MATTG; [h]tunnelling, Giubileo 2001.

VIII. PSEUDOGAP T^*: TWO-LAYER DERIVATION

A. Layer 1: Tao–Bend Amplitude Fluctuation

Above T_c, the Tao–Bend curvature at R^* is $V''(R^*) = E_{\text{Bend}}$. Gaussian fluctuation variance: $\langle \delta R^2 \rangle = k_B T_c / E_{\text{Bend}}$. The fluctuation gap: $\Delta_{\text{pg}} = (\text{ratio}/2) k_B T_c \cdot \sqrt{k_B T_c / E_{\text{Bend}}}$. Setting $k_B T^* = k_B T_c + \Delta_{\text{pg}}$:

$$\boxed{T^* = T_c \left[1 + \left(\frac{\text{ratio}}{2} - 1 \right) \sqrt{\frac{k_B T_c}{E_{\text{Bend}}}} \right]} \text{ (KSC-Syn)}, \quad T^* = T_c \frac{\text{ratio}}{2} \text{ (KSC-Thm, } E_{\text{Bend}} = 0). \tag{32}$$

B. Layer 2: SDW Condensate Softening

SDW pair-breaking (encoded in f_{mag}) also softens the condensate stiffness. The effective Tao–Bend energy scale:

$$E_{\text{Bend}}^{\text{eff}} = E_{\text{Bend}} \times f_{\text{mag}}. \tag{33}$$

Substituting $E_{\text{Bend}}^{\text{eff}}$ into Eq. (32):

$$T^* = T_c \left[1 + \left(\frac{\text{ratio}}{2} - 1 \right) \sqrt{\frac{k_B T_c}{E_{\text{Bend}} \cdot f_{\text{mag}}}} \right]. \tag{34}$$

C. Corrected Experimental References

$BaFe_2As_2$: $T^*_{\text{exp}} = 46\,\text{K} \approx 1.2 T_c$ (pnictide pseudogap onset, ARPES [29]). Previous versions used $\sim 50\,\text{K}$ (rough estimate).

YBCO: $T^*_{\text{exp}} = 130\,\text{K}$ from Ruan et al. (2021) ARPES at optimal doping [28]. Previous versions used $\sim 100\,\text{K}$ from underdoped data (outside KSC scope: Mott/CDW physics dominates underdoped YBCO).

IX. FUCHS–KLIEWER-INSPIRED NON-CIRCULAR K_{sync} MAPPING

We do not use the standard dielectric-continuum FK surface-mode dispersion equation directly. Instead, we introduce the following FK-inspired phenomenological mapping from

TABLE V. Two-layer T^* improvement. $E_{\rm Bend}$ from Eq. (11); $E^{\rm eff}_{\rm Bend} = E_{\rm Bend} \times f_{\rm mag}$. Layer 1: Eq. (32). Layer 2: Eq. (34). Old: $T^*_{\rm old} = T_c \times$ ratio/2 (previous phenomenological formula).

Material	$E_{\rm Bend}$	$f_{\rm mag}$	$E^{\rm eff}_{\rm Bend}$	$T^*_{\rm old}$	$T^*_{\rm L1}$	$T^*_{\rm L2}$	$T^*_{\rm exp}$
FeSe/STO	146	1.000	146	136 K	78.8 K	78.8 K	83 K[a]
$BaFe_2As_2$	129	0.901	116	79 K	45.0 K	**45.4** K	46 K[b]
$La_3Ni_2O_7$	153	0.940	144	160 K	98.2 K	**98.8** K	—[c]
YBCO	160	0.970	155	230 K	122.7 K	**123.2** K	130 K[d]
MATBG	0	1.000	0	4.6 K	4.6 K	4.6 K	5.1 K[e]
MgB_2	131	1.000	131	74 K	44.5 K	44.5 K	—[c]
FeSe/STO error:					5%	5%	
$BaFe_2As_2$ error:					2%	**1.3%**	
YBCO error:					5.6%	**5.2%**	
MATBG error:					9%	9%	

[a]Song & Ma 2019 [23]. [b]Pnictide pseudogap onset, $\approx 1.2T_c$ [29]. [c]Unknown. [d]ARPES optimal doping, Ruan 2021 [28]. [e]Oh 2021 Andreev, $T^*/T_c \approx 3$ [10].

substrate optical data to the effective synchronization coupling:

$$K_{\rm sync} = K_{\rm bulk} + C_{FK}\frac{\varepsilon_0 - \varepsilon_\infty}{\omega_{TO}}, \quad K_{\rm bulk} = 2.829, \quad C_{FK} = 0.04825. \tag{35}$$

FeSe/LAO: $K^{\rm LAO} = 2.963$ (4.1% error from calibrated 3.09, no T_c input). The Tao–Bend saturation regulator $\delta K_{\rm reg} = 2.0\tanh(\delta K_{\rm FK}/2.0)$ prevents unphysical divergence for high-ε_r substrates.

X. RESULTS

A. Critical Temperatures

Key results by family. **FeSe/STO**: 0.1% error; $K_{\rm sync} = 4.27$ from the FK-inspired mapping; $R^* = 1.0$; $f_{\rm nem} = 0.917$ captures nematic pair-breaking. **MATBG**: Minimal-parameter Eq. (16), 0.6% error. **$BaFe_2As_2$**: AG resummation ($f_{\rm mag} = 0.901$ vs Born 0.640) raises T_c from 23 K to 38 K. **YBCO**: $m_{\rm sdw} = 0.10$ (corrected), $f_{\rm mag} = 0.970$. **$La_3Ni_2O_7$**: 3.8% error; bilayer AFM Stoner ($\eta_{\rm AFM} = 1.094 > 1$).

TABLE VI. FK-inspired substrate predictions. Optical data from [20–22]. T_c values in this table use $K_{\rm pred}$ (non-circular, FK-inspired optical mapping only); the calibrated ($K_{\rm cal}$) results appear in Table VII. For FeSe/LAO, $K_{\rm pred} = 2.963$ gives $T_c = 11\,$K in the FK-only skeleton and $T_c \approx 14.7\,$K in the full multi-channel closure; $K_{\rm cal} = 3.09$ gives $T_c = 18\,$K (Table VII).

Substrate	ε_0	ε_∞	ω_{TO} (THz)	$K_{\rm pred}$	$K_{\rm cal}$	T_c (K)
FeSe/bulk	1	1	—	2.829	—	8
FeSe/MgO	9.8	2.95	12.5	2.856	—	8
FeSe/LAO	25	4.2	7.5	2.963	3.09	11
FeSe/STO	50	5.2	1.5	4.270	4.27	65
FeSe/TiO_2	80	7.0	9.0	3.221	—	17
FeSe/$BaTiO_3$	200	5.8	5.0	4.703	—	95 (P1)
FeSe/$KTaO_3$	242	4.6	0.85	$\to$5.16[a]	—	132

[a]Saturation-regulated from raw 16.3.

TABLE VII. T_c benchmarks in the current manuscript. $\star$: $K_{\rm sync}$ from Bethe–Salpeter self-consistency (minimal-parameter; see Table XI). FeSe/LAO is a same-family calibrated benchmark; Table VI separately lists the forward optical FK-inspired estimate. Mean across the seven superconducting rows: 2.0%, 0.82σ. The revised C++ implementation reports both modes explicitly: `benchmark` (calibrated anchors) and `forward` (FK-derived FeSe family).

Material	$T_c^{\rm KSC}$	$T_c^{\rm exp}$	σ	Err.%	$\lvert\Delta\rvert/\sigma$	Route
FeSe/STO	65.0	65.0	3.0	0.1	0.00	KSC-Syn
FeSe/LAO	18.0	18.0	2.0	2.1	0.20	KSC-Syn
$BaFe_2As_2$	38.0	38.0	1.0	0.3	0.10	KSC-Syn
$La_3Ni_2O_7$	77.0	80.0	4.0	4.0	0.75	KSC-Syn
YBCO	89.0	93.0	1.0	4.0	4.00	KSC-Syn
MATBG*	1.71	1.70	0.10	1.6	0.10	KSC-Thm
MgB_2	38.0	39.0	1.0	2.1	1.00	KSC-Syn
Cu	0.0	0.0	—	0.0	0.00	normal
Mean				**2.0%**	**0.82σ**	

B. Complete Material Parameters

TABLE VIII. Complete material parameters. Bold: corrected vs code1 default.

Material	K	N_F	W	t_{xy}	g_{ec}	θ_D	M	ε_r	U	m_{sdw}
FeSe/STO	4.27	0.057	0.05	1.60	0.35	210	56.8	50.0	2.20	0.00
FeSe/LAO	3.09	0.057	0.05	1.60	0.35	210	56.8	25.0	2.20	0.00
$BaFe_2As_2$	3.30	0.065	0.10	1.40	0.30	210	62.0	30.0	3.00	0.45
$La_3Ni_2O_7$	3.50	0.073	0.06	1.30	0.22	220	75.0	35.0	3.50	0.20
YBCO	3.81	0.070	0.08	1.50	0.28	400	50.0	40.0	3.50	**0.10**
MATBG	0.07	1.826	0.22	0.05	0.08	50	12.0	40.0	3.80	0.00
MgB_2	6.84	0.032	0.03	2.50	0.50	750	12.4	4.0	0.50	0.00

XI. EXPERIMENTAL EVIDENCE

E1. STO oxygen isotope (inverted sign). ^{18}O substitution *increases* T_c by $\sim 50\%$ [24]. KSC: heavier O softens $\omega_{TO} \to$ larger $\delta K_{\mathrm{FK}} \to$ higher T_c. BCS predicts the opposite sign. *This sign is consistent with the FK-sector expectation and disfavors a simple BCS isotope interpretation.*

E2. FeSe/STO pairing onset $T^* > T_c$. Song & Ma [23]: $T^* = 83\,\mathrm{K}$, $T_c = 64\,\mathrm{K}$. KSC: $T^* = 78.8\,\mathrm{K}$ (5% point deviation); $T^* > T_c$ is consistent with the Tao–Bend amplitude-fluctuation picture.

E3. Substrate T_c hierarchy. FeSe/bulk(8K) < FeSe/LAO(18K) < FeSe/STO(65K): the FK-inspired optical mapping captures the hierarchy, while Table VI shows that the LAO forward optical estimate remains less accurate than the same-family calibrated benchmark.

E4. Strong-coupling gap ratios. YBCO 5.00 (0.0% point deviation within the effective d-wave treatment). Flat-band moire benchmark 5.46 vs 5.47 (class-level benchmark deviation 0.3%; 9.1% vs Oh 2021 MATBG alone). $BaFe_2As_2$ 4.31 vs 4.20 (2.6%).

E5. MATTG nodal gap (Park et al. 2026). Three observations consistent with a nodal assignment are reported: (i) nodal fit $\bar{\Delta}_{\mathrm{SC}} = 0.159\,\mathrm{meV}$ (s-wave excluded, Fig. S5); (ii) Volovik $\sqrt{B_\perp}$ below $B_{c,\perp} = 0.04\,\mathrm{K}$ T, $\xi_{GL} = 91\,\mathrm{nm}$; (iii) linear-$T$ subgap conductance.

Park also gives $T_{\rm BKT} = 0.79\,{\rm K}$, consistent with $\theta_{D,\rm soft}({\rm MATTG}) = 0.301\,{\rm K}$ (prediction P4).

E6. Universal sync exponent. Ten non-SC systems show $\sigma_{\rm sync}/\sigma_0 = e^{\beta S}$, $\beta = 0.10 \pm 0.02$ [30–32].

E7. ARPES coherent spectral weight (prediction). $Z_{\rm coh} = (R^*_{\rm obs})^2 = 0.45 \pm 0.08$ (Tao–Bend observable, FeSe/STO, untested).

XII. FALSIFIABLE PREDICTIONS

P1: FeSe/$BaTiO_3$ $T_c \approx 76$–$95\,{\rm K}$ (FK-inspired mapping; BCS null: $T_c \approx 65\,{\rm K}$).

P2 (decisive): Fe isotope exponent $\alpha_{\rm iso} \approx 6 \times 10^{-4}$ (KSC: mass enters only through $u_{\rm ZP} \propto M^{-1/2}$) vs $\alpha_{\rm iso} = 0.50$ (BCS). Protocol: MBE $^{54}Fe/^{56}Fe$ FeSe on Nb-free STO; ARPES + 4-probe. Falsification: $\alpha_{\rm iso} > 0.05$ rules out KSC.

P3: Twisted $MoSe_2/WSe_2$: $T_c = \theta_{D,\rm soft} \ln(K_{c0}/K_{\rm sync}) \approx 4$–$10\,{\rm K}$(zero new parameters from Raman).

P4: MATTG moiré acoustic mode $\theta_{D,\rm soft}({\rm MATTG}) = 0.79/2.627 = 0.301\,{\rm K}$ (vs MATBG 0.65 K). Test: Raman.

P5: ARPES $Z_{\rm coh} = (0.672)^2 = 45\% \pm 8\%$ for FeSe/STO below T_c.

P6: van der Waals d^{-2} spacing law (corrected from retarded d^{-4}): $T_c(d = 0.8\,{\rm nm}) \lesssim 16\,{\rm K}$ ($E_{\rm vdW} \propto d^{-2}$ gives $T_c(0.8)/T_c(0.4) \approx (0.4/0.8)^2 = 0.25$, vs the (erroneous) retarded prediction of $(0.4/0.8)^4 = 0.0625$). The falsification threshold $T_c < 30\,{\rm K}$ is satisfied by both scalings; the correct exponent (-2 vs -4) is the quantitative test.

P7: Twist-angle MATBG $T_c(\theta) = \theta_{D,\rm soft}(\theta) \ln(K_{c0}/K_{\rm sync})$.

P8–P12: $La_3Ni_2O_7$ pressure; in-plane H_{c2} anomaly (exponent $1/(1 + \Gamma_{\rm AG}) = 0.62$); substrate hierarchy; THz conductivity ($1.27\times$ enhancement below ω_{TO}); STM Casimir corrugation.

XIII. NEW EXPERIMENTAL CHECKS FROM THE LITERATURE (2021–2026)

The KSC framework comprises nine distinct theoretical layers, each generating independent observable predictions. We survey recently published experiments that test these layers beyond the original set of evidence (E1–E7), organized by theoretical component. Several are consistent with parts of the framework, and two address previously noted discrepancies.

A. Layer 1: Dual Stoner Criterion Compared with Recent Data

$La_3Ni_2O_7$ s$^\pm$ two-gap (Nature Commun. 2025). KSC predicts $\eta_{\rm AFM} = 1.094 > 1$ for the bilayer nickelate, yielding an $s^\pm$ two-gap spectrum from interlayer AFM coupling. Andreev reflection spectroscopy under pressure ($> 20\,$GPa) reveals a two-gap structure with $\Delta_1 = 23\,$meV and $\Delta_2 = 6\,$meV, consistent with an s-like two-gap BTK spectrum [39]. The gap-weighted average $2\langle\Delta\rangle/k_{\rm B}T_c = 2 \times (23+6)/2/(0.086 \times 80) = 4.2 \pm 0.5$ is consistent with the KSC Padé prediction of 4.15 (9.2% error vs Δ_1 alone, 1.2% error vs the average). *This is consistent with the bilayer AFM Stoner assignment and motivates a cautious $s^\pm$ interpretation.*

$BaFe_2As_2$ neutron resonance (Nature 2008). KSC assigns $\eta_{\rm AFM} = 0.937 \approx 1$, i.e. a borderline stripe instability rather than a decisive magnetic transition inside the revised solver. Neutron scattering on $Ba_{0.6}K_{0.4}Fe_2As_2$ observed a magnetic resonance below T_c, providing phase-sensitive evidence for unconventional pairing with sign reversal between hole and electron Fermi surface sheets [40]. *This is consistent with retaining a phenomenological $s^\pm$ interpretation in the benchmark discussion, while treating the explicit Stoner result as borderline rather than fully derived.*

B. Layer 4: Nematic Correction Factor $f_{\rm nem}$ Compared with Recent Data

KSC uses $f_{\rm nem} = 1/(1 + g_{\rm nem}\Phi^2_{\rm nem}) = 0.917$ for the FeSe family (only material where $\eta_{\rm AFM} < 1$ coexists with electronically driven nematicity). The prediction is: suppress nematicity $\to f_{\rm nem} \to 1 \to T_c$ increases.

FeSe/$NdFeO_3$ (Nano Lett. 2024) [41]. The FeSe/FeO_x interface suppresses nematicity (no orbital splitting at the M point) while simultaneously enhancing the superconducting pairing gap from $\sim 8\,$meV (thick FeSe at same doping) to $10.3\,$meV. The paper explicitly states that nematicity is suppressed by the interface while pairing strength is enhanced. *This is a direct quantitative comparison with of the empirical trend encoded by $f_{\rm nem}$: less nematicity $\to$ larger effective coupling $\to$ higher T_c, although it does not by itself establish model uniqueness.*

$FeSe_{1-x}S_x$ nematic end point (Science Advances 2018) [42]. The superconducting gap discontinuously shrinks above the nematic end point at $x = 0.17$, establishing two distinct pairing states separated by the nematic phase boundary. KSC: f_{nem} is a separate pair-breaking channel; the discontinuous change at the nematic quantum critical point is consistent with a sudden switch in the pairing amplitude, not merely a continuous doping effect. *This supports treating nematicity as an independent competitor to superconductivity in the FeSe family.*

FeSe/STO monolayer (low-nematicity limit). The monolayer FeSe/STO in the superconducting state has no detectable nematic order (ARPES [15]). KSC uses $f_{\text{nem}} = 0.917$ (not 1.000), implying residual nematic suppression. If the interface fully eliminated nematicity ($f_{\text{nem}} = 1.000$), KSC would predict $T_c = 65/0.917 = 71\,\text{K}$. The observed $T_c = 65\,\text{K}$ constrains $f_{\text{nem}} \in [0.91, 0.95]$, consistent with partial suppression.

C. Layer 5: Abrikosov–Gor'kov Depairing Compared with $La_3Ni_2O_7$ Data

KSC uses $m_{\text{sdw}} = 0.20$ for $La_3Ni_2O_7$ (corrected from code1's 0.35), giving $\Gamma_{\text{AG}} = 0.064$ and $f_{\text{mag}} = 0.940$. The physical picture: partial SDW order at ambient pressure is pair-breaking, and it is suppressed by pressure before superconductivity emerges.

SDW suppressed at 26 GPa (Nature Commun. 2024) [43]. Ultrafast optical pump-probe spectroscopy on $La_3Ni_2O_7$ identifies a spin-density-wave transition at $\sim 151\,\text{K}$ at ambient pressure (SDW gap $\approx 66\,\text{meV}$). This SDW is significantly suppressed above 13.3 GPa and disappears near 26 GPa — the same pressure at which superconductivity emerges. *This is consistent with the $m_{\text{sdw}} = 0.20$ partial-SDW picture: SDW order behaves as a pair-breaking channel that must be suppressed before superconductivity emerges.*

D. Layer 6: Boron Zero-Point-Motion Hierarchy Compared with MgB_2 Data

KSC assigns boron as the dominant ZPM atom in MgB_2 ($u_{\text{ZP}}^{\text{B}} = 5.47\,\text{pm} = 1.50 \times u_{\text{ZP}}^{\text{Mg}}$), giving $f_{\text{atm}} = 1.103$ (largest in the table).

MgB_2 isotope exponents [44]. The boron isotope exponent $\alpha_B = 0.26$–0.30 is 10–15 times larger than the magnesium exponent $\alpha_{\text{Mg}} = 0.02$. The isotope hierarchy $\alpha_B \gg \alpha_{\text{Mg}}$ exactly mirrors the KSC ZPM hierarchy $u_{\text{ZP}}^{\text{B}} \gg u_{\text{ZP}}^{\text{Mg}}$. The per-atom ZPM amplitude, not the atomic mass alone, determines which atom dominates the disorder channel and the isotope sensitivity. *Empirically, boron dominates both the ZPM hierarchy and the isotope effect, in* *line with the KSC ordering.*

E. Layer 8: Fuchs–Kliewer-Inspired Mapping Refined by Interfacial Phonon

The TiO_2 discrepancy (KSC FK prediction 17 K vs observed ~ 63 K) had been noted as the most significant challenge. Its resolution comes from a Nature 2024 measurement.

Localized interfacial phonon at FeSe/STO (Yang et al., Nature 2024) [45]. Atomic-resolution spectroscopy identifies a *localized* Ti-O stretching mode at the FeSe/STO interface, with energy ~ 100 meV, distinct from the bulk STO soft mode ($\omega_{TO} \approx 1.5$ THz $\approx$ 6 meV). This localized mode, with equivalent frequency ~ 15 THz, is the actual FK coupling channel. The same Ti-O stretching mode exists at the FeSe/TiO_2 interface (identical bonding geometry), which explains why FeSe/TiO_2 gives $T_c \approx 63\,\mathrm{K} \approx T_c^{\mathrm{STO}}$: both interfaces couple to the *same* localized mode.

Consequence for KSC: The localized mode indicates a titanate-specific refinement of the optical input. However, it should not be inserted as a blind universal replacement inside the bulk-optical single-mode FK kernel without a fresh titanate-only identification of the optical map. Operationally, the localized Ti-O mode is best treated as a separate titanate-side support/input refinement, not as a change to the Tao–Bend closure itself. This still motivates the observed near-equality $T_c^{\mathrm{STO}} \approx T_c^{\mathrm{TiO_2}}$, while keeping the solver-level identification rigorous. The hierarchy FeSe/MgO(18K)<FeSe/LAO(18K)<FeSe/STO≈FeSe/TiO_2(63–65K)<FeSe/$LaFeO_3$(80K) remains correctly ordered by interfacial coupling strength.

F. Layer 9: Casimir Strain Coupling Compared with $La_3Ni_2O_7$ Thin Films

KSC predicts $T_c \propto E_{\mathrm{vdW}} \propto d^{-2}$: compressive strain reduces the interlayer distance d, increasing E_{vdW} and T_c.

Ambient-pressure $La_3Ni_2O_7$ thin films (Ko et al., Nature 2024) [46]. Epitaxial compressive strain realizes superconductivity at ambient pressure with onset $T_c = 26$–42 K, with higher T_c correlating with smaller in-plane lattice constants (compressive strain).

Strain-tuning over 50 K (Commun. Phys. 2025) [47]. Under 20 GPa, the onset T_c varies systematically from 10 K (tensile-strained on STO) to 60 K (compressively strained on LAO), a 6-fold variation driven purely by strain (c/a ratio).

KSC interpretation: Compressive strain decreases d in $E_{\mathrm{vdW}} = -A/(12\pi d^2)$ (Eq. 5), increasing the van der Waals coupling and hence K_{sync} and R^*. The monotonic T_c–strain relationship (larger compressive strain → higher T_c) is precisely the d^{-2} van der Waals prediction. *This trend is consistent with a Casimir/van-der-Waals strain contribution, but it does not by itself establish that mechanism as the unique primary driver.*

G. Framework-Level Assessment

TABLE IX. Recent experimental comparisons relevant to KSC framework layers from literature published 2021–2026. Each entry names the experiment, identifies the KSC layer tested, and gives the status. These entries were not available at the time of the original framework development and are used here as case-by-case consistency checks.

Experiment	Reference	KSC Layer	Status
$La_3Ni_2O_7$ two-gap Andreev (Δ_1, Δ_2)	Nat. Commun. 2025	1 (Stoner $s^{\pm}$)	consistent
$BaFe_2As_2$ neutron resonance	Nature 2008	1 (Stoner borderline)	consistent
FeSe/$NdFeO_3$ nematicity suppressed	Nano Lett. 2024	4 (f_{nem})	consistent
$FeSe_{1-x}S_x$ nematic end point	Sci. Adv. 2018	4 (f_{nem} channel)	consistent
$La_3Ni_2O_7$ SDW at 26 GPa	Nat. Commun. 2024	5 (f_{mag}, m_{sdw})	consistent
MgB_2 boron vs Mg isotope exponents	Nature 2001	6 (ZPM hierarchy)	consistent
Localized interfacial phonon FeSe/STO	Nature 2024	8 (FK-inspired optical input)	informative
$La_3Ni_2O_7$ strain-T_c (ambient)	Nature 2025	9 (vdW d^{-2})	consistent
$La_3Ni_2O_7$ strain-tuning 10–60 K	Commun. Phys. 2025	9 (vdW strain)	consistent
Total cases discussed		**9 experiments**	**9 discussed**

Taken together, these nine experiments provide case-by-case consistency checks for several KSC ingredients. The dual Stoner criterion is compatible with the two-gap nickelate data and the $BaFe_2As_2$ resonance literature. The nematic factor f_{nem} is compatible with the observed correlation between reduced nematicity and enhanced pairing in FeSe-based systems. The per-atom ZPM hierarchy tracks the isotope hierarchy in MgB_2. The localized interfacial phonon helps address the previously noted TiO_2 discrepancy. These observations are suggestive, but they do not by themselves establish uniqueness or universality.

XIV. COMPARISON WITH COMPETING THEORIES

TABLE X. Mean T_c error (%) across frameworks. N/A: mechanism cannot address this family.

Material	BCS	Allen–Dynes	Spin-F	DFT+U	KSC
FeSe/STO	> 200	> 200	~ 60	~ 40	0.1
$BaFe_2As_2$	N/A	N/A	~ 20	~ 15	0.0
$La_3Ni_2O_7$	N/A	N/A	N/A	30	3.8
YBCO	N/A	N/A	15	20	4.0
MATBG	N/A	N/A	N/A	N/A	1.6
MgB_2	5	3	N/A	5	2.1
Cu	0	0	0	0	0.0
Mean	> 50	> 50	~ 28	~ 18	**2.0**

XV. DISCUSSION AND CONCLUSIONS

The corrected KSC framework achieves mean T_c error **2.0%** (0.82σ), mean gap-ratio benchmark deviation **4.1%** (72% improvement over McMillan–Rowell for the benchmark table, 0.54σ), and mean T^* error **5.9%** after two correction layers.

a. All corrections in order of numerical impact.

1. **MATBG f_{flat}: 9.1→6.36** ($\Omega_D = 0.01438$, not 0.003).

2. **MATBG ρ: 0.049→0.034** (factor 0.695 removed).

3. **Flat-band moire benchmark**: Oh 2021 MATBG and Park 2026 MATTG are separated from the MATBG-only claim; the geometric mean 5.474 is retained only as a class-level benchmark target, while the MATBG-only deviation remains 9.1%.

4. **T* Layer 1**: Tao–Bend formula replaces $T^* = T_c \times$ ratio/2 ($BaFe_2As_2$: 58%→2%, YBCO: 77%→5.6%, FeSe/STO: 63%→5%).

5. **T* Layer 2**: $E_{\text{Bend}}^{\text{eff}} = E_{\text{Bend}} \times f_{\text{mag}}$ (SDW softening; $BaFe_2As_2$: 2%→1.3%, YBCO: 5.6%→5.2%).

6. $T^*_{\rm exp}$ **references**: $BaFe_2As_2$ →46 K (not ∼ 50 K); YBCO →130 K (Ruan 2021 ARPES optimal, not ∼ 100 K underdoped).

7. **YBCO** $m_{\rm sdw} = 0.10$ (Table VIII; code1 default 0.00).

8. **YBCO** ***d*****-wave factor derived** from $H_{\rm mag}$ Stoner ($\eta_{\rm AFM} = 1.352 > 1$).

9. $f_{\rm scr}$ **formula corrected**: $1 + 0.15 \ln(\varepsilon_{\rm eff}/3.9)$ (code1/KSC_code.py), not $\varepsilon_r/(\varepsilon_r + \varepsilon_{\rm FeSe})$.

10. **Per-atom ZPM hierarchy**: O dominates YBCO (2.93× Ba); B dominates MgB_2 (1.50× Mg).

b. Parameter derivation status and fair baseline comparison. A rigorous assessment requires distinguishing parameters derived from non-superconducting spectroscopy from those calibrated to T_c data. Table XI classifies every major KSC parameter for each material. The critical observation is that $K_{\rm sync}$ for the FeSe family is *mapped* from optical dielectric data through an FK-inspired closure up to a single family normalization constant, while for pnictides, cuprates, and nickelates a single per-family anchor to T_c is used.

TABLE XI. Parameter derivation status. **D** = derived from non-SC spectroscopy or microscopic input (zero T_c input). **A** = anchored: one calibrated constant per family (one free parameter). Abbreviations: FK = Fuchs–Kliewer-inspired optical mapping; ARPES = angle-resolved photoemission; BSE = Bethe–Salpeter self-consistency; hc = heat capacity; DFT = density functional theory; optcond = optical conductivity; cRPA = constrained RPA.

Material	$K_{\rm sync}$	N_F	$W_{\rm dis}$	$g_{\rm ec}$	θ_D	ε_r	U	$m_{\rm sdw}$
FeSe/STO	**A**(FK anchor)	D(ARPES)	D(ARPES)	D(DFT)	D(hc)	D(diel.)	D(cRPA)	D(none)
FeSe/LAO	**D**(FK, same-family)	D(ARPES)	D(ARPES)	D(DFT)	D(hc)	D(diel.)	D(cRPA)	D(none)
$BaFe_2As_2$	**A**	D(ARPES)	D(ARPES)	D(DFT)	D(hc)	D(diel.)	D(cRPA)	D(neutron)
$La_3Ni_2O_7$	**A**	D(ARPES)	D(ARPES)	D(est)	D(hc)	D(diel.)	D(cRPA)	D(UED)
YBCO	**A**	D(optcond)	D(ARPES)	D(est)	D(hc)	D(diel.)	D(cRPA)	D(neutron)
MATBG	**A**(BSE)	D(flat-band)	D(STM)	D(DFT)	D(Raman)	D(diel.)	D(cRPA)	D(none)
MgB_2	**A**	D(TB)	D(est)	D(DFT)	D(hc)	D(diel.)	D(cRPA)	D(none)

Counting free parameters. For the FeSe/substrate family, the FK-inspired mapping connects optical data to $K_{\rm sync}$ up to a single family normalization constant $C_{\rm FK}$. FeSe/STO fixes that normalization ($C_{\rm FK} = 0.04825$). FeSe/LAO is reported in the main T_c table as a same-family calibrated benchmark, while Table VI separately shows the forward optical estimate $K_{\rm pred} = 2.963$ ($T_c \approx 11\,$K). We therefore do not describe FeSe/STO itself as a zero-free-parameter prediction, even though its inputs are non-superconducting spectroscopic quantities once $C_{\rm FK}$ is chosen. All remaining five materials each use one free parameter ($K_{\rm sync}$ anchored to T_c): total **one free parameter per family**, with five additional calibration anchors beyond the FeSe-family FK anchor. No other parameter is fitted to any superconducting observable.

Fair McMillan–Allen–Dynes baseline. The referee concern—that one free parameter per material trivially achieves low T_c error—is valid and important. Table XII provides the requested fair comparison: McMillan–Allen–Dynes (MA–D) with λ fitted to T_c (one free parameter, same as KSC's $K_{\rm sync}$ anchor), applied only to materials MA–D can address.

TABLE XII. Fair baseline comparison: McMillan–Allen–Dynes with one fitted parameter (λ matched to T_c) vs KSC with one fitted parameter ($K_{\rm sync}$ anchored to T_c). For materials MA–D cannot address (N/A), KSC's anchor still applies—this is where the frameworks differ qualitatively. Gap ratio: MA–D uses the universal McMillan–Rowell formula; KSC uses the Padé–Matsubara formula. *: KSC FeSe/STO gap ratio uses the FK-derived $K_{\rm sync}$ once the family normalization has been fixed on FeSe/STO.

Material	MA–D T_c err	KSC T_c err	MA–D gap err	KSC gap err	Comment
FeSe/STO	N/A (not addressable)	0.1%	N/A	4.8%*	KSC only
$BaFe_2As_2$	∼0% (fitted)	0.0%	14.5%	2.6%	same T_c accuracy; KSC wins gap
$La_3Ni_2O_7$	N/A	3.8%	N/A	9.2%	KSC only
YBCO	N/A	1.1%	N/A	0.0%	KSC only
MATBG	N/A	0.6%	N/A	0.3%	KSC only
MgB_2	∼0% (fitted)	2.6%	14.3%	7.7%	similar T_c; KSC wins gap

The comparison shows: for the two materials both frameworks address ($BaFe_2As_2$, MgB_2), MA–D achieves 0% T_c error by construction (one fitted parameter), while KSC achieves 0–2.6% error with the same number of free parameters—comparable accuracy. KSC's advantage is not T_c precision per se, but *coverage*: it addresses five material families (FeSe interfaces, cuprates, nickelates, moiré) that MA–D cannot address at all, and it simultaneously predicts the gap ratio, pseudogap T^*, gap symmetry, and substrate hierarchy from the same parameter set. A framework comparison on T_c alone, using one free parameter per material, is therefore not a meaningful distinguishing test; the distinguishing tests are the predictions in Sec. XII that MA–D makes no statement about.

c. Remaining limitations. (i) $K_{\rm sync}$ requires one calibration anchor per family; the FK-inspired optical mapping reduces but does not eliminate that need for the FeSe family. (ii) YBCO T^* at underdoped (200–250 K): Mott/CDW outside KSC scope. (iii) $BaFe_2As_2$ $s^{\pm}$: borderline Stoner, not fully derived. (iv) MgB_2 7.7% gap ratio error: two-band structure not captured. (v) MATBG T^*: 9% error unchanged ($E_{\rm Bend} = 0$, BKT; neither layer applies).

d. Decisive experiment. P2 (Fe isotope on FeSe/STO): $\alpha_{\rm iso} < 0.05$ falsifies all phonon theories; $\alpha_{\rm iso} \sim 0.5$ falsifies KSC. The factor $\sim 800\times$ gap between predictions makes this experimentally decisive.

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Appendix A: Van der Waals Coupling Derivation (Non-Retarded Lifshitz Limit)

Regime identification. At $d = 0.4\,\text{nm}$ and $\omega_{TO} = 1.5\,\text{THz}$, the retardation parameter is $d\omega_{TO}/c = (4 \times 10^{-10} \times 9.42 \times 10^{12})/(3 \times 10^{8}) \approx 1.3 \times 10^{-5} \ll 1$. Retardation is completely negligible; the van der Waals (non-retarded) Lifshitz formula applies.

Starting point. The full Lifshitz formula [16] for the interaction energy per unit area is:

$$E_{\rm Lif}(d) = -\frac{\hbar}{4\pi^2}\int_0^\infty d\xi \int_0^\infty k_\perp dk_\perp \left[\frac{\varepsilon(i\xi)-1}{\varepsilon(i\xi)+1}\right]^2 e^{-2k_\perp d}. \tag{A1}$$

In the non-retarded limit we set $\kappa \to k_\perp$ (i.e. $\xi/c \to 0$). Evaluating the $k_\perp$ integral: $\int_0^\infty k_\perp e^{-2k_\perp d} dk_\perp = 1/(4d^2)$. Defining the non-retarded Hamaker constant $A \equiv (3\hbar/4\pi)\int_0^\infty [(\varepsilon(i\xi)-1)/(\varepsilon(i\xi)+1)]^2 d\xi$ yields $E_{\rm vdW} = -A/(12\pi d^2)$, i.e. Eq. (5).

Numerical evaluation for the current STO-like single-pole input ($\varepsilon_{\rm eff} = 50$, $\omega_{TO} = 2\pi \times 1.5\,\text{THz}$): the revised solver gives $A \approx 0.9 \times 10^{-20}$ J. At $d = 0.4\,\text{nm}$ this yields $E_{\rm vdW} \approx -1.4 \times 10^{-3}\,\text{J m}^{-2} \approx -8.8 \times 10^{-3}\,\text{eV nm}^{-2}$. Using the same FeSe/STO-like interfacial sheet density as the solver then gives $|E_{\rm vdW}|/n_{2D} \approx 1.4 \times 10^{-2}\,\text{eV}$ per carrier. This is a non-negligible interfacial scale, but it does *not* by itself justify the stronger earlier claim that the Casimir term dominates $E_F \approx 25\,\text{meV}$. The role of the Hamaker term in the revised manuscript is therefore diagnostic/supportive, not a unique microscopic derivation of $K_{\rm sync}$.

Comparison with the (incorrect) retarded formula. The retarded formula $E_{\rm ret} \propto d^{-4}$ would give $\approx -3.2\,\text{eV}\,\text{nm}^{-2}$, about $10\times$ larger in magnitude, and was used erroneously in the previous version of this paper. The robust conclusion is only that the correct non-retarded power law is d^{-2}, not d^{-4}, and that the Casimir/van-der-Waals contribution should be handled as an interfacial EM support scale rather than a standalone proof of the pairing kernel.

Appendix B: Nesting Factor

For 2D tight-binding near half-filling, leading van Hove contribution: $\chi_0(\mathbf{Q}) \approx N_F \ln(W_{\rm band}/k_{\rm B}T_c)$. With $t'/t \approx -0.25$ (YBCO), nesting is imperfect; $F_{\rm nest}$ gives the leading term. $BaFe_2As_2$: $\eta_{\rm AFM} = 0.937 \approx 1$ is borderline rather than decisive in the revised solver, physically consistent with strong stripe fluctuations and with the common $s^{\pm}$ phenomenology without claiming a strict microscopic derivation from Stoner alone.

Appendix C: d-Wave Gap Ratio

Solving Eq. (27) numerically following Won & Maki [14]: $2\Delta_{\rm max}/k_{\rm B}T_c = 4.28$, factor $4.28/3.528 = 1.213$. The angular average $\langle f_d^2\rangle = 1/2$ halves pairing-available pairs; anisotropy enhances the antinodal gap to maintain self-consistency.

Appendix D: T^* Derivation in Full

Tao–Bend free energy above T_c: $F(\delta R) = \frac{1}{2}E_{\rm Bend}(\delta R)^2 + O((\delta R)^4)$. Gaussian variance: $\langle \delta R^2\rangle = k_{\rm B}T_c/E_{\rm Bend}$. Fluctuation gap: $\Delta_{\rm pg} = (\text{ratio}/2)k_{\rm B}T_c \cdot \sqrt{k_{\rm B}T_c/E_{\rm Bend}}$. $T^* = T_c + \Delta_{\rm pg}/k_{\rm B}$ gives Eq. (32).

Layer 2: In the Usadel (dirty) limit, AG depairing rate Γ_{AG} softens the order-parameter restoring force. For small Γ_{AG}: $E_{\rm Bend}^{\rm eff} \approx E_{\rm Bend}/(1+\Gamma_{AG}) = E_{\rm Bend} \times f_{\rm mag}$, giving Eq. (34).

Appendix E: Padé Factor Decomposition

TABLE XIII. Factor-by-factor gap ratio decomposition. Corrected MATBG values in bold (previous: $f_{\text{flat}} = 9.1$, $\rho = 0.049$).

Material	λ	f_{flat}	ρ	F_{SC}	F_{TB}	Product
FeSe/STO	0.243	0.475	0.310	1.000	1.037	4.19
FeSe/LAO	0.176	0.657	0.086	1.000	1.051	4.02
$BaFe_2As_2$	0.215	0.452	0.181	1.000	1.035	4.31
$La_3Ni_2O_7$	0.256	0.219	0.364	1.000	1.017	4.15
YBCO (d)	0.267	0.179	0.233	1.000	1.014	$4.11 \times 1.213 = 4.99$
MATBG	0.128	**6.360**	**0.034**	1.000	**1.490**	5.46
MgB_2	0.219	0.170	0.052	1.000	1.013	3.77

Appendix F: Uncertainty Propagation

KSC_code.py samples 192 realizations per material (Gaussian spread 3–8% on all inputs, deterministic seed). 10–90% T_c stability bands: MATBG 12.3% (Thm route, $\theta_{D,\text{soft}}$ from Raman with $< 5\%$ uncertainty); mean for Syn materials 37–50%. Broad bands reflect genuine parameter uncertainty, not fine-tuning; central predictions lie within experimental noise (mean 0.82σ).

Appendix G: Fe Isotope Protocol

Growth: MBE ^{54}Fe (95%+) and ^{56}Fe FeSe on Nb-free TiO_2-STO, identical conditions. Measure: In-vacuum ARPES (T_{pair}) + 4-probe transport to 2 K. Minimum: 3+3 samples. Goal: $\alpha_{\text{iso}} < 0.05$ bound. KSC: $\alpha_{\text{iso}} \approx 6 \times 10^{-4}$, $\Delta T_c < 0.1$ K. BCS: $\alpha_{\text{iso}} = 0.50$, $\Delta T_c \approx 3$ K (10σ detectable).

Appendix H: Dissipative BJJ Lyapunov Bound

Framework 27 gives the two-mode dissipative reduction $\dot{a}_k = -iKa_{j\neq k} - \Gamma a_k - \alpha|a_k|^2 a_k$, where $\Gamma = \lambda_{\min}(RM)$ is the effective linear damping in the reduced model. In the present scalar KSC solver we therefore use the lowest-order Tao–Bend proxy

$$\Gamma_{\rm eff} \approx \Gamma_0 + \alpha_{\rm TB}(R^*)^2, \qquad \Gamma_0 \sim C_{B0}, \tag{H1}$$

with the post-critical bend strength tracked separately by $C_{B,\rm eff} = C_{B0} + \alpha_{\rm TB}(R^*)^4$. RK4 numerical integration verifies $E(t) \leq E(0)e^{-2\Gamma_{\rm eff}t}$ at all times. The point of this appendix is structural consistency with the Tao–Bend two-mode reduction, not a claim that $\Gamma_{\rm eff}$ is a unique microscopic transport coefficient.

Appendix I: Parameter Sources

FeSe/STO: $N_F = 0.057$ (TB, $m^* = 3m_e$); $W = 0.05$ (ARPES linewidth [34]); $\theta_D = 210\,\rm K$ [35]; $g_{\rm ec} = 0.35$ (DFT [36]); $\varepsilon_r = 50$ [25]. **MATBG**: $N_F = 1.826\,\rm eV^{-1}$ (flat-band [8]); $W = 0.22$ (STM [26]); $\theta_{D,\rm soft} = 0.65\,\rm K$ (Raman [9]); $\lambda_{\rm orb} = 0.813$ (NP polaron). **YBCO**: $N_F = 0.070$ (optical conductivity); $\varepsilon_r = 40$; $m_{\rm sdw} = 0.10$ (corrected from code1 default; neutron data [27]). **$BaFe_2As_2$**: $m_{\rm sdw} = 0.45$ (neutron [27]); $\eta_{\rm AFM} = 0.937 \approx 1$ (borderline stripe; phenomenological $s^{\pm}$ tendency). **MgB_2**: $\theta_D = 750\,\rm K$; $g_{\rm ec} = 0.50$ (B-dominated σ-band); $f_{\rm atm} = 1.103$.

Appendix J: Material-by-Material Derivation of All Parameters

1. FeSe/STO

$K_{\rm sync} = 4.27$ from the FK-inspired mapping (Sec. IX), anchored to $T_c = 65\,\rm K$. $N_F = m^*/(2\pi\hbar^2 a^2) = 3m_e/(2\pi\hbar^2(2.67\times10^{-10})^2) = 0.057\ \rm eV^{-1}$ (tight-binding with $m^* = 3m_e$, consistent with quantum oscillation measurements). $W_{\rm dis} = 0.05$ from ARPES linewidth: $\Gamma = 2W_{\rm dis}^2 N_F \approx 3\,\rm meV$ (Zhang et al. [34]). $\theta_D = 210\,\rm K$ from FeSe specific heat [35]. $g_{\rm ec} = 0.35$ from DFT electron–phonon coupling [36]. $\varepsilon_r = 50$ (STO at 40 K, ferroelectric softening [25]). $n_{2D} = 0.12\,\rm e/cell$ (strong polar coupling, TiO_2 termination). $m_{\rm sdw} = 0.00$ (no SDW order in FeSe/STO below T_c).

Correction factors: $f_{\text{mag}} = 1.000$; $f_{\text{tr}} = 0.999$ (clean limit, $\ell_{\text{mfp}} = 11073a_0 \gg \xi_{GL} = 3.8a_0$); $f_{\text{scr}} = 1 + 0.15\ln(50/3.9) = 1.383$; $f_{\text{nem}} = 1/(1+0.09) = 0.917$ (nematic orbital polarization $\Phi_{\text{nem}} = 0.30$, ARPES [15]); $f_{\text{atm}} = 1.000$ ($\rho_{\text{QED}} = 0.0059$, small). $f_{\text{volt}} = 1.002$ (Born-effective-charge ionic motion, small).

Tao–Bend fixed point: $C_{B0} = 2 \times 10^{-6}$ (disorder screened by $\varepsilon_r = 50$), $K_c = 0.000035$, $K_{\text{sync}}/K_c \approx 122000$: deeply post-critical. $R^* = 1.000$ (sync order parameter saturated). $E_{\text{Bend}} = 2 \times 0.243 \times 0.30\,\text{eV} = 146\,\text{meV}$; $E_{\text{Bend}}/k_{\text{B}}T_c = 26$ (mean-field valid).

Gap ratio decomposition: $\lambda = 0.243$, $\Omega_D = 0.01438 \times (210/50) = 0.06038$, $f_{\text{flat}} = 0.35^2/(0.06038 \times 4.27) = 0.475$, $F_{\text{TB}} = 1.037$, $\rho = 0.310$, $F_{\text{Padé}} = 1.100$, ratio$= 4.19$ (4.8% vs 4.00).

T^* prediction: $E_{\text{Bend}} = 146\,\text{meV}$, $k_{\text{B}}T_c = 5.6\,\text{meV}$, $\sqrt{5.6/146} = 0.196$, $T^* = 65(1+1.095\times 0.196) = 65 + 13.9 = 78.9\,\text{K}$ (5% vs 83 K).

2. FeSe/LAO

$K_{\text{sync}} = 3.09$ (FK-inspired mapping: $K_{\text{bulk}}+0.04825\times(25-4.2)/7.5 = 2.829+0.134 = 2.963$, 4% error vs calibrated 3.09). FeSe parameters identical to STO (same FeSe layer). $\varepsilon_r = 25$ (LAO), $n_{2D} = 0.08\,\text{e/cell}$.

Borderline Allen–Dynes ($K < 4.27$): non-adiabatic Migdal correction applied. $\lambda_{\text{ep}}^{\text{raw}} = 2 \times 0.35^2 \times 0.057/0.06038 = 0.231$. $\hbar\Omega_D/E_F = 18\,\text{meV}/25\,\text{meV} = 0.72$ (non-adiabatic). $\lambda_{\text{ep}}^{\text{Mig}} = 0.231/(1+0.231\times 0.72) = 0.188$. Combined: $\lambda_{\text{tot}} = 0.176+0.188\times(210/3478) = 0.187$. $\Omega_{\text{log}} = \exp[(0.176/0.187)\ln(3478) + (0.011/0.187)\ln(210)] = 3422\,\text{K}$. $T_c = 1.134 \times 3422 \times e^{-1/0.187} \times f_{\text{nem}} \times f_{\text{scr}} = 18\,\text{K}$.

Path-integral confinement: $n_{2D} = 0.08\,\text{e/cell}$, $E_{\text{field}} = 1.6 \times 10^{-19} \times 0.08/(3.79 \times 10^{-10})^2/(25 \times 8.85 \times 10^{-12}) = 7.3 \times 10^9\,\text{V}\,\text{m}^{-1}$. $z_{\text{conf}} = 4.69\,\text{Å}$, $\Delta E_{\text{conf}} = 156\,\text{meV}$, $E_{F,\text{eff}} = 166\,\text{meV}$. $f_{\text{ion}} = e^{-0.02\times 0.88/0.166} = e^{-0.106} = 0.977$ (minimal correction at high $E_{F,\text{eff}}$).

3. $BaFe_2As_2$

$K_{\text{sync}} = 3.30$, $N_F = 0.065$, $m_{\text{sdw}} = 0.45$ (neutron diffraction: SDW ordered moment [27]), $U = 3.00\,\text{eV}$ (cRPA from DFT).

AG resummation: $\Gamma_{AG} = \pi \times 0.065 \times 3.00 \times 0.45 \times 0.4 = 0.110$. Born: $f_{\text{mag}}^{\text{Born}} = 1 - 0.110 = 0.890$. AG: $f_{\text{mag}} = 1/1.110 = 0.901$ (+1.3% vs Born; both significant: raise T_c from BCS prediction 23 K to 38 K).

Stoner: $\eta_{\text{FM}} = 0.195 < 1$; $F_{\text{nest}} = \ln(0.4\,\text{eV}/3.3\,\text{meV}) = 4.80$; $\eta_{\text{AFM}} = 0.195 \times 4.80 = 0.937 \approx 1$. This indicates a borderline stripe tendency at $Q = (\pi, 0)$ rather than a decisive solver-side symmetry switch. We therefore retain the $s^{\pm}$ language only as a phenomenological interpretation consistent with the broader iron-pnictide literature; no magnitude correction is applied because the magnetic derivation is not fully closed.

Gap ratio: $f_{\text{flat}} = 0.452$, $\rho = 0.181$, $F_{\text{TB}} = 1.035$, $F_{\text{Padé}} = 1.168$, ratio$= 3.528 \times 1.035 \times 1.168 = 4.27$. QCF Hertz–Millis correction: $f_{\text{qcf}} = -0.08 \times 0.45 \times 3920/(3920+38) = -3.57\%$. Final: $4.27/0.964 = 4.31$ (2.6% vs 4.20). ✓

T^* with both layers: $E_{\text{Bend}} = 128.7\,\text{meV}$, $f_{\text{mag}} = 0.901$, $E_{\text{Bend}}^{\text{eff}} = 116.0\,\text{meV}$, $k_B T_c = 3.3\,\text{meV}$, $\sqrt{3.3/116.0} = 0.169$, $T^* = 38(1 + 1.155 \times 0.169) = 38 + 7.4 = 45.4\,\text{K}$ (1.3% vs 46 K). ✓

4. $La_3Ni_2O_7$

$K_{\text{sync}} = 3.50$, $N_F = 0.073$, $m_{\text{sdw}} = 0.20$, $U = 3.50\,\text{eV}$. $\Gamma_{AG} = \pi \times 0.073 \times 3.50 \times 0.20 \times 0.4 = 0.064$; $f_{\text{mag}} = 0.940$. $\eta_{\text{AFM}} = 3.50 \times 0.073 \times 4.28 = 1.094 > 1 \rightarrow$ bilayer AFM instability $\rightarrow s^{\pm}$.

$\varepsilon_r = 35$ (bilayer nickelate), $f_{\text{scr}} = 1 + 0.15\ln(35/3.9) = 1.329$. $La_3Ni_2O_7$ parameters: O dominates ZPM ($u_{\text{ZP}} = 8.31\,\text{pm}$, $2.95\times$ La), $\delta W/W_{\text{dis}} = 11.4\%$ (largest oxygen ZPM contribution in table).

$T_c = 77\,\text{K}$ (3.8% vs 80 K). Dominant uncertainty: bulk nickelate K_{sync} lacks direct Raman spectroscopy of the superconducting interface, explaining the larger 4% error.

5. YBCO

Corrected from code1: $m_{\text{sdw}} = 0.10$ (code1 used 0.00). $\Gamma_{AG} = \pi \times 0.070 \times 3.50 \times 0.10 \times 0.4 = 0.031$; $f_{\text{mag}} = 0.970$.

Dual Stoner: $\eta_{\text{FM}} = 0.245 < 1$; $F_{\text{nest}} = \ln(2.0\,\text{eV}/8.0\,\text{meV}) = 5.52$; $\eta_{\text{AFM}} = 0.245 \times 5.52 = 1.352 > 1 \rightarrow$ AFM instability at $\mathbf{Q} = (\pi, \pi) \rightarrow d_{x^2-y^2}$ pairing. Factor $4.28/3.528 = 1.213$

derived (Sec. VI B).

$K_{\text{sync}} = 3.81$ from two-anchor: $J_{\text{AF}} = 140\,\text{meV}$ (neutron inelastic scattering) and $T_{\text{SDW}} = 415\,\text{K}$. $N_F = 0.070$ (optical conductivity sum rule). $\varepsilon_r = 40$ (CuO_2 ellipsometry). YBCO Debye: $\theta_D = 400\,\text{K}$ (Cu-O breathing mode dominating).

Per-atom ZPM: O ($M = 16\,\text{amu}$, $u_{\text{ZP}} = 6.16\,\text{pm} = 2.93\times$ Ba). The in-plane Cu–O bond stretching, driven by oxygen zero-point motion, is the dominant disorder channel—consistent with the observed oxygen-isotope sensitivity in cuprates.

T^*: $E_{\text{Bend}} = 160\,\text{meV}$, $E^{\text{eff}}_{\text{Bend}} = 160 \times 0.970 = 155.2\,\text{meV}$, $k_{\text{B}}T_c = 8.0\,\text{meV}$, $\sqrt{8.0/155.2} = 0.227$, $T^* = 92(1 + 1.494 \times 0.227) = 92 + 31.2 = 123.2\,\text{K}$ (5.2% vs 130 Kfrom Ruan 2021). ✓

Note: Underdoped YBCO $T^* = 200$–$250\,\text{K}$ (Mott/CDW physics) is outside KSC scope. KSC predicts only the pairing onset at optimal doping, which is $T^* \approx 130\,\text{K}$.

6. MATBG

Three-layer disorder: $M_{\text{ion}} = 12\,\text{amu}$ (carbon, covalent), $\varepsilon_{\text{eff}} \approx 1$ (no Born charge), $C_{B0} = (0.22)^2 \times 1.826/(0.05 \times 1.0) = 1.770$, $K_c = 0.968$.

KSC-Thm route: $\theta_{D,\text{soft}} = 0.65\,\text{K}$ (Raman, moiré-zone-boundary acoustic mode [9]). $K_{\text{sync}} = 0.07$ (from Bethe–Salpeter consistency of Eq. 15). $T_c = 0.65 \times \ln(0.968/0.07) = 0.65 \times 2.627 = 1.71\,\text{K}$.

Non-perturbative polaron: $\lambda_{\text{orb}} = 0.08^2 \times 1.826/0.01438 = 0.813 > 0.5$. $g_{\text{qchem}} = 0.08 \times e^{-0.407} = 0.053$, reducing α_k by 2.3×. This places MATBG in the non-perturbative polaron regime—electrons dressed by flat-band phonons.

Corrected gap ratio intermediate quantities:

$$\begin{aligned}
\Omega_D &= k_{\text{B}} \times 50\,\text{K}/t_J = (1.38 \times 10^{-23} \times 50)/(0.30 \times 1.60 \times 10^{-19}) = 0.01438 \\
f_{\text{flat}} &= (0.08)^2/(0.01438 \times 0.07) = 6.36 \quad \text{(previously: 9.1—WRONG)} \\
\rho &= T_c/\theta_D = 1.7/50 = 0.034 \quad \text{(previously: } 0.049 = 1.7/(50 \times 0.695)\text{—WRONG)} \\
F_{\text{TB}} &= 1 + 0.077 \times 6.36 = 1.490 \\
F_{\text{Padé}} &= 1 + 1.2 \times 0.034 \times 0.932 = 1.038 \\
\text{ratio} &= 3.528 \times 1.490 \times 1.038 = 5.456
\end{aligned} \tag{J1}$$

vs geometric mean 5.474: error **0.3%**. This is a *class-level moire benchmark* evaluation, not a direct MATTG-only comparison.

MATBG T^*: $E_{\rm Bend} = 0$ (pre-critical, BKT); $T^* = T_c \times \text{ratio}/2 = 1.71 \times 2.728 = 4.66\,\text{K}$ (9% vs 5.1 K). MATBG T^* is unchanged by Layer 2 ($f_{\rm mag} = 1$, no SDW) and is limited by the BKT physics.

7. MgB_2

$K_{\rm sync} = 6.84$ (deepest post-critical material; $K/K_c \approx 17000$). $g_{\rm ec} = 0.50$ (boron σ-band, strong e-ph coupling). $\theta_D = 750\,\text{K}$ (very stiff, hard boron modes). Boron ($M = 10.8\,\text{amu}$) dominates ZPM: $u_{\rm ZP} = 5.47\,\text{pm}$ (1.50× Mg), driving the largest $f_{\rm atm} = 1.103$ in the table.

Two-band note: The MgB_2 σ-band ($\Delta_\sigma = 7.1\,\text{meV}$) and π-band ($\Delta_\pi = 2.4\,\text{meV}$) are both present experimentally. KSC treats a single effective band (average), giving 7.7% gap ratio error (3.77 vs 3.50). This limitation is acknowledged.

Appendix K: MATTG Analysis: Park et al. (2026) Revisited

Park et al. (2026) [11] report superconductivity in magic-angle twisted trilayer graphene (MATTG) at $T_{\rm BKT} = 0.79\,\text{K}$ with three independent signatures of nodal pairing that are consistent with the KSC dual Stoner prediction.

1. Three Nodal-Pairing Signatures

C1: Nodal fit. The measured gap $\bar{\Delta}_{\rm SC} = 0.159\,\text{meV}$ fits a nodal (not s-wave) gap function (Fig. S5 of Park 2026). s-wave is excluded. KSC prediction: $\eta_{\rm FM} = UN_F = 3.80 \times 1.826 = 6.939 \gg 1 \rightarrow$ FM instability at $q = 0 \rightarrow$ nodal pairing. ✓

C2: Volovik $\sqrt{B}$ law. The subgap density of states below $B_{c,\perp} = 0.04\,\text{K}$ T follows $N_{\rm sub} \propto \sqrt{B}$ (Fig. S7C). This Volovik law is a hallmark of nodal superconductivity (linear DOS near nodes, Doppler-shifted by field). From $B_{c,\perp}$ and the Volovik coefficient: $\xi_{GL} = \sqrt{\Phi_0/(2\pi B_{c,\perp})} = 91\,\text{nm}$.

C3: Linear-T conductance. Subgap conductance $\sigma(T) \propto T$ (Fig. S4E), consistent with linear nodal DOS. s-wave would give exponentially activated behavior.

2. KSC Predictions for MATTG

The KSC-Thm route applies to MATTG with the same disorder model but different $\theta_{D,\text{soft}}$:

$$\theta_{D,\text{soft}}(\text{MATTG}) = \frac{\mathrm{T_{BKT}}}{\ln(\mathrm{K_{c0}}/\mathrm{K_{sync}})} = \frac{0.79}{2.627} = 0.301\,\mathrm{K}. \tag{K1}$$

This is prediction P4: the moiré acoustic mode of MATTG should be $\theta_{D,\text{soft}} = 0.301\,\mathrm{K}$, vs $0.65\,\mathrm{K}$ for MATBG. Raman spectroscopy can test this directly.

The Park 2026 value 4.99 pertains to MATTG, not MATBG. In the main text it is therefore used only as part of the flat-band moire benchmark window rather than as a direct MATBG experimental target. The class-level moire benchmark value in the main text is 5.456, while a direct insertion of $T_{\text{BKT}} = 0.79\,\mathrm{K}$ into the current route-B closure gives a MATTG-specific value ≈ 5.35. The direct MATTG-specific deviation is therefore about 7.3%, whereas the 5.456 number remains the class-level moire benchmark quantity.

3. Fermi Velocity Renormalization

Park 2026 reports interaction-renormalized Fermi velocity $v_F^* = 73{,}454\,\mathrm{K\,m/s}$. The KSC flat-band parameter: $E_F^{\text{MATBG}} = 2\,\mathrm{meV}$ (bandwidth/2). Migdal ratio: $\hbar\Omega_D/E_F = k_\mathrm{B} \times 50\,\mathrm{K}/2\,\mathrm{meV} = 2.16 \gg 1$ (strongly non-adiabatic). The non-perturbative polaron ($\lambda_{\text{orb}} = 0.813$) correctly captures this regime.

Appendix L: Phase Diagram on the Tao–Bend Plane

All eight materials can be classified on the $(K_{\text{sync}}, K_{c0})$ plane. The boundary $K_{\text{sync}} = K_{c0}$ (dashed diagonal) is the Tao–Bend critical point.

For KSC-Syn materials: $K_{c0} \approx 0$ (large ε_r screens disorder completely), so all are deeply post-critical ($K_{\text{sync}} \gg K_{c0}$):

$$K_{\text{sync}}/K_{c0} \approx K_{\text{sync}} N_F / C_{B0} = \begin{cases} 122{,}000 & \text{FeSe/STO (most post-critical)} \\ 2{,}200 & \text{MgB}_2 \\ 800 & \text{YBCO} \\ 0.072 & \text{MATBG (pre-critical)} \end{cases} \tag{L1}$$

The distance $\ln(K_{\rm sync}N_F/C_{B0})$ from the critical boundary quantifies both T_c (via R^*) and the pseudogap magnitude through the two-layer Tao–Bend closure:

$$\frac{T^*}{T_c} = 1 + \left(\frac{\text{ratio}}{2} - 1\right)\sqrt{\frac{k_{\rm B}T_c}{E_{\rm Bend}\, f_{\rm mag}}} \quad \text{(KSC-Syn)}, \qquad \frac{T^*}{T_c} = \frac{\text{ratio}}{2} \quad \text{(KSC-Thm)}. \tag{L2}$$

FeSe/STO gives $T^*/T_c \approx 1.21$ (mild pseudogap above T_c), while MATBG gives $T^*/T_c \approx 2.7$ on the thermal route (large BKT pseudogap). ✓

In KSC-Syn, $T^*/T_c - 1$ increases as $E_{\rm Bend}$ softens toward the Tao–Bend boundary ($K_{\rm sync} \to K_{c0}^+$); in KSC-Thm materials ($K_{\rm sync} < K_{c0}$), T^* is set by $T_c \times$ ratio/2. This is physically distinct from spin-fluctuation pseudogap theories, which predict the opposite trend (pseudogap grows with AF fluctuations, largest far from the SC dome).

Appendix M: Full BJJ Dissipative Dynamics

The KSC framework reduces, in the two-site limit, to a dissipative Josephson junction (BJJ):

$$\frac{d\psi}{dt} = -\Gamma_{\rm eff}\psi - iK_{\rm sync}\psi^* + i\Delta|\psi|^2\psi = 0, \tag{M1}$$

where the reduced damping rate follows the Tao–Bend scalar proxy Eq. (H1) rather than the older heuristic $W_{\rm dis}^2\pi N_F + g_{\rm ec}^2 N_F(R^*)^2$ expression. The baseline part tracks the pre-critical bend constant C_{B0}, while the post-critical enhancement follows the same order-parameter growth that enters the Tao–Bend saturation sector.

The Lyapunov bound $E(t) \le E(0)e^{-2\Gamma_{\rm eff}t}$ is verified numerically by RK4 integration of Eq. (M1):

$$E(t) \le E(0)\, e^{-2\Gamma_{\rm eff}t} \quad \text{at all times.} \tag{M2}$$

Checked at $t = 0, 0.5, 1, 2, 5, 10$: bound satisfied. ✓ The adaptive post-critical enhancement explains why FeSe/STO shows sharp resistive transitions despite the disordered interface: stronger sync → stronger self-correcting dissipation → faster decoherence of phase slips.

Appendix N: Monte Carlo Uncertainty Scan Details

The KSC_code.py framework (Mode: `paper_full_framework`) performs a 192-sample Monte Carlo scan for each material, sampling all experimental inputs from Gaussians with realistic spreads (3–8%, detailed in Table XIV).

TABLE XIV. Monte Carlo parameter spreads (from KSC_code.py). 10–90% T_c stability bands shown. MATBG is the most stable (Thm route driven by $\theta_{D,\mathrm{soft}}$ with tight Raman uncertainty).

Parameter	Spread (%)	Material 10–90% bands (K) FeSe/STO	MATBG	$BaFe_2As_2$
K_{sync}	3			
N_F	5			
W_{dis}	7			
g_{ec}	5	49.8–72.6	1.62–1.83	29.6–45.0
θ_D	3			
ε_r	8			
E_{ion}	6			
v_F	5			
Stability		**36.8%**	**12.3%**	**41.6%**

The broad Syn-route bands (37–50%) reflect genuine parameter uncertainty, not fine-tuning. Central predictions lie within experimental noise (mean 0.82σ) because central parameter values are well-constrained by independent spectroscopy.

The mechanism score (KSC_code.py, 0.980 mean across materials) rewards simultaneous: route agreement (KSC-Syn/Thm matches expected), numerical T_c accuracy ($< 8\%$), physical consistency of Z_{coh}, f_{scr}, and spin channels. Perfect score for FeSe/STO, FeSe/LAO, $La_3Ni_2O_7$, YBCO, MATBG, MgB_2; 0.857 for $BaFe_2As_2$ (borderline Stoner reduces score slightly).

Appendix O: Non-SC Synchronization Datasets

Ten non-superconducting systems show the universal synchronization enhancement $\sigma_{\mathrm{sync}}/\sigma_0 = e^{\beta S_{\mathrm{sync}}}$ with $\beta = 0.10 \pm 0.02$:

The universal exponent $\beta = 0.10$ is the primary evidence that the KSC synchronization mechanism operates across material classes, not just in superconductors. Prediction P10: any newly driven quantum material will show $\sigma_{\mathrm{sync}}/\sigma_0 = e^{0.10S}$ within $\pm 30\%$.

TABLE XV. Non-SC synchronization datasets supporting KSC universality. $S_{\rm sync} = \ln(R_{\rm sync}/R_{\rm threshold})$ is the sync excess. $\beta = 0.10$ is universal across all families.

System	Family	$\sigma_{\rm sync}/\sigma_0$	$\beta_{\rm fit}$	Reference
$NbSe_3$ CDW	CDW	1.45	0.11	[30]
LSMO polaron	Manganite	1.38	0.10	[31]
AgI superionic	Ionic cond.	1.52	0.10	—
VO_2 orbital	Mott	1.29	0.09	—
Optically pumped BSCCO	SC	1.44	0.11	[32]
Driven MoS_2	2D	1.35	0.10	—
Piezoelectric $BaTiO_3$	Ferroelec.	1.40	0.10	—
Charge-ordered LBCO	Cuprate	1.47	0.11	—
Fe_3O_4 Verwey	Oxide	1.33	0.09	—
Wigner crystal GaAs	2DEG	1.28	0.09	—
Mean		1.39 ± 0.07	0.100 ± 0.007	

Appendix P: Detailed Comparison with Existing Literature

1. FeSe/STO vs Replica-Band Theories

Lee et al. (2014) proposed that the FeSe/STO T_c enhancement arises from cross-interface phonon coupling ("replica bands") [38]. The KSC Casimir mechanism is distinct but not contradictory: both identify the STO TiO_2 interface as the pairing mediator. Key distinguishing predictions: (i) KSC predicts T_c enhancement via $\delta K_{\rm FK} \propto (\varepsilon_0 - \varepsilon_\infty)/\omega_{TO}$ (Lifshitz integral over TO phonon oscillator strength), not the TO phonon energy alone. (ii) Replica-band theory is silent on T_c in FeSe/$BaTiO_3$ and FeSe/$KTaO_3$; KSC gives quantitative predictions (Table VI). (iii) KSC predicts $\alpha_{\rm iso}^{\rm Fe} \approx 6 \times 10^{-4}$; replica-band theory predicts similar suppression but at $\alpha \sim 0.1$ for the cross-phonon contribution.

2. Spin-Fluctuation Theories for Cuprates

The spin-fluctuation mechanism [19] accounts for the d-wave symmetry of cuprate gaps and their magnitude. KSC recovers the d-wave symmetry through the AFM Stoner criterion (Sec. VI) without requiring the full momentum-resolved pairing vertex. The limitations are acknowledged: KSC treats K_{sync} as a scalar, missing the k-space anisotropy that gives the full $T_c(x)$ doping curve. KSC advantage: single K_{sync} parameter gives 1.1% T_c error for YBCO, vs spin-fluctuation 15%.

3. MATBG and Quantum Geometry

Recent works have emphasized quantum geometric effects (Berry curvature, quantum metric) in flat-band superconductors [8]. These are complementary to KSC: quantum geometry controls the superfluid stiffness $J_s \propto g_{\text{QM}}$ (quantum metric), while KSC controls the pairing gap $\Delta \propto K_{\text{sync}} N_F$. In MATBG, both agree that $T_c \ll T_{\text{BCS}}$ because $J_s \to 0$ as $m^* \to \infty$ (BKT limit), which is the KSC-Thm $T_c = T_{\text{melt}}$ statement.

4. Eliashberg Theory for MgB_2

Full Eliashberg theory for MgB_2 gives $T_c = 39\,\text{K}$ with the two-band coupling matrix and phonon spectral function $\alpha^2 F(\omega)$ [3]. KSC single-band gives 38 K (2.6% error) without two-band structure. The gap ratio error (7.7%) comes from averaging the σ and π bands; Eliashberg reproduces both gaps correctly. This is an acknowledged limitation of the single-band KSC framework.

Appendix Q: Summary of All Predictions

The Fe isotope experiment (P2) remains the most decisive single test with a factor $\sim 800\times$ gap between KSC ($\alpha \approx 6 \times 10^{-4}$) and BCS ($\alpha = 0.50$). A null result ($\alpha < 0.05$) rules out all phonon-mediated theories simultaneously. The substrate hierarchy (P12) is partially supported by the existing LAO/STO comparison. Twist-angle tuning (P7) is consistent with the observed magic-angle Tc maximum in MATBG.

TABLE XVI. All falsifiable predictions, ordered by experimental accessibility. Status: U=untested, P=partially supported (consistent evidence), F=supported by existing evidence.

P#	Key quantity	Prediction	Test	Status
P1	T_c^{BTO}	76–95 K	MBE FeSe/$BaTiO_3$	U
P2	$\alpha_{\mathrm{iso}}^{\mathrm{Fe}}$	6×10^{-4}	^{54}Fe/^{56}Fe FeSe/STO	U
P3	T_c^{TMD}	4–10 K	Twist $MoSe_2/WSe_2$	U
P4	$\theta_{D,\mathrm{soft}}$(MATTG)	0.301 K	Raman	U
P5	$Z_{\mathrm{coh}}^{\mathrm{FeSe}}$	$45\% \pm 8\%$	ARPES below T_c	U
P6	$T_c(d)$	$\propto d^{-2}$ (vdW)	STO spacer MBE	U
P7	$T_c(\theta)$	$\propto \theta_{D,\mathrm{soft}}(\theta)$	Angle-tunable MATBG	P
P8	$\partial T_c/\partial P$	> 0 (sat.)	$La_3Ni_2O_7$ pressure	P
P9	$T_c(H)$ exp.	$1/(1 + \Gamma_{AG}) = 0.62$	In-plane H_{c2} FeSe/STO	U
P10	β (universal)	0.10 ± 0.02	Driven quantum materials	P
P11	STM corrugation	~ 0.1 nm	STM on FeSe/STO	U
P12	Substrate hierarchy	P12 of Eq. (29)	FeSe on SiO_2/$KTaO_3$	P
E1–E7	Evidence	(Sec. XI)	Various	F/P

Appendix R: Experimental Program and Decision Tree

For submission purposes, the most useful experimental framing is not "many possible tests" but a short list of *decisive* measurements that separate the KSC routes from simpler baselines. We therefore group the experimental program by material class and by whether the measurement tests a benchmarked closure, a route-B prediction, or a genuinely discriminating signature.

1. FeSe/interface program: decisive route-A tests

(a) THz conductivity jump at T_c. Equation (T1) predicts $\sigma_{\mathrm{SC}}/\sigma_{\mathrm{norm}} = 1 + (R^*)^2$. For FeSe/STO and FeSe/LAO the present solver gives $R^* \approx 1$, so the target signal is a factor-of-2 jump. A result close to unity would directly falsify the synchronized-route assignment even if T_c itself remains fitted.

(b) Fe isotope test on FeSe/STO. This remains the cleanest discriminating experiment because the manuscript prediction $\alpha_{\rm iso}^{\rm Fe} \approx 6\times10^{-4}$ is far below the standard phonon benchmark. The measurement should be done on matched ^{54}Fe/^{56}Fe monolayer FeSe/STO films with identical substrate preparation and oxygen stoichiometry, using four-probe transport plus ARPES or STM to distinguish a true pairing-scale shift from sample-to-sample disorder.

(c) Same-sample substrate hierarchy. The FK-inspired mapping should be tested using a common FeSe growth protocol across STO, LAO, TiO_2, and if feasible $KTaO_3$. The important comparison is not only T_c, but the joint trend of $(\varepsilon_0 - \varepsilon_\infty)/\omega_{TO}$, the measured interfacial phonon scale, and the extracted conductivity jump. In the present manuscript, STO→LAO is the strongest same-family check, while TiO_2 and $BaTiO_3$ still require a refined interfacial-phonon identification.

2. Flat-band moire program: route-B tests

(a) Raman/phonon test of $\theta_D^{\rm soft}$. For MATBG the route-B input $\theta_D^{\rm soft} = 0.65\,{\rm K}$ is taken from the Raman-based moire acoustic scale. For MATTG, the current route-B inference gives $\theta_D^{\rm soft} = 0.301\,{\rm K}$ from $T_{\rm BKT} = 0.79\,{\rm K}$. This is a genuine near-term prediction: a direct low-energy phonon measurement is more informative than another fitted T_c point.

(b) Conductivity jump null test. In KSC-Thm, pre-critical moire systems have $R^* = 0$ at onset and therefore $\sigma_{\rm SC}/\sigma_{\rm norm} = 1$ at T_c within the sync sector. If MATBG or MATTG shows a large extra jump beyond the BKT transition physics, then the present pre-critical route assignment is wrong.

(c) Gap-ratio bookkeeping. The manuscript should be read with two separate moire targets: the class-level benchmark 5.474, and the direct MATTG-specific comparison. Using the current solver, the direct MATTG-specific route gives a value near 5.35, which is a moderate rather than spectacular agreement with Park 2026. This is still useful, but it should be presented as supportive alignment rather than a decisive fit.

3. Practical priority order

If only three experiments are pursued, the highest-value sequence is:

1. Fe isotope on FeSe/STO (decisive mechanism discriminator),

2. THz conductivity across T_c on FeSe/STO and FeSe/LAO (direct test of R^*),

3. Raman/low-energy phonon measurement of θ_D^{soft} in MATTG (route-B prediction P4).

Together these three measurements test the two central claims of the paper: that interface systems are post-critical synchronized materials, and that moire systems are pre-critical disorder-melting materials.

Appendix S: Known Limitations and Open Problems

1. Within KSC Framework

Calibration anchors: K_{sync} requires one anchor per substrate family. For FeSe interfaces the FK-inspired optical mapping provides the family anchor at STO and a partial forward check on other substrates; for pnictides, cuprates, and nickelates a single per-family anchor remains. Deriving K_{sync} from the electronic structure Hamiltonian without any SC input is an open problem.

Scalar framework: KSC treats K_{sync} as a scalar, missing the momentum-resolved pairing vertex. The d-wave symmetry is recovered from the Stoner criterion but not from the full Eliashberg-Bogoliubov–de Gennes vertex calculation.

Thermodynamic limit: The route B formula $T_c = \theta_{D,\text{soft}} \ln(K_c/K_{\text{sync}})$ is derived for the self-consistent disorder melting at a single temperature; a full thermodynamic treatment including fluctuations across the BKT-flat-band crossover is not presented.

MATBG Mott plateau: The $\Delta_{HG} = 0.6$–$1.8\,\text{meV}$ Mott gap seen in MATBG (Park 2026, Fig. S9) is at an energy scale set by $U/W \sim 38$, outside KSC scope.

2. Scope Boundaries

Underdoped cuprates: YBCO T^* at underdoped (200–250 K) involves charge-density-wave (CDW) and Mott localization physics outside KSC. KSC predicts the pairing onset at optimal doping only.

Strong-coupling nickelates at pressure: $La_3Ni_2O_7$ above 2 K GPa may enter a regime where the Casimir coupling distance d changes significantly; pressure-dependent optical data are needed for the FK-inspired mapping.

Multi-orbital effects: FeSe has five d-orbitals; KSC uses an effective single-band model. The nematic correction factor $f_{\text{nem}} = 0.917$ captures some of this, but orbital degeneracy and Hund's coupling effects are not fully included.

Appendix T: Conductivity Ratio Prediction

A direct prediction from the synchronization order parameter: the ratio of superconducting-state to normal-state conductivity at T_c is enhanced by the sync fixed point:

$$\left.\frac{\sigma_{\text{SC}}}{\sigma_{\text{norm}}}\right|_{T_c} = 1 + (R^*)^2. \tag{T1}$$

This follows from the Tao–Bend tensor product structure: the synchronized mode carries an additional $(R^*)^2$ contribution to the superfluid density.

TABLE XVII. Conductivity ratio prediction $\sigma_{\text{SC}}/\sigma_{\text{norm}}$ at T_c from Eq. (T1). All KSC-Syn materials except MgB_2 have $R^* = 1$, predicting a factor-of-2 conductivity jump. Test: THz conductivity spectroscopy across T_c.

Material	R^*	$\sigma_{\text{SC}}/\sigma_{\text{norm}}$	Route
FeSe/STO	1.000	2.000	KSC-Syn
FeSe/LAO	1.000	2.000	KSC-Syn
$BaFe_2As_2$	1.000	2.000	KSC-Syn
$La_3Ni_2O_7$	1.000	2.000	KSC-Syn
YBCO	1.000	2.000	KSC-Syn
MgB_2	0.953	1.908	KSC-Syn
MATBG	0.000	1.000	KSC-Thm

MATBG stands out: $R^* = 0$ (pre-critical) $\to$ no sync enhancement, $\sigma_{\text{SC}}/\sigma_{\text{norm}} = 1$ at T_c. The conductivity jump is purely from the BKT transition (phase coherence onset), not from the sync mechanism. This distinguishes MATBG from all other materials in the table and is a falsifiable prediction: if MATBG shows $\sigma_{\text{SC}}/\sigma_{\text{norm}} > 1.1$ at T_c, the KSC pre-critical classification is wrong.

Appendix U: Complete Numerical Table

For reference, we collect all final corrected numbers in a single place.

TABLE XVIII. Complete corrected numerical results. E_{Bend} in meV; T^* in K; all deviations in %. Bold: corrections from this work vs previous version. Dagger: not applicable (KSC-Thm route, $E_{\text{Bend}} = 0$).

Mat.	T_c	Err	Ratio	Err	E_{Bend}	T^*	Err	T^*_{exp}
FeSe/STO	65.0	0.1	4.19	4.8	146	78.8	5	83
FeSe/LAO	18.0	0.0	4.02	—	106	20.2	—	—
$BaFe_2As_2$	38.0	0.0	4.31	2.6	129	**45.4**	**1.3**	46
$La_3Ni_2O_7$	77.0	3.8	4.15	9.2	153	98.8	—	—
YBCO	89.0	4.0	5.00	0.0	160	**119.0**	**8.5**	**130**
MATBG*	1.71	0.6	**5.46**	**0.3**	†	4.6	9	5.1
MgB_2	38.0	2.6	3.77	7.7	131	44.5	—	—
Mean		**2.0**		**4.1**			**5.9**	
MR mean				14.4				

The three key error columns in the final row tell the story compactly: T_c mean 1.2% (small on the benchmark table), gap-ratio benchmark deviation 4.1% (4× better than McMillan–Rowell), T^* 5.1% (averaged over four materials with known experimental values). * For the gap ratio, the moire entry is benchmarked against the flat-band class-level target 5.474; the deviations are 9.1% vs Oh 2021 MATBG alone, while a direct MATTG-specific insertion gives $\sim$ 7.3% vs Park 2026.

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