

# Triality-Resolved Spectral Update Theory: A Minimal Transition-Operator Route to Standard-Model Structure and Gravity

Dr Casey McGrath<sup>1</sup>

<sup>1</sup>University of Queensland

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We present a minimal theorem stack in which low-energy physics emerges from a single causal transition operator acting on a microscopic configuration Hilbert space. Proper time is defined as accumulated transition information along causal histories. An exact order-three invariant sector of the microscopic dynamics yields three fermion generations. In the infrared, the theory is described by an almost-commutative spectral triple whose visible finite algebra reproduces the Standard Model gauge group. The finite Dirac operator acts on a 96-dimensional fermionic Hilbert space and admits a triality-circulant Yukawa core diagonalized by a discrete Fourier basis, producing exact zeroth-order harmonic flavor states. Minimal non-circulant defects generate CKM and PMNS mixing, while a compatible singlet Majorana block closes the neutrino sector by the seesaw mechanism. The bosonic spectral action yields Einstein–Hilbert, Yang–Mills, and Higgs terms from a single operator expansion. The fine-structure sector reduces to one microscopic spectral invariant. We isolate the remaining genuinely downstream inputs to explicit microscopic Yukawa coefficients, defect blocks, and the computation of the relevant spectral invariant.

## MICROSCOPIC FOUNDATION

We begin from a minimal dynamical premise.

**Axiom 1** (Configuration Hilbert space). *There exists a microscopic configuration Hilbert space  $H_{\text{conf}}$  and a single linear transition operator*

$$T : H_{\text{conf}} \rightarrow H_{\text{conf}}. \quad (1)$$

*This operator is the sole primitive dynamical object.*

**Axiom 2** (Causal restriction). *The dynamics is causally filtered: if  $j \notin J^+(i)$ , then*

$$\langle j | T^n | i \rangle = 0, \quad \forall n \geq 1, \quad (2)$$

*where  $J^+(i)$  denotes the causal future of  $i$ .*

**Definition 1** (Emergent time). *For a causal chain*

$$\Gamma = (i_0 \prec i_1 \prec \cdots \prec i_n), \quad (3)$$

*define the emergent-time functional*

$$\mathcal{T}[\Gamma] = - \sum_{k=0}^{n-1} \log |\langle i_{k+1} | T | i_k \rangle|^2. \quad (4)$$

Equation (4) identifies proper time with accumulated transition information along a realized causal history.

## EXACT TRIALITY SECTOR

**Definition 2** (Invariant triality subspace). *Assume there exists an invariant subspace  $H_{\text{tri}} \subset H_{\text{conf}}$  such that the restricted operator*

$$T_{\text{tri}} := T|_{H_{\text{tri}}} \quad (5)$$

*satisfies*

$$T_{\text{tri}}^3 = 1. \quad (6)$$

**Theorem 1** (Exact three-sector decomposition). *If Eq. (6) holds and  $T_{\text{tri}}$  is diagonalizable over  $\mathbb{C}$ , then*

$$\text{Spec}(T_{\text{tri}}) = \{1, \omega, \omega^2\}, \quad \omega = e^{2\pi i/3}, \quad (7)$$

*and*

$$H_{\text{tri}} = H_0 \oplus H_1 \oplus H_2, \quad (8)$$

*with*

$$T_{\text{tri}}|_{H_a} = \omega^a I, \quad a = 0, 1, 2. \quad (9)$$

*Hence the theory contains an exact three-sector structure.*

*Proof.* The polynomial  $x^3 - 1 = (x - 1)(x - \omega)(x - \omega^2)$  has distinct roots over  $\mathbb{C}$ , so any diagonalizable solution of Eq. (6) has eigenvalues among  $\{1, \omega, \omega^2\}$ . The corresponding spectral projectors yield the direct-sum decomposition.  $\square$

## INFRARED SPECTRAL GEOMETRY

The infrared envelope of the microscopic dynamics is assumed to admit an almost-commutative spectral description.

**Definition 3** (Infrared spectral triple). *The low-energy theory is described by a spectral triple*

$$(A, H, D, J, \Gamma) \quad (10)$$

*with algebra*

$$A = C^\infty(M) \otimes A_F, \quad (11)$$

*commutative manifold*

$$M = S^3 \times S^1, \quad (12)$$

*and Dirac operator*

$$D = D_M \otimes 1 + \gamma_5 \otimes D_F. \quad (13)$$

**Lemma 1** (Volume of the commutative sector). *For unit radii,*

$$\text{Vol}(S^3 \times S^1) = \text{Vol}(S^3) \text{Vol}(S^1) = (2\pi^2)(2\pi) = 4\pi^3. \quad (14)$$

### FINITE ALGEBRA AND VISIBLE GAUGE STRUCTURE

**Definition 4** (Finite algebra). *Let*

$$A_F = (\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})) \otimes A_{\text{tri}}. \quad (15)$$

*Define the visible finite algebra by*

$$A_{\text{vis}} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}). \quad (16)$$

**Theorem 2** (Visible gauge group). *The continuous gauge group derived from  $A_{\text{vis}}$  is*

$$G_{\text{vis}} = SU(3)_c \times SU(2)_L \times U(1)_Y \quad (17)$$

*up to the usual finite quotient.*

*Proof.* The unitary group of  $A_{\text{vis}}$  is

$$U(A_{\text{vis}}) = U(1) \times SU(2) \times U(3), \quad (18)$$

where  $U(\mathbb{H}) \simeq SU(2)$ . Using

$$U(3) \cong \frac{SU(3) \times U(1)}{\mathbb{Z}_3}, \quad (19)$$

one obtains two abelian factors. The unimodularity condition removes the redundant overall  $U(1)$ , leaving  $SU(3) \times SU(2) \times U(1)$ .  $\square$

### MATTER CONTENT AND EXACT FAMILY COUNT

A single Standard Model family with a right-handed neutrino contains

$$L_\ell = (\nu_L, e_L), \quad Q_L = (u_L, d_L), \quad e_R, \nu_R, u_R, d_R. \quad (20)$$

The particle-state count is

$$2 + 6 + 1 + 1 + 3 + 3 = 16. \quad (21)$$

Including antiparticles yields 32 states per family.

**Theorem 3** (Finite Hilbert-space dimension and exact replication). *Let  $H_{\text{one family}}$  denote the 32-dimensional one-family space including antiparticles. If the triality sector contributes a factor  $\mathbb{C}^3$ , then*

$$H_F \cong H_{\text{one family}} \otimes \mathbb{C}^3, \quad (22)$$

*and therefore*

$$\dim H_F = 32 \times 3 = 96. \quad (23)$$

*The theory contains exactly three generations.*

### FAMILY-BY-FAMILY ANOMALY CANCELLATION

**Theorem 4** (Exact anomaly cancellation). *For one family, the anomaly coefficients for*

$$SU(2)^2U(1), \quad SU(3)^2U(1), \quad \text{grav-}U(1), \quad U(1)^3 \quad (24)$$

*all vanish.*

*Proof.* Using the standard hypercharge assignments:

$$SU(2)^2U(1) : \quad 3 \left( \frac{1}{6} \right) + \left( -\frac{1}{2} \right) = 0, \quad (25)$$

$$SU(3)^2U(1) : \quad 2 \left( \frac{1}{6} \right) - \frac{2}{3} + \frac{1}{3} = 0, \quad (26)$$

$$\begin{aligned} \text{grav-}U(1) : \quad & 6 \left( \frac{1}{6} \right) + 2 \left( -\frac{1}{2} \right) + 3 \left( -\frac{2}{3} \right) \\ & + 3 \left( \frac{1}{3} \right) + 1 + 0 = 0, \end{aligned} \quad (27)$$

$$\begin{aligned} U(1)^3 : \quad & 6 \left( \frac{1}{6} \right)^3 + 2 \left( -\frac{1}{2} \right)^3 + 3 \left( -\frac{2}{3} \right)^3 \\ & + 3 \left( \frac{1}{3} \right)^3 + 1^3 + 0^3 = 0. \end{aligned} \quad (28)$$

Hence one family is anomaly-free, and so is the exact three-family theory.  $\square$

### FINITE DIRAC OPERATOR AND FLAVOR STRUCTURE

**Definition 5** (Finite Dirac operator). *In particle/conjugate grading, take*

$$D_F = \begin{pmatrix} 0 & Y & 0 & 0 \\ Y^\dagger & 0 & 0 & M \\ 0 & 0 & 0 & Y^* \\ 0 & M^\dagger & Y^T & 0 \end{pmatrix}, \quad (29)$$

*where*

$$Y = \text{diag}(Y_\nu, Y_e, Y_u \otimes I_3, Y_d \otimes I_3). \quad (30)$$

*The block  $M$  is nonzero only on the singlet-neutrino sector.*

**Definition 6** (Triality-circulant Yukawa core). *At zeroth order, each flavor block is taken to be*

$$Y_f^{(0)} = y_{f,0} I_3 + y_{f,1} T_{\text{tri}} + y_{f,2} T_{\text{tri}}^2. \quad (31)$$

*Using the standard cyclic representation*

$$T_{\text{tri}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (32)$$

*each  $Y_f^{(0)}$  is circulant.*

**Theorem 5** (Exact harmonic flavor basis). *The discrete Fourier matrix*

$$F_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \quad (33)$$

diagonalizes every triality-circulant Yukawa block:

$$F_3^\dagger Y_f^{(0)} F_3 = \text{diag}(\lambda_{f,0}, \lambda_{f,1}, \lambda_{f,2}), \quad (34)$$

with eigenvalues

$$\lambda_{f,0} = y_{f,0} + y_{f,1} + y_{f,2}, \quad (35)$$

$$\lambda_{f,1} = y_{f,0} + \omega y_{f,1} + \omega^2 y_{f,2}, \quad (36)$$

$$\lambda_{f,2} = y_{f,0} + \omega^2 y_{f,1} + \omega y_{f,2}. \quad (37)$$

*Proof.* Equation (32) is diagonalized by Eq. (33):

$$F_3^\dagger T_{\text{tri}} F_3 = \text{diag}(1, \omega, \omega^2). \quad (38)$$

Applying this to Eq. (31) gives the result.  $\square$

**Corollary 1** (Zeroth-order masses). *After electroweak symmetry breaking,*

$$m_{f,a}^{(0)} = \frac{v}{\sqrt{2}} |\lambda_{f,a}|, \quad a = 0, 1, 2. \quad (39)$$

## MIXING CLOSURE

**Definition 7** (Minimal non-circulant completion). *Let*

$$Y_f = Y_f^{(0)} + \varepsilon_f \Delta_f, \quad (40)$$

where  $\Delta_f$  is the smallest allowed non-circulant perturbation in the corresponding flavor sector.

**Theorem 6** (First-order CKM and PMNS closure). *To first order in  $\varepsilon_f$ ,*

$$U_f = F_3 (I + \varepsilon_f K_f + O(\varepsilon_f^2)), \quad (41)$$

where for  $i \neq j$ ,

$$(K_f)_{ij} = \frac{(F_3^\dagger \Delta_f F_3)_{ij}}{\lambda_{f,j} - \lambda_{f,i}}. \quad (42)$$

Hence

$$V_{\text{CKM}} = U_u^\dagger U_d = I + \varepsilon_d K_d - \varepsilon_u K_u + O(\varepsilon^2), \quad (43)$$

$$U_{\text{PMNS}} = U_e^\dagger U_\nu = I + \varepsilon_\nu K_\nu - \varepsilon_e K_e + O(\varepsilon^2). \quad (44)$$

## NEUTRINO SECTOR

**Definition 8** (Triality-compatible heavy singlet block). *Let the Majorana singlet block be*

$$M_R = m_{R,0} I_3 + m_{R,1} T_{\text{tri}} + m_{R,2} T_{\text{tri}}^2 + \varepsilon_R \Delta_R. \quad (45)$$

**Theorem 7** (Seesaw closure). *If  $m_D$  is the Dirac neutrino block, then in the heavy-singlet limit,*

$$m_\nu \approx -m_D^T M_R^{-1} m_D. \quad (46)$$

## FINITE SPECTRAL INVARIANTS

Define

$$t_0 = \text{Tr}(1_{H_F}), \quad t_2 = \text{Tr}(D_F^2), \quad t_4 = \text{Tr}(D_F^4), \quad (47)$$

and sector traces

$$y_{2,f} = \text{Tr}(Y_f^\dagger Y_f), \quad y_{4,f} = \text{Tr}[(Y_f^\dagger Y_f)^2]. \quad (48)$$

**Theorem 8** (Exact finite traces). *For the explicit operator (29),*

$$t_0 = 96, \quad (49)$$

and

$$t_2 = 4 \text{Tr}(Y^\dagger Y) + 2 \text{Tr}(M^\dagger M). \quad (50)$$

Hence

$$t_2 = 4(y_{2,\nu} + y_{2,e} + 3y_{2,u} + 3y_{2,d}) + 2 \text{Tr}(M_R^\dagger M_R). \quad (51)$$

If  $M_R = 0$ , then

$$t_4 = 4 \text{Tr}[(Y^\dagger Y)^2] = 4(y_{4,\nu} + y_{4,e} + 3y_{4,u} + 3y_{4,d}). \quad (52)$$

*Proof.* The block structure of Eq. (29) gives four copies of  $Y^\dagger Y$  in  $D_F^2$  and two copies of  $M^\dagger M$  in the Majorana sector. The  $t_4$  formula follows similarly when  $M_R = 0$ .  $\square$

## UNIFIED BOSONIC ACTION

**Theorem 9** (Spectral-action expansion). *The bosonic action*

$$S_B = \text{Tr} f(D_A/\Lambda) \quad (53)$$

has the asymptotic expansion

$$S_B \sim f_4 \Lambda^4 a_0 + f_2 \Lambda^2 a_2 + f_0 a_4 + \dots \quad (54)$$

The coefficients generate, respectively, the cosmological term, Einstein–Hilbert gravity, and Yang–Mills plus Higgs dynamics.

Equation (54) gives a unified operator origin for gauge and gravitational sectors.

## EFFECTIVE LORENTZIAN SPACETIME

**Theorem 10** (Coarse-grained Lorentzian form). *If the causal restriction selects one monotone update direction and the commutative sector supplies an isotropic  $S^3$  spatial slice, then the effective coarse-grained metric takes the Lorentzian form*

$$ds^2 = c^2 d\tau^2 - a(\tau)^2 d\Omega_3^2, \quad \tau \propto \mathcal{T}[\Gamma]. \quad (55)$$

This establishes the intended infrared interpretation: one emergent time direction and three isotropic spatial directions.

## CLASSICAL LOW-ENERGY EQUATIONS

The standard classical equations arise in the low-energy limit. Squaring the Dirac operator yields the Klein–Gordon relation

$$(\square + m^2)\psi = 0. \quad (56)$$

The gauge sector gives Yang–Mills, reducing in the abelian case to Maxwell:

$$D_\mu F^{\mu\nu} = J^\nu, \quad \partial_\mu F^{\mu\nu} = J^\nu. \quad (57)$$

The free Dirac spectrum gives

$$E^2 = p^2 + m^2, \quad (58)$$

and at rest,

$$E = mc^2. \quad (59)$$

## FINE-STRUCTURE SECTOR

**Theorem 11** (Reduction of  $\alpha$  to a single spectral invariant). *Assume the electromagnetic coupling reduces to one microscopic invariant  $\Gamma_\Omega$ . Then*

$$\alpha^{-1} = \frac{\pi}{2}\Gamma_\Omega + \frac{\pi^2}{12} + \frac{\pi^2}{2\zeta(3)}. \quad (60)$$

Equivalently,

$$\Gamma_\Omega = \frac{2}{\pi} \left( \alpha^{-1} - \frac{\pi^2}{12} - \frac{\pi^2}{2\zeta(3)} \right). \quad (61)$$

## EFFECTIVE COSMOLOGICAL DYNAMICS

At coarse scale, discrete spectral updating suggests the expansion law

$$\frac{\dot{a}}{a} = H_* + A \cos(\Omega t + \phi) + R(t), \quad (62)$$

with residual mode

$$\dot{R} + \gamma R = S(t). \quad (63)$$

To leading order, the built-distance relation becomes logarithmic,

$$d(z) \sim \frac{c}{H_*} \ln(1+z), \quad (64)$$

up to bounded oscillatory and residual corrections.

## QUANTUM INTERPRETATION

**Theorem 12** (Measurement as causal-path realization). *A measurement outcome is interpreted as the realization of a causal history selected from transition amplitudes of  $T$ , with path weights determined by*

$$|\langle i_{k+1} | T | i_k \rangle|^2. \quad (65)$$

*Thus amplitudes, probabilities, causal progression, and emergent time are all inherited from the same microscopic operator.*

## MASTER THEOREM

**Theorem 13** (Triality-resolved spectral update unification). *Assume:*

1. a microscopic configuration Hilbert space  $H_{\text{conf}}$ ;
2. a single causal transition operator  $T$ ;
3. the emergent-time functional  $\mathcal{T}[\Gamma]$ ;
4. an invariant triality sector  $T_{\text{tri}}^3 = 1$ ;
5. the infrared spectral triple

$$A = C^\infty(S^3 \times S^1) \otimes ((\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})) \otimes A_{\text{tri}}); \quad (66)$$

6. the finite Dirac operator  $D_F$  with a triality-circulant Yukawa core and minimal non-circulant completion.

Then the infrared theory yields:

1. an effective Lorentzian  $(3+1)$  spacetime;
2. visible gauge group  $SU(3)_c \times SU(2)_L \times U(1)_Y$ ;
3. exactly three generations;
4. finite Hilbert-space dimension 96;
5. family-by-family anomaly cancellation;
6. a unified bosonic spectral action;
7. an exact harmonic flavor basis and zeroth-order mass eigenvalues;
8. perturbative CKM and PMNS mixing from defect misalignment;
9. a triality-compatible heavy Majorana sector;
10. recovery of standard classical field equations;
11. reduction of the fine-structure sector to a single spectral invariant;
12. update-based effective cosmological dynamics.

## STATUS OF THE CONSTRUCTION

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The following are derived in closed structural form from the theorem stack above: (i) causal update dynamics and emergent time; (ii) visible gauge group; (iii) exact three-generation replication and  $\dim H_F = 96$ ; (iv) anomaly cancellation; (v) finite Dirac block structure; (vi) harmonic flavor basis; (vii) unified bosonic spectral action.

The following are reduced to explicit downstream microscopic inputs rather than left conceptually open: (i) numerical Yukawa coefficients; (ii) defect blocks governing CKM and PMNS values; (iii) heavy-singlet scales in the neutrino sector; (iv) microscopic evaluation of  $\Gamma_\Omega$ .

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