

The Principle of Induction

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ABSTRACT

Since ancient Greece the possibility of defining natural numbers was considered, but, unlike what happened in Geometry in the Euclid's Elements, all efforts were in vain. After 2000 years it was the Italian mathematician Giuseppe Peano who was recognized for the historical merit of having provided a rigorous definition of the natural numbers and their properties. His five postulates represent the first well-defined axiomatic foundation of arithmetic. Peano's fifth postulate, known as the Principle of Induction, has provided an indispensable tool in countless mathematical proofs and has enabled significant progress in understanding numbers and their secrets.

This paper contains numerous solved exercises on the application of Induction Principle.

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1. Peano's postulates

Giuseppe Peano (Spinetta (Cuneo) 1858 – Turin 1932) in 1889, later officially published in the book *Arithmetica Generale e Algebra Generale* (Paravia, Turin 1902), based arithmetic on three primitive concepts (one, number, successor) and five postulates:

- 1) One is a number.
- 2) Every number has a successor.
- 3) One is not the successor of any number.
- 4) Two different numbers do not have the same successor.
- 5) (Axiom of Induction) : If the properties that pertain to one pertain to a number and also to its successor, then they pertain to all numbers.

If we want to write Peano's postulates using the symbols of set theory, which was already widely developed at the time, we can indicate with \mathbb{N} the set of natural numbers, with \emptyset the empty set, and with n^* the successor of n . Assuming $\mathbb{N} \neq \emptyset$ we have:

- 1) $\exists 1 \in N$
- 2) $\forall n \in N : \exists n^* \in N$
- 3) $\nexists n \in N : n^* = 1$
- 4) $m^* = n^* \leftrightarrow m = n ; \forall m, n \in N$
- 5) If $A \subseteq N : 1 \in A \wedge \forall n \in A : n^* \in A \rightarrow A \equiv N$

It is noteworthy that Peano never mentions the number zero in his postulates. He introduces it by convention into the base, assuming that $0^* = 1$. He also assumes that $n^* = n + 1$.

1.1. Principle of Induction

From the fifth postulate derives the so-called *principle of mathematical induction*, which can be formulated as follows: consider a proposition p such that:

- If
- 1) $p(1)$ is true,
 - 2) assuming $p(n)$ to be true, prove that $p(n^*)$ is true.
- Then
- 3) $p(n)$ is true $\forall n \in N$

1.2. Algebraic structures

In addition to stating the five postulates, Peano defined the law of internal composition of addition in \mathbb{N} as follows:

- 1) $n + 0 = n \quad \forall n \in N$
- 2) $m + n^* = (m + n)^*$

The sum s is the result of this operation.

In this way he created the structure $(\mathbb{N}, +)$ in which zero is the neutral element and addition has the properties:

- 1) Closure : $\forall m, n \in N : s = m + n \in N$
- 2) Associative : $m + (n + p) = (m + n) + p$
- 3) Commutative : $m + n = n + m$

Let's prove them using the principle of induction.

1) Closure :

$p(1)$: $m + 1 = m^* \in N$ it's true.

Assuming $p(n)$ to be true: $m + n = s \in N$,

We have $p(n^*)$: $m + n^* = (m + n)^* = s^* \in N \rightarrow$ QED

2) Associative :

In the associative property $p(1)$: $m + (n + 1) = m + n^* = (m + n)^* = (m + n) + 1$ it's true.

Assuming $p(p)$ to be true: $m + (n + p) = (m + n) + p$; we prove $p(p^*) = m + (n + p^*) =$

$[m + (n + p)^*] = [m + (n + p)]^* = [(m + n) + p]^* = (m + n) + p^* \rightarrow$ QED

3) Commutative :

In this property, it should be noted that the definition of addition implies that $m + 0 = 0 + m$. Hence, $m + 1 = 1 + m$. In fact, $p(1)$: $1 + 1 = 1 + 0^* = (1 + 0)^* = 1^* = 2$

Assuming that $p(m)$: $m + 1 = 1 + m$, we have $p(m^*)$: $m^* + 1 = (m + 1)^* = (1 + m)^* = 1 + m^*$.

Having demonstrated the commutative property for $n = 1$, it is easy to prove it in general.

In fact, assuming true $p(n)$: $m + n = n + m$, we have:

$p(n^*)$: $m + n^* = m + (n + 1) = (m + n) + 1 = (n + m) + 1 =$

$n + (m + 1) = n + (1 + m) = (n + 1) + m = n^* + m \rightarrow$ QED

In N it's valid the law of cancellation : If $m + x = n + x \rightarrow m = n$

We prove the property by resorting, once again, to the principle of mathematical induction.

For $x = 1$, we have $m + 1 = n + 1$, from which $m^* = n^*$ and therefore, by the 4th postulate, $m = n$.

Assuming $p(x)$ to be true: $m + x = n + x \rightarrow m = n$.

We prove $p(x^*)$: $m + x^* = n + x^* \rightarrow m = n$.

In fact: $m + x^* = n + x^* \rightarrow (m + x)^* = (n + x)^*$ and therefore (again by the 4th postulate) we have:

$m + x = n + x$, from which ($p(x)$ being true) we have $m = n \rightarrow$ QED

Moreover Peano defined the multiplication operation in N as follows:

1) $m \cdot 1 = \forall m \in N$

2) $m \cdot n^* = m \cdot n + m$

The product p is the result of this operation.

The following properties hold in the structure (N, \cdot) :

- 1) Closure: $m \cdot n = p \in N$
- 2) Associative: $(m \cdot n) \cdot p = m \cdot (n \cdot p)$
- 3) Commutative: $m \cdot n = n \cdot m$
- 4) Cancellation: $m \cdot x = n \cdot x \rightarrow m = n$

In the structure $(N, +, \cdot)$ are valid both the right and left distributive properties :

$$m \cdot (n + t) = m \cdot n + m \cdot t$$

$$(m + n) \cdot t = m \cdot t + n \cdot t$$

Moreover it's interesting to note :

$$\begin{aligned} (m + n^*)^* &= m^* + n^* \\ m^* + n^* &= (m + n)^* + 1 \\ (m \cdot n^*)^* &= m \cdot n + m^* \\ m^* \cdot n^* &= (m \cdot n)^* + m + n \\ (m^* \cdot n^*)^* &= m^* + m \cdot n + n^* \end{aligned}$$

All of the above properties can be proven with a few logical steps.

Let's prove the first two :

$$(m + n^*)^* = m + n^* + 1 = m + n + 1 + 1 = (m + 1) + (n + 1) = m^* + n^*$$

Example:

$$(5 + 2^*)^* = 5^* + 2^* = 6 + 3 = 9$$

$$m^* + n^* = m + 1 + n^* = m + n^* + 1 = (m + n)^* + 1$$

Example:

$$4^* + 2^* = (4 + 2)^* + 1 = 8$$

... ..

We would like to point out that with the aid of the *Principle of Induction* and the five postulates (and, of course, to the definitions of the operations of addition and multiplication), it is possible to rigorously demonstrate properties in \mathbb{N} that are taken for granted, considered obvious, or empirically accepted. For example, $2 + 2 = 4$. In fact, $2 + 2 = 2 + 1^* = (2 + 1)^* = 3^* = 4$.

Given the results of sums that do not exceed nine, the addition of any two natural numbers can be reduced to the addition of numerical polynomials. One example :

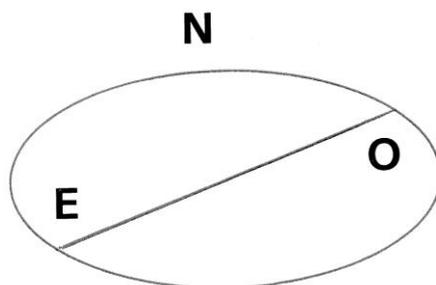
$$32 + 24 = (3 \cdot 10^1 + 2 \cdot 10^0) + (2 \cdot 10^1 + 4 \cdot 10^0) = (3 + 2) \cdot 10^1 + (2 + 4) \cdot 10^0 = 5 \cdot 10^1 + 6 \cdot 10^0 = 56$$

The calculation can be simplified with the so-called column method.

In short, Peano's postulates allow us to rigorously systematize all the operations in \mathbb{N} and their properties.

1.3. Even and Odd numbers

In \mathbb{N} there are the even numbers E and the odd numbers O . They are subsets of \mathbb{N} : $E \cup O = N$, $E \cap O = \emptyset$, that is to say that they constitute a partition of \mathbb{N} .



An even number can be written $e = 2n$, an odd number $o = 2n + 1$.

It is shown that:

- 1) $e_1 + e_2 = e_3$
- 2) $o_1 + o_2 = e$
- 3) $e_1 + o_1 = o_2$
- 4) $e_1 \cdot e_2 = 4e_3$ (doubly even)
- 5) $o_1 \cdot o_2 = o_3$.

We demonstrate :

- 1) $2n_1 + 2n_2 = 2(n_1 + n_2) = 2s \in E$.
- 2) $(2n_1 + 1) + (2n_2 + 1) = 2n_1 + 2n_2 + 2 = 2(n_1 + n_2 + 1) = 2s \in E$.
- 3) $2n_1 + (2n_2 + 1) = 2n_1 + 2n_2 + 1 = 2(n_1 + n_2) + 1 = 2s + 1 \in O$
- 4) $2n_1 \cdot 2n_2 = 2 \cdot 2 \cdot n_1 n_2 = 4p \in E$
- 5) $(2n_1 + 1)(2n_2 + 1) = 4n_1 n_2 + 2n_1 + 2n_2 + 1 = 2(2n_1 n_2 + n_1 + n_2) + 1 = 2p + 1 \in O$

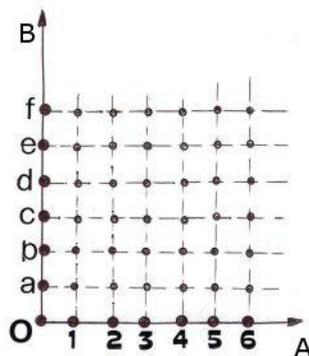
2. Cartesian product

The Cartesian product between two sets A and B is defined, and is denoted by $A \times B$, as the set of all pairs (a, b) such that the first element a belongs to A and the second b belongs to B. That is :

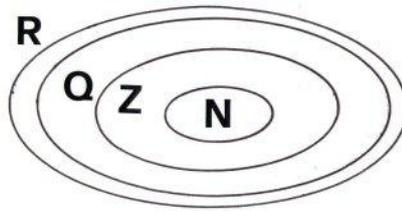
$$A \times B = \{(a, b) : \forall a \in A, \forall b \in B\}$$

The number of pairs is given by the product resulting from the multiplication of $\text{card}(A) \times \text{card}(B)$, where the cardinality of a set is the number of elements it contains. The figure shows the Cartesian product $A \times B$, with $A = \{1, 2, 3, 4, 5, 6\}$ $\text{card}(A) = 6$ and $B = \{a, b, c, d, e, f\}$

$\text{Card}(B) = 6$.



From \mathbb{N} we obtain the other numerical sets. From the Cartesian product $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, by defining an equivalence relation in it (see App. 2./1) we create the relative integers \mathbb{Z} (see App. 2./2), from them, with an equivalence relation in $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ we obtain the rational numbers \mathbb{Q} (see App. 2./3), and from these, with the Dedekind sections (see App. 2./4) the irrational numbers which combined with the rational numbers give the real numbers \mathbb{R} . We can therefore affirm that all numbers originate from Peano's postulates.



Appendix 2./1

A binary relation R on a set A is defined as a subset of the Cartesian product $A \times A$, $R \subset A \times A$. Two elements a and b of A are in relation if: $(a,b) \in R$, and is written aRb .

A relation R is an equivalence relation (generally denoted by the symbol \sim) if it satisfies the properties: reflexive, symmetric, and transitive. That is:

- 1) Reflexive: $a \sim a, \forall a \in A$
- 2) Symmetric: $a \sim b \rightarrow b \sim a, \forall a, b \in A$
- 3) Transitive: $a \sim b$ and $b \sim c \rightarrow a \sim c, \forall a, b, c \in A$

All pairs of $A \times A$ satisfying \sim constitute a quotient set of A and it is denoted by $R = A/\sim$.

Appendix 2./2

The set of relative integers \mathbb{Z} is defined from the set \mathbb{N} of natural numbers using the concept of quotient set. Consider the Cartesian product $\mathbb{N} \times \mathbb{N}$ and the relation R within it:

$$(m,n)R(m_1,n_1) \leftrightarrow m + n_1 = m_1 + n$$

R is an equivalence relation. In fact :

- 1) Reflexive: $(m,n) \sim (m,n)$ because $m + n = m + n$
- 2) Symmetric: if $(m,n) \sim (m_1,n_1)$ then $m + n_1 = m_1 + n$, from which (by the commutability of the equality) $m_1 + n = m + n_1$ and therefore $(m_1,n_1) \sim (m,n)$
- 3) Transitive: if $(m,n) \sim (m_1,n_1)$ and $(m_1,n_1) \sim (m_2,n_2)$ then: $m + n_1 = m_1 + n$ and $m_1 + n_2 = m_2 + n_1$, from which, adding, we have: $m + n_1 + m_1 + n_2 = m_1 + n + m_2 + n_1$ and, by the cancellation law, we have: $m + n_2 = m_2 + n$, and therefore $(m,n) \sim (m_2,n_2)$.

So $\mathbb{Z} = \mathbb{N} \times \mathbb{N} / R(\sim)$.

It is easy to show that in every equivalence class $[(m,n)]$ there is one and only one element of the type (m_1,n_1) with $m_1 = 0$ or $n_1 = 0$ therefore for the relative integers we can use the well-known representation: $+m = [(m,0)]$ and $-m = [(0,m)]$ and $0 = [(0,0)]$.

In \mathbb{Z} , the operations of addition and multiplication are defined as follows:

$$\text{Addition: } (m_1,n_1) + (m_2,n_2) = (m_1 + m_2, n_1 + n_2).$$

$$\text{Multiplication: } (m_1,n_1) \cdot (m_2,n_2) = (m_1m_2 + n_1n_2, m_1n_2 + n_1m_2)$$

The operations thus defined are compatible with the equivalence relation. They translate into the usual notations for integers. From the definition of multiplication, the well-known sign rules can be derived: $(+a) \cdot (-b) = (-a) \cdot (+b) = -ab$; $(+a) \cdot (+b) = +ab$; $(-a) \cdot (-b) = +ab$.

Appendix 2./3

Let $Z \times Z$ be the Cartesian product of the relative integers. Consider the relation R : $(a,b)R(c,d) \leftrightarrow ad = bc$. It is easy to show that R is an equivalence relation (\sim). Therefore, the rational numbers are defined by: $\mathbb{Q} = Z \times Z / R(\sim)$. In \mathbb{Q} the addition and multiplication operations are defined as follows:

Addition: $(a,b) + (c,d) = (ad + bc, bd)$

Multiplication: $(a,b) \times (c,d) = (ac, bd)$

Which correspond to the usual fraction operations. With the above definition of equivalence, we get $0 = (0,1)$; $1 = (1,1)$; $-p = -a/b = (-a,b)$; if $p = a/b = (a,b) \rightarrow p^{-1} = 1/p = b/a = (b,a)$.

The equivalence class allows representation as a fraction of the same rational number: $a/b = ka/kb$: $k \neq 0$. Finally, in \mathbb{Q} an ordering can be defined as follows:

$(a,b) \leq (c,d) \leftrightarrow (bd > 0 \text{ and } ad \leq bc) \text{ or } (bd < 0 \text{ and } ad \geq bc)$.

Appendix 2./4

It was Dedekind (Julius Wilhelm Richard Dedekind; Braunschweig 1831–1916), a German mathematician, who gave a rigorous definition of irrational numbers. He defined an irrational number α as the separating element of two sections of rational numbers: $\alpha = (A', A'')$. A pair (A', A'') of rational numbers constitutes a section of the set \mathbb{Q} if:

- a) Every rational number belongs either to A' or to A'' .
- b) Every element of A' is less than every element of A'' .
- c) The set A' has no maximum, and the set A'' has no minimum. That is, however many a' of A' are chosen, there is always an a'^* of A' such that $a'^* > a'$, and however any a' of A' are chosen, there is always an a'^* of A' such that $a'^* < a'$.

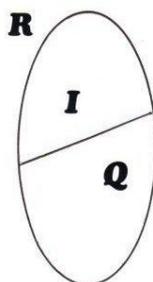
Given this, let $\alpha = (A', A'')$ and $\beta = (B', B'')$, then $\alpha + \beta = (A' + B', A'' + B'')$.

If $\alpha = (A', A'')$ then $-\alpha = (-A'', -A')$, and therefore $\alpha - \beta = \alpha + (-\beta) = (A' - B'', A'' - B')$.

Furthermore $\alpha \cdot \beta = (A' \cdot B', A'' \cdot B'')$.

If $\alpha = (A', A'')$ then $1/\alpha = (1/A'', 1/A')$, and therefore $\alpha/\beta = (A'/B'', A''/B')$.

The rational numbers \mathbb{Q} and the irrational numbers I are such that: $\mathbb{Q} \cup I = \mathbb{R}$ and $\mathbb{Q} \cap I = \emptyset$. That is, they constitute a partition of the Real numbers \mathbb{R} , so called when imaginary numbers were discovered.



The set \mathbb{R} of real numbers is uncountable. To demonstrate that it is not possible to establish a one-to-one correspondence between \mathbb{N} and \mathbb{R} , we reason as follows. Suppose, for the hypothesis, that all real numbers can be listed, for example, in the interval $(0,1)$, and that they are:

- $r_1 = 0.4379542\dots$
- $r_2 = 0.673123\dots$
- $r_3 = 0.865271\dots$
- $r_4 = 0.317181\dots$
- $\dots \dots \dots$

And we establish the correspondence in such a way that each natural number corresponds to the first decimal digit, the second, the third, and so on. It is always possible to construct another real number not included in the list by choosing 0 as the integer part and 1 as the decimal part if the first decimal digit of r_1 is less than 5, or 6 if it is greater than or equal to 5, and so on for the second, the third, and so on. In our case we would have the number $x = 0.1661\dots$ which is different from all the numbers in the list because it has at least one different decimal digit: the one that occupies place $n + 1$. Therefore, the above correspondence does not exhaust all the real numbers in the interval $(0,1)$. Even more so, the set \mathbb{R} .

3. Applications of Induction Principle

Remembering : If p is the proposition to be proved:

- 1) Prove $p(1)$ true
- 2) Assume $p(n)$ true
- 3) Prove $p(n+1)$ true

Then the proposition p is true for $\forall n \in \mathbb{N}$.

Sometimes the property (or proposition) you want to prove holds only for natural integers greater than a certain n_0 . In this case, the principle of induction can be formulated by assuming the domain $\mathbb{N}^* = \{ \forall n \in \mathbb{N} : n > n_0 \}$

Exercise 1

Prove proposition p : the sum of the first n natural numbers (excluding zero) is:
 $s(n) = n \cdot (n+1) / 2$.

- $p(1)$ is true. In fact, $s(1) = 1 \cdot (1+1) / 2 = 1 \cdot 2 / 2 = 1$
- $p(2)$ is true. In fact, $s(2) = 1+2 = 2 \cdot (2+1) / 2 = 6 / 2 = 3$.
- $p(3)$ is true. In fact, $s(3) = 1+2+3 = 3 \cdot (3+1) / 2 = 12 / 2 = 6$
- $p(4)$ is true. In fact, $s(4) = 1+2+3+4 = 4 \cdot (4+1) / 2 = 20 / 2 = 10$
- \dots

But by proceeding in this way, we would have to perform infinite checks on the truth of proposition p . Let's apply the principle of induction.

- 1) We saw that $p(1)$ is true.
- 2) We assume $p(n)$ is true : $s(n+1) = 1+2+3+4+ \dots +n+(n+1) = (n+1) \cdot (n+2) / 2$.
- 3) Having assumed $s(n) = 1+2+3+4+ \dots +n = n \cdot (n+1) / 2$, adding $(n+1)$ to both sides of the equality we have :

$$s(n+1) = 1+2+3+4+ \dots +n+(n+1) = n \cdot (n+1)/2 + (n+1) = [n \cdot (n+1)+2 \cdot (n+1)]/2 = (n+1) \cdot (n+2)/2$$

Since $p(n+1)$ is true, then the proposition $p(n)$ is true $\forall n \in N : n \neq 0 \rightarrow$ QED

Exercise 2

Prove the proposition $p : 1 + 3 + 5 + 7 + \dots + (2n-1) = n^2 \quad \forall n \geq 1$

$$p(1) \text{ is true : } 1 = 1^2 = 1$$

$$p(2) \text{ is true : } 1 + 3 = 2^2 = 4$$

$$p(3) \text{ is true : } 1 + 3 + 5 = 3^2 = 9$$

$$p(4) \text{ is true: } 1 + 3 + 5 + 7 = 4^2 = 16$$

...

$$\text{Suppose } p(n) \text{ is true : } 1 + 3 + 5 + 7 + \dots + (2n-1) = n^2$$

$$\text{Prove } p(n+1) \text{ is true : } 1 + 3 + 5 + 7 + \dots + (2n-1) + (2n+1) = (n+1)^2$$

$$\text{Since } p(n) \text{ is true : } 1 + 3 + 5 + 7 + \dots + (2n-1) = n^2$$

Adding $2n+1$ to both sides of the equality we will have:

$$1+3+5+7+ \dots + (2n-1)+(2n+1) = n^2+2n+1 = (n+1)^2 \rightarrow \text{QED}$$

Exercise 3

Prove the proposition $p : 1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2 = n(n+1)(2n+1)/6 \quad n \geq 1$

$$1) \ p(1) \text{ is true, in fact: } 1^2 = 1 = 1 \cdot 2 \cdot 3/6 = 1$$

$$2) \ p(n) \text{ is true by hypothesis}$$

$$3) \ p(n+1): 1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2 + (n+1)^2 = (n+1)(n+2)(2n+3)/6.$$

$$\begin{aligned} \text{Since } p(n) \text{ is true: } 1^2+2^2+3^2+4^2+ \dots +n^2+(n+1)^2 &= n(n+1)(2n+1)/6 + (n+1)^2 = \\ [n(n+1)(2n+1)+6(n+1)^2]/6 &= (n+1)[n(2n+1)+6(n+1)]/6 = (n+1)(2n^2+n+6n+6)/6 = \\ (n+1)(2n^2+7n+6)/6 &= (n+1)(n+2)(2n+3)/6 \rightarrow \text{QED} \end{aligned}$$

note: applying Ruffini's rule the trinomial $2n^2+7n+6 = (n+2)(2n+3)$

Exercise 4

Prove the proposition $p : 1^3 + 2^3 + 3^3 + 4^3 + \dots + n^3 = (1+2+3+4+ \dots +n)^2 = n^2 \cdot (n+1)^2/4 \quad \forall n \geq 1$

Some examples :

$$p(1) : 1^3 = 1 = 1^2 = 1 \cdot 2^2/4 = 1$$

$$p(2) : 1^3+2^3 = 9 = (1+2)^2 = 2^2 \cdot 3^2/4 = 9$$

$$p(3) : 1^3+2^3+3^3 = 36 = (1+2+3)^2 = 3^2 \cdot 4^2/4 = 36$$

$$p(4) : 1^3+2^3+3^3+4^3 = 100 = (1+2+3+4)^2 = 4^2 \cdot 5^2/4 = 100$$

...

Demonstration :

$$1) \ p(1) \text{ true}$$

$$2) \ p(n): 1^3+2^3+3^3+4^3+ \dots +n^3 = (1+2+3+4+ \dots +n)^2 = n^2 \cdot (n+1)^2/4 \text{ supposed true}$$

$$3) \ \text{We prove } p(n+1) \text{ true:}$$

$$1^3+2^3+3^3+4^3+ \dots +n^3+(n+1)^3= (1+2+3+4+ \dots +n+(n+1))^2 = (n+1)^2 \cdot (n+2)^2/4$$

With reference exercise 1, where : $1+2+3+4+ \dots +n = n(n+1)/2$

Then :

$$1^3+2^3+3^3+4^3+ \dots +n^3+(n+1)^3 = (1+2+3+4+ \dots +n+(n+1))^2 = \{n(n+1)/2 + (n+1)\}^2 = \{n(n+1) + 2(n+1)\}^2/4 = \{(n+1)(n+2)\}^2/4 = (n+1)^2 \cdot (n+2)^2/4 \rightarrow \text{QED}$$

Or :

$$\text{If } p(n) \text{ is true : } 1^3+2^3+3^3+4^3+ \dots +n^3 = n^2 \cdot (n+1)^2/4$$

We have :

$$1^3+2^3+3^3+4^3+ \dots +n^3+(n+1)^3 = n^2 \cdot (n+1)^2/4 + (n+1)^3 = (n+1)^2(n^2/4+n+1) = (n+1)^2(n^2+4n+4)/4 = (n+1)^2(n+2)^2/4 \rightarrow \text{QED}$$

Exercise 5

Prove the proposition $p : 2^3+4^3+6^3+8^3+ \dots +(2n)^3 = 2n^2(n+1)^2 \quad \forall n \geq 1$

Some examples :

$$p(1) : 2^3 = 8 = 2 \cdot 1^2 \cdot 2^2 = 8$$

$$p(2) : 2^3+4^3 = 72 = 2 \cdot 2^2 \cdot 3^2 = 72$$

$$p(3) : 2^3+4^3+6^3 = 288 = 2 \cdot 3^2 \cdot 4^2 = 288$$

$$p(4) : 2^3+4^3+6^3+8^3 = 800 = 2 \cdot 4^2 \cdot 5^2 = 800$$

...

Demonstration :

1) $p(1)$ true (see examples)

2) $p(n)$ $2^3+4^3+6^3+8^3+ \dots +(2n)^3 = 2n^2(n+1)^2$ it's supposed true

3) $p(n+1)$: $2^3+4^3+6^3+8^3+ \dots +(2n)^3+(2n+2)^3 = 2(n+1)^2(n+2)^2$

From $p(n)$ true we have: $2^3+4^3+6^3+8^3+ \dots +(2n)^3+(2n+2)^3 = 2n^2(n+1)^2+8(n+1)^3 = 2(n+1)^2[n^2+4(n+1)] = 2(n+1)^2(n^2+4n+4) = 2(n+1)^2(n+2)^2 \rightarrow \text{QED}$

Exercise 6

We prove the proposition $p: 1 \cdot 5^1+2 \cdot 5^2+3 \cdot 5^3+4 \cdot 5^4+ \dots +n \cdot 5^n = \{5+(4n-1) \cdot 5^{n+1}\}/16$

We show some examples :

$$p(1) : 1 \cdot 5^1 = 5 = (5+3 \cdot 5^2)/16 = (5+75)/16 = 80/16 = 5$$

$$p(2) : 1 \cdot 5^1+2 \cdot 5^2 = 55 = (5+7 \cdot 5^3)/16 = (5+875)/16 = 880/16 = 55$$

$$p(3) : 1 \cdot 5^1+2 \cdot 5^2+3 \cdot 5^3 = 430 = (5+11 \cdot 5^4)/16 = (5+6875)/16 = 6880/16 = 430$$

$$p(4) : 1 \cdot 5^1+2 \cdot 5^2+3 \cdot 5^3+4 \cdot 5^4 = 2930 = (5+15 \cdot 5^5)/16 = (5+46875)/16 = 46880/16 = 2930$$

...

Demonstration :

1) $p(1)$ is true from example

2) $p(n)$ it's supposed true

3) We demonstrate that $p(n+1)$ it's true

$$i. e. : 1 \cdot 5^1 + 2 \cdot 5^2 + 3 \cdot 5^3 + 4 \cdot 5^4 + \dots + n \cdot 5^n + (n+1) \cdot 5^{n+1} = \{5 + (4n+3) \cdot 5^{n+2}\} / 16$$

From $p(n)$ true, result :

$$1 \cdot 5^1 + 2 \cdot 5^2 + 3 \cdot 5^3 + 4 \cdot 5^4 + \dots + n \cdot 5^n + (n+1) \cdot 5^{n+1} = \{5 + (4n-1) \cdot 5^{n+1}\} / 16 + (n+1) \cdot 5^{n+1} = \\ \{5 + (4n-1) \cdot 5^{n+1} + 16 \cdot (n+1) \cdot 5^{n+1}\} / 16 = \{5 + (4n-1+16n+16) \cdot 5^{n+1}\} / 16 = \{5 + (20n+15) \cdot 5^{n+1}\} / 16 = \\ \{5 + 5(4n+3) \cdot 5^{n+1}\} / 16 = \{5 + (4n+3) \cdot 5^{n+2}\} / 16 \rightarrow \text{QED}$$

Exercise 7

We prove the proposition $p : 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \dots + n \cdot (n+1) = n \cdot (n+1) \cdot (n+2) / 3 \quad \forall n \geq 1$

We check the proposition for some examples :

$$p(1) : 1 \cdot 2 = 2 = 1 \cdot (1+1) \cdot (1+2) / 3 = 6 / 3 = 2 \\ p(2) : 1 \cdot 2 + 2 \cdot 3 = 8 = 2 \cdot (2+1) \cdot (2+2) / 3 = 24 / 3 = 8 \\ p(3) : 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 = 20 = 3 \cdot (3+1) \cdot (3+2) / 3 = 60 / 3 = 20 \\ p(4) : 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 = 40 = 4 \cdot (4+1) \cdot (4+2) / 3 = 40 \\ \dots$$

Demonstration :

- 1) $p(1)$ true (see first example)
- 2) $p(n)$ it's supposed true
- 3) We demonstrate that $p(n+1)$ it's true :
i.e. : $p(n+1) : 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \dots + n \cdot (n+1) + (n+1) \cdot (n+2) = (n+1) \cdot (n+2) \cdot (n+3) / 3$

From $p(n)$ true, result :

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \dots + n \cdot (n+1) + (n+1) \cdot (n+2) = n \cdot (n+1) \cdot (n+2) / 3 + (n+1) \cdot (n+2) = \\ [n \cdot (n+1) \cdot (n+2) + 3 \cdot (n+1) \cdot (n+2)] / 3 = (n+1) \cdot (n+2) \cdot (n+3) / 3 \rightarrow \text{QED}$$

Exercise 8

Prove the proposition $p :$

$$1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \dots + 1/(2n-1) - 1/2n = 1/(n+1) + 1/(n+2) + 1/(n+3) + \dots + 1/2n \quad \forall n \geq 1$$

Check p for $n = 1, n = 2, n = 3, n = 4$

$$p(1) : 1 - 1/2 = 1/2 = 1/2 \\ p(2) : 1 - 1/2 + 1/3 - 1/4 = 1/2 + 1/12 = 7/12 = 1/3 + 1/4 = 7/12 \\ p(3) : 1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 = 1/2 + 1/12 + 1/30 = 37/60 = 1/4 + 1/5 + 1/6 = 37/60 \\ p(4) : 1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + 1/7 - 1/8 = 1/2 + 1/12 + 1/30 + 1/56 = 533/840 = 1/5 + 1/6 + 1/7 + 1/8 = \\ 533/840 \\ \dots$$

Demonstration :

- 1) $p(1)$ it's true from example
- 2) $p(n)$ it's supposed true
- 3°) We prove that $p(n+1)$ results true : $1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \dots + 1/(2n-1) - 1/2n + 1/(2n+1) - 1/(2n+2) = 1/(n+1) + 1/(n+2) + 1/(n+3) + \dots + 1/2n + 1/(2n+2)$

From hypothesis $p(n)$ true, result : $(1-1/2)+(1/3-1/4)+(1/5-1/6)+ \dots + (1/(2n-1)-1/2n) + (1/(2n+1)+1/(2n+2)) = 1/2+1/12+1/30+1/56+ \dots + [1/2n(2n-1)] + [1/(2n+2)(2n+1)] = 1/(n+1)+1/(n+2)+1/(n+3)+ \dots +1/2n+1/(2n+2) \rightarrow \text{QED}$

Exercise 9

Prove $p : 1/2+2/2^2+3/2^3+4/2^4 \dots +n/2^n = 2 - (n+2)/2^n \quad \forall n \geq 1$

We check the proposition for some values :

$$n = 1 \rightarrow 1/2 = 2 - 3/2 = 1/2$$

$$n = 2 \rightarrow 1/2+1/2 = 2 - 4/4 = 1$$

$$n = 3 \rightarrow 1/2+1/2+3/8 = 11/8 = 2 - 5/8 = 11/8$$

$$n = 4 \rightarrow 1/2+1/2+3/8+1/4 = 13/8 = 2 - 3/8 = 13/8$$

...

Demonstration :

1) $p(1)$ it's true

2) $p(n)$ it's supposed true

3) we prove that results true $p(n+1) : 1/2+2/2^2+3/2^3+4/2^4 \dots +n/2^n+(n+1)/2^{n+1} = 2 - (n+3)/2^{n+1}$

Being true $p(n)$ we have :

$$1/2+2/2^2+3/2^3+4/2^4 \dots +n/2^n+(n+1)/2^{n+1} = 2 - (n+2)/2^n + (n+1)/2^{n+1} = 2 - (n+2)/2^n + (n+1)/2 \cdot 2^n = (4 \cdot 2^n - 2n - 4 + n + 1)/2 \cdot 2^n = (4 \cdot 2^n - n - 3)/2 \cdot 2^n = 4 \cdot 2^n / 2 \cdot 2^n - (n+3)/2 \cdot 2^n = 2 - (n+3)/2^{n+1} \rightarrow \text{QED}$$

Exercise 10

Prove the proposition $p : \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \dots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)} \quad \forall n \geq 1$

We check it for $n = 1, n = 2, n = 3$

$$p(1) : \frac{1}{1 \cdot 2 \cdot 3} = \frac{1(1+3)}{4(1+1)(2+1)} = 4/24 = 1/6$$

$$p(2) : \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} = 1/6 + 1/24 = 5/24 = \frac{2(2+3)}{4(2+1)(2+2)} = 10/48 = 5/24$$

$$p(3) : \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} = 1/6 + 1/24 + 1/60 = 27/120 = 9/40 = \frac{3(3+3)}{4(3+1)(3+2)} = 18/80 = 9/40$$

Demonstration :

1) $p(1)$ it's true as checked in the example

2) $p(n)$ it's supposed true

3) We prove that it's true also $p(n+1) : \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \dots + \frac{1}{n(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} = \frac{(n+1)(n+4)}{4(n+2)(n+3)}$

From $p(n)$ we have : $p(n+1) : \frac{n(n+3)}{4(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} = \frac{n(n+3)(n+3)+4}{4(n+1)(n+2)(n+3)} = \frac{n(n+3)^2 + 4}{4(n+1)(n+2)(n+3)} = \frac{n^3+6n^2+9n+4}{4(n+1)(n+2)(n+3)}$, factoring with Ruffini $n^3+6n^2+9n+4 = (n+1)(n^2+5n+4) = (n+1)(n+1)(n+4)$

$$\rightarrow \frac{(n+1)(n+1)(n+4)}{4(n+1)(n+2)(n+3)} = \frac{(n+1)(n+4)}{4(n+2)(n+3)} \rightarrow \text{QED}$$

Exercise 11

Prove the proposition $p : (1-1/4) \cdot (1-1/9) \cdot (1-1/16) \cdot (1-1/25) \cdot \dots \cdot (1-1/n^2) = (1+n)/2n \quad \forall n \geq 2$

$$n = 2 \rightarrow p(2) : (1-1/4) = 3/4 = (1+2)/4 = 3/4$$

$$n = 3 \rightarrow p(3) : (1-1/4) \cdot (1-1/9) = 3/4 \cdot 8/9 = 2/3 = (1+3)/6 = 2/3$$

$$n = 4 \rightarrow p(4) : (1-1/4) \cdot (1-1/9) \cdot (1-1/16) = 3/4 \cdot 8/9 \cdot 15/16 = 5/8 = (1+4)/16 = 5/16$$

...

Demonstration :

1) $p(2)$ it's true by example

2) $p(n)$ it's supposed true

3) We prove true $p(n+1) : (1-1/4) \cdot (1-1/9) \cdot (1-1/16) \cdot (1-1/25) \cdot \dots \cdot (1-1/n^2) \cdot (1-1/(n+1)^2) = (2+n)/2(n+1)$

$$\text{From 2) we have : } \frac{1+n}{2n} \cdot \left[1 - \frac{1}{(n+1)^2} \right] = \frac{1+n}{2n} \cdot \left[\frac{n^2+2n}{(n+1)^2} \right] = \frac{1+n}{2n} \cdot \frac{n(n+2)}{(n+1)^2} = \frac{2+n}{2(n+1)} \rightarrow \text{QED}$$

Exercise 12

We prove the proposition $p : (1-1/2) \cdot (1-1/3) \cdot (1-1/4) \cdot \dots \cdot (1-1/n) = 1/n \quad \forall n \geq 2$

$$n = 2 \rightarrow p(2) : (1-1/2) = 1/2 = 1/2$$

$$n = 3 \rightarrow p(3) : (1-1/2) \cdot (1-1/3) = 1/2 \cdot 2/3 = 1/3 = 1/3$$

$$n = 4 \rightarrow p(4) : (1-1/2) \cdot (1-1/3) \cdot (1-1/4) = 1/2 \cdot 2/3 \cdot 3/4 = 1/4 = 1/4$$

...

Dimostrazione:

1) $p(2)$ it's true

2) $p(n)$ it's supposed true

3) We prove true $p(n+1) : (1-1/2) \cdot (1-1/3) \cdot (1-1/4) \cdot \dots \cdot (1-1/n) \cdot (1-1/(n+1)) = 1/(n+1)$

$$\text{From 2) we have : } \frac{1}{n} \cdot \left(1 - \frac{1}{n+1} \right) = \frac{1}{n} \cdot \frac{n}{n+1} = \frac{1}{n+1} \rightarrow \text{QED}$$

Exercise 13

Prove the proposition $p(n) : 1-x^n = (1-x)(1+x+x^2+x^3+\dots+x^{n-1}) \quad \forall n \in \mathbb{R}, \forall n \geq 2$

$$n = 2 \rightarrow 1-x^2 = (1-x)(1+x) = 1-x^2$$

$$n = 3 \rightarrow 1-x^3 = (1-x)(1+x+x^2) = 1+x+x^2-x-x^2-x^3 = 1-x^3$$

$$n = 4 \rightarrow 1-x^4 = (1-x)(1+x+x^2+x^3) = (1-x)(1+x) + (1-x)(x^2+x^3) = 1-x^2+x^2(1-x)(1+x) = 1-x^2+x^2(1-x^2) = 1-x^2+x^2-x^4 = 1-x^4$$

...

Demonstration :

- 1) $p(2)$ it's true
- 2) $p(n)$ it's supposed true
- 3) We prove true $p(n+1)$: $1-x^{n+1} = (1-x)(1+x+x^2+x^3+ \dots +x^{n-1}+x^n)$

From hypothesis $p(n)$ true, we have : $1-x^{n+1} = (1-x)(1+x+x^2+x^3+ \dots +x^{n-1}+x^n) = (1-x)(1+x+x^2+x^3+ \dots +x^{n-1}) + (1-x)x^n = 1-x^n+x^n-x^{n+1} = 1-x^{n+1} \rightarrow$ QED

Exercise 14

Prove the proposition p : n^3-n+6 è divisibile per 3 $\forall n \geq 0$

- $p(0)$: $6/3 = 2$
 $p(1)$: $(1-1+6)/3 = 2$
 $p(2)$: $(8-2+6)/3 = 12/3 = 4$
 $p(3)$: $(27-3+6)/3 = 30/6 = 5$
 $p(4)$: $(64-4+6)/3 = 66/6 = 11$
 $p(5)$: $(125-5+6)/3 = 126/3 = 42$
...

Demonstration :

- 1) $p(0)$ it's true
- 2) $p(n)$ it's supposed true
- 3°) we prove true $p(n+1)$: $(n+1)^3-(n+1)+6$ è divisibile per 3

$(n+1)^3-(n+1)+6 = n^3+3n^2+3n+1-n-1+6 = n^3+3n^2+2n+6 = n^3+3n^2+3n-n+6 = n^3-n+6+3n(n+1)$;
being true $p(n) \rightarrow n^3-n+6$ it's divisible by 3, hence, being $3n(n+1)$ divisible by 3 \rightarrow
 $n^3-n+6+3n(n+1)$ it's divisible by 3 (*) $\rightarrow (n+1)^3-(n+1)+6$ it's divisible by 3 \rightarrow QED

(*) : If two natural numbers are both divisible by the same number, then their sum is divisible by the same number : $m/q + n/q = (m + n)/q$.

Exercise 15

Prove the proposition p : $11^n - 1$ it's divisible by 10 $\forall n \geq 1$

Demonstration :

- 1) $p(1)$ it's true : $11 - 1 = 10/10 = 1$
- 2) $p(n)$ it's supposed true : $11^n - 1$ it's divisible by 10
- 3) we demonstrate that is true $p(n+1)$: $11^{n+1} - 1$ it's divisible by 10

$11^{n+1} - 1 = 11^n \cdot 11 - 1 = 11^n \cdot (10 + 1) - 1 = 11^n \cdot 10 + 11^n - 1$, but $11^n \cdot 10$ it's surely divisible by 10 and $11^n - 1$ divisible by 10 from $p(n)$ true, it follows that their sum is also divisible by 10 (see ex. 14) \rightarrow QED

Exercise 16

Prove the proposition $p : 4^{2n+1} + 3^{n+2}$ it's divisible by 13 $\forall n \geq 0$

$$p(0) : 4 + 9 = 13/13 = 1$$

$$p(1) : 4^3 + 3^3 = 4^2 \cdot 4 + 3 \cdot 9 = 64 + 27 = 91/13 = 7$$

$$p(2) : 4^5 + 3^4 = 4^4 \cdot 4 + 3^2 \cdot 9 = 1024 + 81 = 1105/13 = 85$$

$$p(3) : 4^7 + 3^5 = 4^6 \cdot 4 + 3^3 \cdot 9 = 16384 + 243 = 16627/13 = 1279$$

...

Demonstration :

1) $p(0)$ it's true

2) $p(n)$ it' supposed true

3) We prove true $p(n+1) : 4^{2n+3} + 3^{n+3}$ it's divisible by 13

$4^{2n+1} + 3^{n+2} = 4^{2n} \cdot 4 + 3^n \cdot 9$ it's divisible by 13 being true $p(n) \rightarrow 4^{2n+3} + 3^{n+3} = 4^{2n} \cdot 4^2 \cdot 4 + 3^n \cdot 3 \cdot 9$
It's divisible by 13 (*).

$$(*) : 4 + 9 = 13/13 = 1$$

$$4^2 \cdot 4 + 3 \cdot 9 = 64 + 27 = 91/13 = 7$$

Exercise 17 (Mengoli series) [v. § 4.]

Prove the proposition $p : 1/1 \cdot 2 + 1/2 \cdot 3 + 1/3 \cdot 4 + \dots + 1/n(n+1) = 1 - 1/(n+1) \quad \forall n \geq 1$

$$p(1) : 1/2 = 1 - 1/2 = 1/2 \quad \text{it's true}$$

$$p(n) : 1/1 \cdot 2 + 1/2 \cdot 3 + 1/3 \cdot 4 + \dots + 1/n(n+1) = 1 - 1/(n+1) \quad \text{it's supposed true for hypothesis}$$

$$p(n+1) : 1/1 \cdot 2 + 1/2 \cdot 3 + 1/3 \cdot 4 + \dots + 1/n(n+1) + 1/(n+1)(n+2) = 1 - 1/(n+2)$$

Bearing in mind that : $1/1 \cdot 2 = 1 - 1/2$; $1/2 \cdot 3 = 1/2 - 1/3$; $1/3 \cdot 4 = 1/3 - 1/4$; $1/n(n+1) = 1/n - 1/(n+1)$;
we have : $1 - 1/2 + 1/2 - 1/3 + 1/3 - 1/4 + \dots + 1/n - 1/(n+1) = 1 - 1/(n+1)$.

Then :

$$p(n+1) : 1/1 \cdot 2 + 1/2 \cdot 3 + 1/3 \cdot 4 + \dots + 1/n(n+1) + 1/(n+1)(n+2) = 1 - 1/(n+1) + 1/(n+1) - 1/(n+2) = 1 - 1/(n+2)$$

Hence \rightarrow QED

Exercise 18

Prove Bernoulli's inequality : $p : (1+a)^n \geq 1+na \quad \forall a \in R, \forall n \geq 0$

$$p(1) : 1+a \geq 1+a \quad \text{it's true}$$

$$p(n) : (1+a)^n \geq 1+na \quad \text{it's supposed true}$$

$$\text{We prove } p(n+1) : (1+a)^{n+1} \geq 1+(n+1)a$$

From $(1+a)^n \geq 1+na$ multiplying both members by $(1+a) \rightarrow (1+a)^n \cdot (1+a) \geq (1+na) \cdot (1+a) = 1+a+na+na^2 \geq 1+a+na = 1+(n+1)a \rightarrow$ QED

Particular case : prove the proposition $p : (1+\sqrt{2})^n \geq 1+n\sqrt{2} \quad \forall n \geq 0$

$$p(0) : 1 \geq 1 \quad \text{it's true}$$

$p(n) : (1+\sqrt{2})^n \geq 1$ it's supposed true for hypothesis

We prove true $p(n+1) : (1+\sqrt{2})^{n+1} \geq 1+(n+1)\sqrt{2}$

$$(1+\sqrt{2})^{n+1} = (1+\sqrt{2})^n (1+\sqrt{2}) \geq (1+n\sqrt{2})(1+\sqrt{2}) = 1+\sqrt{2}+n\sqrt{2}+2n = 1+2n+(n+1)\sqrt{2} > 1+(n+1)\sqrt{2} \rightarrow (1+\sqrt{2})^{n+1} \geq 1+(n+1)\sqrt{2} \rightarrow \text{QED}$$

Exercise 19

Prove the proposition $p : 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + n \cdot n! = (n+1)! - 1 \quad \forall n \geq 0$

$p(1) : 1! = (1+1)! - 1 = 2! - 1 = 1$ it's true

$p(n)$ it's supposed true

We prove $p(n+1) : 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + n \cdot n! + (n+1) \cdot (n+1)! = (n+2)! - 1$

Being true $p(n)$ result :

$$p(n+1) : [1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + n \cdot n!] + (n+1) \cdot (n+1)! = (n+1)! - 1 + (n+1) \cdot (n+1)! = (1+n+1) \cdot (n+1)! - 1 = (n+2) \cdot (n+1)! - 1 = (n+2)! - 1 \rightarrow \text{QED}$$

Exercise 20

Prove the proposition $p : n! > 2^n \quad \forall n \geq 4$

$p(4) : 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 > 2^4 = 16$ it's true

$p(n)$ it's supposed true

We prove $p(n+1) : (n+1)! > 2^{n+1}$

Being true $p(n)$ we have : $(n+1)! = (n+1) \cdot n! > (n+1) \cdot 2^n$ but $(n+1) > 2$ for $n \geq 2 \rightarrow (n+1) \cdot n! > 2 \cdot 2^n = 2^{n+1} \rightarrow (n+1)! > 2^{n+1} \rightarrow \text{QED}$

Exercise 21

Prove the proposition $p : n! > 3^n \quad \forall n \geq 7$

$p(7) : 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040 > 3^7 = 2187$ it's true

$p(n)$ it's supposed true

We prove $p(n+1) : (n+1)! > 3^{n+1}$

Being true $p(n)$ we have : $(n+1)! = (n+1) \cdot n! > (n+1) \cdot 3^n$ but $(n+1) > 3$ for $n \geq 3 \rightarrow (n+1) \cdot n! > 3 \cdot 3^n = 3^{n+1} \rightarrow (n+1)! > 3^{n+1} \rightarrow \text{QED}$

By similar procedures we can demonstrate the following inequalities :

$$n! > 4^n \quad \forall n \geq 9$$

$$n! > 5^n \quad \forall n \geq 12$$

... ..

Exercise 22

Prove that the proposition $p : n! > n^n$ it's false $\forall n \geq 2$

$p(2) : 2! > 2^2 \rightarrow 2 > 4$ it's false

$p(n) : n! > n^n$ it's supposed false
 dimostriamo che $p(n+1) : (n+1)! > (n+1)^{n+1}$ it's also false

$(n+1)! > (n+1)^{n+1} \rightarrow (n+1) \cdot n! > (n+1) \cdot (n+1)^n \rightarrow n! > (n+1)^n$ being $n! > n^n$ false, also $n! > (n+1)^n$ it's false $\rightarrow (n+1)! > (n+1)^{n+1}$ it's false $\rightarrow n! > n^n$ it's false \rightarrow QED

4. Mengoli's series

Mengoli's series (Pietro Mengoli; Bologna 1626 – 1686), has $a_n = 1/n(n+1)$ which can be written $(1+n-n)/n(n+1) = (1+n)/n(n+1) - n/n(n+1) = 1/n - 1/(n+1)$. So the partial sum results :

$$S_n = 1/1 \cdot 2 + 1/2 \cdot 3 + 1/3 \cdot 4 + \dots + 1/n(n+1) = 1 - 1/2 + 1/2 - 1/3 + 1/3 - 1/4 + \dots 1/(n-1) + 1/n - 1/(n+1) = 1 - 1/(n+1) = 1 \text{ if } n \rightarrow +\infty$$

If we compared Mengoli's series with harmonic series of order 2 we have :

$$1/2^2 + 1/3^2 + 1/4^2 + \dots + 1/(n+1)^2 + \dots : 1/2^2 < 1/2 \cdot 3; 1/3^2 < 1/3 \cdot 4; \dots$$

being $1/(n+1)^2 < 1/n(n+1)$, as is easy to verify, it's a minority of the Mengoli's series, and therefore it is also convergent and its sum is : $s - 1 < 1$, i.e. $s < 2$.

Euler, for the harmonic series of order 2 has found : $s = \pi^2/6 = 1,6432\dots$

5. Historical note

Giuseppe Peano, born in Spinetta (CN) in 1858, after high school, studied mathematics at the University of Turin, where he taught infinitesimal calculus starting in 1890. An eccentric by nature, he was repeatedly removed from teaching despite his international fame. In 1932, Bertrand Russell said: *"I felt great admiration for him when I first met him at the 1900 Philosophy Congress, and I was dominated by the precision of his mind."*

He also studied fractals. The broken line known as the Peano curve is famous. A constellation among the stars in the sky is dedicated to him. His most famous phrase was : *"It turns out that the words of mathematics, and in general of every science, are international, that is, common to European languages, from Italian to English, from Spanish to Russian."*

Memories of the great mathematician and his family life are recounted with delight and admiration in the biographical novel *Giovinezza inventata* by his great-niece Lalla Romano, a famous writer and poet. He died at night in 1932 of a heart attack in his country home in Cavoretto, near Turin.

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