

# Integer Encoding Symmetries of the Riemann Zeta Function

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## Abstract:

Here I present a derivation of an equation whose solution sets are the trivial and nontrivial zeros of the Riemann Zeta Function. I demonstrate how the trivial solutions are directly encoded by integer inputs and how these can be mapped by a symmetry to positive odd integers. I extend this insight to encode the even integers, and map these to the negative odd integers, which provides an explicit connection between particular values of the Riemann Zeta Function which have historical and ongoing research interest. I then extend this symmetry to the nontrivial zeroes, and demonstrate the dependence of the critical line in producing this symmetry. Finally, I note the distribution of the nontrivial zeroes have a correspondence with the distribution of trivial zeroes, and provide a first order approximation of this correspondence.

## Introduction:

The zeroes of the Riemann Zeta Function rely on the unique property of the ratio (1):

$$\frac{a + b i}{(a + b i) - 1} = \frac{2 b - i}{2 b + i} \text{ where } a = \frac{1}{2} \quad (1)$$

In that for any choice of b, this ratio produces a unit scaled complex number with decreasing exponent c for increasing b.

$$\frac{2 b - i}{2 b + i} = e^{-c i}$$

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$$c = i \log \left( \frac{2 b - i}{2 b + i} \right) \quad (2)$$

This exponent  $c$ , can be associated with a unique value  $q$ :

$$\begin{aligned}\log(c + 2\pi) &= \log\left(\frac{1}{q} + 1\right) \\ i \log\left(\frac{2b-i}{2b+i}\right) + 2\pi &= \frac{1}{q} + 1 \\ q &= \frac{1}{i \log\left(\frac{2b-i}{2b+i}\right) + 2\pi - 1}\end{aligned}\tag{3}$$

This value  $q$ , can be replicated with the equation (4):

$$q = \frac{1}{-1 + e^{i(2\pi+x)} e^{i(2\pi+y)}}\tag{4}$$

Where,

$$\operatorname{Im}\left(\frac{1}{-1 + e^{i(2\pi+x)} e^{i(2\pi+y)}}\right) = 0$$

Which gives an expression for  $x$  in terms of  $y$ :

$$x = -2\pi + 2\pi \sec(y)\tag{5}$$

Substituting (5) for  $x$  in (4).

$$q = \frac{1}{-1 + e^{2i e^{iy} \pi \sec(y)}}\tag{6}$$

We can then equate (3) and (6):

$$\frac{1}{i \log\left(\frac{2b-i}{2b+i}\right) + 2\pi - 1} = \frac{1}{-1 + e^{2i e^{iy} \pi \sec(y)}} \quad (7)$$

Is it possible to solve for b in terms of y:

$$\begin{aligned} i \log\left(\frac{2b-i}{2b+i}\right) + 2\pi - 1 &= -1 + e^{2i e^{iy} \pi \sec(y)} \\ i \log\left(\frac{2b-i}{2b+i}\right) &= e^{-2\pi \tan(y)} - 2\pi \\ b &= \frac{i \left(1 + e^{i e^{-2\pi \tan(y)}}\right)}{2 \left(-1 + e^{i e^{-2\pi \tan(y)}}\right)} \end{aligned} \quad (8)$$

Likewise, you can solve for y in terms of b:

$$y = -\tan^{-1} \left( \frac{\log\left(2\pi - i \log\left(\frac{i+2b}{-i+2b}\right)\right)}{2\pi} \right) \quad (9)$$

A list of corresponding y values for the first 40 zeroes of the Zeta Function is included in Appendix A.

## Integer mapping of the Trivial Zeroes of the Zeta Function

Finally, the point being that it is possible to construct the function (10) from (4) and (5):

$$f(y) = \frac{1}{1 - e^{i e^{-2\pi \tan(y)}}} \quad (10)$$

Such that it generates the solutions to the zeroes of the zeta function.

$$\zeta(f(y)) = 0$$

This function is particularly useful because it has 2 valid solution sets. The first solution set encodes the trivial zeroes of the Zeta Function through integer inputs:

$$y = \pi - i \tanh^{-1} \left( \frac{1}{2\pi} \left( 2\pi - i \log \left( 2\pi - i \log \left( 1 + \frac{1}{2n} \right) \right) \right) \right) \quad (11)$$
$$f(y) = -2n$$

Most remarkable is that it is possible to map from the trivial zeroes to the odd integers with single sign change for y:

$$y = \pi - i \tanh^{-1} \left( \frac{1}{2\pi} \left( 2\pi - i \log \left( 2\pi + i \log \left( 1 + \frac{1}{2n} \right) \right) \right) \right)$$
$$f(y) = 2n + 1$$

Likewise, the positive even integers (although not zeroes) can be encoded by:

$$y = \pi - i \tanh^{-1} \left( \frac{1}{2\pi} \left( 2\pi - i \log \left( 2\pi + i \log \left( 1 + \frac{1}{-1 + 2n} \right) \right) \right) \right)$$
$$f(y) = 2n$$

And mapped to the negative even integers with a similar single sign change:

$$y = \pi - i \tanh^{-1} \left( \frac{1}{2\pi} \left( 2\pi - i \log \left( 2\pi - i \log \left( 1 + \frac{1}{-1 + 2n} \right) \right) \right) \right)$$


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$$f(y) = 1 - 2n$$


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### Dependence on Critical line

The second solution set encodes the nontrivial zeroes through the ratio (1), where a  $\rho_n$  is the  $n$ th nontrivial zero of the Zeta Function.

$$y = \pi - i \tanh^{-1} \left( \frac{1}{2\pi} \left( 2\pi - i \log \left( 2\pi - i \log \left( 1 - \frac{1}{\rho_n} \right) \right) \right) \right) \quad (12)$$

However, we cannot rely on the cyclical knowledge of already knowing the  $n$ th zero. We can, however, explore what this solution set maps to following the same method as with the trivial zeroes.

We find that:

$$y = \pi - i \tanh^{-1} \left( \frac{1}{2\pi} \left( 2\pi - i \log \left( 2\pi - i \log \left( 1 - \frac{1}{a + ib} \right) \right) \right) \right)$$


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$$f(y) = a + bi$$

and:

$$y = \pi - i \tanh^{-1} \left( \frac{1}{2\pi} \left( 2\pi - i \log \left( 2\pi + i \log \left( 1 - \frac{1}{a + ib} \right) \right) \right) \right)$$


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$$f(y) = -a - bi + 1$$

Which reveals the dependence on  $a=1/2$  in mapping values to their horizontal reflection.

## Distribution of Nontrivial Zeroes

The symmetry of the trivial and nontrivial solutions sets indicates a correspondence between the distribution of the trivial zeroes and the nontrivial zeroes. In particular, the trivial zeroes generated by integer inputs in (11):

$$i \log \left( 1 + \frac{1}{2n} \right)$$

Seems to have some correspondence with the nontrivial zeroes in (12):

$$i \log \left( 1 - \frac{1}{\rho_n} \right)$$

## Predicting Zeroes of the Zeta Function

Here I present an alternate form of (10) which is a simpler version and less cumbersome to use. Denoted here as  $f(x)$  to distinguish it from the  $f(y)$  of earlier focus:

$$f(x) = \frac{1}{1 - 2^{ix}} \tag{13}$$

And again, we want to know when zeta of the function produces a zero:

$$\zeta(f(x)) = 0$$

The solution sets meeting these requirements show the same pattern and symmetry as in (10):

With the first solution set encoding the trivial zeroes through integer inputs

$$x = \frac{2\pi - i \log\left(\frac{0.5}{n} + 1\right)}{\log(2)} \quad (14)$$

$$f(x) = -2n$$

And the second solution set encoding the nontrivial zeroes, through the nontrivial zeroes.

$$x = \frac{2\pi - i \log\left(1 - \frac{1}{\rho_n}\right)}{\log(2)} \quad (15)$$

My intuition about this correspondence is that (15) is approaching  $\frac{2\pi}{\log(2)}$  with incremental reductions in length along the same axis, while (14) approaches  $\frac{2\pi}{\log(2)}$  by reducing the angle. Thus, it seems that the correspondence should be based upon the change in magnitude of these complex numbers, and if that correspondence can be identified exactly, it would be possible to predict the nth nontrivial zero using the nth trivial zero.

Equation (16) is a quick, empirical attempt at doing just that, using the change in magnitude of the trivial zeros to map the nontrivial zeroes.

$$x = \left\| \left\{ \frac{1}{\log(2)} \left( -2\pi - i \log\left(\frac{0.5}{n} + 1\right) \right) \right\} \right\| + \frac{0.08316}{n^{0.43}} \quad (16)$$

Table 1. is a list of first 9 predicted nontrivial x values:

Table 1. Using the distribution of trivial zeroes to predict nontrivial zeros of the Riemann Zeta Function

| <i>n</i> | (14) Trivial x | (15) Nontrivial x | (16) Predicted Nontrivial x |
|----------|----------------|-------------------|-----------------------------|
| 1        | 9.08358        | 9.16674           | 9.16674                     |
| 2        | 9.07044        | 9.13333           | 9.13217                     |
| 3        | 9.06745        | 9.12239           | 9.11930                     |
| 4        | 9.06631        | 9.11213           | 9.11213                     |
| 5        | 9.06576        | 9.10852           | 9.10739                     |
| 6        | 9.06546        | 9.1031            | 9.10395                     |
| 7        | 9.06527        | 9.09997           | 9.10129                     |
| 8        | 9.06514        | 9.09801           | 9.09915                     |
| 9        | 9.06506        | 9.09477           | 9.09739                     |



## References

- Meyer, D. (n.d.). Euler's product formula and the Riemann zeta function.  
[https://davidmeyer.github.io/qc/Euler\\_product\\_formula\\_for\\_the\\_Riemann\\_zeta\\_function.pdf](https://davidmeyer.github.io/qc/Euler_product_formula_for_the_Riemann_zeta_function.pdf)
- Riemann, B. (1859). On the Number of Prime Numbers less than a Given Quantity. (D. R. Wilkins, Trans.). *Monatsberichte Der Berliner Akademie*.

### Appendix A. Solutions for (8) and (10)

| ZERO | B        | Y            |
|------|----------|--------------|
| 1    | 14.13473 | -0.286208711 |
| 2    | 21.02204 | -0.285673812 |
| 3    | 25.01086 | -0.285498204 |
| 4    | 30.42488 | -0.285333282 |
| 5    | 32.93506 | -0.285275162 |
| 6    | 37.58618 | -0.285187938 |
| 7    | 40.91872 | -0.285137607 |
| 8    | 43.32707 | -0.285106044 |
| 9    | 48.00515 | -0.285053767 |
| 10   | 49.77383 | -0.285036558 |
| 11   | 52.97032 | -0.285008366 |
| 12   | 56.44625 | -0.284981327 |
| 13   | 59.34704 | -0.284961183 |
| 14   | 60.83178 | -0.284951615 |
| 15   | 65.11254 | -0.284926468 |
| 16   | 67.07981 | -0.284915986 |
| 17   | 69.5464  | -0.284903681 |
| 18   | 72.06716 | -0.284891974 |
| 19   | 75.70469 | -0.284876454 |
| 20   | 77.14484 | -0.284870713 |
| 21   | 79.33738 | -0.284862373 |
| 22   | 82.91038 | -0.284849726 |
| 23   | 84.73549 | -0.284843677 |
| 24   | 87.42527 | -0.284835222 |
| 25   | 88.80911 | -0.284831071 |
| 26   | 92.4919  | -0.28482063  |
| 27   | 94.65134 | -0.284814885 |
| 28   | 95.87063 | -0.284811755 |
| 29   | 98.83119 | -0.284804478 |
| 30   | 101.3179 | -0.284798693 |
| 31   | 103.7255 | -0.284793357 |
| 32   | 105.4466 | -0.284789691 |
| 33   | 107.1686 | -0.284786141 |
| 34   | 111.0295 | -0.284778582 |
| 35   | 111.8747 | -0.284776997 |
| 36   | 114.3202 | -0.284772542 |
| 37   | 116.2267 | -0.284769199 |
| 38   | 118.7908 | -0.284764872 |
| 39   | 121.3701 | -0.284760704 |
| 40   | 122.9468 | -0.284758242 |

