Theory of Quadratic Triadic Relations for Prime Numbers

Youssouf Ouédraogo
Email: youssoufouedraogo23@gmail.com
ORCID iD: 0009-0005-4602-0722
Date: January 1, 2026

Abstract
This paper proposes a new structural approach to the study of consecutive prime numbers based on a quadratic relation linking three successive primes. A stability ratio is introduced and shown to converge asymptotically to unity using explicit bounds for the k-th prime number. This convergence induces a constraint on the local variation of prime gaps, leading to an asymptotic smoothness law for their relative fluctuations. The analysis is fully deterministic and avoids heuristic arguments based on average asymptotics. Numerical validations using verified large prime datasets confirm the theoretical predictions and illustrate the progressive regularization of local gap variations as the prime index increases.

Keywords: Prime numbers, quadratic discriminant, Bertrand's theorem, triadic relations, prime gaps, consecutive primes, asymptotic analysis, local bounds of prime numbers.
MSC (2020) codes: 11N05, 11N37, 11A41.

Contents

1 Introduction

The distribution of prime numbers remains one of the deepest and most extensively studied topics in mathematics. While their average density is now well understood through the Prime Number Theorem, the local behavior of the gaps between consecutive primes is still marked by strong irregularity, making the identification of fine structural laws particularly difficult.

Most classical approaches model prime numbers using global analytic methods or probabilistic frameworks, such as those inspired by Cramér. These models efficiently explain average trends but often fail to reveal deterministic local relations directly linking several consecutive prime numbers.

In this work, we propose a different approach based on the study of a quadratic relation linking three consecutive prime numbers. We introduce the stability ratio

$$C_k = \frac{P_{k+1}^2}{P_k P_{k+2}},$$

and show that this quantity converges asymptotically to unity. This convergence reveals the existence of an internal quadratic equilibrium within the sequence of prime numbers, independently of the local fluctuations of the gaps.

Unlike heuristic approaches relying solely on the asymptotic equivalence $P_k \sim k \ln k$, we establish this result using fully rigorous explicit bounds, thereby providing a solid analytical proof. We then show that this equilibrium law implies a progressive regularization of the relative variation of the gaps between consecutive prime numbers, which we call the *law of asymptotic smoothness*.

The aim of this article is not to solve a major classical problem in number theory, but rather to propose a new structural framework that connects local irregularity with global asymptotic regularity in the sequence of prime numbers.

2 Quadratic Relations

2.1 Construction of the equation

Let us consider the quadratic equation:

$$P_k X^2 + P_{k+1} X + P_{k+2} = 0 \tag{1}$$

P_k, P_{k+1}, and P_{k+2} denote three consecutive primes.

The discriminant associated with this equation is:

$$\Delta = P_{k+1}^2 - 4 P_k P_{k+2} \tag{2}$$

2.2 Interpretation of the discriminant

- If $\Delta > 0$, the equation has two distinct real roots.
- If $\Delta = 0$, the equation has a double root.
- If $\Delta < 0$, the equation has no real root.

3 Empirical Verification

Table 1: Empirical verification.

$\mathbf{P_k}$	$\mathbf{P_{k+1}}$	$\mathbf{P_{k+2}}$	$\mathbf{\Delta}$	Sign of Δ
2	3	5	-31	< 0
3	5	7	-59	< 0
5	7	11	-171	< 0
7	11	13	-243	< 0

4 Proof of the Negativity of the Discriminant

Bertrand's theorem states that for any integer $n > 1$, there exists at least one prime number p such that [1]:

$$n < p < 2n. \tag{2}$$

According to Bertrand's postulate, we know that for any prime number $P_k > 2$, there exists another prime number P_{k+1} such that:

$$P_k < P_{k+1} < 2P_k \tag{3}$$

Thus, $P_k > \frac{1}{2}P_{k+1}$.

Since prime numbers are strictly increasing, $P_{k+2} > P_{k+1}$, therefore:

$$P_k P_{k+2} > \frac{1}{2}P_{k+1}^2 \tag{4}$$

$$\Rightarrow 4P_k P_{k+2} > 4\left(\frac{1}{2}P_{k+1}^2\right) \Rightarrow 4P_k P_{k+2} > 2P_{k+1}^2 \tag{5}$$

Since $2P_{k+1}^2 > P_{k+1}^2$,

$$\Rightarrow 4P_k P_{k+2} > P_{k+1}^2 \tag{6}$$

Consequently:

$$P_{k+1}^2 - 4P_k P_{k+2} < 0 \Rightarrow \Delta < 0 \tag{7}$$

5 Deduction of the Fundamental Inequality

From Δ <0, we immediately deduct:

$$P_{k+1}^2 < 4P_k P_{k+2} \tag{8}$$

We isolate P_{k+2}:

$$P_{k+2} > \frac{P_{k+1}^2}{4P_k} \tag{9}$$

Then, combining this lower bound with the inequality from Bertrand ($P_{k+2} < 2P_{k+1}$), we obtain the general framework:

$$\frac{P_{k+1}^2}{4P_k} < P_{k+2} < 2P_{k+1} \tag{10}$$

This inequality expresses a local regularity of the sequence of prime numbers:

- A lower bound resulting from the negativity of the discriminant.
- An upper bound from Bertrand's theorem.

6 Prime Growth Rate

Let

$$r_k = \frac{P_{k+1}}{P_k} \quad \text{and} \quad r_{k+1} = \frac{P_{k+2}}{P_{k+1}}.$$

According to the previous inequality,

$$\frac{P_{k+1}^2}{4P_k} < P_{k+2} < 2P_{k+1}, \tag{11}$$

We deduce that

$$\frac{r_k}{r_{k+1}} < 4. \tag{12}$$

This ratio establishes, analytically, that the ratio of the growth rate between three consecutive primes is strictly less than 4 for any $k \geq 1$.

6.1 Numerical Verification

A numerical verification was carried out to confirm the theoretical results obtained previously.

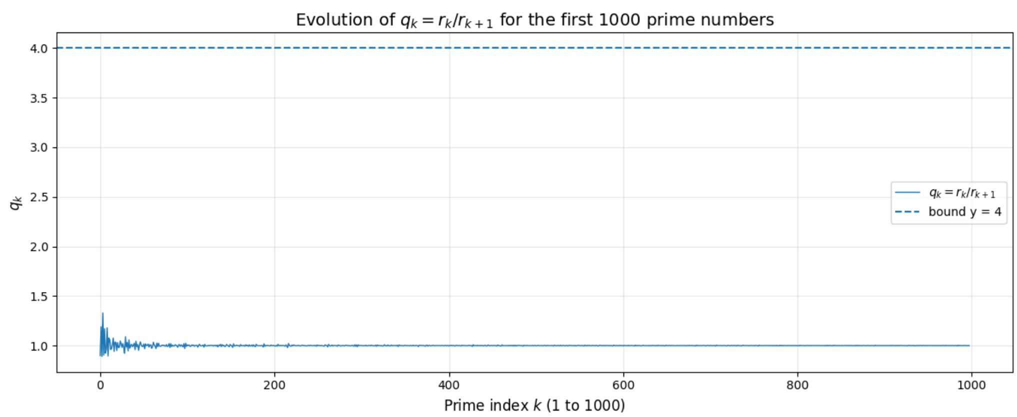

Figure 1: Evolution of r_k/r_{k+1} for the first 1000 primes.

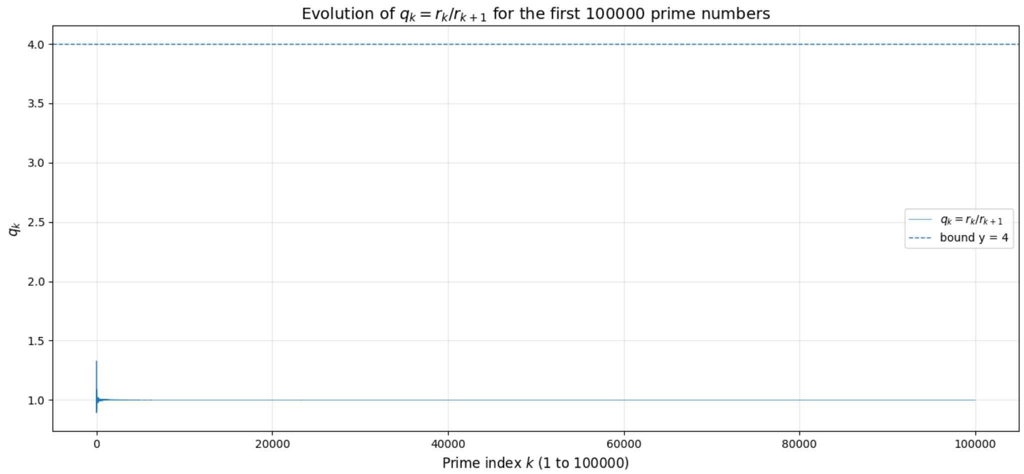

Figure 2: Evolution of r_k/r_{k+1} for the first 100,000 primes.

A simulation on the first 10^3 prime numbers, then on the first 10^5 prime numbers confirms that:

$$\max\left(\frac{r_k}{r_{k+1}}\right) < 4.$$

The constant 4 represents a structural bound of stability. As long as $\frac{r_k}{r_{k+1}} < 4$, the discriminant remains negative and the progression of the prime numbers remains consistent.

7 General Quadratic Equation

The bound C and the growth rate of the primes are related by the following relationship:

$$\frac{r_k}{r_{k+1}} < C \qquad \left(C \in \mathbb{R}_+^*\right).$$

From $\frac{r_k}{r_{k+1}} < C$, we deduce that:

$$P_{k+1}^2 - CP_kP_{k+2} < 0 \implies \Delta < 0 \tag{13}$$

With

$$\Delta = P_{k+1}^2 - CP_kP_{k+2} \implies \Delta = P_{k+1}^2 - 4P_k\lambda P_{k+2}, \qquad C = 4\lambda \tag{14}$$

We deduce that the general quadratic equation is of the form:

$$P_kX^2 + P_{k+1}X + \lambda P_{k+2} = 0 \tag{15}$$

Where λ is called *weight* $\left(\lambda \in \mathbb{R}_+^*\right)$.

The discriminant Δ will be used as a measure of the stability of the progression of prime numbers.

8 Asymptotic Analysis of the Stability of the Prime Numbers Growth Rate

The quadratic equation

$$P_kX^2 + P_{k+1}X + \lambda P_{k+2} = 0, \tag{16}$$

allowed us to obtain a quadratic constraint:

$$P_{k+1}^2 < 4P_kP_{k+2}. \tag{17}$$

However, the results of the digital verification of

$$\frac{r_k}{r_{k+1}}$$

on the first 100 000 prime numbers proves that the bound $C = 4$ is too wide. Asymptotic analysis will allow us to establish, by analytical demonstration and numerical verification, the existence of an optimal lower bound and an optimal upper bound.

8.1 Finding the critical interval

The critical interval is the interval that contains the values of C for which Δ is both positive, negative or zero: this is the critical instability zone.

$$\Delta = 0 \implies P_{k+1}^2 - CP_kP_{k+2} = 0 \iff C = \frac{P_{k+1}^2}{P_kP_{k+2}}. \tag{18}$$

The representation of the distribution of

$$C_k = \frac{P_{k+1}^2}{P_kP_{k+2}}, \tag{19}$$

for the first 10^n prime numbers, will allow the determination of critical instabilities (critical interval) and regular stability zones.

8.2 Numerical validation

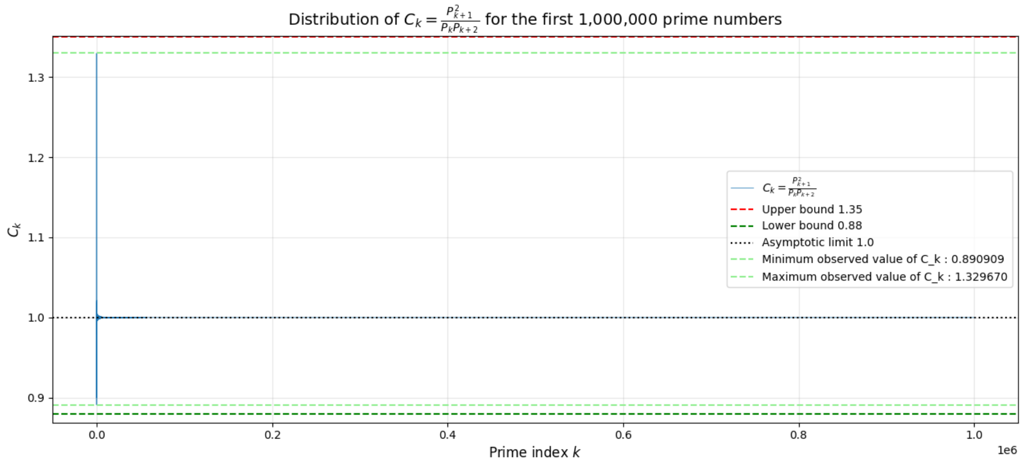

Figure 3: Distribution of $C_k = \dfrac{P_{k+1}^2}{P_k P_{k+2}}$ for the first 1,000,000 prime numbers.

Observation of the numerical validation data allows the existence of a:

- **Critical instability zone:** The values of C_k are confined to the interval $[0.89\,;\; 1.33]$.
- **Stability zone:** for the values of $C_k \geq 1.35$ and $C_k \leq 0.88$.
- **Asymptotic stability:** the asymptotic mean of $C_k \approx 1$.

8.3 Discussion

The current refinement, which isolates the interval $[0.89\,;\; 1.33]$, allows a more precise description of the equilibrium regime of the sequence of prime numbers.

The interval $[0.89\,;\; 1.33]$ is the critical instability zone where the discriminant Δ can change sign from one triplet to another, defining the set of values of C_k for which the quadratic relation does not impose a unique sign on Δ. Outside this area, observation is simple and robust:

- **For** $C_k \geq 1.35$, the discriminant is systematically negative, and the quadratic relation ensures the lower bound (threshold $C_k = 1.35$).

  Proof:

$$\Delta = P_{k+1}^2 - 1.35\, P_k P_{k+2} = P_k P_{k+2} \left(\frac{P_{k+1}^2}{P_k P_{k+2}} - 1.35 \right) \tag{20}$$

$$\rightarrow \Delta = P_k P_{k+2}\,(C_{\max} - 1.35) = P_k P_{k+2}\,(1.33 - 1.35) \tag{21}$$

$$\Longrightarrow \Delta < 0$$

  Consequently:

$$P_{k+2} > \frac{P_{k+1}^2}{1.35\, P_k} \tag{22}$$

- **For** $C_k \leq 0.88$, the discriminant is systematically positive, and the quadratic relation ensures the upper bound (threshold $C_k = 0.88$).

  Proof:

$$\Delta = P_{k+1}^2 - 0.88\, P_k P_{k+2} = P_k P_{k+2} \left(\frac{P_{k+1}^2}{P_k P_{k+2}} - 0.88 \right) \tag{23}$$

$$\to \Delta = P_k P_{k+2} \, (C_{\min} - 0.88) = P_k P_{k+2} \, (0.89 - 0.88) \tag{24}$$

$$\Longrightarrow \Delta > 0$$

  Consequently:

$$P_{k+2} < \frac{P_{k+1}^2}{0.88\, P_k} \tag{25}$$

The instability is therefore confined to the critical interval. The absolute stability (a constant sign for Δ) is found outside this interval. This distinction is essential for the rigorous formulation of the conjecture.

9 Asymptotic Bounds Consistency Analysis

In order to confirm the robustness of our triadic framing conjecture, we check its compatibility with the classical asymptotic behavior of prime numbers.

The Prime Number Theorem implies that the local growth ratio tends to unity when $k \to \infty$ (see detailed proof in Section 12):

$$\lim_{k\to\infty} \frac{P_{k+1}^2}{P_k P_{k+2}} = 1. \tag{26}$$

Our conjectural bounds, defined by constants

$$C_{\min} = 0.88, \qquad C_{\max} = 1.35,$$

imposes the following inequality for P_{k+2}:

$$\frac{1}{1.35} \left(\frac{P_{k+1}^2}{P_k} \right) < P_{k+2} < \frac{1}{0.88} \left(\frac{P_{k+1}^2}{P_k} \right). \tag{27}$$

Dividing this double inequality by P_{k+2} and passing to the limit when $k \to \infty$ (where $P_{k+2} \approx P_{k+1}^2 / P_k$), we obtain the following equivalent condition:

$$\frac{1}{1.35} < 1 < \frac{1}{0.88}. \tag{28}$$

Numerically:

$$0.7407 < 1 < 1.1363. \tag{29}$$

Conclusion: Since these inequalities are strictly satisfied, the proposed bounding is fully consistent with the Prime Number Theorem. The constants 1.35 and 0.88 constitute *safety margins* that absorb the initial irregularities of the low values of k. When k becomes large, the ratio C_k converges to 1, moving away from the critical bounds: the bounding then becomes asymptotically more flexible while remaining universally valid, without contradicting the average structure of the prime number distribution.

10 Extension of Bertrand's Theorem

10.1 Upper bound comparison

Bertrand's theorem (1845) states that, for any integer $n > 1$, there exists at least one prime number p such that [1]:

$$n < p < 2n \tag{30}$$

In the triadic framework of consecutive primes $P_k,\ P_{k+1},\ P_{k+2}$, this inequality can be interpreted analytically as:

$$P_{k+2} < 2P_{k+1} \tag{31}$$

Our previous study established a much tighter upper bound:

$$P_{k+2} < \frac{P_{k+1}^2}{0.88\,P_k} \tag{32}$$

To determine which of the two is more constraining, let's compare the expressions analytically:

$$\frac{P_{k+1}^2}{0.88\,P_k} \qquad \text{and} \qquad 2P_{k+1}.$$

We have:

$$\frac{P_{k+1}^2}{0.88\,P_k} < 2P_{k+1} \quad \iff \quad \frac{P_{k+1}}{P_k} < 1.76 \tag{33}$$

However, for all $k \geq 1$, the ratio $\frac{P_{k+1}}{P_k}$ tends to 1 and never exceeds 1.7 (even for small triplets).

Thus:

$$\frac{P_{k+1}^2}{0.88\,P_k} < 2P_{k+1} \tag{34}$$

The upper bound obtained in this new theory is therefore stricter than that of Bertrand's theorem. thus, it refines the Bertrand bound by replacing the factor 2 with a coefficient dependent on the squared coherence of the prime numbers.

10.2 Lower bound comparison

Bertrand's theorem does not provide any explicit lower bound on P_{k+2}.

The lower bound obtained in this new theory is written:

$$P_{k+2} > \frac{P_{k+1}^2}{1.35\,P_k} \tag{35}$$

To assess its scope, let's compare it to P_{k+1}:

$$\frac{P_{k+1}^2}{1.35\,P_k} > P_{k+1} \quad \iff \quad \frac{P_{k+1}}{P_k} > 1.35 \tag{36}$$

However, this condition is not universally verified:

- for small triples like $(2, 3, 5)$, we have $1.5 > 1.35$;

- but from a certain rank, the ratio $\dfrac{P_{k+1}}{P_k}$ becomes strictly less than 1.35 and tends asymptotically towards 1 when $k \longrightarrow \infty$.

Thus, depending on the rank k, it may happen that:

$$\frac{P_{k+1}^2}{1.35\,P_k} \leq P_{k+1} \tag{37}$$

which would imply that the quadratic lower bound is less than or equal to P_{k+1}.

To guarantee the universal validity of inequality, it is therefore necessary to consider the maximum lower bound between the two expressions:

$$P_{k+2} > \max\left(\frac{P_{k+1}^2}{1.35\,P_k},\ P_{k+1}\right) \tag{38}$$

The lower bound square becomes dominant when $\dfrac{P_{k+1}}{P_k} > 1.35$, i.e. for the first small ranks. As the sequel progresses, P_{k+1} becomes the natural reference again.

The maximum formulation thus guarantees a valid and continuous framework for any $k \geq 1$, unifying the two growth regimes.

11 Local Quadratic Triadic Bounding Conjecture of Prime Numbers

For any integer $k \geq 1$, let P_k, P_{k+1} and P_{k+2} denote three consecutive primes. The $(k+2)$-th prime number P_{k+2} is enclosed by two quadratic bounds depending on the two preceding prime numbers P_k and P_{k+1}.

Conjecture [Ouédraogo – Local Quadratic Triadic Bounding Frame]
For any $k \geq 1$, the prime number P_{k+2} verifies the following frame:

$$\max\left(\frac{P_{k+1}^2}{1.35\,P_k}\ \ P_{k+1}\right)\ <\ P_{k+2}\ <\ \frac{P_{k+1}^2}{0.88\,P_k}. \tag{39}$$

12 Law of Asymptotic Quadratic Equilibrium

In this section, we rigorously establish the convergence of the stability ratio

$$C_k = \frac{P_{k+1}^2}{P_k P_{k+2}}, \tag{40}$$

where P_k is the k-th prime number. The proof is based exclusively on explicit bounds for prime numbers, in order to avoid any heuristic approximation derived from the prime number theorem.

12.1 Dusart's explicit inequalities

For all $k \geq 396\,738$, Dusart (2010) [2] has established the limits

$$k\big(\ln k + \ln\ln k - 1\big) < P_k < k\big(\ln k + \ln\ln k\big). \tag{41}$$

Let

$$f(k) = \ln k + \ln\ln k. \tag{42}$$

The bounds are then rewritten as:

$$k(f(k) - 1) < P_k < kf(k). \tag{43}$$

12.2 Bounds for the C_k ratio

We construct two explicit bounds $C_k^{\min}$ and $C_k^{\max}$ such that

$$C_k^{\min} < C_k < C_k^{\max}. \tag{44}$$

12.2.1 Upper bound $C_k^{\max}$

To maximize C_k, we maximize P_{k+1} and minimize P_k and P_{k+2}:

$$P_{k+1}^{\max} = (k+1)f(k+1), \tag{45}$$

$$P_k^{\min} = k(f(k) - 1), \qquad P_{k+2}^{\min} = (k+2)(f(k+2) - 1). \tag{46}$$

Thus

$$C_k^{\max} = \frac{\big[(k+1)f(k+1)\big]^2}{k(k+2)\big(f(k) - 1\big)\big(f(k+2) - 1\big)}. \tag{47}$$

12.2.2 Lower bound $C_k^{\min}$

To minimize Ck, we minimize the numerator and maximize the denominator:

$$P_{k+1}^{\min} = (k+1)(f(k+1) - 1), \tag{48}$$

$$P_k^{\max} = kf(k), \qquad P_{k+2}^{\max} = (k+2)f(k+2). \tag{49}$$

Where from

$$C_k^{\min} = \frac{\big[(k+1)(f(k+1) - 1)\big]^2}{k(k+2)f(k)f(k+2)}. \tag{50}$$

12.3 Asymptotic analysis of the bounds

Each bound is factored into a rational part

$$R(k) = \frac{(k+1)^2}{k(k+2)}, \tag{51}$$

and a logarithmic part grouping the various $f(k+a)$.

12.3.1 Rational part behavior

A simple development shows that

$$R(k) = 1 + \frac{1}{k^2 + 2k} = 1 + O\left(\frac{1}{k^2}\right). \tag{52}$$

12.3.2 Logarithmic part behavior

Since

$$f(k+1) = f(k) + O\left(\frac{1}{k}\right), \qquad f(k+2) = f(k) + O\left(\frac{1}{k}\right), \tag{53}$$

we obtain

$$\frac{f(k+1)}{f(k)} = 1 + O\left(\frac{1}{k \ln k}\right), \tag{54}$$

and the same for $\frac{f(k+2)}{f(k)}$. Thus, all the logarithmic components converge to the unit:

$$L(k) = 1 + o\left(\frac{1}{k^2}\right). \tag{55}$$

12.4 Application of the squeeze theorem

We have

$$C_k^{\min} = R(k)\, L_{\min}(k), \qquad C_k^{\max} = R(k)\, L_{\max}(k). \tag{56}$$

Since

$$\lim_{k\to\infty} R(k) = 1, \qquad \lim_{k\to\infty} L_{\min}(k) = \lim_{k\to\infty} L_{\max}(k) = 1, \tag{57}$$

the squeeze theorem implies that

$$\boxed{\lim_{k\to\infty} C_k = 1}. \tag{58}$$

The rate of convergence is dominated by $R(k)$:

$$\boxed{C_k - 1 = O\left(\frac{1}{k^2}\right)}. \tag{59}$$

We refer to this result as the *Law of Asymptotic Quadratic Equilibrium.*

This relation expresses that the triadic stability ratio tends towards unity, with an inverse hyperbolic correction, reflecting a rapid quadratic decay of the irregularity.

This result leads to the quadratic equilibrium law, stated as follows:

$$P_{k+2} \approx \frac{P_{k+1}^2}{P_k} \tag{60}$$

This law constitutes the asymptotic form of the triadic bounding conjecture and formalizes the natural tendency of prime numbers to organize themselves according to a local quadratic relation.

13 Asymptotic Smoothness Law of Prime Gaps

The quadratic convergence of the stability ratio

$$C_k = \frac{P_{k+1}^2}{P_k P_{k+2}}$$

established in the previous section imposes a strong analytical constraint on the local variation of the gaps between consecutive prime numbers

$$g_k = P_{k+1} - P_k.$$

This constraint can be formulated independently of any global statistical hypothesis on the distribution of prime numbers.

13.1 Exact algebraic identity

Using the trivial identities

$$P_{k+1} = P_k + g_k, \qquad P_{k+2} = P_k + g_k + g_{k+1}, \tag{61}$$

and substituting them into the definition of the stability ratio, one obtains after simplification the following exact identity:

$$C_k - 1 = \frac{\dfrac{g_k - g_{k+1}}{P_k} + \left(\dfrac{g_k}{P_k}\right)^2}{1 + \dfrac{g_k + g_{k+1}}{P_k}}. \tag{62}$$

This relation is purely algebraic and does not rely on any approximation.

13.2 Asymptotic control without global assumptions

We use only two elementary and rigorously established facts:

(i) The sequence of prime numbers is strictly increasing; hence $P_k \to +\infty$ as $k \to \infty$.

(ii) The gaps g_k are positive and necessarily satisfy $g_k = o(P_k)$, since $P_{k+1}/P_k \to 1$ is a direct consequence of the convergence $C_k \to 1$.

It follows that

$$\frac{g_k}{P_k} \longrightarrow 0, \qquad \frac{g_{k+1}}{P_k} \longrightarrow 0 \quad (k \to \infty). \tag{63}$$

Consequently,

$$\left(\frac{g_k}{P_k}\right)^2 = o\left(\frac{g_k}{P_k}\right), \qquad \frac{g_k + g_{k+1}}{P_k} = o(1). \tag{64}$$

The denominator of expression (62) therefore tends to 1, and the exact identity reduces asymptotically to

$$C_k - 1 = \frac{g_k - g_{k+1}}{P_k} + o\left(\frac{g_k}{P_k}\right). \tag{65}$$

13.3 Derivation of the smoothness law

The previous section established analytically that

$$C_k - 1 = O\left(\frac{1}{k^2}\right). \tag{66}$$

Injecting this estimate into the asymptotic expression (65) necessarily yields

$$\frac{g_{k+1} - g_k}{P_k} = O\left(\frac{1}{k^2}\right). \tag{67}$$

This relation expresses that the relative variation of the gaps between consecutive prime numbers decreases quadratically with the index.

Relation (67) will be referred to as the *Asymptotic Smoothness Law of Prime Gaps.*

13.4 Arithmetic granularity

Since all prime gaps are even integers, their variations are necessarily even as well. Consequently, one has the exact lower bound (if $g_{k+1} - g_k \neq 0$):

$$|g_{k+1} - g_k| \geq 2. \tag{68}$$

Dividing by P_k yields the corresponding relative inequality

$$\frac{|g_{k+1} - g_k|}{P_k} \geq \frac{2}{P_k}. \tag{69}$$

Using the known growth of the k-th prime number, namely $P_k \sim k \ln k$ as $k \to \infty$, inequality (69) implies the asymptotic lower bound

$$\frac{|g_{k+1} - g_k|}{P_k} = \Omega\left(\frac{1}{k \ln k}\right). \tag{70}$$

On the other hand, the asymptotic quadratic equilibrium established in Section 13.2 yields the upper bound

$$\frac{|g_{k+1} - g_k|}{P_k} = O\left(\frac{1}{k^2}\right). \tag{71}$$

Since

$$\frac{1}{k^2} \ll \frac{1}{k \ln k}, \tag{72}$$

the analytical constraint imposed by the quadratic equilibrium is asymptotically stronger than the minimal observable variation imposed by the discrete nature of integers.

This shows that the sequence of relative gap variations is regulated by a theoretical force of order $O(1/k^2)$, while the observable fluctuations are bounded below by $\Omega(1/(k \ln k))$. The discrepancy between these two scales reflects the arithmetic granularity of prime gaps rather than a failure of the analytical model.

Therefore, the sequence of deviations is asymptotically as smooth as possible, given its intrinsic discrete structure.

Statement: The law of asymptotic smoothness quantifies the impact of the triadic quadratic constraint on the variation of gaps between prime numbers. It establishes that, for the major indices k, the relative change in the

$$\frac{|g_{k+1} - g_k|}{P_k} \tag{73}$$

is dominated by a rapidly decreasing function.

The law of asymptotic smoothness is directly derived from the quadratic equilibrium law and demonstrates the following order-of-magnitude equivalence:

$$C_k - 1 = O\left(\frac{1}{k^2}\right) \quad \Longleftrightarrow \quad \frac{|g_{k+1} - g_k|}{P_k} = O\left(\frac{1}{k^2}\right). \tag{74}$$

13.5 Interpretation

The law of asymptotic smoothness provides insight into the internal dynamics of the sequence of prime numbers. It specifies how the fluctuations of the successive gaps $(g_k,\ g_{k+1})$ are gradually absorbed by the growth of the prime number P_k, leading to a relative regularity that becomes more and more marked as k becomes large.

14 Quadratic Stability Increment

The relationship

$$C_k = \frac{P_{k+1}^2}{P_k P_{k+2}} \tag{75}$$

indicates that when $C_k = 1$, the progression P_k, P_{k+1}, P_{k+2} is said to be in *perfect quadratic equilibrium*, which corresponds to the equilibrium law:

$$P_{k+2} \approx \frac{P_{k+1}^2}{P_k}. \tag{76}$$

Any deviation of C_k from unity reflects a local disturbance in the regularity of the sequence of prime numbers. To analyze this dynamic, we introduce the *normalized variation* of the stability ratio:

$$\delta_k = \frac{g_k - g_{k+1}}{P_k} \approx C_k - 1 \tag{77}$$

where $g_k = P_{k+1} - P_k$ and $g_{k+1} = P_{k+2} - P_{k+1}$.

14.1 Definition and interpretation

The increment δ_k measures the relative change of two consecutive prime deviations (g_k and g_{k+1}) relative to the size of the prime number P_k.

In other words:

- If $\delta_k > 0$, then $g_k > g_{k+1}$: the sequence contracts locally and the discriminant becomes negative ($\Delta < 0$).
- If $\delta_k < 0$, then $g_k < g_{k+1}$: the sequence is in local expansion and the discriminant becomes positive ($\Delta > 0$).
- If $\delta_k = 0$, then $g_k = g_{k+1}$: perfect quadratic equilibrium and $\Delta = 0$.
  The gaps are temporarily expanding.

14.2 Domain of validity and asymptotic condition

As $g_k \ll P_k$ for large k, the magnitude of δ_k is extremely small. Numerically, we observe that:

$$\delta_k = O\left(\frac{1}{k^2}\right) \tag{78}$$

Thus, the sequence of prime numbers becomes asymptotically stable, since:

$$\lim_{k\to\infty} \delta_k = 0 \quad \text{and therefore} \quad \lim_{k\to\infty} C_k = 1. \tag{79}$$

This property gives the triadic system a self-regulating structure: the phases of expansion ($\delta_k > 0$) and contraction ($\delta_k < 0$) alternate and cancel each other asymptotically, ensuring convergence towards global equilibrium.

15 Numerical Validations of Equilibrium and Smoothness Laws

This section presents a series of numerical validations aimed at confirming, for different orders of magnitude, the asymptotic quadratic equilibrium law and the asymptotic smoothness law derived in the previous sections. Tests are made from consecutive triples of prime numbers (P_k, P_{k+1}, P_{k+2}) covering sizes ranging from 10^3 to 10^9. The numerical values are calculated from the exact identities:

$$P_{k+1}^2 - P_k P_{k+2} = P_k(g_k - g_{k+1}) + g_k^2, \tag{80}$$

$$C_k - 1 = \frac{P_{k+1}^2}{P_k P_{k+2}} - 1 = \frac{P_k(g_k - g_{k+1}) + g_k^2}{P_k P_{k+2}}. \tag{81}$$

The differences $g_k = P_{k+1} - P_k$ and $g_{k+1} = P_{k+2} - P_{k+1}$ allow to examine simultaneously the quadratic equilibrium and the local smoothness.

The numerical verifications presented in this work were performed on consecutive prime triplets extracted from public lists, notably those available on the Compoasso website [3].

15.1 Numerical validation of the asymptotic quadratic equilibrium law

For each triplet, we compare the real value of P_{k+2} with Quadratic Estimation

$$E_k = \frac{P_{k+1}^2}{P_k}. \tag{82}$$

The following table shows E_k, the difference $E_k - P_{k+2}$, and the stability increment $C_k - 1$.

Interpretation: The data show that the difference $E_k - P_{k+2}$ is approximated by the gap variation $g_k - g_{k+1}$, (a small, discrete integer value, often ± 2, ± 8), plus a corrective term g_k^2/P_k of negligible order. The stability increment $C_k - 1$ is extremely small, decreasing rapidly as P_k increases:

$$|C_k - 1| \sim 10^{-4},\ 10^{-5},\ 10^{-6}, \ldots, 10^{-16}.$$

Table 2: Numerical validation of the asymptotic quadratic equilibrium law.

P_k	P_{k+1}	P_{k+2}	$E_k \approx$	$E_k - P_{k+2}$	$C_k - 1$
7919	7927	7933	7935.008087	$+2.008087$	2.53×10^{-4}
104729	104743	104759	104757.001872	-1.998128	-1.91×10^{-5}
1299811	1299817	1299821	1299823.000028	$+2.000028$	1.54×10^{-6}
1300021	1300027	1300031	1300033.000028	$+2.000028$	1.54×10^{-6}
1000000009	1000000021	1000000033	1000000033.00000014	$+1.44 \times 10^{-7}$	1.44×10^{-16}
1000000427	1000000433	1000000439	1000000439.00000004	$+3.6 \times 10^{-8}$	3.6×10^{-17}
1000003153	1000003157	1000003163	1000003161.00000002	-2.0	-2.0×10^{-9}
1000004507	1000004519	1000004539	1000004531.00000014	-8.0	-8.0×10^{-9}

This decrease empirically confirms that

$$C_k - 1 = O\left(\frac{1}{k^2}\right), \tag{83}$$

as demonstrated analytically in the previous section.

15.2 Numerical validation of the asymptotic smoothness distribution

The smoothness law states that the normalized variation of the deviations

$$\delta_k = \frac{|g_{k+1} - g_k|}{P_k} \tag{84}$$

is rapidly moving towards zero. The following table shows the values of the variances and their normalized variation.

Table 3: Numerical validation of the asymptotic smoothness distribution.

P_k	g_k	g_{k+1}	$\lvert g_{k+1} - g_k \rvert$	δ_k
7919	8	6	2	2.53×10^{-4}
104729	14	16	2	1.91×10^{-5}
1299811	6	4	2	1.54×10^{-6}
1300021	6	4	2	1.54×10^{-6}
1000000009	12	12	0	0
1000000427	6	6	0	0
1000003153	4	6	2	2.0×10^{-9}
1000004507	12	20	8	8.0×10^{-9}

Interpretation: There is a systematic decrease of δ_k, dropping from 10^{-4} for $P_k \approx 8 \times 10^3$ to 10^{-9} for $P_k \approx 10^9$. The relative variation in the differences therefore quickly becomes negligible. Cases where $\delta_k = 0$ correspond to equal consecutive gaps ($g_k = g_{k+1}$), which maximizes quadratic stability and confirms the local structure predicted by the theory.

15.3 Conclusion of the numerical validation

Both tables consistently demonstrate the following properties:

- The asymptotic quadratic equilibrium law is numerically verified over several orders of magnitude: the estimate $P_{k+2} \approx P_{k+1}^2/P_k$ has a controlled gap, entirely determined by the structure of the local gaps.
- The law of asymptotic smoothness observed experimentally shows that the irregularity of the spreads is absorbed by the growth of P_k, with a rapid decrease in the relative change $|g_{k+1} - g_k|/P_k$.
- The numerical results fully confirm the predictions established in the previous sections, in particular the quadratic convergence $C_k - 1 = O(1/k^2)$ and its consequence on the local regularity.

These validations reinforce the empirical soundness of the theory and demonstrate that the local behaviors of prime number triplets are in agreement with the structural laws derived analytically.

16 Note on the Robustness of the Bounds in the Face of Pathological Cases

It is essential to note that the constants $C_{\min}$ =0.88 and $C_{\max}$ =1.35 are not mere statistical artifacts, but structural barriers determined by the initial volatility zone (k <10). A *stress-test* on the most extreme configurations confirms their robustness:

- **Maximum contraction:** the triplet (7, 11, 13), characterized by an abrupt reduction in the gap (from 4 to 2), generates a coefficient $C_k \approx 1.33$, grazing the upper bound without crossing it.
- **Maximum expansion:** the triplet (5, 7, 11), where the gap suddenly doubles, produces a coefficient $C_k \approx 0.89$, testing the lower bound.

For the large indices k, the stability of the bounds is guaranteed by the system's inertia. Indeed, even the hypothetical appearance of record gaps (exceptional *gaps*) could not break this framework, because the impact of the local variation $|g_{k+1} - g_k|$ is diluted by the magnitude of P_k. Thus, the bounds set by the initial chaos of small prime numbers remain inviolable for any subsequent k.

17 Synthesis and Scope of the Results

The results obtained in this article highlight the existence of an asymptotic quadratic constraint governing the relationship between three consecutive prime numbers. The rigorously established convergence of the stability ratio C_k toward unity shows that the sequence of prime numbers does not evolve arbitrarily at the local scale, but is instead subject to an internal balancing mechanism.

This constraint does not eliminate the irregularity of the absolute gaps between prime numbers, which remain discrete integers subject to significant fluctuations. However, it imposes a strong form of regularity on relative quantities, in particular on the normalized variation of the gaps, which asymptotically decreases toward zero.

The resulting law of asymptotic smoothness neither contradicts existing probabilistic models nor classical results from analytic number theory. Rather, it provides a complementary viewpoint that reveals a stable quadratic structure coexisting with the seemingly chaotic nature of prime numbers.

From a conceptual perspective, this approach suggests that the sequence of prime numbers may be interpreted as a discrete system subject to an asymptotic restoring force, ensuring a form of global regularity despite persistent local fluctuations.

18 Conclusion

We have introduced and investigated a new triadic quadratic relation linking three consecutive prime numbers. Using rigorous explicit bounds and the squeeze theorem, we have shown that the associated stability ratio converges to unity with an error rate of order $O(1/k^2)$, thereby establishing the law of asymptotic quadratic equilibrium.

This convergence naturally induces a strong constraint on the relative variation of the gaps between consecutive prime numbers, leading to the formulation of a law of asymptotic smoothness. Although the discrete nature of prime numbers imposes a minimal granularity on the gaps, our analysis shows that this granularity is dominated by the asymptotic quadratic constraint, making the sequence "as smooth as possible" at large scales.

Numerical validations performed on strictly consecutive prime triplets confirm the consistency of the analytical results and illustrate their stability across several orders of magnitude.

This work opens several perspectives, including the study of analogous relations involving a larger number of consecutive prime numbers, as well as the exploration of potential links between this quadratic approach and other analytical frameworks in number theory. Above all, it proposes a new structural viewpoint that may contribute to a deeper global understanding of the distribution of prime numbers.

References

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