GENERALIZED CRUDE BRAUER INEQUALITY ON ADDITION CHAINS

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Abstract. We extend the inequality due to Alfred Brauer on standard addition chains to a sequence of additions leading to a finite number where at most at most $d \geq 2$ previous terms can be added to generate each term in the sequence.

1. Introduction

An addition chain of length h leading to n, first introduced by the German mathematician Arnold Scholz in 1937 [1], is a sequence of numbers $s_0 = 1, s_1 = 2, \ldots, s_h = n$ where $s_i = s_k + s_s$ for $i > k \geq s \geq 0$. The number of terms (excluding the first term) in an addition chain leading to n is the length of the chain. We call an addition chain leading to n with a minimal length an *optimal* addition chain leading to n. In standard practice, we denote by $\ell(n)$ the length of an optimal addition chain that leads to n.

Example 1.1. Consider the addition chain $\mathcal{C}_{16} : s_0 = 1, s_1 = 2, s_2 = 4, s_3 = 8, s_4 = 9, s_5 = 16$ with sums

$$\begin{aligned} s_1 &= 2 = 1 + 1 \\ s_2 &= 4 = 2 + 2 \\ s_3 &= 8 = 4 + 4 \\ s_4 &= 9 = 8 + 1 \\ s_5 &= 16 = 8 + 8. \end{aligned}$$

In [1], Arnold Scholz asked if the crude inequality $\ell(n) \leq 2\frac{\log n}{\log 2}$ could be improved. In the paper [2], Alfred Brauer proved the class of inequality:

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Theorem 1.2.
$$\ell(n) \leq \min_{1\leq r\leq m}\left[\left(1+\frac{1}{r}\right)\frac{\log n}{\log 2}+2^r-2\right]$$
for $2^m \leq n < 2^{m+1}$.

Brauer obtained Theorem 1.2 by showing that there exists an addition chain leading to n for $2^{rs} \leq n < 2^{r(s+1)}$, starting with $1, 2, \ldots, 2^r-1$ and having length at most $(1+r)s+2^r-2$. Fixing r, he argued basically by the method of contradiction with s. Using the inequality $s \leq \frac{\log n}{r\log 2}$, which can easily be deduced from the constraint on n, one obtains Theorem 1.2. By choosing $r := \lfloor \log\log n\rfloor + 1$ in Theorem 1.2 - which comes from optimizing the quantity on the right-hand side - he obtained the upper bound as a consequence

Proof of Theorem 1.2. Theorem 1.2 is obtained in the spirit of showing that for any positive integer r and any non-negative integer s there exists an addition chain leading to $2^{rs} \leq n < 2^{r(s+1)}$ that starts with the first 2^r-1 consecutive positive integers $1, 2, \cdots, 2^r-1$ and has length
$$\ell(n) \leq (1+r)s+2^r-2.$$
In the case $r=1$, we obtain as a consequence
$$\ell(n) \leq 2s$$
for $2^s \leq n < 2^{s+1}$. The claim holds for this choice of r since this is the right-hand side inequality of Arnold Scholz established by Alfred Brauer.

In the case $s=0$, we obtain the statement: There exists an addition chain that leads to $n < 2^r$ starting with the first 2^r-1 consecutive integers and has length $\ell(n) \leq 2^r-2$. This is obviously true since with the target $n = 2^r-1$ the first 2^r-1 consecutive integers is an addition chain leading to 2^r-1 with length 2^r-2. This serves as an upper bound for the optimal length of all targets $n < 2^r$.

Fix $r > 1$ and suppose that there exists some s for which there are no addition chains leading to $n \in [2^{rs}, 2^{r(s+1)})$ satisfying the length requirement $\ell(n) \leq (1+r)s+2^r-2$ and starts with the first 2^r-1 consecutive integers. Let s_o be the smallest among those s that satisfy this observation. This implies that there are no addition chains leading to n with $2^{rs_o} \leq n < 2^{r(s_o+1)}$ satisfying $\ell(n) \leq (1+r)s_o+2^r-2$ and containing the first 2^r-1 consecutive positive integers. Now divide n by 2^r, then we can write $n = a2^r+b$ for $2^{r(s_o-1)} \leq a < 2^{rs_o}$ and $b < 2^r$. It follows that there is an addition chain
$$a_o = 1, a_1 = 2, \ldots, a_l = a$$

that starts with the first $2^r - 1$ consecutive positive integers and has length $\ell(a) \leq (1+r)(s_o - 1) + 2^r - 2$.
We now construct an addition chain leading to n using the addition chain leading to a as follows

$$1 = a_o, 2 = a_1, \ldots, a_l = a, 2a, \ldots, 2^r a, 2^r a + b = n$$

since $b \leq 2^r - 1$ and must be a term in the chain leading to a. The length of this addition chain is at most

$$(1+r)(s_o - 1) + (r+1) + 2^r - 2 = (1+r)s_o + 2^r - 2$$

which violates the earlier assumption that s_o is the smallest positive integer satisfying the observation. Therefore, the claim holds for all nonnegative integers s for a fixed r. Using the inequality $s \leq \frac{\log n}{r \log 2}$, we deduce

$$\ell(n) \leq \left(1 + \frac{1}{r}\right) \frac{\log n}{\log 2} + 2^r - 2$$

for all $r > 1$. The inequality holds for the minimum over all $r > 1$. Optimizing

$$\left(1 + \frac{1}{r}\right) \frac{\log n}{\log 2} + 2^r - 2$$

we obtain an ideal choice for $r := \lfloor \log \log n \rfloor + 1$ which yields the following inequality.

Corollary 1.3.

$$\ell(n) < \frac{\log n}{\log 2} \left\{ 1 + \frac{1}{\log \log n} + \frac{2 \log 2}{(\log n)^{1 - \log 2}} \right\}$$

for $n \geq n_o$ for some fixed $n_o > 0$

a result which significantly improves the upper bound $\ell(n) \leq 2\frac{\log n}{\log 2}$ for all sufficiently large n.

2. A crude Generalization

Definition 2.1. Let $n \geq 3$ and $d \geq 2$ be positive integers. We say that the sequence of positive integers

$$s_0 = 1 < s_1 < \cdots < s_h = n$$

is an addition chain with fixed degree $d \geq 2$ leading to n of length h if for each $1 \leq i \leq h$ the representation

$$s_i = \sum_{\substack{i_j \in [0, i-1] \cap \mathbb{Z} \\ 1 \leq j \leq d}} s_{i_j}, \quad (s_{i_j} < s_i)$$

holds.

In other words, each term term in an addition chain with a fixed degree $d \geq 2$ is the sum of at most d previous terms in the chain, with repetition allowed. We call the shortest addition chain with fixed degree $d \geq 2$ leading to a target an *optimal* addition chain with fixed degree d. We denote the length of an optimal addition chain with fixed degree $d \geq 2$ leading to a target n with $\ell^d(n)$. The special case where the fixed degree $d = 2$ recovers the well-known concept of an addition chain.

Example 2.2. Choose the target $n = 21$ and fix the degree $d = 3$. The sequence

$$s_0 = 1, s_1 = 2, s_2 = 4, s_3 = 8, s_4 = 16, s_5 = 21$$

is an addition chain with fixed degree $d = 3$, because $s_2 = 2s_1, s_3 = 2s_2, s_4 = 2s_3, s_5 = s_4 + s_2 + s_0$. However, it is not of minimal length. An example chain of fixed degree 3 and of minimal length is

$$s_0 = 1, s_1 = 3, s_2 = 9, s_3 = 21.$$

Theorem 2.3 (Generalized Brauer-type bound)**.** *Let $d \geq 2$ be fixed and let n be a positive integer satisfying $d^m \leq n < d^{m+1}$. Then*

$$\ell^d(n) \leq \min_{1 \leq r \leq v \leq m} \left[\left(1 + \frac{1}{r}\right) \frac{\log n}{\log d} + d^v - 2 \right].$$

Proof. Let $d \geq 2$ be fixed and let n be a positive integer such that $d^m \leq n < d^{m+1}$. We choose $r \geq 1$ to be a positive integer and set

$$k := \left\lceil \frac{m}{r} \right\rceil$$

and

$$v := \max_{0 \leq i \leq k-1} (r(i+1) - i).$$

Let $u_1, u_2, \ldots, u_k$ be a sequence of positive integers such that $0 \leq u_j < d^v$ for $1 \leq j \leq k$ and such that

$$n = \sum_{j=0}^{k-1} u_{k-j} d^{r(k-(j+1))}$$

where

$$u_{k-j} < d^{r(j+1)-j} \leq d^v.$$

This representation exists and is unique. We check this by considering the base-d expansion of n as

$$n = a_0 + a_1 d + \cdots + a_m d^m$$

where $0 \leq a_i < d$. Because $(k-1)r \leq m < kr$, the representation can be rewritten in the form

$$n = b_0 + b_1 d^{kr-(kr-1)} + \cdots + b_p d^{kr-1}$$

with $0 \leq b_i < d$ for $0 \leq i \leq p$. We can therefore write

$$\begin{aligned} b_{p-j} d^{kr-(j+1)} &= b_{p-j} d^{r(k-(j+1))} d^{r(j+1)-(j+1)} \\ &= d^{r(k-(j+1))} u_{k-j} \end{aligned}$$

where

$$u_{k-j} := b_{p-j} d^{r(j+1)-(j+1)} < d^{r(j+1)-j}.$$

Therefore, the representation

$$n = \sum_{j=0}^{k-1} u_{k-j} d^{r(k-(j+1))}$$

with

$$u_{k-j} := b_{p-j} d^{r(j+1)-(j+1)} < d^{r(j+1)-j}$$

exists for each n such that $d^m \leq n < d^{m+1}$ and is unique. Now, we construct an addition chain with fixed degree $d \geq 2$ starting with the sequence of consecutive integers

$$s_0 = 1, s_1 = 2, \ldots, s_{d^v-2} = d^v - 1$$

and continue the degree d chain as

$$\begin{aligned} s_{d^v-1} &= u_k, \\ s_{d^v-1+r} &= d^r u_k, \\ s_{d^v+r} &= d^r u_k + u_{k-1}, \\ s_{d^v+2r} &= d^{2r} u_k + d^r u_{k-1}, \\ s_{d^v+2r+1} &= d^{2r} u_k + d^r u_{k-1} + u_{k-2}. \end{aligned}$$

Continue the degree $d \geq 2$ addition chain in a similar fashion until we reach

$$s_{d^v+(k-1)r+k-2} = \sum_{j=0}^{k-1} u_{k-j} d^{r(k-(j+1))} = n.$$

It follows that $\ell^d(n) \leq d^v - 2 + (k-1)r + (k-1) \leq d^v - 2 + m + \frac{m}{r}$. Using the fact $m \leq \frac{\log n}{\log d}$ and the inequality holds for all $r \geq 1$, the claim follows immediately. □

We can specialize Theorem 2.3 by taking $v = r$ and setting

$$F(r) := \left[\left(1 + \frac{1}{r}\right)\frac{\log n}{\log d} + d^r - 2\right]$$

as a continuous function of r. We minimize $F(r)$ by setting $\frac{dF}{dr}=0$. A quick elementary calculation yields

$$\frac{dF}{dr}=-\frac{1}{r^2}\frac{\log n}{\log d}+d^r\log d=0$$

which gives the equation

$$r\log d+2\log\log d=\log\log n-2\log r.$$

The integer choice of r which gives the local minimum for $F(r)$ can be approximated by

$$r:=\left\lfloor\frac{\log\log n}{\log d}\right\rfloor.$$

By plugging this choice of r into Theorem 2.3, we deduce the general upper bound

Corollary 2.4. *Let $d\geq 2$ be fixed and denote $\{\cdot\}$ as the fractional part of $\cdot$. Then*

$$\ell^d(n)<\frac{\log n}{\log d}\left(1+\frac{\log d}{(\log 2)(\log\log n)}+\frac{\log d}{d^{\{\frac{\log n}{\log d}\}}}\right)$$

for all $n\geq n_0$ for some fixed $n_0>0$.

Remark 2.5 (Improvement communicated by N. Doyon). Immediately after obtaining this result, the author received a private communication from his research director, N. Doyon, containing a substantially stronger upper bound than the one stated in Theorem 2.3. The bound communicated to the author dramatically improves the asymptotic upper bound for $\ell^d(n)$. The sharper inequality communicated is a manuscript in preparation for a journal submission and therefore kept confidential.

References

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