Approximation of the Viswanath's constant in closed form.

Miroslav Súkeník and Magdaléna Súkeníková
Slovak University of Technology, Radlinského 9, 812 37 Bratislava. Slovakia
(sukenik4@gmail.com)

Abstract: *The calculation of this constant requires advanced numerical methods such as interval mathematics, using specialized software for high accuracy and tight error limits. In this article, we present a simpler method for deriving the Viswanath's constant.*

1: Introduction

We define the Fibonacci sequence for n > 1

$$F_{(n)} = F_{(n-1)} + F_{(n-2)} \tag{1}$$

Using Binet's relation, we get:

$$F_{(n)} = \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{\varphi^n - \psi^n}{\sqrt{5}} \tag{2}$$

For the golden ratio and its conjugate, the following applies:

$$\varphi = \frac{1}{2}(1+\sqrt{5}) = 1.61803$$

$$\psi = \frac{1}{2}(1-\sqrt{5}) = -0.61803 \tag{3}$$

A random Fibonacci sequence is given by the recurrence relation f(n-1) +- f(n-2), where the signs + or – are chosen randomly, with equal probability independent for different n. In 1999, Divakar Vishwanath proved that the growth rate of a random Fibonacci sequence is equal to 1.131988248 [1]. In the same year, it was proved that the sequence (4)

$$f_n = \pm f_{n-1} \pm \beta f_{n-2} \tag{4}$$

also decreases if β is less than or approximately equal to 0.70258 [2]. This value is known as the Embree Trefethen constant. It has also been shown that the asymptotic ratio σ(в) between consecutive terms converges for each value of β with a local minimum approximately equal to 0.36747. σ represents the Lyapunov coefficient. Random Fibonacci sequences (RFS) represent a complex system – causality controlled by chance (probability). Random Hermitian matrices (RHM) are used to study it, which play a role both in the distribution of prime numbers and in determining the quantum energy levels of atoms. They also occur in the Riemann zeta function. Now it is necessary to define the Fibonacci zeta function ξf(s).

$$\xi_{f(s)} = \sum_{n=1}^{\infty} (\frac{1}{F_n})^s \tag{5}$$

s can be both real or complex. For all complex numbers s, with real part greater than zero, the function converges [3]. The value for s = 1 is known and this is the reciprocal of the Fibonacci constant with the value 3.359……., which is probably an irrational number. The value for s = 0.757…. is also known and this has the value 4.

2: Control function g(x) and derivation of Viswanath's constant

We write the function g(x) as the sum of all integer dimensions in the interval 0 to infinity.

$$g_{(x)} = x^0 + x^1 + x^2 + ... + x^n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \tag{6}$$

Only the interval (-1, 1) is of interest to us. Figure 1 shows the graph of this function.

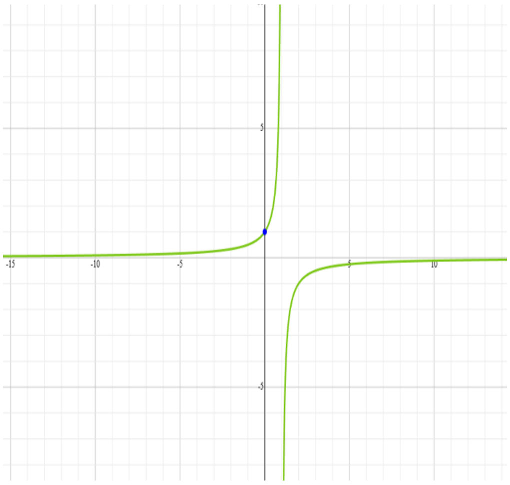

Figure 1: *Graph of the function g(x)*

This function has the following properties [4] applies:

$$\int_{-1}^{0} g_{(x)} dx = \int_{0}^{0.5} g_{(x)} dx = \ln 2 \tag{7}$$

$$\int_{-1}^{0} g_{(x)}^2 dx = \int_{0}^{0.5} g_{(x)}^2 dx = 1 \tag{8}$$

$$\int_{-1}^{0}\sqrt{1+(\frac{dg(x)}{dx})^2}\,dx=\int_{0}^{0.5}\sqrt{1+(\frac{dg(x)}{dx})^2}\,dx=1.132.... \quad (9)$$

Both intervals have the same area, the same normalization (we assume a causal connection there) and the same length of the curve and in the case of the second interval we assume that it is a Viswanath’s constant. The increment is related to the initial value of the function g(x) which is equal to one at point 0.
There are several clues to this. The Embree Trefetheten constant -see (4) has a value of 0.70258. If we put this value as the x-coordinate in the function g(x) in relation (6), we get a value of 3.362 which is very close to the value of 3.359…, the reciprocal of the Fibonacci constant, which represents the value of the Fibonacci zeta function for s = 1. Furthermore, if we substitute the value of 0.382… into equation (6) for x, we get the value of the golden ratio 1.618…. The value of 0.382 is close to the beta value of the local minimum of 0.367… And finally, at s = 0.757… the value of the Fibonacci zeta function is 4, which is close to our function, which has a value of 4.11 at the point 0.757. All this shows that the function g(x) is closely related to the Fibonacci series. Finally, we repeat the derivation of Viswanath's constant.

$$f_{(n)}=f_{(n-2)}\pm f_{(n-1)} \quad (10)$$

$$g_{(x)}=\sum_{n=0}^{\infty}x^n=\frac{1}{1-x} \quad (11)$$

$$\lim_{n\to\infty}\left|f_{(n)}\right|^{\frac{1}{n}}=\int_{0}^{0.5}\sqrt{1+(\frac{dg(x)}{dx})^2}\,dx=1.13209 \quad (12)$$

NOTE: The control function g(x) has an interesting relationship to the Riemann zeta function (see Fig. 2)

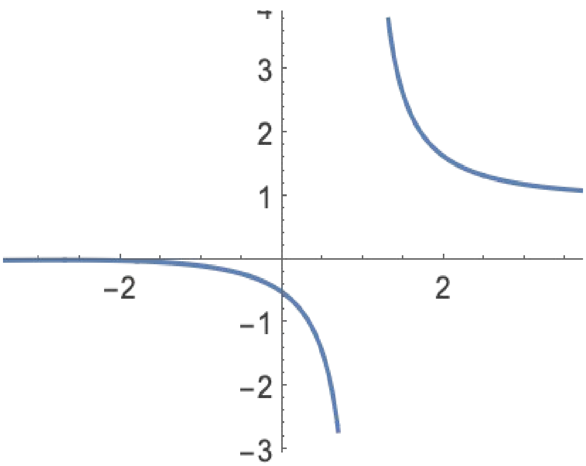

Figure 2: *Graph of the Riemann zeta function*

In the interval (0,1) the following holds:

$$g_{(x)} = -(\zeta_{(s)} + \sigma_{(s)}) \tag{13}$$

The second term on the right is a small deviation depending on s.
There are known relationships between the zeta function, the Moebius function, and the Euler function. Interestingly, the Euler and Moebius functions can be expressed using the golden ratio [5]. We express the golden ratio as t. Then:

$$t = 2\cos(\frac{\pi}{5}) = -\sum_{n=1}^{\infty} \frac{\varphi_{(n)}}{n} \log(1 - \frac{1}{t^n}) \tag{14}$$

$$\frac{1}{t} = 2\sin(\frac{\pi}{10}) = -\sum_{n=1}^{\infty} \frac{\mu_{(n)}}{n} \log(1 - \frac{1}{t^n}) \tag{15}$$

3: Conclusions

In the article, we derived the value of Viswanath's constant equal to 1.13209…. using the control function g(x). This value lies between the value calculated by Viswanath (1.1319882...) and the value printed by Eric Weisstein (1.13215069...) [6]. Finally it is valid:

$$\lim_{n\to\infty} |f_n|^{\frac{1}{n}} = \int_0^{0.5} \sqrt{1 + (\frac{d\sum_{n=0}^{\infty} x^n}{dx})^2}\, dx = 1.13209... \tag{16}$$

There is also the possibility that the Viswanath’s constant corresponds to some fractal dimension of an as yet unknown shape (for instance a fractal dimension equal to 1.26 is known…)

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