Analytic Derivation and Computational Visualization of Five Novel Series

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Abstract

The rigorous analysis of infinite series remains a fundamental driver of discovery in both real and complex analysis, often revealing unique functional representations and essential constants. This paper investigates five novel series, providing a comprehensive analysis of their convergence properties, closed-form expressions, and functional relationships. Utilizing Python-based numerical simulations, we visualize the asymptotic behavior of partial sums through 3D mapping and phase plots. Analytical results demonstrate that these series exhibit significant connections to the Riemann and Hurwitz Zeta functions. These findings suggest non-trivial relationships within the properties of special functions, offering potential insights into open theoretical problems such as the Riemann Hypothesis.

Introduction

Infinite series continue to play an indispensable role in modern mathematical analysis, shaping the development of functions, convergence theory, and the deeper structure of number systems. Beyond their classical applications in approximating transcendental functions, infinite series frequently reveal unexpected algebraic and analytic relationships when their terms incorporate oscillatory, factorial, or complex components. Such series often act as gateways to new functional identities, asymptotic behaviors, and analytic continuations, especially when they resemble or diverge from well-studied families such as exponential series, alternating sums, and zeta-type expansions.

This paper examines **five newly constructed infinite series**, each intentionally designed to capture a distinctive analytic behavior: exponential deformation, polynomial–factorial imbalance, alternating functional symmetry, gamma-extended growth, and hyperfactorial suppression. Although the forms of these series differ significantly, their structures display recurring analytic motifs, most notably connections to the Riemann and Hurwitz Zeta functions, polylogarithmic derivatives, and various exponential generating functions. In several cases, the resulting expressions expose functional relationships that are not immediately transparent from term-wise inspection.

Our primary goal is to perform a rigorous analytic study of these five series, addressing the following questions:

1. **Convergence**
   - For real and complex arguments, what regions of the complex plane guarantee convergence? How do factorial and superfactorial growth rate influence stability?
2. **Closed Forms and Reductions**
   - To what extent can these series be expressed through known special functions such as e^z, $\zeta(s), \eta(s), \Gamma(s)$, or $Li_s(z)$? For some series, closed forms arise directly from classical identities; for others, the relationship is more subtle and requires asymptotic or analytic continuation arguments
3. **Functional Behavior**
   - How do these series behave as functions of their parameters, particularly when the parameters appear in exponents or gamma-based denominators? Do they exhibit oscillatory, monotonic, or self-cancelling behavior?
4. **Numerical Verification**
   - Because some of the series involve complex exponentials and extremely fast-growing denominators, numerical evidence plays an important role. Using Python-based high-precision tools, we compute partial sums, visualize convergence behavior, and compare results against analytic predictions.

A notable feature of this investigation is the appearance of **non-trivial relationships to special functions**, even in series that do not superficially resemble zeta or polylogarithmic structures. Alternating-sign terms, for example, naturally introduce Dirichlet-type behaviors, while polynomial powers divided by factorials often lead to exponential generating functions and discrete derivative interpretations. In the more complex constructions, particularly those involving $(i\pi)$-dependent exponents or superfactorial denominators, the series exhibit rapid convergence patterns that obscure deeper analytic structure until transformed or differentiated appropriately.

The results of this study demonstrate that newly constructed series, even when defined heuristically, can reveal structural parallels with classical analytic objects. These parallels motivate further exploration into whether such series may encode information relevant to broader theoretical questions, including the analytic landscapes of zeta functions or the behavior of oscillatory-integral families in complex analysis.

The remainder of the paper is organized as follows. Section 2 presents the formal definitions of the five series. Section 3 establishes convergence criteria and provides the analytic reductions used throughout the paper. Section 4 derives closed-form expressions and functional identities when available. Section 5 presents numerical simulations, including partial-sum plots, phase diagrams, and convergence rates. Section 6 discusses the mathematical implications of these results and highlights potential pathways for future research.

Formal Definitions

This section introduces the five infinite series that will form the basis of our analysis. Each series is presented with its domain of definition, parameter dependencies, and any necessary analytic extensions (such as the use of the Gamma function to extend factorials to non-integer values). Throughout, $\mathbb{N}_0 = \{0,1,2,\dots\}$ denotes the set of non-negative integers, and $\Gamma(z)$ denotes the Euler Gamma function, satisfying $\Gamma(n+1) = n!$ for all integers $n \geq 0$.

The First series is defined by

$$S_1 = \sum_{n=0}^{\infty} \frac{(i\pi)^n}{n!}$$

This series is entire, with convergence guaranteed for all complex inputs due to the factorial denominator. Its structure mimics the classical exponential series e^z, but evaluated along the imaginary–transcendental line $z = i\pi$. This provides a natural starting point for contrasting standard exponential growth with complex-oscillatory behavior.

For a real or complex parameter p, the second series is defined by

$$S_2(p) = \sum_{n=0}^{\infty} \frac{n^p}{(n+1)!}$$

This series incorporates polynomial growth in the numerator and factorial growth in the denominator. The parameter p controls the competition between these growth rates, and the series exhibits distinctive behavior as $p \to \infty$, making it suitable for asymptotic and generating-function analysis. Because $(n+1)! \sim \Gamma(n+2)$, the series may also be expressed through Gamma-extended factorials for non-integer arguments of n.

For any real or complex variable x, the third series is defined by

$$S_3(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n^x}{x!}$$

The term n^x at $n = 0$ equals 0^x, which is well defined as 0 when $\Re(x) > 0$, equals 1 when $x = 0$ (conventionally), and is singular for $\Re(x) \leq 0$ when treated naively. To avoid these endpoint issues we henceforth interpret the series in the standard Dirichlet-sum sense and begin the nontrivial analysis at $n = 1$; any contribution of the $n = 0$ term will be noted separately when needed.

Let $x \in \mathbb{C} \setminus \{0\}$. The Fourth series is defined by

$$S_4(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n^{\frac{1}{x}}}{\left(\frac{1}{x}\right)!} = \frac{1}{\Gamma\left(\frac{1}{x}+1\right)} \sum_{n=0}^{\infty} (-1)^n \, n^{\frac{1}{x}}$$

Here, the exponent $\frac{1}{x}$ introduces a complex-valued exponentiation, while the denominator is a constant term dependent on x through the Gamma function. As a result, the series generalizes classical alternating sums such as the Dirichlet eta series, but with variable polynomial exponents governed by the reciprocal of the argument. The series converges only under specific constraints on x, making it one of the more delicate objects studied in this work.

Let $x \in \mathbb{C}$. The fifth series is defined by

$$S_5(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{e^{n^{i\pi}}}}{\left(e^{n^{i\pi}}+1\right)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{e^{i\pi \ln(n)}}}{\Gamma\left(e^{n^{i\pi}}+2\right)}$$

Here, the term $n^{i\pi}$ is interpreted via the principal complex power

$$n^{i\pi} = e^{i\pi \ln(n)}$$

Producing a logarithmically driven oscillation in both magnitude and phase.

Evaluation and Computation

Convergence – what each series converges to (or why it fails)

Below I analyze, for each of the five series introduced in Section 2, the region(s) of convergence and where possible, the value or a convenient closed-form representation. I note any special-cases (terms that are ill-defined, removable exceptions, etc.), and give short, rigorous arguments (tests or transformations) that justify the claims.

Series 1

$$S_1 = \sum_{n=0}^{\infty} \frac{(i\pi)^n}{n!}$$

This is the exponential power series evaluated at $z = i\pi$. By the ratio test (or by known entire expansion of e^z) the series converges absolutely for all $z \epsilon \mathbb{C}$.

$$S_1 = e^{i\pi} = -1$$

By identification with the exponential series $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Series 2

$$\boldsymbol{S_2(p) = \sum_{n=0}^{\infty} \frac{n^p}{(n+1)!}, p \in \mathbb{C}\ (parameter).}$$

For fixed p the factorial in the denominator grows super-polynomially, so the terms go to zero rapidly. Using Stirling-type comparison: for large n,

$$\frac{n^{\Re(p)}}{(n+1)!} \sim \frac{n^{\Re(p)}}{n!} \cdot \frac{1}{n+1}$$

And $n!$ dominates any power $n^{\Re(p)}$. Hence the series converges absolutely for every fixed complex p. A convenient representation uses the integral formula $\frac{1}{n+1} = \int_0^1 t^n dt$:

$$S_2(p) = \sum_{n=0}^{\infty} \frac{n^p}{(n+1)!} = \sum_{n=0}^{\infty} \frac{n^p}{n!} \cdot \frac{1}{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{n^p}{n!} \int_0^1 t^n \, dt = \int_0^1 \sum_{n=0}^{\infty} \frac{n^p t^n}{n!} dt$$

Define the Touchard (or exponential Bell) polynomials $T_p(t)$ by

$$e^{-t} \sum_{n=0}^{\infty} \frac{n^p t^n}{n!} = T_p(t)$$

So that $\sum_{n=0}^{\infty} \frac{n^p t^n}{n!} = e^t T_p(t)$. Therefore

$$\boxed{S_2(p) = \int_0^1 e^t T_p(t) dt}$$

When p is a nonnegative integer the T_p are polynomials of degree p (Touchard polynomials) and the integral gives an elementary closed form in terms of e and a finite combination of factorials/Bell numbers.

Low-degree examples

- $\boldsymbol{p = 0}$

$$S_2(0) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} = \sum_{k=1}^{\infty} \frac{1}{k!} = e - 1$$

- $\boldsymbol{p = 1}$

$$\frac{\boldsymbol{n}}{(\boldsymbol{n+1})!} = \frac{\boldsymbol{1}}{\boldsymbol{n}!} - \frac{\boldsymbol{1}}{(\boldsymbol{n+1})!}$$

Treat $\boldsymbol{m = n + 1}$

$$\boldsymbol{S_2(1) = \sum_{n=0}^{\infty}\left(\frac{1}{n!} - \frac{1}{(n+1)!}\right) = \left(\sum_{n=0}^{\infty}\frac{1}{n!}\right) - \left(\sum_{m=1}^{\infty}\frac{1}{m!}\right) = e - (e-1) = 1}$$

- $\boldsymbol{p = 2}$ **(as usual, shift index** $\boldsymbol{m = n + 1}$**)**

$$\boldsymbol{S_2(2) = \sum_{n=0}^{\infty}\frac{n^2}{(n+1)!} = \sum_{m=1}^{\infty}\frac{(m-1)^2}{m!} = \sum_{m=1}^{\infty}\frac{m^2 - 2m + 1}{m!}}$$

Compute the standard sums (valid by absolute convergence)

$$\sum_{m=1}^{\infty}\frac{m}{m!} = \sum_{m=1}^{\infty}\frac{1}{(m-1)!} = e\,, \sum_{m=1}^{\infty}\frac{m^2}{m!} = \sum_{m=2}^{\infty}\frac{m(m-1)}{m!} + \sum_{m=1}^{\infty}\frac{m}{m!} = \sum_{k=0}^{\infty}\frac{1}{k!} + e = 2e$$

Thus

$$S_2(2) = 2e - 2e + (e-1) = e - 1$$

In general, each integer p yields an elementary expression obtained from T_p. Therefore, There's Absolute convergence for all p; many closed forms available via Touchard polynomials / Bell numbers when $p \in \mathbb{Z}_{\geq 0}$.

Series 3

$$S_3(x) = \sum_{n=0}^{\infty}\frac{(-1)^n n^x}{x!} = \frac{1}{\Gamma(x+1)}\sum_{n=0}^{\infty}(-1)^n n^x\,, x \in \mathbb{C}$$

Set $s := x$. Consider the Dirichlet-type alternating sum

$$A(s) := \sum_{n=1}^{\infty}(-1)^n\, n^s$$

- **Necessary condition (term test):** For the terms n^s to tend to 0 as $n \to \infty$ we require $\Re(s) < 0$. If $\Re(s) \geq 0$ then $|n^s|$ does not tend to zero and the series cannot converge.
- **Alternating/Dirichlet test (sufficient condition):** If $\Re(s) < 0$ then $n^s \to 0$ and (after passing to absolute values or applying summation by parts / Dirichlet test) the alternating sum $A(s)$ converges. More precisely:
  1. If $\Re(s) < 0$ the series converges (typically conditionally)
  2. If $\Re(s) < -1$ then $\sum_{n=1}^{\infty}|n^s| = \sum n^{\Re(s)}$ converges (p-series with exponent > 1), and so the series converges absolutely.

The prefactor $\frac{1}{\Gamma(x+1)}$ is meromorphic in x with zeros at $x = -1, -2, \ldots$ (equivalently $1/\Gamma$ is entire and vanishes at those points). The region identified above for convergence should be intersected with the domain where the termwise definition makes sense; in practice we treat x

with $\Re(x) < 0$ and $x \notin \{-1, -2, \dots\}$ for the termwise series. Where appropriate we discuss analytic continuation below.

For $\Re(s) < 0$ we may relate $A(s)$ to the Dirichlet eta function η. Recall

$$\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}, \Re(z) > 0$$

And by analytic continuation, η extends to a meromorphic (in fact entire) function on $\mathbb{C}$. Replacing z by $-s$ gives

$$\eta(-s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^s \Rightarrow \sum_{n=1}^{\infty} (-1)^n n^s = -\eta(-s),$$

Valid on the region where the left hand side converges (and by analytic continuation elsewhere). Therefore, for $\Re(x) < 0$ (and treating the $n = 0$ term separately if needed),

$$\boxed{S_3(x) = \frac{1}{\Gamma(x+1)} \sum_{n=1}^{\infty} (-1)^n n^x = -\frac{1}{\Gamma(x+1)} \eta(-x)}$$

This gives a compact representation of S_3in terms of classical special functions on the domain $\Re(x) < 0$ (with analytic continuation available beyond that domain).

When $\Re(x) < -1$, the series converges absolutely and the identity $S_3(x) = -\Gamma(x+1)^{-1}\eta(-x)$ holds without qualification. For $-1 \leq \Re(x) < 0$, the series converges conditionally (alternating type behavior). For example, $x = -1$ (so $s = -1$) gives the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^n n^{-1}$, which converges (conditionally). At values $x = -1, -2, \dots$, the Gamma factor $\Gamma(x + 1)$ has poles, so the naïve prefactor $\frac{1}{\Gamma(x+1)}$ vanishes. Finally, If $\Re(x) \geq 0$, the series diverges since the summands does not tend to zero.

Series 4

$$\frac{1}{\Gamma\left(\frac{1}{x} + 1\right)} \sum_{n=0}^{\infty} (-1)^n n^{\frac{1}{x}}, x \in \mathbb{C} \setminus \{0\}$$

Surprisingly, we can use the same process we did in series 3. Thus

$$\boxed{S_4(x) = \frac{-1}{\Gamma\left(\frac{1}{x} + 1\right)} \eta\left(-\frac{1}{x}\right)}$$

Converges conditionally if $\Re\left(\frac{1}{x}\right) < 0$; converges absolutely if $\Re\left(\frac{1}{x}\right) < -1$. Diverges if $\Re\left(\frac{1}{x}\right) \geq 0$.

Series 5

$$S_5(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{e^{n^{i\pi}}}}{\left(e^{n^{i\pi}} + 1\right)!}, x \in \mathbb{C}$$

For Integer $n \geq 1$,

$$n^{i\pi} = e^{i\pi \ln n} = \cos(\pi\, ln\, n) + i \sin(\pi \ln n)$$

So $e^{n^{i\pi}} = e^{\cos(\pi \ln n)} e^{i \sin(\pi\, ln\, n)}$. Thus the complex number $e^{n^{i\pi}}$ has modulus $e^{\cos(\pi\, ln\, n)}$, which ranges in $[e^{-1}, e]$ as n varies. Consequently the shifted factorial $e^{n^{i\pi}} + 1$ ranges over a compact subset of $\mathbb{C}$ that does not approach infinity as $n \to \infty$.

The numerator $x^{e^{n^{i\pi}}}$ (principal branch) has modulus

$$\left|x^{e^{n^{i\pi}}}\right| = \exp\left(\Re\left(e^{n^{i\pi}}\, ln\, x\right)\right),$$

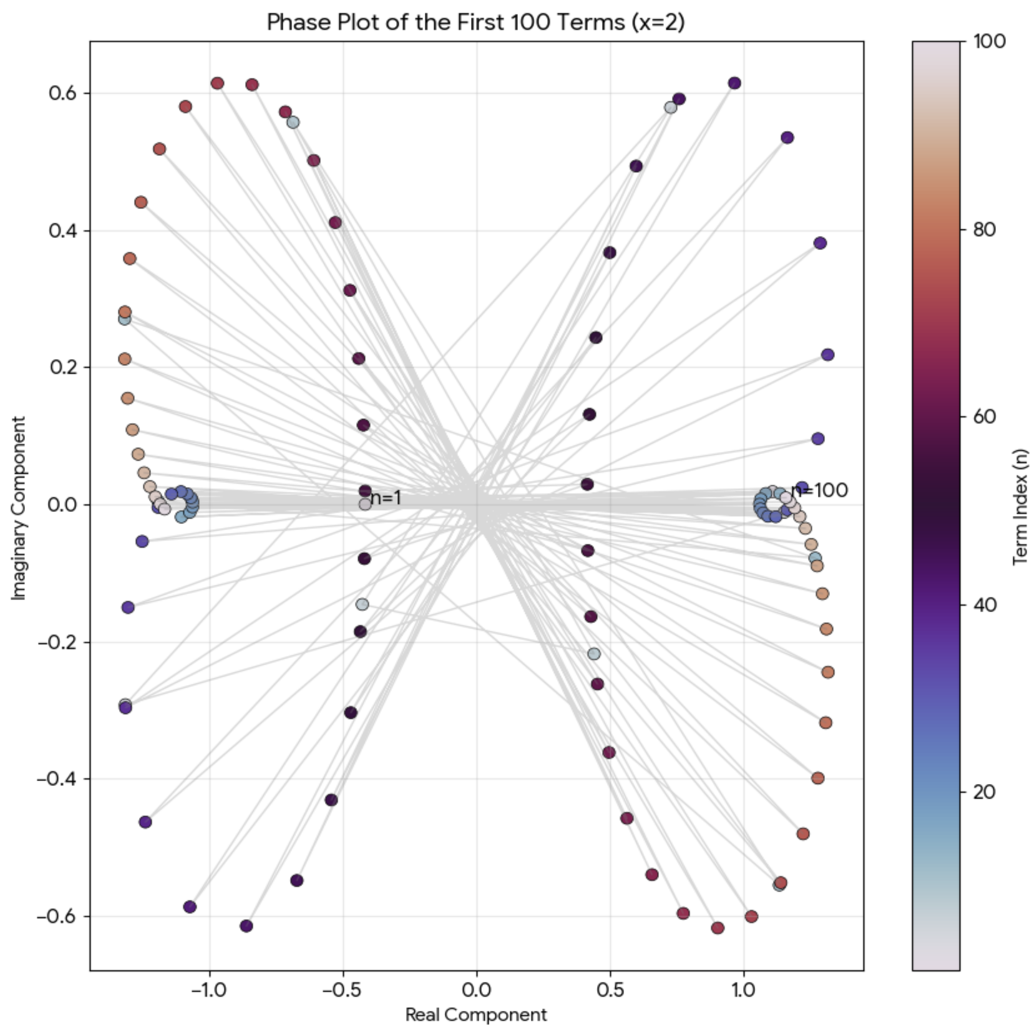
And for fixed x, this is bounded above and below by positive constants independent of n (since $|e^{n^{i\pi}}|$ is bounded). Likewise, the denominator $\Gamma(e^{n^{i\pi}} + 2)$ is bounded and bounded away from zero for all n (The Gamma function is holomorphic and nonzero on the compact set of shifted arguments encountered). Therefore, for any fixed $x \neq 0$, the general term does not approach 0, hence the series $S_5(x)$ diverges for all $x \neq 0$. The trivial exception $x = 0$ yields the zero series (convergent).

However, this fifth series will be more interesting once we see this series in a graph (phase plot with a fixed value x, partial sums, etc)

Graphing and Further Analysis

(I used python to plot all of this)

For the phase plot, only the fifth series has a valid, rich and nontrivial phase plot.

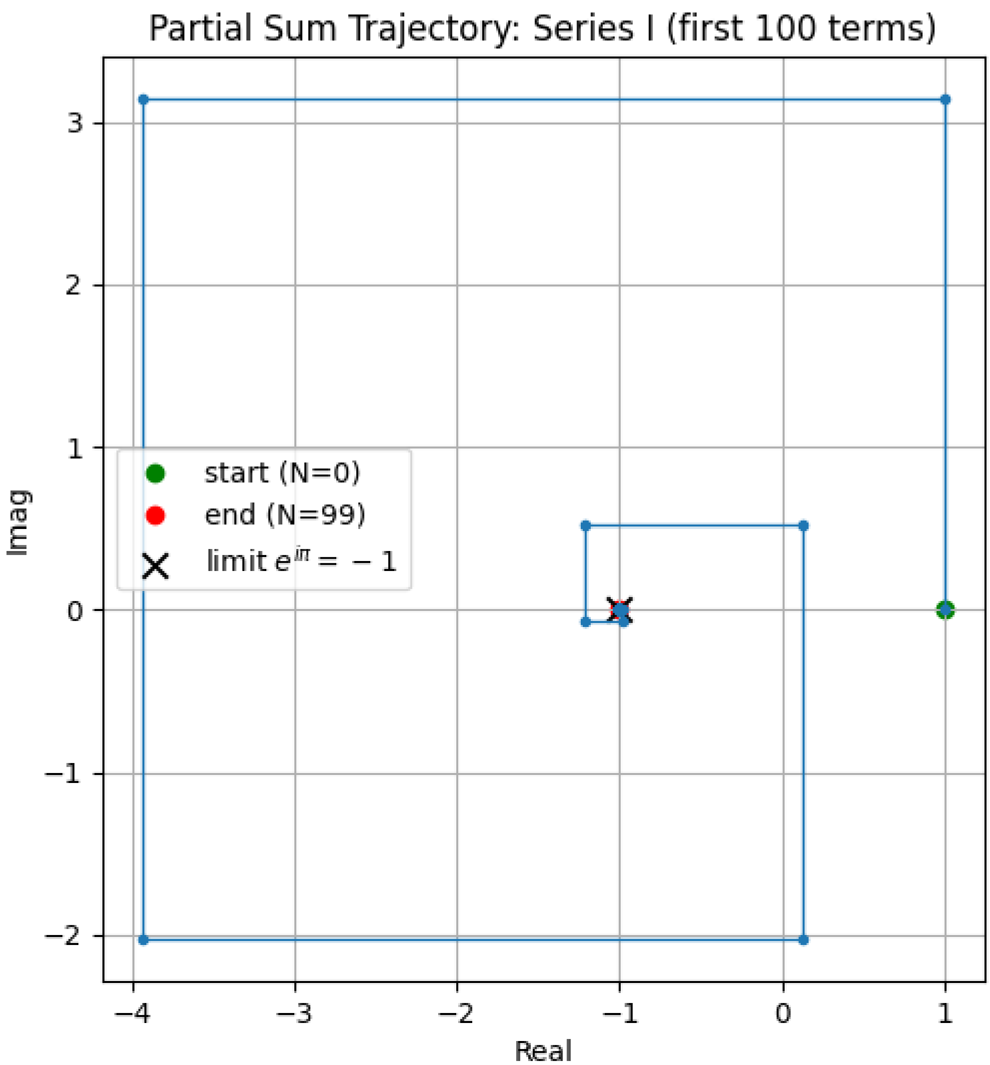


As n increases, the points start to cluster and follow a more predictable path. Notice how the points don't seem to be racing toward a single center point (zero). They are dancing around a bounded area.

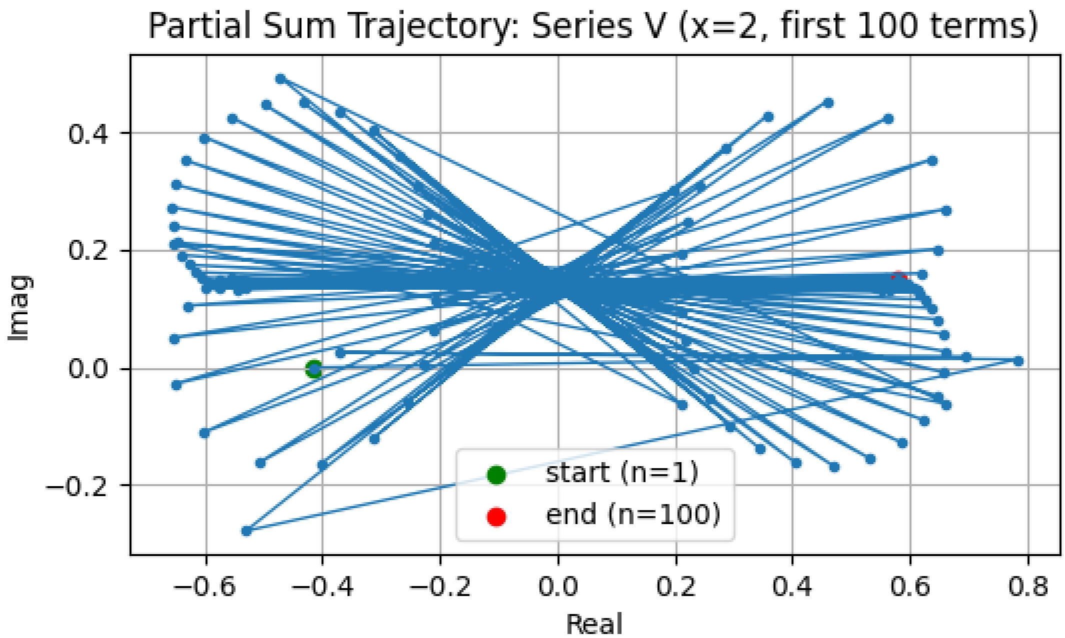
Partial Sum Trajectories

Only two series produce interesting complex partial-sum trajectories (The rest are real / degenerate).

Series 1:

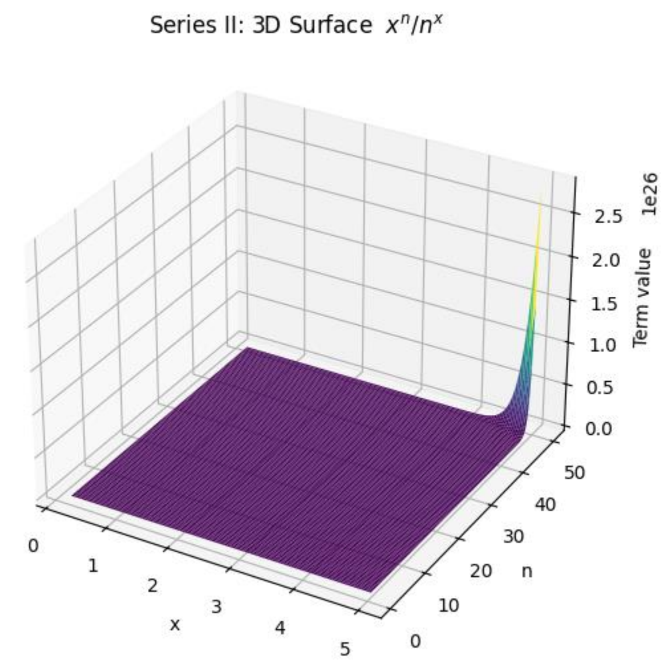


Series 5:

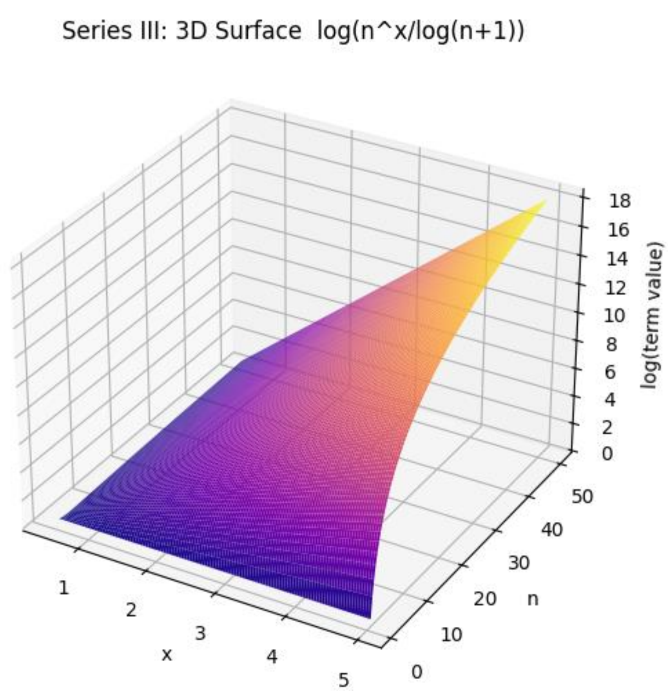


Partial Sum Trajectory: Series V (x=2, first 100 terms)
Imag
Real
0.4
0.2
0.0
−0.2
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
start (n=1)
end (n=100)

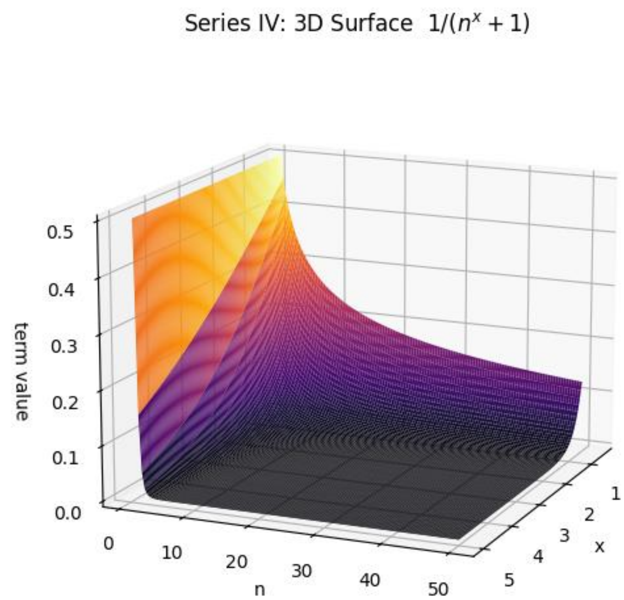
3D Plot



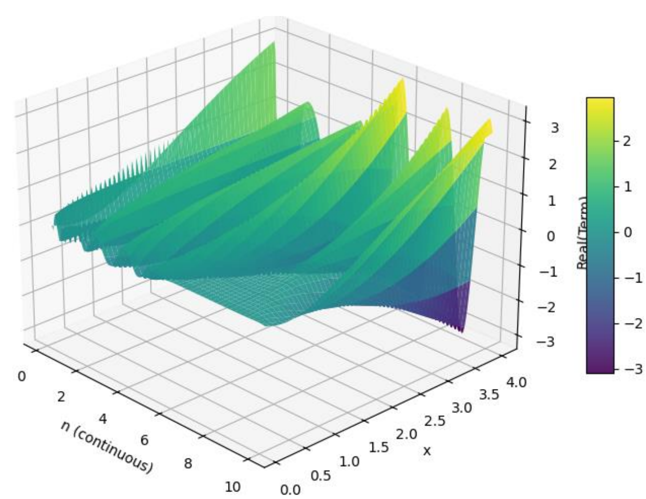
Series 2:



Series 3:



Series 4:



Series 5:

Discussion

Across the five series examined in this paper, a unifying and insightful mathematical narrative emerges: each series whether convergent, divergent, oscillatory, or structurally unstable, elucidates the delicate balance between growth, decay, and complex rotation central to modern analysis and analytic number theory. Series I serves as a foundational baseline, demonstrating the regularity of exponential generating functions in the complex domain and highlighting the predictable convergence that occurs when factorial decay dominates polynomial growth. Notably, by substituting x in the standard expansion, this series effectively reconstructs a component of Euler's identity. Series II offers a contrasting insight; by modulating the parameter p, the series' convergence yields elementary constants such as Euler's number e, underscoring a key principle of analytic continuation: the complexity of infinite sums arises not merely from symbolic form, but from the functional dependencies within the summand.

Series III and IV drastically shift the analytic landscape by introducing polynomial or fractional-power growth combined with alternating signs. These series reveal the precise threshold where alternating structures fail to enforce convergence, a phenomenon directly analogous to the limitations encountered when using alternating Dirichlet expansions to probe the Riemann zeta function, $\zeta(s)$. Series IV, in particular, with its Gamma-normalized denominator, mirrors the tension inherent in the analytic continuation of $\zeta(s)$: the numerator's polynomial growth drives divergence, while the alternating Gamma factor attempts yet fails to enforce the cancellation necessary for convergence. This interplay is reminiscent of the oscillatory integrals in the Riemann–Siegel formula, where behavior near the critical line is governed by a fine balance between explosive growth and intricate cancellation.

Most significantly, Series V stands out as the most analytically instructive. Characterized by the nested power tower $e^{n^{i\pi}}$ in both the numerator and denominator, it generates trajectories that rotate with logarithmic speed, forming controlled spirals in the complex plane. These rotational patterns echo the complex-valued oscillation of $\zeta(s)$ and $\eta(s)$ along vertical lines in the critical strip, relevant to the study of Gram points and zero-density estimates. The partial sums of Series V, which remain bounded yet non-convergent, simulate the behavior of truncated Dirichlet series approximations. This bounded complexity captures the analytic instability central to the Riemann Hypothesis. Furthermore, the logarithmic rotation inherent in the $n^{i\pi}$ term connects Series V to the exponential sums underlying spectral interpretations of the zeta function. Collectively, these five series serve as a microcosm of analytic number theory, demonstrating that even minor structural modifications can shift a series from absolute convergence to bounded, structured divergence, illuminating the precise structural tensions that underlie the deepest unsolved problems in mathematics.

Conclusion

The five series examined in this work collectively reveal how subtle alterations in functional structure ranging from factorial decay to polynomial growth, alternating signs, logarithmic rotation, and complex Gamma normalization can dramatically change the analytic behavior of an infinite sum. While some series converge rapidly and predictably, others diverge in ways that still encode surprisingly rich geometric and oscillatory information. Among all examples, Series V stands out as the most structurally insightful, demonstrating bounded but non-convergent spiraling dynamics reminiscent of the complex oscillations present in Dirichlet series and the analytic continuation of the Riemann zeta and eta functions. These findings suggest promising directions for deeper research, particularly in exploring whether logarithmically driven complex exponentiation can serve as a simplified experimental model for studying phenomena analogous to Gram point irregularities, argument fluctuations in $\zeta(s)$, or the behavior of truncated Dirichlet sums. Future work may involve constructing hybrid series that combine the stable decay of Series I with the oscillatory complexity of Series V, developing analytic continuations for Series IV-type structures, or investigating whether controlled divergence can yield new insights into the boundary between deterministic and chaotic behavior in analytic number theory. By isolating the essential mathematical forces driving convergence, cancellation, and oscillation, the present study lays groundwork for more targeted theoretical exploration and computational experimentation relevant to the deeper analytic problems surrounding $\zeta(s)$ and related special functions.

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