Euler Product of the Dirichlet eta Function

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Let the Dirichlet eta Function representation of the zeta function, also known as the alternating zeta function, be defined as:

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \tag{1}$$

Euler's expansion of the Dirichlet series of the zeta function into an infinite product (2) provides inspiration for a similar product formula for the eta Function.

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=p_n}^{\infty} \frac{1}{1-p^{-s}} \tag{2}$$

Whereby noting the alternating sum corresponds with alternating multiplication, we find a product formula for real input in the eta function (3).

$$\zeta(a) = \frac{1}{1-2^{1-a}} \prod_{p=p_n}^{\infty} \left(\frac{1}{1-p_n^{-a}}\right)^{(-1)^n} \tag{3}$$

Which provides a useful infinite product representation for particular values of the eta function and their relationship with pi and the Bernoulli numbers.

$$\prod_{p=p_n}^{\infty} \left(\frac{1}{1-\frac{1}{p_n^2}}\right)^{(-1)^n} = \frac{\pi^2}{12}$$

$$\prod_{p=p_n}^{\infty} \left(\frac{1}{1-\frac{1}{p_n^4}}\right)^{(-1)^n} = \frac{7\pi^4}{720}$$

This Euler Product (3) of the Dirichlet eta Function is only valid for a > 1, which prevents its use in evaluating the zeroes of the zeta function, all of which are known to occur on the critical line, where a = .5. We can derive an alternative Product formula, by applying Euler's original Product Formula derivation to the Dirichlet eta Function.

Let the Zeta function be defined as :

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \tag{4}$$

Let the Dirichlet eta Function be defined as:

$$\eta(s) = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} \tag{5}$$

It is possible to separate the positive and negative terms in (5) as:

$$\eta(s) = \left(\frac{1}{1^s} + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s}\right) - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s}\right) \tag{6}$$

We can rewrite the positive terms following Euler's factorization of the zeta function:

$$\eta(s) = \left(1 - \frac{1}{2^s}\right)\zeta(s) - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s}\right) \tag{7}$$

We can also factor the negative factors which are all even:

$$\eta(s) = \left(1 - \frac{1}{2^s}\right)\zeta(s) - \frac{1}{2^s}\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s}\right) \tag{8}$$

The negative terms are then just a multiple of the zeta function.

$$\eta(s) = \left(1 - \frac{1}{2^s}\right)\zeta(s) - \frac{1}{2^s}\,\zeta(s) \tag{9}$$

Finally, we can rewrite the zeta function as Euler's product formula and factor it out:

$$\eta(s) = \prod_{p=p_n}^{\infty} \frac{1}{1-p^{-s}} \ \left(1 - \frac{1}{2^s}\right) - \frac{1}{2^s} \tag{10}$$

This inner difference of the infinite product (which I will henceforth refer to as o(s) and e(s) for odd and even respectively), provides deep insight into the behavior of the Dirichlet eta Function and the Riemann Zeta function.

$$\eta(s) = \prod_{p=p_n}^{\infty} \frac{1}{1-p^{-s}} \quad o(s) - e(s) \tag{11}$$

The zeroes of the function occur when:

$$o(s) - e(s) = 0$$

Or in terms of a ratio as:

$$\frac{o(s)}{e(s)} = 1$$

This ratio of even to odd portions of the zeta function shows relative contribution by each and allows direct conversion between the eta and zeta function:

$$\frac{o(2)}{e(2)} = 3$$

$$\frac{o(1)}{e(1)} = 1$$

$$\frac{o(.5)}{e(.5)} = -1 + \sqrt{2}$$

Note that for any input s:

$$o(s) + e(s) = 1$$

For any input s, with real part = ½, this inner difference has the special property that:

$$o(0.5+b\,i)-e\,(0.5+b\,i)=-\frac{o(0.5+b\,i)}{e\,(0.5+b\,i)} \qquad (12)$$

This reflective point of a symmetry can be seen as we move away from the critical line, in that:

$$o(0.4+b\,i)-e\,(0.4+b\,i)=-\frac{o(0.6+b\,i)}{e\,(0.6+b\,i)}$$

$$o(0.3+b\,i)-e\,(0.3+b\,i)=-\frac{o(0.7+b\,i)}{e\,(0.7+b\,i)}$$

$$o(0+b\,i)-e\,(0+b\,i)=-\frac{o(1+b\,i)}{e\,(1+b\,i)}$$

Thus, we can interpret the zeroes of the Zeta Function as when:

$$o(s)+e(s)=1$$

And

$$o(s)-e(s)=-\frac{o(s)}{e(s)} \qquad (13)$$

Of which the only solutions to this hyperbolic cylinder are:

$$\left\{e(s)=1-\frac{1}{2^s},\, o(s)=\frac{1}{2^s}\right\} \text{ where, } \quad s=\frac{1}{2}$$

and

$$\left\{e(s)=1+\frac{1}{2^s},\, o(s)=-\frac{1}{2^s}\right\} \text{ where, } \quad s=\frac{1}{2}$$

References:

Meyer, D. (n.d.). Euler’s product formula and the Riemann zeta function. https://davidmeyer.github.io/qc/Euler_product_formula_for_the_Riemann_zeta_function.pdf