A Structural Resolution to the Halting Problem: Bijective Guarantee of the Collatz Conjecture via Peano-Cantor Integration

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Abstract

This paper identifies the fundamental difficulty of the long-unresolved Collatz Conjecture as stemming from an unconscious self-limitation in conventional mathematics—specifically, a structural "lack of collaboration" between the linear definition of natural numbers (Peano Axioms) and non-linear structural analysis (Cantor's Set Theory). We propose that, to solve a non-linear Halting Problem like the Collatz Conjecture, the proof must prioritize the structural fundamental of "symmetry of the start and stop point (1)" instead of adhering to linear methods. Specifically, we resolve the "bijective definition deficit" of the mapping existing between specific sub-patterns (e.g., $4n + 1 \leftrightarrow 3n + 1$) derived from the Collatz operation, by using Cantor's dimensional expansion pairing function. This method reconstructs the Collatz operation as a closed, bijective structure centered at 1, structurally and completely excluding the possibility of cycles other than 1 and divergence to infinity. This represents a structural solution that, by integrating the Peano Axioms and Cantor's Set Theory, rigorously guarantees the global stability of the Collatz infinite tree for the first time.

Keywords: Collatz, Tree Equivalence Theorem, Peano's Successor Function, Cantor's Pairing Function

1 Introduction

The Collatz Conjecture (CC) asks whether all $n \in \mathbb{N}_+$ reach 1 under the iteration: $n \mapsto n/2$ if even, $n \mapsto 3n + 1$ if odd.

Proposed by Lothar Collatz in 1937, its nonlinear branching structure has long fascinated both mathematicians and the general public. Yet, a complete proof remains elusive.

The author has previously shown, using the axiom of infinite sets and reductio ad absurdum, that the existence of isolated points or nontrivial cycles entails logical contradictions [1, 2, 3]. This approach, based on an Indirect Recursive Proof (IRP), incorporates philosophical, computational, and reductio-based limitations. Although IRP is a powerful tool suggesting the validity of CC, the extrapolation from finite regions to the entire infinite domain inevitably retains ambiguity.

Van Bendegem sharply criticizes these limitations of IRP. In "The Collatz Conjecture: A Case Study in Mathematical Problem Solving" (2005) [4], he characterizes informal proofs as those which fail to meet formal criteria yet still indicate the possibility of correctness, and highlights the insufficient justification for extending finite reasoning

to the infinite. Moreover, Van Bendegem argues that arguments relying solely on computational difficulty cannot reach the structural essence of the problem, thus calling for a fundamental relocation of the underlying framework. For instance, the Erdős Paradox (EP)—symbolized by Paul Erdős's $500 reward for a counterexample—reflects how the potential of divergent orbits exposes the boundaries of mathematics itself. Likewise, Lagarias' approach to computability (LC) refines the understanding of CC's computability and partially demonstrates boundedness of orbits under Turing completeness, but still fails to establish static guarantees.

These perspectives have been organized by Jeffrey Lagarias in "The 3x+1 Problem and Its Generalizations" (1985, updated 2010) [5]. However, the formal foundation for a direct proof of CC remains undeveloped.

This paper introduces a Direct Relocation Proof (DRP) of the CC, grounded in a Four-Dimensional Natural Numbers ($\mathbb{N}^{4d}$). By replacing the successor-based iteration with a Parent Mapping (PM) and Child Root (CR) structure, we construct a static, bijective framework that enables the relocation of CC's dynamic series into a spatially closed system. This approach aims to resolve the conjecture not through extrapolation, but through structural closure.

2　Background to Dimensional Natural Number System

2.1　Collatz Operation Table (2022)

2.1.1　Arrangement in Row Order (Prototype of DRP)

In the 2022 study [1], the bijection between Column B and Column E was used as the basis to derive the other columns as follows:

- **Column A:** If B is odd: $A = B$. If B is even: $A = B/2$ repeatedly until an odd number is obtained.

- **Column B (B-type):** $3n+1$ is $C/2$ in Collatz operation or a base element in inverse operation. This represents natural numbers of the form $3n + 1$, where each row lists: $1, 4, 7, 10, 13, \ldots$

- **Column C (C-type):** $6n+2$ is $D/2$ in Collatz operation or $2B$ in inverse operation.

- **Column D (D-type):** $12n+4$ is $3E+1$ in Collatz operation or $2C$ in inverse operation.

- **Column E (E-type):** $4n + 1$ is a base element in Collatz operation or $(D-1)/3$ in inverse operation. This represents odd numbers of the form $4n + 1$, where each row lists: $1, 5, 9, 13, 17, \ldots$

- **Column F (F-type):** $4n + 3$, defined as $(2E - 1)/3$ if divisible. This represents odd numbers of the form $4n + 3$, and if an integer result is obtained, the iteration continues through subsequent columns.

- :

If we extend the rows infinitely in the $n \geq 0$ order, all odd numbers will be located in Columns **E** and **F** onwards.

This bijection $(B \leftrightarrow E)$ corresponds to the number of nodes N in the tree structure. For $n = 0$, $(B = E)$ is the root R, and only this case is identical; for all other cases $(B \neq E)$, the numbers belong to different $4n + 1$ type odd classes, which correspond to the number of edges E in the tree structure.

2.1.2 Sorting from the R-row (IRP-type)

Using the **0th relocation** of the R-row for $n = 0$ ($E = 4 \cdot 0 + 1 = 1$) as the key, the rows of the **1st relocation** with $A = 1$ are sorted below it.

Next, using the non-$3n$-type odd numbers in Columns **E** and **F** of the **1st relocation** rows as keys, the rows of the **2nd relocation** having the same A-value are sorted below them.

By repeating step 2, all odd numbers are arranged in relocation order of a tree structure as relocations progress.

2.1.3 Note 2022

This sorting is a visualization of the structure, but does not guarantee it. The dimensional natural number system ($\mathbb{N}^d$) proposed in this research was born from the intuition that the difference between Peano's "+1" arithmetic and Collatz arithmetic is determined solely by the arrangement of natural numbers in this row unit.

2.2 CC's IRP (2023)

2.2.1 In the 2023 study

Based on the **2022 preprint**, the Collatz sequence (and its reverse sequence) was classified into the following three dimensional patterns:

1. **Dimensional pattern 1(Even numbers):** Repeatedly divided by $1/2$ until they become odd. In the reverse sequence, this corresponds to a geometric sequence with an initial odd term and common ratio 2.

2. **Dimensional pattern 2(Odd numbers of the form $4n + 3$):** Bijective mappings such as $4n + 3 \leftrightarrow 12n + 10 \leftrightarrow 6n + 5 = 3(2n + 1) + 2$.

3. **Dimensional pattern 3(Odd numbers of the form $4n + 1$):** Bijective mappings such as $4n + 1 \leftrightarrow 12n + 4 \leftrightarrow 6n + 2 \leftrightarrow 3n + 1$.

Among these, Dimensional pattern 1 can be interpreted as a bit operation that removes the lower $0_{(2)}$ in the binary representation, and Dimensional pattern 2 as a bit operation that removes the lower $1_{(2)}$. Therefore, any natural number can be divided into subsets that have a $4n + 1$ type odd number as a child root after a finite number of Collatz operations.

In the remaining Dimensional pattern 3, within the subset rooted at $4 \cdot 0 + 1$, a different odd number of the form $4n + 1$, which is bijective with a natural number of the form $3n + 1$ relocated by the inverse operation, becomes connected, and the Collatz tree expands. If an isolated subset rooted at an odd number of the form $4n + 1$ existed,

it would contradict the bijection with the $3n+1$-type natural numbers. Hence, the CC was established as an IRP through contradiction.

2.2.2   Note 2023

The fundamental difference between the 2023 study [2] and the present paper is that, although the case classification by the three dimensional patterns of an IRP was achieved, the definition, proof, and notation of the $\mathbb{N}^d$ were still lacking.

2.3   Refinement of the CC's IRP (2024)

2.3.1   In the 2024 study

The focus was on refining the 2023 IRP of the CC into a rigorous mathematical proof suitable for journal submission. However, despite multiple submission attempts, the refined proof was repeatedly rejected by various journals.

   Through this process, it became increasingly clear that the root of the "trivial loop" governs the Infinitely Branching (IB) and provides the only explicit path to escape the "devil's proof".

   (1) The Collatz operation applies $x/2$ to even integers and $3x+1$ to odd integers;

   (2) When applied to $4n+3$ type odd integers, the shortcut operation $(3x+1)/2$ successively reduces the trailing $1_{(2)}$ bits in the binary representation, ensuring that repeated application always leads to a $4n+1$ type odd integer;

   (3) When applied to $4n+1$ type odd integers, the shortcut operation $(3x+1)/4$ ultimately converges to the trivial loop $(1 \rightarrow 4 \rightarrow 2 \rightarrow 1)$.

2.3.2   Note 2024

In the 2024 study [3], Therefore, the structure of the trivial loop $(1 \rightarrow 4 \rightarrow 2 \rightarrow 1)$ of (3) itself determines the operational formats of (2), and functions as the " successor function" that governs the relocation of IB.

3   Relocation of Two-, Three-, and Four-Dimensional Natural Numbers

3.1   Peano–Cantor Disunion and the Limitations of ZFC

The difficulty in proving the CC can be traced back to a historical "disunion" in the late 19th century regarding the foundations of natural numbers. Giuseppe Peano, in Arithmetices Principia (1889), axiomatized the natural numbers (existence of 0, successor $S(n) = n+1$, induction) and established a linear sequence as the basis of arithmetic [6]. Influenced by Dedekind, this concise system aligned well with the then analysis-oriented mathematical community, yet regarded Cantor's infinite equipotence (bijections) and diagonal arguments (1870s) as "overly abstract," thereby remaining focused on finite induction [7]. Consequently, the Peano axioms did not actively integrate spatial freedom in infinite sets and solidified a framework bound to linear "+1" succession. This has been a latent obstacle to static analysis of nonlinear tree structures such as those appearing in the CC (i.e., the topological identity $N = E+1$).

   Meanwhile, Georg Cantor, in parallel with Dedekind, founded set theory and demonstrated the nonequipotence between natural numbers and the reals, offering a distinct foundation

based on the "spatial rearrangement" of infinity [8]. However, since Peano's perspective remained dominant, Cantor's notion of bijection—with its potential to generalize Peano's linear structure—was left largely unintegrated. When the CC emerged in 1937, this gap became explicit: infinitely branching trajectories were handled solely through Peano-style induction, making the dynamic ambiguity of IRP unavoidable.

The ZFC axiomatic system (Zermelo 1908, Fraenkel–Skolem 1922) has often been viewed as a later reconciliation of this divide [9]. ZFC axiomatized Cantor's set theory and made Peano arithmetic interpretable inside it (by constructing $\mathbb{N}$ as the von Neumann ordinal ω) [10]. With the Axiom of Choice (C), the existence of bijections was formally guaranteed, ensuring consistency in injecting Cantorian infinity into a Peano-based framework.

However, this remains at the level of interpretability, not constructive integration: it does not actively promote spatial relocations such as establishing a bijection between the address dimension $\mathbb{N}_0$ and the sequence dimension $\mathbb{N}_+$ to build $\mathbb{N}^d$ [11]. While ZFC's Axiom of Infinity supports Peano's "+1," it does not enforce static closure of nonlinear trees (i.e., $N = E + R$). Gödel's incompleteness theorems (1931) further emphasized the inherent limitations of the axiomatic system [12].

As a result, the prevailing IRP approaches to the CC (e.g., Tao's partial result on orbit boundedness, 2019) have remained stalled within ZFC, and the philosophical and computational limitations highlighted by VP / EP / LC continue to be exposed [13].

3.2 Relocation of Two-Dimensional Natural Numbers Based on the Peano Axioms

Whether natural numbers begin from 0 or from 1 is merely a matter of notation. However, in order to treat natural numbers as an infinite set in a structural manner, the following properties must be defined as a bijectively connected two-dimensional configuration:

- Sequential progression: $n \to n + 1$

- Spatial mappings: address space $\mathbb{N}_0$

Through this correspondence, Peano's sequential progression is reinterpreted not merely as an ordering but as a path-graph structure in which addresses and sequences correspond. Consequently, the entire set of natural numbers is redefined as a spatially guaranteed equivalent structure: the two-dimensional natural numbers ($\mathbb{N}^{2d}$).

This structure closes the unresolved lack of targeting between Peano and Cantor, and serves as a $\mathbb{N}^d$ for dealing with nonlinear branching structures—such as the Collatz dynamics—in a static manner.

First, we define the closure of the root in the $\mathbb{N}^{2d}$ using the following mappings:

$$f(x) = \begin{cases} x & \text{if } x = 1 \\ x - 1 & \text{if } x > 1 \end{cases}$$

Based on the above mappings, the following definitions are established:

- **PS (Parent Sequences): A sequence pattern $(n, n+1)$ corresponding to the origin $n = 0$ of $\mathbb{N}_0$**

- **PM (Parent Mappings): A bijection between $n \in \mathbb{N}_0$ and $n+1$ in PS, expanded until it reaches the R**

- **R (Parent Root): The stopping point of PS, Address $0 : (0,1)$; not merely a symbolic origin, but a structurally guaranteed root ensured by PM**

- **CM (Child Mappings): Natural numbers belonging to $\mathbb{N}_+$**

- **CR (Child Roots) / CI (Child Images): Images of natural numbers belonging to $\mathbb{N}_+$**

If we regard $n \leftrightarrow n+1$ as a PS, then the initial origin R is given by $(n = 0, n+1 = 1)$. By extending this in the order of n, the structure of $\mathbb{N}^{2d}$ is relocated.

In this paper, based on $\mathbb{N}^{2d}$, we deductively define the higher dimensional relocation of the CC.

3.3 Simplified CC and the Three-Dimensional Structure of Natural Numbers

First, the shortcut of the simplified CC—defined as follows: if even, $y = x_{\mathbf{odd}}/2$; if odd, $y = (x_{\mathbf{even}} + 1)/2$—always decreases the value in either case, and therefore eventually reaches the trivial loop $(0, 1 \to 2 \to 1)$.

$$f(x) = \begin{cases} \frac{x+1}{2} & \text{if } x \equiv 1 \pmod{2} \quad \textbf{(odd)} \\ \frac{x}{2} & \text{if } x \equiv 0 \pmod{2} \quad \textbf{(even)} \end{cases}$$

Based on the above mappings, the following definitions are established:

- **PS (Parent Sequences): The sequence pattern corresponding to the origin $n = 0$ of $\mathbb{N}_0$, given by $(n, 2n+1 \to 2n+2 \to n+1)$.**

- **PM (Parent Mappings): A bijective mapping via the shortcut operation $(2n+1 \leftrightarrow n+1)$, which unfolds until reaching R.**

- **R (Parent Root): The root of PM and the stopping point of PS, $(0, 1 \to 2 \to 1)$. This structurally guarantees the origin $n = 0$ of $\mathbb{N}_0$. It is not a symbolic starting point, but a root defined by PM.**

- **CM (Child Mappings): A mapping applied to even numbers and expanded until reaching CR.**

- **CR (Child Roots): Roots of CM, which are odd numbers of the $2n + 1$ type. Through PM, they connect not to R but to the CI of another CM.**

- **CI (Child Images): Images under CM, which are natural numbers of the $n + 1$ type. Through PM, they connect not to R but to the CR of another CM.**

When this structure is viewed in reverse—defining natural numbers as $x_{\mathbf{even}} = 2y$ for even and $x_{\mathbf{odd}} = 2y_{\mathbf{even}} - 1$ for odd—a geometric progression of ratio 2 emerges, using $2n + 1$ type odd numbers as initial values, and associating $n + 1$ of $\mathbb{N}_+$.

In this structure, all odd numbers $x = 2n + 1$ constitute CR, and all even numbers $y = 2x$ expand horizontally to form CM.

Gen.0

$\{\,[(1)],\ (2),\ (4),\ (8),\dots\}$ ← initial value 1

Gen.1   ↑   ↕

                    ↓ $\{\,[(7)],(14),(28),(56),\dots\}$ ← initial value 7

          $\{\,[(3)],(6),(12),(24),\dots\}$ ← initial value 3

Gen.2   ↕

          $\{\,[(5)],(10),(20),(40),\dots\}$ ← initial value 5

Gen.3   ↕

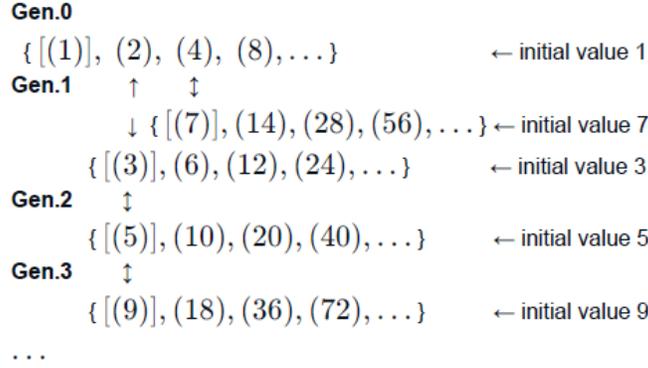 $\{\,[(9)],(18),(36),(72),\dots\}$ ← initial value 9

$\dots$

Figure 1: Odd numbers of the $2n+1$ type constitute **CR**, and even numbers branch horizontally from them to form **CM**. Furthermore, by connecting $(n+1)$ in $\mathbb{N}_+$ vertically with $[2n+1]$ (except for **R**), a $\mathbb{N}^{3d}$ structure is formed.

In addition, natural numbers $y = n+1$ and odd numbers $2y-1 = 2n+1$ are connected vertically by a bijection, making odd numbers the nodes (**N**). That is, among $(n,\ 2n+1 \rightarrow 2n+2 \rightarrow n+1)$, the bijections excluding **R**—such as $(0,\ 2\cdot 0 + 1 \rightarrow 2\cdot 0 + 2 \rightarrow 0+1)$—correspond to the number of edges (**E**).

This bijective mapping yields a $\mathbb{N}^{3d}$ structure satisfying the relation: **N = E + R**, where $\mathbf{R} = (0,\ 1 \rightarrow 2 \rightarrow 1)$.

3.4   Relocation of Four-Dimensional Natural Numbers in the CC

The commonly known mappings of the CC are structured in three dimensions: odd and even number, and the address dimension. However, an additional distinction can be introduced between odd numbers of the $4n+1$ type—which are included in the mappings of the trivial loop $(0,\ 1 \rightarrow 4 \rightarrow 2 \rightarrow 1)$—and those of the $4n+3$ type, which are not. This extension defines the four-dimensional natural numbers $\mathbb{N}^{4d}$.

$$f(x) = \begin{cases} \frac{3x+1}{4} & \text{if } x \equiv 1 \pmod 4 \quad \text{(odd } 4n+1 \text{ type)} \\ \frac{3x+1}{2} & \text{if } x \equiv 3 \pmod 4 \quad \text{(odd } 4n+3 \text{ type)} \\ \frac{x}{2} & \text{if } x \equiv 0 \pmod 2 \quad \text{(even)} \end{cases}$$

Based on the above mappings, the following definitions are established:

- **PS (Parent Sequences):** A sequence pattern originating from $n = 0$ in the address dimension $\mathbb{N}_0$, defined as $(n,\ 4n+1 \rightarrow 12n+4 \rightarrow 6n+2 \rightarrow 3n+1)$.

- **PM (Parent Mappings):** A bijective shortcut mapping between $4n+1$ and $3n+1$ within PS, expanded until it reaches the **R**.

- **R (Parent Root):** The structural root of PM, defined as the stopping point of the trivial loop $(0,\ 1 \rightarrow 4 \rightarrow 2 \rightarrow 1)$. It guarantees the origin $n = 0$ in $\mathbb{N}_0$ not merely symbolically, but structurally.

- **CM (Child Mappings):** Expansion mappings applied to even numbers and odd numbers of the $4n+3$ type, leading toward CR.

- **CR (Child Roots):** The roots of CM, which are odd numbers of the $4n+1$ type. These connect to CI via PM; only the R (1) is common to CI.

- **CI (Child Images):** The images within CM, which are natural numbers of the $3n+1$ type. These connect to CR via PM; only the R (1) is common to CR.

Any natural number, when subjected to Collatz operations, reaches the trivial loop $(0, 1 \rightarrow 4 \rightarrow 2 \rightarrow 1)$ in a finite number of steps. The mappings of $\mathbb{N}_+$ via a geometric sequence with common ratio 2 (starting from odd numbers of the $2n+1$ type) correspond to the structure of $\mathbb{N}^{3d}$. The key difference in $\mathbb{N}^{4d}$ lies in the dimensional generalization of the trivial loop—$n \rightarrow 4n+1 \rightarrow 12n+4 \rightarrow 6n+2 \rightarrow 3n+1$—which excludes $4n+3$ type odd numbers. Instead, the dimension n, $4n+3 \rightarrow 12n+10 \rightarrow 6n+5 = 3(2n+1)+2$ leads toward $4n+1$ type odd numbers, forming an additional relocative dimension.

Gen.0
$\{[(1)],2,(4),8,(16),...\}$ $\leftarrow$ initial value 1
Gen.1 ↑   ↕
      ↓ $\{[21],42,84,168,...\}$ $\leftarrow$ initial value 21
      $\{[<5>],(10),20,(40),...\} \{3,6,12,24,...\}$ $\leftarrow$ initial value 5, 3
Gen.2   ↕
          $\{[(13)],26,(52),104,...\}$ $\leftarrow$ initial value 13
Gen.3   ↕
          $\{[<17>],(34),68,(136),...\} \{<11>,(22),44,(88),...\} \{(7),14,(28),56,...\}$
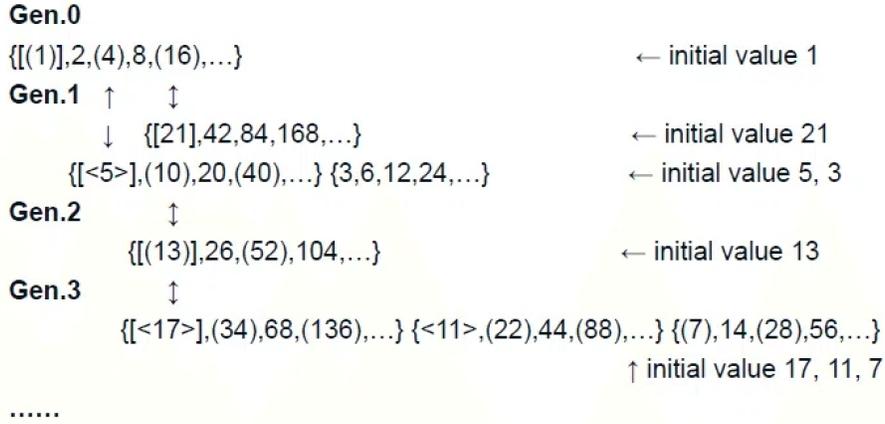   ↑ initial value 17, 11, 7

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Figure 2: For even numbers, the trailing $0_{(2)}$ in binary is removed to yield an odd number. For $4n+3$ type odd numbers, the trailing $1_{(2)}$ is removed. Including inverse operations, any odd number of the form $<3(2n+1)+2>$ connects horizontally via bijection to a $4n+3$ type number. Thus, any natural number reaches a unique $[4n+1]$ type odd number in a finite number of steps, determined by the binary tail of $1_{(2)}$. The remaining dimension—generalizing the trivial loop $(0, 1 \rightarrow 4 \rightarrow 2 \rightarrow 1)$—connects $(3n+1)$ type natural numbers vertically to the leading $[4n+1]$ type odd numbers, thereby relocating the full structure of $\mathbb{N}^{4d}$.

This infinite tree structure is expressed as $\mathbf{N} = \mathbf{E} + \mathbf{R}$, where $\mathbf{R} = (0, 1 \rightarrow 4 \rightarrow 2 \rightarrow 1)$, within $\mathbb{N}^{4d}$.

4 Definition, Proof, and Notation of Dimensional Natural Numbers

4.1 Definition of the Dimensional Natural Number System

The $\mathbb{N}^d$ is defined as a commutative direct product structure consisting of:● the address dimension $\mathbb{N}_0$ the sequence dimension $\mathbb{N}_+$ This structure satisfies the following Cantor-type bijection: $f : \mathbb{N}_0 \times \mathbb{N}_+ \xrightarrow{\text{bijection}} \mathbb{N}^d$

Furthermore, we introduce a relocative operation based on PS (Successor Function), which satisfies the following conditions:

(C1) Halting ConstraintWithin the operation group, 1 is defined as the unique halting point, and any operation resulting in a value less than 1 is excluded.

(C2) CR/CM Inclusion HierarchyAll positive integers are classified as follows: CR type: $(a+1)n+1, \quad a \in \mathbb{N}_0$ Each CR serves as a R, and its corresponding subset is defined as CM. Furthermore, the set $\mathbb{N}_+ \setminus (a+1)n+1$ is assigned to each CM's CR via an order-preserving surjection, forming an inclusion-based relocative structure.

(C3) PM Connectivity Each CR is connected as follows: to the address dimension $\mathbb{N}_0$ to the CI hierarchy (of the form $a \cdot n + 1$) These are connected via bijective mappings, and all CM subsets are ultimately linked to the unique R.

Canonical Bijections for Dimensional Expansion (Simplified Notation)
$\mathbb{N}^{2d} : \ n \leftrightarrow n+1, \qquad \mathbb{N}^{3d} : \ n+1 \leftrightarrow 2n+1, \qquad \mathbb{N}^{4d} : \ 3n+1 \leftrightarrow 4n+1$

	$\mathbb{N}_0$	$\mathbb{N}_+$, CI of $\mathbb{N}^{3d}$	CR of $\mathbb{N}^{3d}$	CI of $\mathbb{N}^{4d}$	CR of $\mathbb{N}^{4d}$
PM	n	$n+1$	$2n+1$	$3n+1$	$4n+1$
R	0	1	1	1	1
	1	2	3	4	5
	2	3	5	7	9
	3	4	7	10	13
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1: The **R** of **PM** in $\mathbb{N}^d$ and the **PM** between $\mathbb{N}_0$, **CR**, and **CI** across each layer of $\mathbb{N}^d$

4.2 Refined Proof of the Dimensional Natural Number System

4.2.1 Theorem (Guarantee of the Dimensional Natural Number System

If, within $\mathbb{N}^d$, the PM connecting CR-type natural numbers of the form $(a+1) \cdot n + 1$ is bijective with the address dimension $\mathbb{N}_0$, then every CM is connected to the R. Therefore, the structure of $\mathbb{N}^d$ is consistently established.

4.2.2 Proof of the Contrapositive Using Contradiction

Assumption: Through PS relocation, the address dimension $\mathbb{N}_0$ is spatially relocated into CR and CI without omission, guaranteeing complete coverage. This relocation ensures that every address in $\mathbb{N}_0$ corresponds to either a CR or CI, structurally—not by inherited mappings, but by relocative placement.

Contradictory Hypothesis: Assume that there exists a CM that is not connected to the R.

Step 3: Since $\mathbb{N}_0$ is fully relocated into CR and CI, every address is structurally assigned. Thus, isolated nodes cannot exist.

Step 4: By the surjectivity condition (C2), the set $\mathbb{N}_+ \setminus \{(a+1) \cdot n + 1\}$ is assigned to CM via an order-preserving surjection. Therefore, every element within CM is necessarily included in or mapped to some CR or CI.

Conclusion: The assumption that "a CM exists which is not connected to R" contradicts both Step 3 and Step 4. Hence, every CM is connected to R. ∎

4.3 Notation of the Dimensional Natural Number System

The $\mathbb{N}^d$ is composed of:

- the number of CM: **N**

- the number of connections: **E**

- the root point: **R**

This structure is expressed by the formula: **N = E + R**. This formula represents a topological invariant that extends the Euler characteristic, incorporating spatial mappings.

4.3.1 Canonical Bijections in Each Dimension

- In $\mathbb{N}^{2d}$, the bijection $n \leftrightarrow n+1$ ensures that any natural number reaches the **R** $(0,1)$ in finitely many mapping steps.

- In $\mathbb{N}^{3d}$, the bijection $n+1 \leftrightarrow 2n+1$ ensures that any natural number reaches the **R** $(0, 1 \to 2 \to 1)$ in finitely many steps.

- In $\mathbb{N}^{4d}$, the bijection $3n+1 \leftrightarrow 4n+1$ ensures that any natural number reaches the **R** $(0, 1 \to 4 \to 2 \to 1)$ in finitely many steps.

4.3.2 N = E + R and Tree Structure Equivalence

The structural formula **N = E + R** resembles the classical tree identity **N = E + 1** [15], but applies only when natural numbers are reconfigured based on **PS** (Successor Function). While the classical tree identity holds as a topological invariant for finite trees, in $\mathbb{N}^d$ the **R** represents an extended element corresponding to a convergence-guaranteed **PS**.

Therefore, $\mathbb{N}^d$ is a dimensional infinite tree structure of natural numbers, developed bijectively based on **PS** and spatially guaranteed by the formula **N = E + R**. It is rooted in the classical tree identity, but incorporates dimensional expansion through bijective mappings.

Conclusion

This study provides a relocative response to the axiomatic limitations faced by conventional IRP methods. Sequential tracing based on Peano's one-dimensional successor fails to host the nonlinear branching required by the CC, relying on visible tracking without guaranteeing termination.

To resolve this, we introduce the dimensional natural number system $\mathbb{N}^d$, which systematizes relocation via **PS** (Parent Sequences). Once relocated, bijective connections and finite mappings guarantee convergence. Peano's "+1" is redefined as the prototype of the root structure (**R**), ensuring that sequences deductively reach **R** in finitely many steps.

Through the relocates of $\mathbb{N}^{2d}, \mathbb{N}^{3d}, \mathbb{N}^{4d}$, the CC is closed as a **DRP** (Direct Relocation Proof). Termination is thus guaranteed not by operational complexity, but by the presence of a defined configuration space.

Previous CC claims lacked explicit bijective construction, leading to rejection due to failure to close infinite sets [16, 17]. This study resolves that by embedding Peano's "+1" into the address dimension $\mathbb{N}_0$ and ensuring bijectivity in the series dimension $\mathbb{N}_+$

via Cantor's equicardinality. Specifically, the continuity $S(k) = k + 1$ is embedded as $\iota(S(k)) = S(\iota(k))$, structurally mapping the spatial freedom of infinite sets.

This deductively overcomes the "ineffectiveness of equicardinality" and establishes a static guarantee of termination. Even for sequences where **R** is undefined (e.g., Conway's generalized problem [18]), the structural guarantee provided by $\mathbb{N}^d$ remains applicable.

By reconciling Peano's absolute generation with Cantor's bijective relocation, the CC ascends from the provisional dynamics of IRP to the DRP. This relocation reactivates a century-long omission and presents $\mathbb{N}^d$ as the legitimate vessel for closure.

Disclosure of Delegation to Generative AI

The authors declare the use of generative AI in the research and writing process. According to the GAIDeT taxonomy (2025), the following tasks were delegated to GAI tools under full human supervision:
- Proofreading and editing
- Translation

The GAI tool used was: Grok 4, ChatGPT(GPT-5), Microsoft Copilot, Gemini. Responsibility for the final manuscript lies entirely with the authors. GAI tools are not listed as authors and do not bear responsibility for the final outcomes. Declaration submitted by: The author (Shinsuke Hamaji) independently decided to use generative AI tools to assist with language refinement and structural formatting.

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