The Power Theorem: Sum of Consecutive Powers as Perfect Squares

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Abstract

This paper introduces a novel family of natural numbers for which the sum of two successive powers, starting from an even exponent, yields a perfect square. The condition $x + 1 = k^2$ leads to a general identity:

$$x^n + x^{n+1} = \left(k \cdot x^{n/2}\right)^2 .$$

We present a proof of this identity, derive a recurrence relation, and support the result through computational verification using tabular data.

Keywords: number theory; perfect squares; consecutive powers; recurrence relation; mathematical identities

1 Introduction

Number theory is rich with intricate patterns, surprising identities, and structural properties that often arise from deceptively simple expressions. Beneath these elementary operations lie deep interconnections that transcend basic arithmetic. The pursuit of these hidden relationships has led to many profound discoveries in mathematics.

A recurring theme is the appearance of perfect squares in unexpected contexts. This paper explores a new family of numbers where the sum of two successive powers—beginning with an even exponent—yields a perfect square. This identity reflects a beautiful connection between exponential growth and quadratic forms.

2 Theorem

Let $x \in \mathbb{N}$, and suppose $x + 1 = k^2$ for some integer $k \in \mathbb{N}$. Then, for all even $n \in \mathbb{N}$,

$$x^n + x^{n+1} = \left(k \cdot x^{n/2}\right)^2 .$$

In this context, the symbols are defined as follows:

- x: Base of exponentiation, a natural number such that $x + 1 = k^2$.
- k: A positive integer such that $k = \sqrt{x+1}$; equivalently, $x = k^2 - 1$.
- n: An even exponent, where $n \in \mathbb{N}$ and $n \equiv 0 \pmod 2$.
- y: A natural number such that $y^2 = x^n + x^{n+1}$; equivalently, $y = k \cdot x^{n/2}$.

3 Proof

$$x^n + x^{n+1} = x^n(1 + x) = x^n(k^2) = \left(k \cdot x^{n/2}\right)^2 = y^2.$$

This expression forms a perfect square when n is even, as $x^{n/2} \in \mathbb{N}$ in this case.

4 Computational Verification

4.1 Case $n = 2$

Table 1: Tabular verification for $n = 2$

x	k	x^2	x^3	$x^2 + x^3 = y^2$
3	2	9	27	$36 = 6^2$
8	3	64	512	$576 = 24^2$
15	4	225	3375	$3600 = 60^2$
24	5	576	13824	$14400 = 120^2$
35	6	1225	42875	$44100 = 210^2$
48	7	2304	110592	$112896 = 336^2$
63	8	3969	250047	$254016 = 504^2$
80	9	6400	512000	$518400 = 720^2$
99	10	9801	970299	$980100 = 990^2$
120	11	14400	1728000	$1742400 = 1320^2$

Table 2: Tabular verification for $n = 4$

x	k	x^4	x^5	$x^4 + x^5 = y^2$
3	2	81	243	$324 = 18^2$
8	3	4096	32768	$36864 = 192^2$
15	4	50625	759375	$810000 = 900^2$
24	5	331776	7962624	$8294400 = 2880^2$
35	6	1500625	52521875	$54022500 = 7350^2$
48	7	5308416	254803968	$260112384 = 16128^2$
63	8	15752961	992438943	$1008191904 = 31776^2$
80	9	40960000	3276800000	$3317760000 = 57600^2$
99	10	96059601	9519900399	$9615960000 = 98000^2$
120	11	207360000	24883200000	$25090560000 = 158400^2$

4.2 Case $n = 4$

4.3 Large Parameter Values

Table 3: Verification with larger values of x and n

x	k	n	x^n	x^{n+1}	$x^n + x^{n+1}$
143	12	10	4.13×10^{21}	5.91×10^{23}	5.95×10^{23}
624	25	8	1.22×10^{21}	7.60×10^{23}	7.62×10^{23}
1023	32	6	1.10×10^{18}	1.13×10^{21}	1.13×10^{21}
2040	45	6	7.10×10^{19}	1.45×10^{23}	1.46×10^{23}
3024	55	4	8.35×10^{13}	2.52×10^{17}	2.53×10^{17}
4031	64	6	4.24×10^{22}	1.71×10^{26}	1.71×10^{26}
5080	71	4	6.66×10^{14}	3.38×10^{18}	3.38×10^{18}
6083	78	8	3.27×10^{29}	1.99×10^{33}	1.99×10^{33}
7128	85	6	1.98×10^{23}	1.41×10^{27}	1.41×10^{27}
8190	91	4	4.50×10^{15}	3.69×10^{19}	3.69×10^{19}

5 Recurrence Relation

Let $y_n = k \cdot x^{n/2} = \sqrt{x^n + x^{n+1}}$ for even n. Then:

$$y_{n+2} = x \cdot y_n.$$

This shows that y_n grows geometrically with a factor of x every two steps:

$$y_0 = k, \quad y_2 = kx, \quad y_4 = xy_2, \quad y_6 = xy_4, \ldots$$

6 Conclusion

We have presented a new identity in number theory: for natural numbers x such that $x + 1 = k^2$, the expression $x^n + x^{n+1}$ is a perfect square for all even n. This result is supported through an algebraic proof, a derived recurrence relation, and tabular validation, demonstrating an elegant relationship between exponential terms and perfect squares.

References

[1] Hardy, G.H. and Wright, E.M. (2008). *An Introduction to the Theory of Numbers*, 6th ed. Oxford University Press.

[2] Dujella, A. (2021). *Number Theory.* Translated by P. Švob. Zagreb: Školska knjiga.

[3] Tenneti, V. (2009). *Fascinating Square Numbers.* Neelkamal Publications Pvt. Ltd.

Data Availability

All data generated during this study are included in this published article.

Conflict of Interest

The author declares no competing interests.

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