BASIC Magicmare

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PayDie Square Year

Abstract
The exploration of incomplete magic square of squares, whose being fully working magic square with less than 9 square entries, leads to the extensive use of Brahmagupta-Fibonacci identity. By only taking account of primitive (irreducible) entries, the umbrella term Brahmagupta's Abacial Slice of Irreducible Calamari (BASIC) **magic**al **ma**rine squa**re** was adopted. Interesting varieties ranging from congruent elliptic curves up to the affine variety $\mathbb{A}6$ along with the K3 surface of degree 8 were encountered when one considers the birational model of Magicmare.

1. In Defense of Incompleteness

Incomplete magic square of squares comes with various forms depending on how severe its fault is. Two instances being considered here are the *standard* magic square with some of its entries being square, and the second type being the all distinct square entries but *invalid* since it possesses a failed sum, which may be a defective diagonal, or any row or column. Of course there are many other classification of "semi-complete" squares [1][2][3][4][5][8], but those other types are outside the scope of this article. The defense being carried in this article only covers the incompleteness of being not all entries being square each, yet putting the strict requirement of all sum (3 rows, 3 columns, and both diagonals) must work. The next two subsections (1.1 and 1.2) detail why the invalid magic square with a faulty diagonal—even if all entries being squares, or evenmore its sum being square (!) as well—is of lesser urgency.

1.1. Michael Schweitzer's Magic Sum

Consider a "void" square, that is an invalid order 3 magic square, since the diagonal sum labelled as v eventually doomed.

v_1	a	b
c	v_2	d
e	f	v_3

Figure 1. Void Draft

After tweaking around, rotating, flipping, scaling, and mirroring, one could easily come up with particular instance of scaled permutation square shown in Fig. 2.

222^2	381^2	6^2
291^2	174^2	282^2
246^2	138^2	339^2

Figure 2. Credits to Michael Schweitzer

Such an example of 9 square entries yet failing one sum, was earlier given by Michael Schweitzer [5] and it is clear why such sample is invalid to be an order 3 magic square (having faulty diagonal sum), the magic constant is 147^2 i.e. a square number. In order to be a fully working order 3 magic square, the magic sum must be thrice of its central entry, say $3v_2$ in this case. But if v_2 itself being a square number, then its squarefree part must be exactly 3, i.e. can never be 1! The fact that 147^2 is of squarefree 1 implies that such construction with magic sum being *again* a square number, will never able to produce any valid order 3 magic square, with central entry—v_2—being perfect square.

To achieve the square exhibited in Fig. 2, start with three **primitive** Pythagorean triples with squarefree part being 105, thus (12,35,37), (15,112,113), (20,21,29), turn these into equidistant triples by performing $(a-b)^2$, a^2+b^2, $(a+b)^2$. So far, the (23,37,47), (97,113,127), (1,29,41) equidistant triplet being obtained. The distance for each is $37^2 - 23^2$, $113^2 - 97^2$, $29^2 - 1^2$, that is 840, 3360, 840, thus need to multiply with the factor 4, 1, 4 for the *squared* distance. When one returned back to (23,37,47), (97,113,127), (1,29,41) triples, i.e. before they got squared, those factors must be powered by half, so 2,1,2 be the scaler. Therefore, the triples are now 2(23,37,47), 1(97,113,127), 2(1,29,41) which is (46, 74, 94), (97, 113, 127), (2,58,82).

97^2	82^2	74^2
94^2	113^2	2^2
58^2	46^2	127^2

Figure 3. Built Schweitzer square from squarefree 105

Andrew Bremner [6] already gave a remark about such construction using the language of congruent elliptic curve $E : y^2 = x(x^2 - c^2)$ up to the step (6) and remarked *sic*. This square fails to be magic only at the non-principal diagonal. Further at step (7) more restriction of the said congruent elliptic curve were added, but this step is unnecessary for the case of invalid magic square. In aforesaid void square, the non-principal diagonal is the v_1,v_2, and v_3 entries. Considering the failure of the $v_1+v_2+v_3$ diagonal, one can discount them for the sum at the moment, thus revisiting Fig. 1, the sum $a + f = b + e = c + d$ must hold, ignoring all v. Such reconsideration allows each triple to be rearranged as follows

74^2	127^2	2^2
97^2	58^2	94^2
82^2	46^2	113^2

Figure 4. Permuted version after switching which faulty diagonal

that is, the failing v_1,v_2, and v_3 entries got permuted to occupy 82^2, 56^2, 2^2, instead. It means that the square shown at Fig. 4 do indeed worked for

$74^2+127^2+2^2 = 21609 = 147^2$

$97^2+58^2+94^2 = 21609$

$82^2+46^2+113^2 = 21609$

row sums, and also

$74^2+97^2+82^2 = 21609$

$127^2+58^2+46^2 = 21609$

$2^2+94^2+113^2 = 21609$

vertical sums, with one diagonal $74^2+58^2+113^2 = 21609$ worked as well. However $82^2+56^2+2^2$ is now not working due to it being the non-principal diagonal after the permutation. There is neither way for mere rotation, flip, mirror, nor shuffling the teal, orange, and white coloured entries shown in Fig. 3 and Fig. 4 in order to resolve the diagonal sum. The doomed v_1,v_2, and v_3 is inherent, even after scaling.

148^2	254^2	4^2
194^2	116^2	188^2
164^2	92^2	226^2

Figure 5. The square of Fig. 4 scaled by 2

After some rearrangement of Fig 5. via shuffling the white, teal, and orange coloured entries, the MS2 square mentioned in Boyer's note [5] will be apparent. This is merely scaled version of Fig. 4, and the magic constant is simply 2^2 147^2 = 294^2.

222^2	381^2	6^2
291^2	174^2	282^2
246^2	138^2	339^2

Figure 6. The square of Fig. 4 scaled by 3

At a glance, one finds out that Fig. 6 is none other than Fig. 2, therefore credit where credit is due. Both squares shown in Fig.2 and Fig. 6 are identical, and has the magic constant of 3^2 147^2 = 441^2.

1.2. Trivial Construction

The aforesaid "lesser urgency" verdict is due to the fact of infinitude of these faulty, i.e. invalid magic square coming into existence. To construct faulty, i.e. invalid magic square, à la Michael Schweitzer, i.e. all 9 entries are squares, along with the magic sum being also square, one need to fulfill the following recipe:

1. Start with three distinct **primitive** Pythagorean triple with exact squarefree constituent, for instance the triple (17, 144, 145), (225, 272, 353), and (1377, 3136, 3425) were chosen to represent *the first*, i.e. smallest, congruent elliptic curve of rank 2.

2. Translate into equidistant triples by performing $(a-b)^2$, a^2+b^2, $(a+b)^2$ transform. For previous instance, this yields (127, 145, 161), (47, 353, 497), and (1759, 3425, 4513).

3. Check the claimed distance, for instance 145^2 - 127^2 vs 353^2 - 47^2 vs 3425^2 - 1759^2 one obtained 4896 vs 124593 vs 8636544.

4. Get the least common multiple (LCM) and adjust them triple. In previous case this is 210 vs 42 vs 5, so multiply each triple by so, yielding 210(127, 145, 161), 42(47, 353, 497), and 5(1759, 3425, 4513). Carrying out the multiplication, the (26670, 30450, 33810), (1974,14826,20874), (8795,17125,22565) was obtained. Set the first as diagonal, the second as lower tip triangle, the last as upper tip triangle, where tip meant the middle of the triplet. Finally square them all.

26670^2	20874^2	17125^2
22565^2	30450^2	1974^2
14826^2	8795^2	33810^2

Figure 7. Credits to Michael Schweitzer

Upon closer inspection of Fig. 7,
$26670^2 + 20874^2 + 17125^2$
$22565^2 + 30450^2 + 1974^2$
$14826^2 + 8795^2 + 33810^2$
be the horizontal sum.

Meanwhile,
$26670^2 + 22565^2 + 14826^2$
$20874^2 + 30450^2 + 8795^2$
$17125^2 + 1974^2 + 33810^2$
be the vertical sum.

And $14826^2 + 30450^2 + 17125^2$ be the lone diagonal. All sum claimed to be 37951^2, of course, ignoring the faulty $26670^2+30450^2+33810^2 \neq 37951^2$.

Does the sum have to be square? Let the squarefree component obtained from congruent elliptic curve with *rank 1*, this avoids the magic sum to be square. For example, the most basic yet nontrivial (3,4,5) Pythagorean triple has squarefree residue of 3. Doing point doubling under Mordell-Weil group the pair (3,4,5), (49,1200,1201), (2066690884801,339252715200,2094350404801) were obtained in succession, and are all *congruent* in the sense of sharing the squarefree component of 3. Following the aforesaid prescription, the next step is to transform them into equidistant triplets, therefore the (1, 5, 7), (1151, 1201, 1249), (1727438169601, 2094350404801, 2405943600001) were obtained. Such triples having difference of 5^2-1^2 vs 1201^2-1151^2 vs $2094350404801^2-1727438169601^2$, that is 24 vs 117600 vs 1402260988295659323350400, with common multiplier being 241717895860 vs 3453112798 vs 1 in order to attain same LCM. Next, simply multiply each triple 241717895860 times (1, 5, 7), 3453112798 times (1151, 1201, 1249), and 1 times (1727438169601, 2094350404801, 2405943600001) . Yielding the fancy looking yet mundane

241717895860^2	2405943600001^2	4147188470398^2
4312937884702^2	1208589479300^2	1727438169601^2
2094350404801^2	3974532830498^2	1692025271020^2

Figure 8. Credits to Michael Schweitzer

since
$241717895860^2 + 2405943600001^2 + 4147188470398^2$
$4312937884702^2 + 1208589479300^2 + 1727438169601^2$
$2094350404801^2 + 3974532830498^2 + 1692025271020^2$
as row sum each,
$241717895860^2 + 4312937884702^2 + 2094350404801^2$
$2405943600001^2 + 1208589479300^2 + 3974532830498^2$
$4147188470398^2 + 1727438169601^2 + 1692025271020^2$
as column sum each,
$2094350404801^2 + 1208589479300^2 + 4147188470398^2$ as the lone diagonal, did check out, albeit one faulty diagonal, **invalidating** the entire affair.

One might be tempted to declare the impossibility of all 9 entries being square based on the condition raised by the nonresolvable diagonal (or arguing that higher rank elliptic curve forces the magic sum to be perfect square, rendering the centre to be non-integral *viz.* algebraic $\mathbb{Q}[a]$, yet a magic constant must be thrice of its centre), however this is counting one's chicken before they are hatched. There are other ways to compose *standard* magic square, with all sums (3 rows, 3 columns, and both diagonals) working, namely by starting from a pair (or even just lone) Pythagorean triples and then forcing the missing entries to be **computed** instead of being assigned from the chosen 3 triplets altogether, therefore the broken diagonal finally able to be resolved. However, being *standard* magic square, means it was never formulated in-mind unto producing square entries, but merely positive integers.

1.3. Magicmare is *Standard* Magic Square

An order 3 magic square, follows the usual definition of 3x3 square array of positive integers, whose sums in each row, each column, and both main diagonals are the same [1][2][8][9]. The third-order magic square was known to Chinese mathematicians as early as 190 BCE [2][9], putting the emphasis on the sum on **all** 9 directions **must be working**, no exception. If one bad sum—faulty diagonal—was allowed, then one succumbed for the triviality of being void square as mentioned in the previous subsections, rendering such thing to be less prioritised as the object for hunting.

🪝	𝒶	𝓁
𝒸	m	m_c
𝓇	m_a	⚓

Figure 9. Magicmare Entries

Using Brahmagupta-Fibonacci identities one can start with the initial construction with m being square of prime congruent to 1 (mod 4), then arranging the obtained equidistant triples in the following possibilities

🪝	**𝒶**	𝓁
𝒸	**m**	m_c
𝓇	**m_a**	**⚓**

Figure 10. BF eel

🪝	**𝒶**	𝓁
𝒸	**m**	**m_c**
𝓇	**m_a**	⚓

Figure 11. BF cross

🪝	𝒶	**𝓁**
𝒸	**m**	m_c
𝓇	m_a	**⚓**

Figure 12. BF diagonals

multiplying m more and more with various prime factors, each being congruent to 1 (mod 4), will astronomically increase the options of equidistant squares pairing.

Duncan [7] proved that in order for the entries 🪝, 𝒶, 𝓁, m, 𝓇, m_a, and ⚓ to be of perfect squares, then m must be at least 10^{24} (C. Boyer [5]). The author originally thought that since the middle entry m deemed as *too restrictive*, therefore by setting m to be non-square, the following order 3 magic square with 8 square entries (blue colour) was sought after:

🪝	𝒶	𝓁
𝒸	m	m_c
𝓇	m_a	⚓

Figure 13. Calamari slice

The origin of the letter c,a,l,m,r is none other inspired from the Magicmare of the Calamari type, which still open for hunting as of the year 45^2.

Magicmare must be of irreducible form, that is all 9 entries has greatest common divisor of 1. At least 6 of those entries are being perfect square.
Magicmare must have entries ordered as 🪝 < m < ⚓, m_c < 𝓇 < 𝒶, m_a < 𝓁 < 𝒸, and $m_c < m_a$. The mnemonic is **micra** (m_c < 𝓇 < 𝒶) for small, must be lesser than **malc** (m_a, 𝓁, 𝒸). The 𝓇 < m < 𝓁 ordering comes for free. The hook emoji (🪝) is used to represent the bootstrap element, i.e. starting value, or initiation, like the troll in fishing. While the anchor emoji (⚓) is used to represent the heavy end of the term. In the following figures, blue square depicts a perfect square entry, while blank square represents entry which may or may not be a square.

2. Catalogue Initiative

Along the journey involving blazing precision for the deduction of Magicmare specimens, such endeavour had left lasting mark of charred memorials of computational passion. Sacrificed not in quiet retirement, but in a brilliant pyroclastic moment of ultimate performance, these results transcended the mere hardware which went toasted. All of those dice are not grilled without meaning, as their incandescent transformation had yield these incomplete magic squares of squares. May this tribute be forever crystallised as their eternally timeless obituary. Today is the day of remembrance that one needs to pay to replace such units.

As of such, this section contains very brief overview of the ongoing project hosted at CodeBerg initiative, as of the current year.

🪝	𝒶	𝓁
𝒸	m	m_c
𝓇	m_a	⚓

Figure 14. Escallop

Magicmare of the Escallop type may look like of having 2 symmetries, the horizontal and the vertical version where 🪝, 𝒸,𝓇,𝓁,m_c,⚓ being squares instead. However, the vertical state can be attained using only the horizontal version when the ordering condition $m_c < m_a$ is being ignored. The horizontal version alone, shown in Fig. 14 above, already have 3 symmetries that yield into different value of 𝒸 and m_c, namely; the original, the swapped (𝒶,𝓁) along with the swapped (𝓇,m_a), and the swapped (m_a, ⚓) along with (🪝,𝒶) swap.

Escallop arises from set of three equations involving the six square entries, 🪝, a, l, r, m_a, ⚓ thus defining a three-dimensional affine variety in $\mathbb{A}6$ with coordinates 🪝, a, l, r, m_a, ⚓ as highlighted blue in Fig. 14 above. Because each coordinate is required to be a perfect square, the actual set of integer solutions is the set of rational points on the intersection of variety formed by sum of opposite pairs simultaneously with 3m as sum of three squares, with the quadric hypersurface. Hence the problem can be viewed as the rational points on the intersection of a linear subspace with the 6-dimensional cone with each coordinates being a square. Geometrically this is a quartic surface of dimension 5 cut by a 3-dimensional linear subspace, thereby giving a surface of dimension 2. That is, the six perfect square entries must be the set of integral points on a rational surface.

149^2	559^2	371^2
272881	157441	42001
421^2	7^4	541^2

Figure 15a. Smallest possible Escallop with positive integers, ordering violated

Upon realisation of its birational model as a conic bundle, the determining integrality conditions can be derived. After projectivising, the smooth quadric surface can be obtained in $\mathbb{P}^3$, in which, over $\mathbb{Q}$ a smooth quartic surface is birational to $\mathbb{P}^1$ x $\mathbb{P}^1$ which leads to a conic bundle with a rational parametrisation. The extra square condition imposed on the conic, renders the surface as a double cover of a quadric, i.e. a K3 surface of degree 8. Evaluating the fibre that arisen in such, a fairly slow search can be deployed to obtain actual integer solutions. One of such result shown in Fig. 15a.

371^2	42001	541^2
559^2	157441	7^4
149^2	272881	421^2

Figure 15b. Rotated Fig. 15a in order to abide the ordering $m_c < m_a$

Proposition: there are infinitely many sponge Magicmare.
Proof, the defunct square shown in Fig. 16 and Fig. 17 can be seen as torsion, or a defective point where the location of such in its affine surface is trivial. Opting for the element of the free group instead, as the rational point, one can perform the point doubling indefinitely, provided the curve is of positive rank.

1^2	7^2	5^2
7^2	5^2	1^2
5^2	1^2	7^2

Figure 16. Not Magicmare

Fig. 16 was trivially made from the Pythagorean triple (a,b,c) = (3,4,5) as $1^2 = (3-4)^2$, $5^2 = 3^2 + 4^2$, $7^2 = (3+4)^2$. This particular triple has squarefree kernel of 3 which is associated to a twist of an elliptic curve, which finally settled down into the congruent elliptic curve as its minimal model. Using the doubling of rational point from the Mordell-Weil group of its congruent elliptic curve, $(a', b') = ((a^2-b^2)^2, 4abc(a^2-b^2))$, one can recursively generate infinitely many **relatively primitive** Pythagorean triple of said kernel. For instance (a,b) = (3,4), leads to (a',b') = (49, 1200) which formed the triple (49,1200,1201). The generated triple following such recursion need not to be directly primitive, it only needs to be relatively primitive against the precursor (a,b). The difference between terms can be adjusted via the least common multiple (LCM). Thus proving the infinitude of sponge.

$(a-b)^2$	$(a+b)^2$	a^2+b^2
$(a+b)^2$	a^2+b^2	$(a-b)^2$
a^2+b^2	$(a-b)^2$	$(a+b)^2$

Figure 17. Invalid square due to duplicates caused by torsion

🪝	$(a+b)^2$	$a'^2+b'^2$
$(a'+b)^2$	m	$(a-b)^2$
a^2+b^2	$(a'-b')^2$	⚓

Figure 18. Sponge

Fig. 18 above illustrate one possible way to obtain Magicmare of the Sponge type, via point doubling. However, if the curve has rank greater than 1, point addition across generators can also yield another construct of Sponge. Below are such exhibits.

265	41^2	37^2
47^2	1105	1^2
29^2	23^2	1945

Figure 20. Smallest sponge without duplicates

19306	206^2	244^2
284^2	40426	14^2
146^2	196^2	61546

Figure 21. Irreducible despite being even

Following similar method of construction, starting with **primitive** Pythagorean triple, and then attempting to make evenly distanced squares from it, namely $(a-b)^2$, a^2+b^2, $(a+b)^2$, then assign this as either the diagonal 🪝, m, ⚓, or the lower tip triangle m_c, 𝓇, 𝒶, or the upper tip triangle m_a, 𝓁, 𝒸. From initial three square entries, further Magicmare types could be produced.

🪝	𝒶	𝓁
𝒸	m	m_c
𝓇	m_a	⚓

Figure 22. Shark tooth

🪝	𝒶	𝓁
𝒸	m	m_c
𝓇	m_a	⚓

Figure 23. Gull (formerly Bluetooth)

Gull (or seagull) Magicmare is characterised with all squares diagonal 🪝, m, ⚓ along with either equidistant squares m_a, 𝓁, 𝒸 (**malc** type) or m_c, 𝓇, 𝒶 (**micra** type). Swapping the diagonal 🪝, m, ⚓ with 𝓇,m,𝓁 will yield the same values only switched sides, thus being redundant. The **micra** type may illustrate the visual of a soaring seagull, viewed from distance, thus the name. Shown in Fig. 23 is Gull of the **malc** type, think of a gliding bird on descent to the left.

🪝	𝒶	𝓁
𝒸	m	m_c
𝓇	m_a	⚓

Figure 24. Barnacle

Magicmare of Barnacle type has two symmetry, in a nutshell, one can think about such symmetry as the upright direction and the sideway direction. The mirror of such, i.e. up vs down, and left vs right are trivial. Trivial here means identical, i.e. not yielding different values. In the catalogue, Barnacles are of the **malc** type and the **micra** type, from the respective equidistance squares m_a, 𝓁, 𝒸 and m_c, 𝓇, 𝒶, Shown in Fig. 24 is of the **malc** type.

🪝	𝒶	𝓁
𝒸	m	m_c
𝓇	m_a	⚓

Figure 25. Shrimp

Magicmare of Shrimp type has two symmetry, the upright direction and the sideway direction. Barnacle and Shrimp can be viewed as originating from indistinguishable “larval” stage. The upright Shrimp resembling the letter J and its mirror, are of the same type albeit mirrored. The side facing one (as of Fig. 25 being facing left) Trivial here means identical, i.e. not yielding different instance of another Shrimp (not in the sense of being trivial due to duplicates, which it will be regarded as invalid instead).

Further are more types of BASIC Magicmare

🪝	𝒶	𝓁
𝒸	m	m_c
𝓇	m_a	⚓

Figure 26. Oyster

The dot on the corner can be viewed as a pearl, like an oyster with a pearl on it.

🪝	𝒶	𝓁
𝒸	m	m_c
𝓇	m_a	⚓

Figure 27. Seaweed

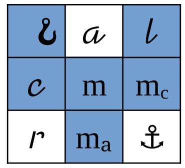

Figure 28. Harpoon

Both Seaweed and Harpoon are obtained in similar way, instead of finding two Pythagorean triples with same squarefree component, it tried to mix and match different squarefree parts, thus experimenting the concoction involving elliptic curves of distinct quadratic twist.

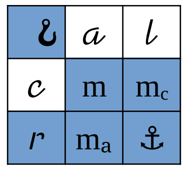

Figure 29. Mussel

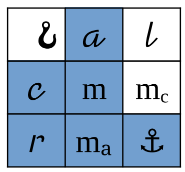

Figure 30. Nautilus

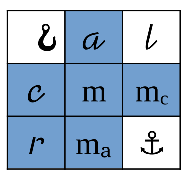

Figure 31. Cod

The following are the kinds of Magicmare obtained from slightly different the construction rather than relying on the Pythagorean triple as the starting ansatz.

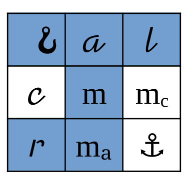

Figure 32. Sea Anemone

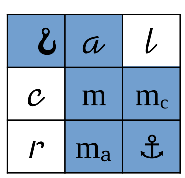

Figure 33. Geoduck

ʖ	𝒶	𝓁
𝒸	m	m_c
𝓇	m_a	⚓

Figure 34. Bivalve

Bivalve comes in 2 symmetries, the upper one, i.e. the entries ʖ, 𝒶, 𝓁, 𝒸,m, m_c are of squares, or the lower one, with ʖ , 𝒶, 𝓁, 𝒸,m, m_c entries are of square. Shown in Fig. 34 is the lower Bivalve type. The vertically stood-up mussel ʖ,𝒸,𝓇,𝒶,m,m_a and its shifted form, 𝒶,m,m_a,𝓁,m_c,⚓ are merely the rotated version of the former, thus being redundant.

Seahorse

The following are the kinds of Magicmare discovered by Lee Sallow and Andrew Bremner, independently to each other.

ʖ	𝒶	𝓁
𝒸	m	m_c
𝓇	m_a	⚓

Figure 35. Seahorse, facing left

With the notorious Sallows-Bremner square depicted in Fig. 36, the right facing Seahorse.

373^2	17^4	565^2
𝒸	425^2	23^2
205^2	527^2	⚓

Figure 36. Sallow-Bremner Seahorse

Is it possible to find the left facing Seahorse? The project continues.

2.1. Remark

BASIC Magicmare of 6 square entries comprise of Escallop, Shark tooth, Gull, Barnacle, Shrimp, Oyster (with pearl), Seaweed, Harpoon, Mussel, Nautilus, Cod, Sea Anemone, Geoduck, and Bivalve.

373^2	17^4	565^2
360721	425^2	23^2
205^2	527^2	222121

Figure 37. Sallow-Bremner Seahorse, in full glory

The only known Magicmare with 7 square entries as of this perfect square year, is depicted in Fig. 37 which assumes the visual of a Seahorse. This is none other than the notorious Sallows-Bremner square SB mentioned in Boyer's note [5]. The search for more Magicmare entries is documented as CodeBerg initiative, please visit https://codeberg.org/Fian/BASIC.Magicmare for updates and news.

Citations

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CodeBerg project for BASIC Magicmare
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