Annals of Mathematics **00** (XXXX), 1–10
https://doi.org/10.4007/annals.XXXX.00.0.0000

Prime Distribution in Specific Intervals and the Generation of Primes via a Prime Indicator

By Pravin Kumar Mishra

Abstract

This paper presents a series of theorems and corollaries in two sections. The 2nd section outlines a method for verifying the existence of prime numbers within specific intervals. Postulates 1 and 2 establish a methodology to verify the existence of primes in the specific intervals shown in Theorem 1 and its corollary and Theorem 2. The 3rd section outlines a prime indicator function that generates all primes sequentially. The construction of this indicator begins from Theorem 3, and Theorem 4 provides insights on the sum of all odd composite numbers, and its corollary produces a prime indicator supported by an Illustration of a few first numbers. This work provides new insights into how to use elementary principles and methods.

1. Introduction

Prime numbers are foundational objects in number theory with complex distribution patterns that remain an active area of research. This paper is structured in two major sections that present a sequence of theorems and corollaries aimed at advancing elementary methods for prime verification and generation. The second section develops a method grounded on Postulates 1 and 2, enabling verification of prime existence within specific intervals detailed in Theorem 1, its corollary, and Theorem 2. The third section constructs a prime indicator function starting from Theorem 3, with Theorem 4 analyzing the sum of all odd composite numbers and its corollary producing a prime indicator supported with illustrative examples. This work introduces novel, yet accessible techniques that combine classical insights and new constructions, providing fresh perspectives on prime number identification and sequential generation.

2. **Existence of Primes within a specific interval**

Postulate 1: The prime factors of an even number a greater than 2 are less than or equal to $a/2$.

Proof. Consider an even number $a > 2$. We know that all even numbers greater than 2 are composite numbers, and every composite number has two or more prime factors. So we can express a as follows:

$$a = p' \cdot p'' \cdot p''' \cdot p'''' \cdot \ldots$$

Let $X = p'' \cdot p''' \cdot p'''' \cdot \ldots$.

$$a = p' \cdot X,$$

$$a/X = p' \quad and \quad a/p' = X$$

Since p' is a prime and X is a prime or a product of primes. Thus, $p', X \geq 2$.

$$a/X \geq 2, \quad and \quad a/p' \geq 2.$$

Therefore, $X \leq a/2$ and $p' \leq a/2$.

□

Postulate 2: The prime factors of an odd composite number a are less than or equal to $a/3$.

Proof. Consider an odd composite number a. We know that every composite number has two or more prime factors. So we can express a as follows:

$$a = p' \cdot p'' \cdot p''' \cdot p'''' \cdot \ldots$$

Let $X = p'' \cdot p''' \cdot p'''' \cdot \ldots$.

$$a = p' \cdot X,$$

$$a/X = p', \quad and \quad a/p' = X$$

Since a is an odd composite, p, and X must be odd numbers greater than or equal to 3.

$$a/X \geq 3 \quad and \quad a/p' \geq 3.$$

Therefore, $X \leq a/3$ and $p' \leq a/3$.

□

Methodology: To determine whether a prime p exists in a specific range. The following are the steps required:

(1) Select a range $a < k$
(2) Consider that all the integers in the interval (a, k) are composites.
(3) Write the sequence $S = (a+0), (a+1), (a+2), ..., (a+b)$, where $b \geq 0$, and $(a+b) \leq k-1$
(4) From Postulates 1 and 2, we have $p \leq (a+b)/2$.
(5) Substitute the lowest values of a and b in step 4 and check $p \leq a/2$.
(6) Conclusion: If $p < 2$, then at least one prime exists in the range such that $a \leq p \leq k$. If $p \leq (2+c)\, where\, c \geq 0$, then the interval has only composite numbers.

Theorem 1: For every integer $n \geq 3$, there exists a prime p such that $n/2 \leq p < n$.

Proof. Consider an integer $n \geq 3$. Thus, $n/2 < n$

Let's assume the interval $(((n/2)+k), n-1)$, all the integers are composite numbers. The sequence of these integers is as follows:

$$((n/2)+1), \quad ((n/2)+2), \quad ((n/2)+3), \cdot \ldots \; ((n/2)+k), \quad where\, ((n/2)+k) \geq (n-1),\, and\, k \geq 0$$

Since $((n/2)+k)$ is a composite number, we can use the result from Postulates 1 and 2 that all the prime factors of a composite number a are less than or equal to $a/2 \quad and \quad a/3$ respectively, where $a = ((n/2)+k)$.

Since $(a/2) > (a/3)$, we can say all the prime factors of a composite number are less than or equal to $(a/2)$. Now we have the following:

$$p \leq (((n/2)+k)/2 < ((n/2)+k$$

Substituting the lowest values and maintaining the inequalities, we get,

$$p \leq (((n/2)+k)/2 \geq (((3/2)+0)/2 = 3/4$$

$\Rightarrow p \leq 3/4$. Thus, we obtained $p \leq 3/4$. But this contradicts the known fact that prime $p \geq 2$.

It implies that our original assumption was false, that the integers in the given range are composites. It suggests that at least one prime must exist in the given range such that $n/2 \leq p < n$.

□

Corollary: For every integer $n \geq 3$, a prime exists such that $(n/k) \leq p < n$.

Proof. Consider an integer $n \geq 3$ and $k \geq 2$ such that $(n/k) < n$.
Since Theorem 3 shows the proof for $k = 2$. Now let's consider $k > 2$.

$$\Rightarrow (n/k) < (n/2)$$

$$(n/k) < (n/2) \leq p < n$$

$$(n/k) < p < n$$

But if we use the methodology to determine the existence of primes, then we will obtain,

$$(n/k) \leq p' < n$$

□

Illustrative Example: Determine whether a prime exists in the given interval $a! + 2$ and $a! + b$, where $2 \leq b \leq a$.
Solution: Consider all the integers in the interval $a! + 2$ and $a! + b$ are composites.

$$\Rightarrow p \leq ((a! + 2) + k)/2 \quad where \quad k \geq 0.$$

Substituting the lowest value we get, $p \leq ((2! + 2) + 0)/2 = 2$
It implies that the given range has no primes.

Theorem 2: Every even integer greater than 2 can be written as the sum of two prime numbers.

Proof. Consider $2c$, where $c > 1$ is a composite number. So we have $1 < c < 2c$.
So from Postulates 1 and 2, when c is composite, its prime factors exist such that $p \geq (c/2)$ and $p \geq c/3$. So we have $p \leq c/2 < c$.

Now let's check whether there exists any prime p' such that $(2c - c/2 = 3c/2) \leq p' < 2c$.
Suppose the interval does not have primes. That means from Postulates 1 and 2, we can conclude a prime q must exist such that $q \leq (3c/2 + k)/2$, where $k \geq 0$.

Substituting the smallest values we have

$$q \leq (3(2)/2 + 0)/2 = 3/2$$

But this contradicts the fact that $q \geq 2$.
Therefore, $3c/2 \leq p' < 2c$ is true.

Now adding $c/2 + 3c/2 = 2c$

Substituting the values in the LHS

$p + p' = 2c$, Because $p \leq c/2$ and $p' \geq 3c/2$.

From Postulate 1 we have $(c/2) \geq r > c$, where r is prime. Let's check whether primes exist in the interval $c, (3c/2)$ or not.

Using the methodology, we will obtain $r' \leq (c + k)/2$, where r' is a prime factor.

Substituting the lowest value, we get $r' \leq 1$. This implies that $c < r'' \leq (3c/2)$ is true, where r'' is prime. Because c is composite according to our assumptions and $c + k \leq (3c/2)$.

Now adding $c/2 + 3c/2 = 2c$

Substituting the values in the LHS

$r + r'' = 2c$, Because $r \geq c/2$ and $r'' \leq 3c/2$.

Since $p \leq c/2$, $p' \geq 3c/2$, $r \geq c/2$, and $r'' \leq 3c/2$.

We have $p \leq c/2 \leq q$ and $q'' \leq (3c/2) \leq p'$.

This shows that for every $c > 1$, there exists at least one pair of primes such that:

$$p + p' = 2c$$

□

3. Construction of Prime Indicator that generates Primes in sequence

Theorem 3: For all integers $n > 1$, if n^2 is the sum of all first positive odd integers, then the first odd multiples of a positive odd integer k within the original sum can be expressed as:

$$k * (\lfloor(\lfloor(2n-1)/k)\rfloor + 1)/2)\rfloor)^2, where\, k \leq 2n-1$$

Proof. Consider the sum of all first odd integers:

$$1 + 3 + 5 + ... + (2n-1) = n^2,\ n \geq 2$$

Now consider an odd integer k such that $k \leq (2n-1)$. Now consider a sequence of odd multiples of k such that:

k, $3k$, $5k$, ..., $k(2m-1) \leq (2n-1)$, where m is the count of the odd multiples of k within the original sum n^2.

$\Rightarrow m = \lfloor(\lfloor(2n-1)/k)\rfloor + 1)/2)\rfloor$, where floor functions ensures m to be an integer.

Now, let's take the sum of all first odd multiples of k

$$k + 3k + 5k + ... + k(2m-1) = km^2$$

Substituting m, we get,

$$km^2 = k * (\lfloor(\lfloor(2n-1)/k)\rfloor + 1)/2)\rfloor)^2$$

□

Theorem 4: The sum of all odd composites equals the sum of odd multiples of all odd primes, minus the sum of all odd primes and the sum of the odd multiples of their prime products.

Proof. Let's consider the sum of all first positive odd integers.

$$1 + 3 + 5 + ... + (2n-1) = n^2$$

The above series has all the odd primes and odd composites. We know that all the prime numbers are odd (excluding 2). Let's consider the sequence of all first odd primes:

$$p_1, \; p_2, \; p_3, \; ..., \; p_k$$

Composite numbers are the product of 2 or more primes. Now we can use the property from Theorem 3 to obtain the sum of odd multiples of each prime as follows:

Let $f(k) = k * (\lfloor(\lfloor(2n-1)/k)\rfloor + 1)/2)\rfloor)^2$

So our first prime odd multiple sum from the given odd numbers sum is below, and after subtracting the prime, we get the following, which is the composite sum of the first odd multiples of the first odd prime.

$$f(p_1) - p_1$$

Similarly, we can obtain the sum of odd multiples of the second prime. But we are obtaining the sum of all first odd multiples of each prime from the sum of the given odd numbers. To get the sum of all first composite numbers, we must subtract the sum of odd multiples of the product of previous primes and current prime in all unique possible ways as follows:

$$C(p_1) = 0$$
$$C(p_2) = p_1 * p_2$$
$$C(p_3) = p_1p_3, \; p_2p_3, \; p_1p_2p_3$$
$$C(p_k) = p_1p_k, \; p_2p_k, \; ..., \; (p_1p_2...p_k)$$

Where $C(p_k)$ denotes all the unique possible combinations of products.

Since $f(k) = k * (\lfloor(\lfloor(2n-1)/k)\rfloor + 1)/2)\rfloor)^2$

So, $k = p_1, \; p_2, \; ..., \; p_1p_2, \; p_1p_3, \; ..., \; (p_1p_2 \cdots p_k)$

So the sum of odd multiples of each prime is from the sum of odd numbers n^2 given as follows:

$$f(p_1) - p_1$$
$$f(p_2) - f(p_1p_2) - p_2$$
$$f(p_3) - f(p_1p_3) - f(p_2p_3) - f(p_1p_2p_3) - p_3$$
$$f(p_k) - f(p_1p_k) - f(p_2p_k) - ... - f(p_1p_2...p_k) - p_k$$

Postulate 2 states that every prime factor of an odd composite number is in the interval of $p_k \leq (2n-1)/3$.

To obtain the sum of all first composite numbers from the odd number sum, add all the sums of odd multiples of each prime number. So we have,

$$\begin{aligned}S_c = (f(p_1) - p_1) + (f(p_2) - f(p_1p_2) - p_2) + \\ (f(p_3) - f(p_1p_3) - f(p_2p_3) - f(p_1p_2p_3) - p_3) + \\ \cdots + (f(p_k) - f(p_1p_k) - f(p_2p_k) - \cdots - f(p_1p_2\cdots p_k) - p_k)\end{aligned}$$

Where S_c is the sum of all odd multiples of odd composites from the given odd numbers sum.

$$\begin{aligned}S_c = (f(p_1)) + (f(p_2) - f(p_1p_2)) + \\ (f(p_3) - f(p_1p_3) - f(p_2p_3) - f(p_1p_2p_3)) + \\ \cdots + (f(p_k) - f(p_1p_k) - f(p_2p_k) - \cdots - f(p_1p_2\cdots p_k)) \\ - (p_1 + p_2 + p_3 + \cdots + p_k)\end{aligned}$$

□

Corollary : For every positive odd integer greater than 1, the following expression is a prime indicator and produces primes in sequence.

$$\begin{aligned}p_{k+1} = (f(p_1)) + (f(p_2) - f(p_1p_2)) + \\ (f(p_3) - f(p_1p_3) - f(p_2p_3) - f(p_1p_2p_3)) + \\ \cdots + (f(p_k) - f(p_1p_k) - f(p_2p_k) - \cdots - f(p_1p_2\cdots p_k))\end{aligned}$$

Proof. Let's consider the sum of all first positive odd integers.

$$1 + 3 + 5 + ... + (2n - 1) = n^2$$

Subtracting 1 from both sides, we get the sum of all first odd primes S_p and composites S_c.

$$3 + 5 + 7 + ... + (2n - 1) = n^2 - 1 \Rightarrow n^2 - 1 = S_c + S_p$$

$$S_p = n^2 - (1 + S_c) \quad ...[Equation\, 1]$$

Postulate 2, Theorem 1 and its corollary together state $p_k \leq n/2 \leq p_{k+1} < n$

Thus, $S_p = p_1 + p_2 + p_3 + ... + p_k + p_{k+1}$

We have S_c from Theorem 4. Substituting all in Equation 1, we get,

$$p_{k+1} = (f(p_1)) + (f(p_2) - f(p_1p_2)) + (f(p_3) - f(p_1p_3) - f(p_2p_3) - f(p_1p_2p_3)) + \cdots + (f(p_k) - f(p_1p_k) - f(p_2p_k) - \cdots - f(p_1p_2\cdots p_k))$$

Where $f(k) = k*(\lfloor(\lfloor(2n-1)/k)\rfloor+1)/2)\rfloor)^2$ and $k = p_1, p_2, ..., p_1p_2, p_1p_3, ..., (p_1p_2\cdots p_k)$ □

Illustration :

For $n = 2$
Previous odd primes do not exist.
We have $p_1 = 2^2 - (1 + 0) = 3$

For $n = 3$
Previous prime: 3
We have $p_2 = 3^2 - (1 + (3(1)^2)) = 5$

For $n = 4$, Previous primes: 3, 5
We have $p_3 = 4^2 - (1 + (3(1)^2) + (5(1)^2)) = 7$

For $n = 5$
Previous primes: 3, 5, 7
We have $p_4 = 5^2 - (1 + (3(2)^2) + (5(1)^2 - 15(0)^2) + (7(1)^2) - 0))) = 0.$
p_4 does not exist for $n = 4$

For $n = 6$
Previous primes: 3, 5, 7
We have $p_4 = 6^2 - (1 + (3(2)^2) + (5(1)^2 - 15(0)^2) + (7(1)^2) - 0))) = 11.$

For $n = 7$
Previous primes: 3, 5, 7, 11
We have $p_5 = 7^2 - (1 + (3(2)^2) + (5(1)^2) + (7(1)^2) + (11(1)^2)) = 13.$

For $n = 8$
Previous primes: 3, 5, 7, 11, 13
We have $p_6 = 8^2 - (1 + (3(3)^2) + (5(2)^2 - 15(1)^2) + (7(1)^2) + (11(1)^2) + (13(1)^2)) = 0.$
p_6 does not exist for $n = 8$

For $n = 9$
Previous primes: 3, 5, 7, 11, 13
We have $p_6 = 9^2 - (1 + (3(3)^2) + (5(2)^2 - 15(1)^2) + (7(1)^2) + (11(1)^2) + (13(1)^2)) = 17$

For $n = 10$
Previous primes: 3, 5, 7, 11, 13, 17
We have $p_7 = 10^2 - (1 + (3(3)^2) + (5(2)^2 - 15(1)^2) + (7(1)^2) + (11(1)^2) + (13(1)^2) + (17(1)^2)) = 19$.

For $n = 11$
Previous primes: 3, 5, 7, 11, 13, 17, 19
We have $p_8 = 11^2 - (1 + (3(4)^2) + (5(2)^2 - 15(1)^2) + (7(2)^2 - 21(1)^2) + (11(1)^2) + (13(1)^2) + (17(1)^2) + (19(1)^2)) = 0$.
p_8 does not exist for $n = 11$

If $p_{k+1} = 0$ then $2n-1$ is composite. If $p_{k+1} > 0$, then it is the next prime number in the sequence. n has been tested till $n = 3000$, and the results are accurate.

References

[1] David M. Burton, *Elementary Number Theory*, 7th Edition, McGraw Hill Education India, 2023.

Pravin Kumar Mishra