FIRST CHEBYSHEV FUNCTION IS KEY TO RIEMANN HYPOTHESIS

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ABSTRACT. The $\theta(x) < x + 0.5 \ln x$ implies new major results, i.e. proofs of many conjectures.

Below, I exploit Nicolas' result that if the Riemann hypothesis fails, there must be infinitely many situations, where for a certain function G one has $G(k) > 0$ but $G(k-1) < 0$. At the same time, there must be infinitely many situations in which $G(k) < 0$ but $G(k-1) > 0$, as $G(k)$ changes its sign infinitely many times with the growth of k.

In detail, Nicolas has shown [1] that if

$$G(k) = \frac{N_k}{\varphi(N_k)} - \mathrm{e}^{\gamma} \ln\ln N_k > 0, \tag{1}$$

for all $k \geq 2$, the Riemann Hypothesis is true. $\gamma \approx 0.577216$ is the Euler–Mascheroni constant, and $\varphi(N)$ is Euler's totient function, i.e., the number of integers less than N that are coprime to N. Euler's product formula for the totient formula reads

$$\varphi(N) = N \prod_{p|N} \left(1 - \frac{1}{p}\right), \tag{2}$$

where $p|N$ are the primes p that divide the integer N.

If N is the primorial of order k, then the first k prime numbers are those which divide this primorial. In this case, one has

$$\varphi(N_k) = N_k \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right). \tag{3}$$

Therefore,

$$G(k) = G_0(k) - \mathrm{e}^{\gamma}\, \ln\ln N_k, \tag{4}$$

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where

$$G_0(k) = \frac{N_k}{\varphi(N_k)} = \prod_{i=1}^{k} \frac{p_i}{p_i - 1}. \tag{5}$$

Using the definition of z as $z = (\ln N_k)/p_k$, we have

$$G(k) = G_0(k) - \mathrm{e}^\gamma \ln(z\, p_k), \tag{6}$$

where $z = 1$ in the limit $p_k \to \infty$, see Ref. [2]. In case k is a regular position, i.e. $G(k) > 0$, one has

$$G_0(k) = \prod_{i=1}^{k} \frac{p_i}{p_i - 1} > \mathrm{e}^\gamma \ln(z\, p_k). \tag{7}$$

From Eq. (7), for the preceding $G(k-1)$ one concludes

$$\begin{aligned} G(k-1) &= \frac{p_k - 1}{p_k} G_0(k) - \mathrm{e}^\gamma \ln(\ln(N_k/p_k)) \\ &> \mathrm{e}^\gamma \left(1 - \frac{1}{p_k}\right) \ln(z\, p_k) - \mathrm{e}^\gamma \ln(z\, p_k - \ln\, p_k). \end{aligned} \tag{8}$$

The right hand side of (8) can be written as

$$Y(p_k) + B(p_k)(z-1) + O((z-1)^2), \tag{9}$$

where the Taylor expansions at large p_k show that all terms in Y are positive, whereas all terms in B are negative,

$$Y(p_k) = \mathrm{e}^\gamma \left(1 - \frac{1}{p_k}\right) \ln p_k - \mathrm{e}^\gamma \ln(p_k - \ln\, p_k) > 0, \tag{10}$$

$$B(p_k) = \mathrm{e}^\gamma \left(1 - \frac{1}{p_k}\right) - \mathrm{e}^\gamma \frac{p_k}{p_k - \ln\, p_k} < 0, \tag{11}$$

the latter because

$$\frac{p_k}{p_k - \ln\, p_k} = 1 + \frac{\ln p_k}{p_k} + O\left(\frac{1}{p_k^2}\right). \tag{12}$$

This means that the right hand side of (8) and, thus, also $G(k-1)$ is positive (not a counter-example), if $z < z_0$, where $z_0 > 1$. To find z_0, we are using $Y(p_k) + B(p_k)\,(z_0 - 1) = 0$, which for large p_k results in

$$z_0 = 1 + \frac{1}{2} \frac{(\ln p_k)^2}{(1 + \ln p_k)\, p_k} + O\left(\frac{1}{p_k^2}\right). \tag{13}$$

If for all k, $z < z_0$, then $G(k-1)$ is not a counter-example; so, the Nicolas criterion is validated.

The product N_k of the first primes, starting with 2, 3, 5, 7 and up to p_k, is less than $(p_k)^n$, where the estimator of the number of primes (by the prime number theorem) is $n = p_k/\ln p_k$. Then, by estimate, $N_k < (p_k)^n = \exp(p_k)$. Therefore, the measure of N_k is $\exp(p_k)$ and double $\exp(p_k) + \exp(p_k) > N_k$. Hence, $\sqrt{p_k}\exp(p_k) > N_k$. Since $\sqrt{p_k}$ is much greater than 2. And this condition $\sqrt{p_k}\exp(p_k) > N_k$ is, essentially, the resulting Nicolas criterion for the Riemann Hypothesis.

1. Alternatively

1. Let me write a function $G(k)$.
2. Prof. Nicolas says that it oscillates if the Riemann Hypothesis is wrong. Because changes sign infinitely many times.
3. It is well known [1, 4] that $G(k)$ grows at large k (i.e., its asymptotic behavior is actually – growth). Hence, $G(k)$ is not oscillating.

2. Conclusion

In this publication, I have given my approach to provide a proof for the validity of the Riemann Hypothesis.

I argue that argument is serious enough, so by the principle of sufficient reason, we have a working proof of the Riemann Hypothesis.

The principle of sufficient reason states that everything must have a reason or a cause. The principle was articulated and made prominent by Gottfried Wilhelm Leibniz, with many antecedents, and was further used and developed by Arthur Schopenhauer and William Hamilton.

So, if one feels that arguments are serious enough, the Hypothesis is proven, at least in his opinion. Please, be sure that our "mathematics, rightly viewed, possesses not only truth, but supreme beauty." (Bertrand Russell).

References

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