An Operational Formula for $(xV)^n$

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Abstract

We show that the operator $(xV)^n$ can be expressed as a sum of powers of the Volterra operator V, with coefficients given by the Bessel coefficients A001498.

1 Introduction

Motivation

The Euler operator xD, where $D = \frac{d}{dx}$, plays a central role in special functions, combinatorics, and operational calculus. Its powers admit the well-known Stirling expansion [3]

$$(xD)^n = \sum_{k=0}^{n} S(n,k)\, x^k D^k, \tag{1}$$

where the coefficients are the Stirling numbers of the second kind $S(n,k)$.

In this paper, we pose a natural but, to the best of our knowledge, unexplored question: what happens to this expansion when the differentiation operator D is replaced by the Volterra operator

$$V(f)(x) = \int_0^x f(t)\, dt\,?$$

Does an analogous decomposition exist, and if so, what are the corresponding coefficients?

To address this problem, we define the operator $(xV)^n$ recursively by

$$(xV)^0(f)(x) := f(x), \quad (xV)^n(f)(x) := x\, V\big((tV)^{n-1} f\big)(x), \quad n \geq 1.$$

Bessel coefficients

The Bessel coefficient $a(n,k)$ (see [1]; A001498) is defined as the coefficient of x^k in the Bessel polynomial

$$y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} \left(\frac{x}{2}\right)^k.$$

Explicitly,

$$a(n,k) = \frac{(n+k)!}{2^k k!(n-k)!}, \qquad 0 \le k \le n.$$

This coefficient satisfies the recurrence relation [2]

$$a(n,k) = a(n-1,k) + (n-k+1)a(n,k-1), \qquad a(0,0) = 1. \tag{2}$$

$n\backslash k$	0	1	2	3	4	5
0	1					
1	1	1				
2	1	3	3			
3	1	6	15	15		
4	1	10	45	105	105	
5	1	15	105	420	945	945

Table 1: Bessel coefficients $a(n,k)$ for $0 \le k \le n \le 5$.

A simple symmetry also holds [2]:

$$a(n,n-1) = a(n,n), \qquad n \ge 1. \tag{3}$$

Main result

Our main result provides an integral analogue of Formula (1):

$$(xV)^n = \sum_{k=0}^{n-1} (-1)^k a(n-1,k)\, x^{n-k}\, V^{n+k}. \tag{4}$$

Here V^m denotes the m-fold composition

$$V^0(f)(x) := f(x), \quad V^{m+1}(f)(x) := V(V^m f)(x).$$

Unlike the differential case, the operator V^m admits the classical Cauchy formula for repeated integration:

$$V^m(f)(x) = \int_0^x \frac{(x-t)^{m-1}}{\Gamma(m)} f(t)\, dt.$$

Substituting this into (4) yields the explicit identity

$$(xV)^n(f)(x) = \int_0^x \frac{x(x^2-t^2)^{n-1}}{2^{n-1}\Gamma(n)} f(t)\, dt. \tag{5}$$

This shows that $(xV)^n$ is a Volterra-type integral operator with kernel given by

$$\frac{x(x^2-t^2)^{n-1}}{2^{n-1}(n-1)!}.$$

In the remainder of the paper, we prove Formula (4) and use it to derive new identities for the Bessel coefficients.

2 Proofs

Lemma 1. *The Bessel coefficients satisfy*

$$a(n,k) = \sum_{i=0}^{\min(n-1,k)} (n-i)_{k-i}\, a(n-1,i), \tag{6}$$

for all $n \geq 1$ *and* $0 \leq k \leq n$*, where* $(x)_m = x(x-1)\cdots(x-m+1)$ *denotes the falling factorial.*

Proof. **Case 1:** $1 \leq k \leq n-1$. We claim that

$$a(n,k) = \sum_{i=0}^{k} (n-i)_{k-i}\, a(n-1,i).$$

The proof is by induction on k. The base case is immediate. Suppose the claim holds for some $k \geq 1$. Using the recurrence (2),

$$a(n,k+1) = a(n-1,k+1) + (n-k)\sum_{i=0}^{k} (n-i)_{k-i}\, a(n-1,i).$$

Since $(n-k)(n-i)_{k-i} = (n-i)_{k+1-i}$, this simplifies to

$$a(n,k+1) = \sum_{i=0}^{k+1} (n-i)_{k+1-i}\, a(n-1,i),$$

completing the induction step.

Case 2: $k = n$. We must show that

$$a(n,n) = \sum_{i=0}^{n-1} (n-i)_{n-i}\, a(n-1,i).$$

From Case 1,

$$a(n,n-1) = \sum_{i=0}^{n-1} (n-i)_{n-1-i}\, a(n-1,i).$$

By the symmetry relation (3) and the fact that $(n-i)_{n-i} = (n-i)_{n-1-i}$, the result follows. □

Remark 2. Define

$$\Omega(n) = \sum_{i=0}^{\min(n-1,k)} (n-i)_{k-i}\, B^{k-i}, \qquad \text{where } B\, f(k) = f(k-1).$$

Equation (6) can then be expressed in operator form as

$$a(n,k) = \Omega(n)\, a(n-1,k). \tag{7}$$

Iterating yields

$$a(n,k) = \Big(\prod \Omega(i)\Big)\delta_{k0},$$

with

$$\begin{aligned}\prod \Omega(i) &= \Omega(n)\Omega(n-1)\cdots\Omega(1)\\ &= \sum_{\substack{0\le i_1\le\cdots\le i_n\le k\\ i_j\le j-1}} \left(\prod_{j=1}^{n}(j-i_j)_{\,i_{j+1}-i_j}\right) B^{\,k-i_1},\end{aligned}$$

where δ denotes the Kronecker delta.

Hence,

$$a(n,k) = \sum_{\substack{0=i_1\le i_2\le\cdots\le i_n\le k\\ i_j\le j-1}} \prod_{j=1}^{n}(j-i_j)_{\,i_{j+1}-i_j}. \tag{8}$$

Introducing increments

$$m_j := i_{j+1}-i_j, \quad j=1,\ldots,n,$$

we have $\sum_{j=1}^{n} m_j = k$ and $i_j = \sum_{r=1}^{j-1} m_r$. Substituting gives

$$(j-i_j)_{m_j} = \Big(j - \sum_{r=1}^{j-1} m_r\Big)_{m_j},$$

so that

$$a(n,k) = \sum_{\substack{m_1+\cdots+m_n=k\\ m_j\in\mathbb{N}}} \prod_{j=1}^{n}\left(j-\sum_{r=1}^{j-1} m_r\right)_{m_j}. \tag{9}$$

As an example, consider $a(3,2)$. Since $m_1+m_2+m_3=2$,

$$a(3,2) = \sum_{m_1+m_2+m_3=2} (1)_{m_1}\,(2-m_1)_{m_2}\,(3-m_1-m_2)_{m_3}.$$

Enumerating all solutions gives:

m_1	m_2	m_3	Contribution
2	0	0	0
1	1	0	1
1	0	1	2
0	2	0	2
0	1	1	4
0	0	2	6
Total			15

Thus $a(3,2) = 15$, as expected.

Combinatorially, $a(n,k)$ counts the ways to sequentially distribute k labeled elements across n ordered steps. At step j, a nonnegative number m_j of elements is placed, with $\sum m_j = k$. The factor

$$\Big(j - \sum_{r=1}^{j-1} m_r\Big)_{m_j}$$

records the number of placements at step j, given the prior choices. Taking the product aggregates contributions over all steps, while summing over $(m_1, \ldots, m_n)$ accounts for all possible distributions.

Lemma 3. *For all $m \in \mathbb{N}$ and $m' \in \mathbb{N}^*$,*

$$V\big(t^m V^{m'}(f)(t)\big)(x) = \sum_{j=0}^{m} (-1)^j (m)_j \, x^{m-j} \, V^{m'+j+1}(f)(x). \tag{10}$$

Proof. We argue by induction on m.

For $m = 0$,

$$V\big(V^{m'}(f)\big)(x) = V^{m'+1}(f)(x),$$

which matches the sum with $j = 0$.

Assume the formula holds for some $m \geq 0$:

$$V\big(t^m V^{m'}(f)(t)\big)(x) = \sum_{j=0}^{m} (-1)^j (m)_j x^{m-j} V^{m'+j+1}(f)(x).$$

Consider $m+1$. Integration by parts gives

$$V\big(t^{m+1} V^{m'}(f)(t)\big)(x) = x^{m+1} V^{m'+1}(f)(x) - (m+1) V\big(t^m V^{m'+1}(f)(t)\big)(x).$$

Apply the induction hypothesis to the second term:

$$V\big(t^m V^{m'+1}(f)(t)\big)(x) = \sum_{j=0}^{m} (-1)^j (m)_j x^{m-j} V^{m'+j+2}(f)(x).$$

Multiply by $-(m+1)$ and combine with the first term:

$$x^{m+1} V^{m'+1}(f)(x) + \sum_{j=0}^{m} (-1)^j (m+1)(m)_j x^{m-j} V^{m'+j+2}(f)(x).$$

Rewrite the sum using $(m+1)(m)_j = (m+1)_{j+1}$ and shift indices as needed to match the formula

$$\sum_{j=0}^{m+1} (-1)^j (m+1)_j x^{m+1-j} V^{m'+j+1}(f)(x).$$

This completes the induction.

□

Theorem 4. *For all $n \in \mathbb{N}^*$,*

$$(x\,V)^n(f)(x) = \sum_{k=0}^{n-1}(-1)^k a(n-1,k)\, x^{n-k}\, V^{n+k}(f)(x). \tag{11}$$

Proof. We proceed by induction on n.

For $n = 1$,

$$(xV)^1(f)(x) = x\,V(f)(x),$$

which coincides with the right-hand side of (11).

Assume it holds for some $n \geq 1$, i.e.,

$$(t\,V)^n(f)(t) = \sum_{k=0}^{n-1}(-1)^k a(n-1,k)\, t^{n-k}\, V^{n+k}(f)(t).$$

Then

$$\begin{aligned}(x\,V)^{n+1}(f)(x) &= xV\big((t\,V)^n\big)(x)\\ &= x\sum_{k=0}^{n-1}(-1)^k a(n-1,k) V\big[t^{\,n-k}V^{n+k}(f)(t)\big](x).\end{aligned}$$

By Lemma 3,

$$V\big[t^{\,n-k}V^{n+k}(f)(t)\big](x) = \sum_{j=0}^{n-k}(-1)^j (n-k)_j\, x^{\,n-k-j}\; V^{n+k+j+1}(f)(x).$$

Re-indexing and interchanging sums, and then applying Lemma 1, which gives

$$\sum_{k=0}^{\min(n-1,i)} (n-k)_{i-k}\, a(n-1,k) = a(n,i),$$

we obtain

$$(x\,V)^{n+1}(f)(x) = \sum_{i=0}^{n}(-1)^i a(n,i)\, x^{\,n+1-i}\; V^{n+1+i}(f)(x),$$

which completes the induction. □

3 Two applications

Power function

Let $x > 0$.

Let $\alpha \in \mathbb{R}^*$. We compute $(xV)^n(t^{\alpha-1})$ in two ways.

Method 1: Closed-form kernel.

From Formula (5),

$$(xV)^n(t^{\alpha-1}) = \frac{x}{2^{\,n-1}(n-1)!} \int_0^x (x^2 - t^2)^{\,n-1} t^{\alpha-1}\, dt.$$

Set $t = xu^{1/2}$, so that $dt = \frac{x}{2} u^{-1/2} du$. Then

$$\begin{aligned}(xV)^n(t^{\alpha-1}) &= \frac{x}{2^{\,n-1}(n-1)!} \int_0^1 \big(x^2(1-u)\big)^{n-1} (xu^{1/2})^{\alpha-1} \frac{x}{2} u^{-1/2}\, du \\ &= \frac{x^{\alpha+2n-1}}{2^n(n-1)!} \int_0^1 (1-u)^{n-1} u^{\frac{\alpha}{2}-1}\, du.\end{aligned}$$

The integral is the Beta function:

$$\int_0^1 (1-u)^{n-1} u^{\frac{\alpha}{2}-1}\, du = \mathrm{B}\big(\tfrac{\alpha}{2}, n\big) = \frac{\Gamma(\alpha/2)\,\Gamma(n)}{\Gamma(n+\alpha/2)}.$$

Since $\Gamma(n) = (n-1)!$, this yields

$$(xV)^n(t^{\alpha-1}) = \frac{x^{\alpha+2n-1}}{2^n\,(\frac{\alpha}{2})^{(n)}},$$

where $x^{(m)} = x(x+1)\cdots(x+m-1)$ denotes the rising factorial.

Method 2: Series expansion.

From Formula (4),

$$(xV)^n(t^{\alpha-1}) = \sum_{k=0}^{n-1} (-1)^k a(n-1,k)\, x^{n-k}\, V^{n+k}(t^{\alpha-1})(x).$$

By the repeated integration formula,

$$V^m(t^{\alpha-1})(x) = \frac{1}{\Gamma(m)} \int_0^x (x-t)^{m-1} t^{\alpha-1}\, dt.$$

Substituting $m = n+k$, we get

$$(xV)^n(t^{\alpha-1}) = \sum_{k=0}^{n-1} \frac{(-1)^k a(n-1,k)}{(n+k-1)!}\, x^{n-k} \int_0^x (x-t)^{\,n+k-1} t^{\alpha-1} dt.$$

The integral is a Beta-type integral. Setting $t = xu$ gives

$$\begin{aligned}\int_0^x (x-t)^{n+k-1} t^{\alpha-1} dt &= x^{\alpha+n+k-1} \int_0^1 (1-u)^{n+k-1} u^{\alpha-1} du \\ &= x^{\alpha+n+k-1}\, \mathrm{B}(\alpha, n+k) \\ &= x^{\alpha+n+k-1} \frac{\Gamma(\alpha)\,(n+k-1)!}{\Gamma(\alpha+n+k)}.\end{aligned}$$

Hence

$$\begin{aligned}(xV)^n(t^{\alpha-1}) &= \sum_{k=0}^{n-1} (-1)^k a(n-1,k)\, x^{n-k}\, \frac{x^{\alpha+n+k-1}\,\Gamma(\alpha)}{\Gamma(\alpha+n+k)} \\ &= x^{\alpha+2n-1} \sum_{k=0}^{n-1} \frac{(-1)^k a(n-1,k)}{\alpha^{(n+k)}}.\end{aligned}$$

Equating the two expressions gives

$$\frac{1}{2^n \left(\frac{\alpha}{2}\right)^{(n)}} = \sum_{k=0}^{n-1} \frac{(-1)^k a(n-1,k)}{\alpha^{(n+k)}}. \tag{12}$$

Exponential function

We now evaluate $(xV)^{n+1}(e^t)$. From Formula (4), we have

$$(xV)^{n+1}(e^t) = \sum_{k=0}^{n} \frac{(-1)^k}{2^k k!(n-k)!}\, x^{n+1-k} \int_0^x (x-t)^{n+k} e^t\, dt.$$

The integral evaluates to

$$\int_0^x (x-t)^{n+k} e^t\, dt = e^x \gamma(n+k+1, x),$$

where γ is the lower incomplete gamma function. Thus,

$$(xV)^{n+1}(e^t) = e^x \sum_{k=0}^{n} \frac{(-1)^k}{2^k k!(n-k)!}\, x^{n+1-k}\, \gamma(n+k+1, x).$$

Since

$$\gamma(m,x) = (m-1)! \left(1 - e^{-x} \sum_{j=0}^{m-1} \frac{x^j}{j!}\right),$$

this becomes

$$(xV)^{n+1}(e^t) = e^x \sum_{k=0}^{n} (-1)^k a(n,k)\, x^{n+1-k} \left(1 - e^{-x} \sum_{j=0}^{n+k} \frac{x^j}{j!}\right).$$

Expanding gives

$$(xV)^{n+1}(e^t) = e^x \sum_{k=0}^{n} (-1)^k a(n,k)\, x^{n+1-k} - S,$$

where

$$S = \sum_{k=0}^{n} (-1)^k a(n,k)\, x^{n+1-k} \sum_{j=0}^{n+k} \frac{x^j}{j!}.$$

It is shown in [4] that

$$S = \sqrt{\pi/2}\, x^{n+\frac{3}{2}} \left(I_{-n-\frac{1}{2}}(x) - L_{n+\frac{1}{2}}(x)\right), \tag{13}$$

where $I_\nu(z)$ and $L_\nu(z)$ denotes respectively the modified Bessel function of the first kind and the modified Struve function.

Therefore,

$$(xV)^{n+1}(e^t) = e^x \sum_{k=0}^{n} (-1)^k a(n,k)\, x^{n+1-k} - \sqrt{\pi/2}\, x^{n+\frac{3}{2}} \left(I_{-n-\frac{1}{2}}(x) - L_{n+\frac{1}{2}}(x)\right) \tag{14}$$

Connection with OEIS sequences

$$\text{A122850}(n+1,k) = [x^k]\, e^{-x} \left((xV)^{n+1}(e^t)(x) + \sqrt{\pi/2}\, x^{n+\frac{3}{2}} \left(I_{-n-\frac{1}{2}}(x) - L_{n+\frac{1}{2}}(x)\right) \right) \tag{15}$$

$$\text{A000806}(n) = \frac{1}{e} \left((xV)^{n+1}(e^t)(1) \; + \sqrt{\pi/2} \left(I_{-n-\frac{1}{2}}(1) - L_{n+\frac{1}{2}}(1)\right) \right) \tag{16}$$

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References

[1] J. Y. Choi and J. D. H. Smith, On the unimodality and combinatorics of Bessel numbers, *Discrete Math.*, **264**(1–3) (2003), 45–53.

[2] E. Grosswald, *Bessel Polynomials*, Springer, New York, 1978.

[3] P. M. Knopf, The operator $(x\frac{d}{dx})^n$ and its applications to series, *Math. Mag.*, **76**(5) (2003), 364–371.

[4] MathOverflow, "Evaluating $\sum_{k=0}^{n}(-1)^k \frac{(n+k)!}{2^k k!(n-k)!} x^{n-k} \sum_{j=0} \ldots$", MathOverflow link

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