The Integral Analog of $(xD)^n$ and Its Connection to Bessel Numbers

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Abstract

We study the integral analog of the operator $\left(x\frac{d}{dx}\right)^n$, obtained by replacing differentiation with integration. We prove that the resulting operator admits an expansion in powers of the integration operator with coefficients given by the Bessel numbers of the second kind $\{B(n,k)\}$ (OEIS A122848), leading to new explicit formulas and revealing a fundamental role of Bessel numbers in the structure of certain integral operators. The author has contributed these findings to the corresponding entries in the OEIS.

1 Introduction

The Euler operator xD, where $D = \frac{d}{dx}$, is a classical and central object in special functions, combinatorics, and operational calculus. It is well known (see [3]) that the powers of this operator admit an expansion in terms of the Stirling numbers of the second kind $S(n,k)$:

$$(xD)^n = \sum_{k=0}^{n} S(n,k)\, x^k D^k. \tag{1}$$

This formula expresses the operator $(xD)^n$ as a linear combination of derivatives D^k, weighted by the coefficients $S(n,k)$.

In this paper, we investigate the *integral analog* of $(xD)^n$, obtained by replacing the differentiation operator D with the integration operator

$$\mathrm{I}(f)(x) := \int_{x_0}^{x} f(t)\, dt,$$

for a fixed reference point $x_0 \in \mathbb{R}$. We then define the operator $x\mathrm{I}$ and its n-fold composition by

$$(x\mathrm{I})^n(f)(x) := x\int_{x_0}^{x} x_{n-1}\int_{x_0}^{x_{n-1}} \cdots x_1 \int_{x_0}^{x_1} f(t)\,dt\,dx_1\cdots\,dx_{n-1}$$
$$= \int_{x_0}^{x}\cdots\int_{x_0}^{x_1} (x\cdots x_1)\ f(t)\,dt\,dx_1\cdots dx_{n-1}.$$

This operator intertwines multiplication by x with integration from x_0, and thus $(x\,\mathrm{I})^n$ corresponds to a nested sequence of weighted Volterra-type integrals. Our aim is to derive an explicit expansion for $(x\,\mathrm{I})^n$ analogous to (1).

1.1 Bessel numbers

The *Bessel coefficients* $a(n,k)$ (OEIS A001498) naturally appear as the coefficients of x^k in the *Bessel polynomials* [4]

$$y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!}\left(\frac{x}{2}\right)^k,$$

and are explicitly given by

$$a(n,k) = \frac{(n+k)!}{2^k\,k!\,(n-k)!},$$

for $0 \le k \le n$. These numbers satisfy the recurrence relation [1]

$$a(n,k) = a(n-1,k) + (n-k+1)\,a(n,k-1), \tag{2}$$

with the initial condition $a(0,0) = 1$.

Table 1 shows the first few values of the Bessel coefficients.

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	3	3				
3	1	6	15	15			
4	1	10	45	105	105		
5	1	15	105	420	945	945	
6	1	21	210	1260	4725	10395	10395

Table 1: The first Bessel coefficients $a(n,k)$ for $0 \le k \le n \le 6$.

An intriguing symmetry noted in 1 is the equality [1]

$$a(n,n-1) = a(n,n), \qquad n \ge 1, \tag{3}$$

The main result we establish is the explicit expansion for the integral operator powers:

$$(x\,\mathrm{I})^n = \sum_{k=0}^{n-1} (-1)^k\, a(n-1,k)\, x^{n-k}\, \mathrm{I}^{n+k}, \tag{4}$$

Alternatively, one may re–index (4) to work directly with the powers of the integration operator I in ascending or descending order.

Defining [2]

$$B(n,k) := a(k, n-k), \qquad B(n,k) = 0 \text{ if } k < \lceil n/2 \rceil,$$

we can re-express (4) by setting $m = n + k$ as

$$(x\mathrm{I})^n = \sum_{m=n}^{2n-1} (-1)^{m-n}\, B(m-1, n-1)\, x^{2n-m}\, \mathrm{I}^m,$$

Here, the numbers $B(n,k)$ are the *Bessel numbers of the second kind* (OEIS A122848), and in this expansion, they naturally weight the powers of the integration operator I. Similarly, defining

$$b(n,k) := (-1)^{n-k}\, a(n-1, n-k),$$

we have

$$(x\mathrm{I})^n = \sum_{k=1}^{n} b(n,k)\, x^k\, \mathrm{I}^{\,2n-k},$$

where the numbers $b(n,k)$ are the *Bessel numbers of the first kind* (OEIS A122850), now weighting powers of x. Here I^m denotes the m-fold composition of I, defined recursively as

$$\mathrm{I}^0(f)(x) := f(x), \quad \text{and} \quad \mathrm{I}^{m+1}(f)(x) := \mathrm{I}\big(\mathrm{I}^m(f)\big)(x), \quad m \geq 0.$$

A key difference from the differential case is that the iterated integrals I^m can be expressed in a simplified form (see [5]):

$$\mathrm{I}^m(f)(x) = \int_{x_0}^{x} \frac{(x-s)^{m-1}}{(m-1)!} f(s)\, ds,$$

which leads to a transformation of the coefficients of (4) into a simpler rational sequence

$$r_{n-1,k} := \frac{(-1)^k}{2^k\, k!\, (n-1-k)!} = \frac{1}{(n-1)!} \binom{n-1}{k} \left(-\frac{1}{2}\right)^k, \tag{5}$$

revealing a binomial-type structure reminiscent of classical Volterra operators.

This paper explores this integral operator in detail, providing new explicit formulas, establishing connections with Bessel numbers, and extending the framework to fractional orders.

2 Main Results

Lemma 1. *For all $m, m' \in \mathbb{N}$, we have*

$$\int_{x_0}^{x} t^m \int_{x_0}^{t} \frac{(t-s)^{m'-1}}{(m'-1)!} f(s)\, ds\, dt = \sum_{j=0}^{m} (-1)^j (m)_j \, x^{m-j} \int_{x_0}^{x} \frac{(x-s)^{m'+j}}{(m'+j)!} f(s)\, ds, \tag{6}$$

where $(m)_j = m(m-1)\cdots(m-j+1)$ denotes the falling factorial.

Proof. Recall that

$$\mathrm{I}^n(f)(t) = \int_{x_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} f(s)\, ds.$$

So we have to prove that

$$\mathrm{I}\big(t^m \mathrm{I}^{m'}(f)(t)\big)(x) = \sum_{j=0}^{m} (-1)^j (m)_j x^{m-j} \mathrm{I}^{m'+j+1}(f)(x).$$

We proceed by induction on $m \geq 0$.

For the base case $m = 0$, this reduces to

$$\mathrm{I}\big(\mathrm{I}^{m'}(f)\big)(x) = \mathrm{I}^{m'+1}(f)(x),$$

matching the sum with only the $j = 0$ term.

Assuming the formula holds for m, integration by parts gives

$$\mathrm{I}\big(t^{m+1} \mathrm{I}^{m'}(f)(t)\big)(x) = x^{m+1} \mathrm{I}^{m'+1}(f)(x) - (m+1)\, \mathrm{I}\big(t^m \mathrm{I}^{m'+1}(f)(t)\big)(x).$$

Applying the induction hypothesis to the second term and rearranging yields the formula for $m+1$, completing the proof. □

Lemma 2. *The Bessel coefficients $a(n,k)$ satisfy the recurrence*

$$a(n,k) = \sum_{i=0}^{\min(n-1,k)} (n-i)_{k-i} \cdot a(n-1, i), \tag{7}$$

for all integers $n \geq 1$ and $k \geq 0$.

Proof. We proceed by cases.

Case 1: $1 \leq k \leq n-1$. In this range, we want to show

$$a(n,k) = \sum_{i=0}^{k} (n-i)_{k-i} \cdot a(n-1, i).$$

We argue by induction on k. Assume the equality holds for some $k \geq 1$. Using the recurrence (2), we have

$$a(n, k+1) = a(n-1, k+1) + (n-k) \sum_{i=0}^{k} (n-i)_{k-i} \cdot a(n-1, i).$$

Since

$$(n-k)(n-i)_{k-i} = (n-i)_{k+1-i},$$

it follows that

$$a(n, k+1) = \sum_{i=0}^{k+1} (n-i)_{k+1-i} \cdot a(n-1, i),$$

which completes the induction step.

Case 2: $k = n$. We need to show

$$a(n, n) = \sum_{i=0}^{n-1} (n-i)_{n-i} \cdot a(n-1, i).$$

From Case 1, we already have

$$a(n, n-1) = \sum_{i=0}^{n-1} (n-i)_{n-1-i} \cdot a(n-1, i).$$

Using (3) and noting that

$$(n-i)_{n-i} = (n-i)_{n-1-i},$$

we conclude that the formula holds for $k = n$ as well.

This completes the proof. □

Remark 3. This recurrence appears to be new and provides an efficient way to compute $a(n, k)$ recursively from previous coefficients. One may additionally observe that, for all k and n,

$$a(n, k) = \sum_{i=0}^{k} (n-i)_{k-i} \cdot a(n-1, i).$$

Theorem 4 (Main Operational Formula)**.** *For all $n \in \mathbb{N}^*$, we have*

$$(x\,\mathrm{I})^n(f)(x) = \sum_{k=0}^{n-1} (-1)^k a(n-1, k)\, x^{n-k}\, \mathrm{I}^{n+k}(f)(x). \tag{8}$$

Proof. We prove by induction on n the equivalent formula:

$$(x\,\mathrm{I})^n(f)(x) = \sum_{k=0}^{n-1} \frac{(-1)^k a(n-1,k)}{(n-1+k)!}\, x^{n-k} \int_{x_0}^{x} (x-t)^{n+k-1} f(t)\, dt. \tag{9}$$

Base case: For $n = 1$, the formula trivially holds by direct computation.
Inductive step: Assume the formula holds for some $n \geq 1$, i.e.,

$$(x_n\,\mathrm{I})^n(f)(x_n) = \sum_{k=0}^{n-1} \frac{(-1)^k a(n-1,k)}{(n-1+k)!}\, x_n^{n-k} \int_{x_0}^{x_n} (x_n-t)^{n+k-1} f(t)\, dt.$$

Then,

$$(x\,\mathrm{I})^{n+1}(f)(x) = x \int_{x_0}^{x} (x_n\,\mathrm{I})^n(f)(x_n)\, dx_n.$$

Substituting the induction hypothesis into the integral gives

$$(x\,\mathrm{I})^{n+1}(f)(x) = x \sum_{k=0}^{n-1} \frac{(-1)^k a(n-1,k)}{(n-1+k)!} \int_{x_0}^{x} x_n^{n-k} \int_{x_0}^{x_n} (x_n-t)^{n+k-1} f(t)\, dt\, dx_n.$$

By Lemma 1, we can rewrite the nested integral as

$$\int_{x_0}^{x} x_n^{n-k} \int_{x_0}^{x_n} \frac{(x_n-t)^{n+k-1}}{(n+k-1)!} f(t)\, dt\, dx_n = \sum_{j=0}^{n-k} (-1)^j (n-k)_j\, x^{n-k-j} \int_{x_0}^{x} \frac{(x-t)^{n+k+j}}{(n+k+j)!} f(t)\, dt.$$

By interchanging the sums, re-indexing the terms, and using the recurrence relation for the Bessel coefficients $a(n,k)$ (Lemma 2),

$$\sum_{k=0}^{\min(n-1,i)} (n-k)_{i-k} \cdot a(n-1,k) = a(n,i),$$

we arrive at

$$(x\,\mathrm{I})^{n+1}(f)(x) = \sum_{i=0}^{n} \frac{(-1)^i a(n,i)}{(n+i)!}\, x^{n+1-i} \int_{x_0}^{x} (x-t)^{n+i} f(t)\, dt,$$

which matches the formula for $n+1$.

This completes the induction. □

Remark 5 (Generalization and Fractional Extension). Let $f \in C[x_0, x]$ and $n \in \mathbb{N}$. Define the operator $\mathcal{A}^{n-1}$ associated with $(x\mathrm{I})^n$ by

$$\mathcal{A}^{n-1}(f)(x_0;x) := \frac{(x\,\mathrm{I})^n(f)(x)}{x} = \int_{x_0}^{x} \left(\sum_{k=0}^{n-1} r_{n-1,k}\, x^{n-1-k} (x-t)^{n-1+k} \right) f(t)\, dt,$$

where $r_{n-1,k}$ are the coefficients defined in (5). Then $\mathcal{A}^n$ can be expressed as a linear Volterra-type integral operator with polynomial kernel

$$\mathcal{A}^n(f)(x_0;x) = \int_{x_0}^{x} K_n(x,t)f(t)\,dt, \quad K_n(x,t) = \sum_{k=0}^{n} r_{n,k}\,x^{n-k}(x-t)^{n+k}.$$

Fractional generalization: For $\alpha > 0$, not necessarily an integer, define

$$\mathcal{A}^\alpha(f)(x_0;x) := \int_{x_0}^{x} K_\alpha(x,t)f(t)\,dt, \quad K_\alpha(x,t) := \sum_{k=0}^{\infty} r_{\alpha,k}\,x^{\alpha-k}(x-t)^{\alpha+k},$$

with

$$r_{\alpha,k} := \frac{(-1)^k}{2^k k!\Gamma(\alpha-k+1)}.$$

This series generalizes the finite sum for integer $\alpha = n$, since $\Gamma(\alpha-k+1)$ has poles at non-positive integers, so the series truncates automatically when $\alpha \in \mathbb{N}$.

Uniform convergence of the kernel series: For $x \geq x_0 \geq 0$ and $t \in [x_0, x]$, we have $0 \leq x-t \leq x-x_0$. Hence,

$$\left|r_{\alpha,k}\,x^{\alpha-k}(x-t)^{\alpha+k}\right| \leq |r_{\alpha,k}|\,x^\alpha(x-x_0)^\alpha\left(\frac{x-x_0}{x}\right)^k.$$

Choose an integer $N > \alpha$ (e.g., $N = \lfloor\alpha\rfloor + 1$). Using the reflection formula for the Gamma function for $k \geq N$,

$$\Gamma(\alpha-k+1) = \frac{\pi(-1)^k}{\sin(\pi\alpha)\,\Gamma(k-\alpha)}, \quad \Rightarrow \quad |r_{\alpha,k}| = \frac{|\sin(\pi\alpha)|}{\pi}\frac{\Gamma(k-\alpha)}{2^k k!} \leq \frac{|\sin(\pi\alpha)|}{\pi}\frac{1}{2^k k}.$$

Define

$$M_k := \begin{cases} |r_{\alpha,k}|\,x^{\alpha-k}(x-x_0)^{\alpha+k}, & 0 \leq k < N, \\ \dfrac{|\sin(\pi\alpha)|}{\pi}\,x^\alpha(x-x_0)^\alpha\dfrac{1}{2^k k}\left(\dfrac{x-x_0}{x}\right)^k, & k \geq N. \end{cases}$$

Then $\sum_{k=0}^{\infty} M_k < \infty$ and $|r_{\alpha,k}x^{\alpha-k}(x-t)^{\alpha+k}| \leq M_k$ for all $k \geq 0$. By the Weierstrass M-test, $K_\alpha(x,t)$ converges uniformly on $[x_0, x]$.

Interchange of summation and integration: Since $f \in C[x_0, x]$ is bounded, $\|f\|_\infty < \infty$, and

$$|r_{\alpha,k}x^{\alpha-k}(x-t)^{\alpha+k}f(t)| \leq \|f\|_\infty M_k,$$

the series $\sum_{k=0}^{\infty} r_{\alpha,k}x^{\alpha-k}(x-t)^{\alpha+k}f(t)$ converges uniformly. Therefore, summation and integration can be interchanged:

$$\mathcal{A}^\alpha(f)(x_0;x) = \sum_{k=0}^{\infty} r_{\alpha,k}\,x^{\alpha-k}\int_{x_0}^{x}(x-t)^{\alpha+k}f(t)\,dt. \tag{10}$$

Example 6 (Two applications)**.**

1. Application to power functions: Let $f(t) = t^\beta$, $\beta > -1$. Then

$$\begin{aligned}\mathcal{A}^\alpha(t^\beta)(0;x) &= \sum_{k=0}^{\infty} r_{\alpha,k}\ x^{\alpha-k}\int_0^x (x-t)^{\alpha+k} t^\beta\, dt \\ &= \sum_{k=0}^{\infty} r_{\alpha,k}\ x^{2\alpha+\beta+1}\int_0^1 (1-u)^{\alpha+k} u^\beta\, du \quad (u = t/x) \\ &= x^{2\alpha+\beta+1}\sum_{k=0}^{\infty} r_{\alpha,k}\ B(\beta+1, \alpha+k+1) \\ &= x^{2\alpha+\beta+1}\sum_{k=0}^{\infty}(-1)^k a(\alpha,k)\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+k+2)},\end{aligned}$$

where $B(.,.)$ is the Euler Beta function and $a(\alpha,k) = \Gamma(\alpha+k+1)(-1)^k r_{\alpha,k}$.

Specialization $x = 1$

$$\mathcal{A}^\alpha(t^\beta)(0;1) = \sum_{k=0}^{\infty}(-1)^k a(\alpha,k)\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+k+2)}. \tag{11}$$

If $\alpha = n-1$ and $\beta = m-1$ are positive integers, then

$$\mathcal{A}^{n-1}(t^{m-1})(0;1) = \sum_{k=0}^{n-1}\frac{(-1)^k a(n-1,k)}{(m)_{n+k}}.$$

Direct computation also gives

$$\mathcal{A}^{n-1}(t^{m-1})(0;1) = \int_0^1 x_{n-1}\int_0^{x_{n-1}}\cdots x_1\int_0^{x_1} t^{m-1}\, dt\, dx_1\cdots dx_{n-1} = \frac{1}{\prod_{i=0}^{n-1}(m+2i)}.$$

Equating the two expressions, we obtain the identity

$$\frac{1}{\prod_{i=0}^{n-1}(m+2i)} = \sum_{k=0}^{n-1}\frac{(-1)^k a(n-1,k)}{(m)_{n+k}}. \tag{12}$$

2. Application to the exponential function: Let $f(t) = e^t$. We organize the computation in steps:

Change of variables

$$\begin{aligned}\mathcal{A}^{\alpha}(e^t)(0;x) &= \sum_{k=0}^{\infty} r_{\alpha,k}\ x^{\alpha-k}\int_0^x (x-t)^{\alpha+k}e^t\,dt\\ &= e^x\sum_{k=0}^{\infty} r_{\alpha,k}\ x^{\alpha-k}\int_0^x u^{\alpha+k}e^{-u}\,du, \quad (u=x-t)\\ &= e^x\sum_{k=0}^{\infty} r_{\alpha,k}\ x^{\alpha-k}\,\gamma(\alpha+k+1,x),\end{aligned}$$

where $\gamma(.,.)$ is the lower incomplete Gamma function.

Nonnegative Integer $\alpha = n$

$$\mathcal{A}^{n}(e^t)(0;x) = \sum_{k=0}^{\infty}\frac{\mathcal{A}^{n}(t^k)(0;x)}{k!} = \sum_{k=0}^{\infty}\frac{x^{2n+k+1}}{k!\prod_{i=1}^{n+1}(k+2i-1)}.$$

Hence

$$\sum_{k=0}^{\infty}\frac{x^{k}}{k!\prod_{i=1}^{n+1}(k+2i-1)} = \frac{e^x}{x^{n+1}}\sum_{k=0}^{n} r_{n,k}\,x^{-k}\,\gamma(n+k+1,x) \tag{13}$$

$$= \frac{e^x}{x^{n+1}}\sum_{k=0}^{n}(-1)^k a(n,k)\,x^{-k}\left(1-e^{-x}\sum_{j=0}^{n+k}\frac{x^j}{j!}\right), \tag{14}$$

Remark: This expression can be regarded as the exponential generating function of the sequence

$$a_{k,n} = \frac{1}{\prod_{i=1}^{n}(k+2i-1)}.$$

It also addresses, in a sense, a problem we raised on MathOverflow — see [6].

New Expression for $a(n) =$ A000806

$$\sum_{k=0}^{\infty}\frac{x^{2n+k+1}}{k!\prod_{i=1}^{n+1}(k+2i-1)} = e^x\sum_{k=0}^{n}(-1)^k a(n,k)x^{n-k} - \sum_{j=0}^{2n}\frac{x^j}{j!}\sum_{k=\max(0,j-n)}^{n}(-1)^k a(n,k)x^{n-k}.$$

Specialization $x = 1$

$$a(n) = \sum_{k=0}^{n}(-1)^k a(n,k) = \frac{1}{e}\left(\sum_{k=0}^{\infty}\frac{1}{k!\prod_{i=1}^{n+1}(k+2i-1)} + \sum_{j=0}^{2n}\frac{1}{j!}\sum_{k=\max(0,j-n)}^{n}(-1)^k a(n,k)\right).$$

Relation to Lemma 2. Recall from Lemma 2 that $a(n,k)$ satisfies the recursive relation

$$a(n,k) = \sum_{i=0}^{\min(k,n-1)} (n-i)_{k-i}\, a(n-1,i).$$

By iterating this recurrence n times, starting from the initial condition $a(0,k) = \delta_k$ (where δ_k is the Kronecker delta, equal to 1 if $k = 0$ and 0 otherwise), we obtain a nested summation over non-decreasing indices with certain constraints. This unfolding naturally leads to the combinatorial formula given below, expressed as a sum over integer compositions of k.

Theorem 7 (Combinatorial Formula)**.** *For all $n \in \mathbb{N}^*$ and $1 \le k \le n$,*

$$a(n,k) = \sum_{\substack{m_1+\cdots+m_n=k \\ m_1,\ldots,m_n \ge 0}} \prod_{j=1}^{n} \left(j - \sum_{r=1}^{j-1} m_r \right)_{m_j}. \tag{15}$$

Proof. Starting from the recurrence given in Lemma 2,

$$a(n,k) = \sum_{i=0}^{\min(n-1,k)} (n-i)_{k-i}\, a(n-1,i),$$

we iterate this relation recursively to express $a(n,k)$ as a nested sum over indices $i_1 \le i_2 \le \cdots \le i_n$ with constraints $i_j \le j-1$:

$$a(n,k) = \sum_{\substack{0 \le i_1 \le i_2 \le \cdots \le i_n \le k \\ i_j \le j-1}} \left(\prod_{j=1}^{n} (j-i_j)_{i_{j+1}-i_j} \right) a(0,i_1),$$

where we set $i_{n+1} := k$.

Since $i_1 \le 0$, it follows that $a(0,i_1) = a(0,0) = 1$. This simplifies the expression to

$$a(n,k) = \sum_{\substack{0 = i_1 \le i_2 \le \cdots \le i_n \le k \\ i_j \le j-1}} \prod_{j=1}^{n} (j-i_j)_{i_{j+1}-i_j}. \tag{16}$$

Now define the increments (step sizes)

$$m_j := i_{j+1} - i_j, \quad j = 1,\ldots,n.$$

Because $i_1 = 0$ and $i_{n+1} = k$, these satisfy

$$\sum_{j=1}^{n} m_j = k, \qquad \text{and} \qquad i_j = \sum_{r=1}^{j-1} m_r.$$

Substituting back into the product yields

$$(j-i_j)_{m_j} = \left(j - \sum_{r=1}^{j-1} m_r\right)_{m_j},$$

and hence

$$a(n,k) = \sum_{\substack{m_1+\cdots+m_n=k\\ m_j\geq 0}} \prod_{j=1}^{n} \left(j - \sum_{r=1}^{j-1} m_r\right)_{m_j},$$

which is exactly (15). □

Remark 8. The recurrence for $a(n,k)$ can also be expressed in terms of an operator. Define

$$\Omega(n) = \sum_{i=0}^{\min(n-1,k)} (n-i)_{k-i}\, B^{k-i}, \qquad \text{where } B[f(k)] = f(k-1).$$

Then

$$a(n,k) = \Omega(n)\, a(n-1,k), \tag{17}$$

and, by iteration,

$$a(n,k) = \Omega(n)\,\Omega(n-1)\cdots\Omega(1)\, a(0,k). \tag{18}$$

Expanding the product of operators gives

$$\Omega(n)\cdots\Omega(1) = \sum_{\substack{0\leq i_1\leq\cdots\leq i_n\leq k\\ i_j\leq j-1}} \left(\prod_{j=1}^{n}(j-i_j)_{i_{j+1}-i_j}\right) B^{k-i_1},$$

with $i_{n+1} := k$. Applying this to the initial condition $a(0,k) = \delta_k$ recovers Formula (16).

Example 9. Computation of $a(3,2)$ using (15).

Here, $n=3$, $k=2$, and increments satisfy

$$m_1+m_2+m_3 = 2.$$

By the formula,

$$a(3,2) = \sum_{m_1+m_2+m_3=2} (1)_{m_1}\cdot(2-m_1)_{m_2}\cdot(3-m_1-m_2)_{m_3}.$$

Enumerating all solutions:

m_1	m_2	m_3	Terms	Contribution
2	0	0	$(1)_2 \cdot (0)_0 \cdot (1)_0$	0
1	1	0	$(1)_1 \cdot (1)_1 \cdot (1)_0$	1
1	0	1	$(1)_1 \cdot (1)_0 \cdot (2)_1$	2
0	2	0	$(1)_0 \cdot (2)_2 \cdot (1)_0$	2
0	1	1	$(1)_0 \cdot (2)_1 \cdot (2)_1$	4
0	0	2	$(1)_0 \cdot (2)_0 \cdot (3)_2$	6
Total				15

Hence, $a(3,2) = 15$, matching the known value.

Remark 10 (Combinatorial Interpretation). The Bessel number $a(n,k)$ counts the number of ways to sequentially place k labeled elements into n ordered steps. At each step j, a nonnegative number m_j of elements is placed, with $\sum_{j=1}^{n} m_j = k$. The term

$$\left(j - \sum_{r=1}^{j-1} m_r\right)_{m_j}$$

enumerates the ways to arrange these m_j elements at step j, considering the elements already placed in previous steps. The product over all steps accumulates the total number of valid placement sequences for the fixed distribution $(m_1, \ldots, m_n)$, and summing over all such distributions accounts for all valid placements.

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[6] https://mathoverflow.net/questions/477205/properties-of-polynomials-defined-by-p-nx-e-x-sum-k-0-infty-frac

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