Residue classes and stopping time of the 3n+1 problem

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Abstract

This paper presents an analysis of the stopping time of the 3n+1 problem based on the residue class of n.

3n + 1 problem (or conjecture)

In the $3\cdot n+1$ problem[1] it is possible to define the function $s: N \rightarrow N$:

$$s(n)=\begin{cases} 3\cdot n+1 & if\ n\equiv 1 (mod\, 2) \\ \frac{n}{2} & if\ n\equiv 0 (mod\, 2) \end{cases}$$

the sequence $s^k(n)$ for $k\in N$ obtained using the function $s(n)$ is as follows:

$$s^k(n)=\begin{cases} n & for\ k=0 \\ s(s^{k-1}(n)) & for\ k>0 \end{cases}$$

Stopping time

The $3\cdot n+1$ conjecture is equivalent to the conjecture that for each $n\in N$, $n>1$, there exists $k\in N$ such that $s^k(n)<n$. The least $k\in N$ such that $s^k(n)<n$ is called the stopping time of n [1] .

If n is even $n\equiv 0 (mod\, 2)$ then the stopping time $k=1$ and $s^1(n)=\frac{n}{2}$.

Let's analyze the case where n is odd.

Let m be the number of odd terms in the first k terms of the $3\cdot n+1$ sequence, and d_i be the number of consecutive even terms immediately following the *i*-th odd term, then the next term $s^k(n)$ in he $3\cdot n+1$ sequence is[1] :

$$s^k(n)=\frac{3^m}{2^{k-m}}\cdot n+\sum_{i=1}^{m}\frac{3^{m-i}}{2^{d_i+\cdots+d_m}}$$

note that for n odd $k-m=d_1+\cdots+d_m$.

Then

$$s^k(n)=\frac{3^m\cdot n+3^{m-1}+\sum_{i=2}^{m}3^{m-i}\cdot 2^{d_1+\cdots+d_{i-1}}}{2^{k-m}}=\frac{3^m\cdot n+r}{2^{k-m}}$$

where r depends on m and n with $r\geq 3^m-2^m$.

It is possible to observe that if $n \equiv 1 (mod\, 2^2)$ then $m=1$ and the stopping time is $k=3$

$$s^3(n)=\frac{3\cdot n+1}{2^2}$$

so we have $r=1=3^1-2^1$ and $d_1=2$.

If we now consider the numbers $n \equiv 3 (mod\, 2^2)$ we have:

$$n=3+4\cdot a$$

$$s^1(n)=3\cdot n+1=10+12\cdot a$$

$$s^2(n)=\frac{3\cdot n+1}{2}=5+6\cdot a=2+2\cdot a+n$$ odd number

$$s^4(n)=\frac{3\cdot\frac{3\cdot n+1}{2}+1}{2}=2+a+2n$$

if a odd then $n \equiv 7 (mod\, 2^3)$, $d_1=1$, $d_2=1$ and $m>2$ since $s^2(n), s^4(n)$ odd and $s^4(n)>n$

if a even then $a=2\cdot b$

$$s^5(n)=\frac{3\cdot\frac{3\cdot n+1}{2}+1}{4}=1+b+n$$

if b odd then $n \equiv 11 (mod\, 2^4)$, $d_1=1$, $d_2=2$ and $m>2$ since $s^2(n), s^5(n)$ odd and $s^5(n)>n$

if b even $b=2\cdot c$ then $n \equiv 3 (mod\, 2^4)$

$$s^6(n)=\frac{3\cdot\frac{3\cdot n+1}{2}+1}{8}=c+\frac{(n+1)}{2}=\frac{3^2\cdot n+5}{2^4}$$

then $k=6$, $d_1=1$, $d_2=3$ and $m=2$ with $r=5=3+2^1=3^2-2^2$.

As seen for numbers $n \equiv 7 (mod\, 2^3)$

$$s^4(n)=\frac{3\cdot\frac{3\cdot n+1}{2}+1}{2}=\frac{9\cdot n+5}{4}$$ odd number and $d_1=1$ and $d_2=1$

$$s^6(n)=\frac{3\cdot\frac{9\cdot n+5}{4}+1}{2}=\frac{27\cdot n+19}{8}$$

if $n \equiv 7 \left(mod\, 2^4\right)$ then $s^6(n)$ even and

$$s^7(n)=\frac{27\cdot n+19}{16}$$

if $n \equiv 23 \left(mod\, 2^5\right)$ then $s^7(n)$ even and

$$s^8(n)=\frac{27\cdot n+19}{32}<n$$

then $k=8$, $d_1=1$, $d_2=1$, $d_3=3$ and $m=3$ from which

$$r=19=3^3-2^3=3^2+3\cdot 2^1+2^{(1+1)}=3\cdot\left(3+2^1\right)+2^{(1+1)}=3\cdot 5+2^2 \quad .$$

Note that the numbers $n \equiv 15 \left(mod\, 2^4\right)$ and $n \equiv 7 \left(mod\, 2^5\right)$ remain to be analyzed.

For numbers $n \equiv 11 \left(mod\, 2^4\right)$

$$s^5(n)=\frac{3\cdot\frac{3\cdot n+1}{2}+1}{4}=\frac{9\cdot n+5}{8}$$ odd number and $d_1=1$ and $d_2=2$

$$s^7(n)=\frac{3\cdot\frac{9\cdot n+5}{8}+1}{2}=\frac{27\cdot n+23}{16}$$

if $n \equiv 11 \left(mod\, 2^5\right)$ then $s^7(n)$ even and

$$s^8(n)=\frac{3\cdot\frac{9\cdot n+5}{8}+1}{2}=\frac{27\cdot n+23}{32}<n$$

then $k=8$, $d_1=1$, $d_2=2$, $d_3=2$ and $m=3$ from which

$$r=23=3^2+3\cdot 2^1+2^{(1+2)}=3\cdot\left(3+2^1\right)+2^{(1+2)}=3\cdot 5+2^3 \quad .$$

For numbers $n \equiv 27 \left(mod\, 2^5\right)$ then $s^7(n)$ odd and $d_3=1$

$$s^7(n)=\frac{3\cdot\frac{9\cdot n+5}{8}+1}{2}=\frac{27\cdot n+23}{16}$$

$$s^9(n)=\frac{3\cdot\frac{27\cdot n+23}{16}+1}{2}=\frac{81\cdot n+85}{32}$$

if $n \equiv 59 \left(mod\, 2^6\right)$ then $s^9(n)$ even

$$s^{10}(n)=\frac{81\cdot n+85}{64}$$

if $n \equiv 59 \left(mod\, 2^7\right)$ then $s^{10}(n)$ even

$$s^{11}(n)=\frac{81\cdot n+85}{128}<n$$

then $k=11$, $d_1=1$, $d_2=2$, $d_3=1$, $d_4=3$ and $m=4$ from which

$$r=85=3^3+3^2\cdot 2^1+3\cdot 2^{(1+2)}+2^{(1+2+1)}=3\cdot(3^2+3\cdot 2^1+2^{(1+2)})+2^{(1+2+1)}+2^{(1+2)}=3\cdot 23+2^4 \quad .$$

Note that the numbers $n\equiv 27(mod\,2^6)$ and $n\equiv 123(mod\,2^7)$ remain to be analyzed.

For numbers $n\equiv 15(mod\,2^4)$

$$s^6(n)=\frac{3\cdot\frac{9\cdot n+5}{4}+1}{2}=\frac{27\cdot n+19}{8}$$ odd and $d_3=1$

$$s^8(n)=\frac{3\cdot\frac{27\cdot n+19}{8}+1}{2}=\frac{81\cdot n+65}{16}$$

if $n\equiv 15(mod\,2^5)$ then $s^8(n)$ even and

$$s^9(n)=\frac{81\cdot n+65}{32}$$

if $n\equiv 15(mod\,2^6)$ then $s^9(n)$ even and

$$s^{10}(n)=\frac{81\cdot n+65}{64}$$

if $n\equiv 15(mod\,2^7)$ then $s^{10}(n)$ even and

$$s^{11}(n)=\frac{81\cdot n+65}{128}<n$$

then $k=11$, $d_1=1$, $d_2=1$, $d_3=1$, $d_4=4$ and $m=4$ from which

$$r=65=3^4-2^4=3^3+3^2\cdot 2^1+3\cdot 2^{(1+1)}+2^{(1+1+1)}=3\cdot(3^2+3\cdot 2^1+2^{(1+1)})+2^{(1+1+1)}=3\cdot 19+2^3 \quad .$$

For numbers $n\equiv 7(mod\,2^5)$

$$s^7(n)=\frac{27\cdot n+19}{16}$$ odd and $d_3=2$

$$s^9(n)=\frac{3\cdot\frac{27\cdot n+19}{16}+1}{2}=\frac{81\cdot n+73}{32}$$

if $n\equiv 7(mod\,2^6)$ then $s^9(n)$ even and

$$s^{10}(n)=\frac{81\cdot n+73}{64}<n$$

if $n \equiv 7 \left(mod\, 2^7 \right)$ then $s^{10}(n)$ even and

$$s^{11}(n) = \frac{81 \cdot n + 73}{128} < n$$

then $k = 11$, $d_1 = 1$, $d_2 = 1$, $d_3 = 2$, $d_4 = 3$ and $m = 4$ from which

$$r = 73 = 3^3 + 3^2 \cdot 2^1 + 3 \cdot 2^{(1+1)} + 2^{(1+1+2)} = 3 \cdot \left(3^2 + 3 \cdot 2^1 + 2^{(1+1)} \right) + 2^{(1+1+2)} = 3 \cdot 19 + 2^4 \quad .$$

Note that the numbers $n \equiv 39 \left(mod\, 2^6 \right)$, $n \equiv 71 \left(mod\, 2^7 \right)$, $n \equiv 31 \left(mod\, 2^5 \right)$, $n \equiv 47 \left(mod\, 2^6 \right)$ and $n \equiv 79 \left(mod\, 2^7 \right)$ remain to be analyzed.

By continuing with this procedure it is easy to verify the following results:

m	k	$n\left(mod\, 2^{k-m}\right)$	d_m	r
1	3	1	2	1
2	6	3	3	$5 = 3 + 2^1$
3	8	11	2	$23 = 3^2 + 3 \cdot 2^1 + 2^{(1+2)}$
		23	3	$19 = 3^2 + 3 \cdot 2^1 + 2^{(1+1)}$
4	11	7	3	$73 = 3^3 + 3^2 \cdot 2^1 + 3 \cdot 2^{(1+1)} + 2^{(1+1+2)}$
		15	4	$65 = 3^3 + 3^2 \cdot 2^1 + 3 \cdot 2^{(1+1)} + 2^{(1+1+1)}$
		59	3	$85 = 3^3 + 3^2 \cdot 2^1 + 3 \cdot 2^{(1+2)} + 2^{(1+2+1)}$
5	13	39	3	$251 = 3^4 + 3^3 \cdot 2^1 + 3^2 \cdot 2^{(1+1)} + 3 \cdot 2^{(1+1+2)} + 2^{(1+1+2+1)}$
		79	2	$259 = 3^4 + 3^3 \cdot 2^1 + 3^2 \cdot 2^{(1+1)} + 3 \cdot 2^{(1+1+1)} + 2^{(1+1+1+3)}$
		95	4	$211 = 3^4 + 3^3 \cdot 2^1 + 3^2 \cdot 2^{(1+1)} + 3 \cdot 2^{(1+1+1)} + 2^{(1+1+1+1)}$
		123	2	$319 = 3^4 + 3^3 \cdot 2^1 + 3^2 \cdot 2^{(1+2)} + 3 \cdot 2^{(1+2+1)} + 2^{(1+2+1+2)}$
		175	3	$227 = 3^4 + 3^3 \cdot 2^1 + 3^2 \cdot 2^{(1+1)} + 3 \cdot 2^{(1+1+1)} + 2^{(1+1+1+2)}$
		199	2	$283 = 3^4 + 3^3 \cdot 2^1 + 3^2 \cdot 2^{(1+1)} + 3 \cdot 2^{(1+1+2)} + 2^{(1+1+2+2)}$
		219	3	$287 = 3^4 + 3^3 \cdot 2^1 + 3^2 \cdot 2^{(1+2)} + 3 \cdot 2^{(1+2+1)} + 2^{(1+2+1+1)}$
6	16	287	4	$697 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+1)} + 3^2 \cdot 2^{(1+1+1)} + 3 \cdot 2^{(1+1+1+1)} + 2^{(1+1+1+1+2)}$
		347	3	$989 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+2)} + 3^2 \cdot 2^{(1+2+1)} + 3 \cdot 2^{(1+2+1+1)} + 2^{(1+2+1+1+2)}$
		367	4	$745 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+1)} + 3^2 \cdot 2^{(1+1+1)} + 3 \cdot 2^{(1+1+1+2)} + 2^{(1+1+1+2+1)}$
		423	3	$881 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+1)} + 3^2 \cdot 2^{(1+1+2)} + 3 \cdot 2^{(1+1+2+1)} + 2^{(1+1+2+1+2)}$
		507	3	$1085 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+2)} + 3^2 \cdot 2^{(1+2+1)} + 3 \cdot 2^{(1+2+1+2)} + 2^{(1+2+1+2+1)}$
		575	5	$665 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+1)} + 3^2 \cdot 2^{(1+1+1)} + 3 \cdot 2^{(1+1+1+1)} + 2^{(1+1+1+1+1)}$
		583	3	$977 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+1)} + 3^2 \cdot 2^{(1+1+2)} + 3 \cdot 2^{(1+1+2+2)} + 2^{(1+1+2+2+1)}$
		735	3	$761 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+1)} + 3^2 \cdot 2^{(1+1+1)} + 3 \cdot 2^{(1+1+1+1)} + 2^{(1+1+1+1+3)}$
		815	3	$809 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+1)} + 3^2 \cdot 2^{(1+1+1)} + 3 \cdot 2^{(1+1+1+2)} + 2^{(1+1+1+2+2)}$
		923	4	$925 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+2)} + 3^2 \cdot 2^{(1+2+1)} + 3 \cdot 2^{(1+2+1+1)} + 2^{(1+2+1+1+1)}$
		975	3	$905 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+1)} + 3^2 \cdot 2^{(1+1+1)} + 3 \cdot 2^{(1+1+1+3)} + 2^{(1+1+1+3+1)}$
		999	4	$817 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+1)} + 3^2 \cdot 2^{(1+1+2)} + 3 \cdot 2^{(1+1+2+1)} + 2^{(1+1+2+1+1)}$

As seen $d_m=k-m-(d_1+\cdots+d_{m-1})$ and if for a certain value of m if we find a value of r , which we indicate as $r^i_m=3^{m-1}+\sum_{i=2}^{m} 3^{m-i}\cdot 2^{d^i_1+\cdots+d^i_{i-1}}$ depends on the values of $d^i_1,\cdots,d^i_{m-1}$, then from this value of r^i_m for $m+1$ we can obtain $r^j_{m+1}=3\cdot r^i_m+2^{(d_1+\cdots+d_{m-1}+d^j_m)}$ with $1\le d^j_m<d^i_m$ and $x^j_{m+1}\cdot 3^{(m+1)}\equiv -r^j_{m+1}(mod\, 2^{d_1+\cdots d_{m-1}+d^j_m+d^j_{m+1}})$ from which we can obtain x^j_{m+1} and for $n\equiv x^j_{m+1}(mod\, 2^{d_1+\cdots+d_{m-1}+d^j_m+d^j_{m+1}})$ we have $s^{m+1+d_1+\cdots+d_{m-1}+d^j_m+d^j_{m+1}}(n)<n$.

It can be observed that in all the cases examined $k-m=\lfloor 1+m\cdot\log_2(3)\rfloor$.

If $n\equiv x(mod\, 2^{d_1+\cdots+d_{m-1}+d_m})$ with x odd, $m>1$, $s^i(n)>n$ for $1\le i<m+d_1+\cdots+d_{m-1}+d_m$

and $s^{m+d_1+\cdots+d_{m-1}+d_m}(n)=\dfrac{3^m\cdot n+3^{m-1}+\sum_{i=2}^{m} 3^{m-i}\cdot 2^{d_1+\cdots+d_{i-1}}}{2^{d_1+\cdots+d_{m-1}+d_m}}<n$ then if $d_m>1$ for $1\le d'_m<d_m$

$$s^{m+d_1+\cdots+d_{m-1}+d'_m}(x)=\frac{3^m\cdot x+3^{m-1}+\sum_{i=2}^{m} 3^{m-i}\cdot 2^{d_1+\cdots+d_{i-1}}}{2^{d_1+\cdots+d_{m-1}+d'_m}}\quad \text{even}$$

we indicate as $r_m=3^{m-1}+\sum_{i=2}^{m} 3^{m-i}\cdot 2^{d_1+\cdots+d_{i-1}}$

and if $x=x'+b\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}$ with $x'<2^{d_1+\cdots+d_{m-1}+d'_m}$

$$s^{m+d_1+\cdots+d_{m-1}+d'_m}(x)=\frac{3^m\cdot x+3^{m-1}+\sum_{i=2}^{m} 3^{m-i}\cdot 2^{d_1+\cdots+d_{i-1}}}{2^{d_1+\cdots+d_{m-1}+d'_m}}=\frac{3^m\cdot x'+3^{m-1}+\sum_{i=2}^{m} 3^{m-i}\cdot 2^{d_1+\cdots+d_{i-1}}}{2^{d_1+\cdots+d_{m-1}+d'_m}}+b\cdot 3^m\quad \text{even}$$

for $y=x+(1+2\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}=x'+b\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}+(1+2\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}$

$$s^{m+d_1+\cdots+d_{m-1}+d'_m}(y)=\frac{3^m\cdot x+3^{m-1}+\sum_{i=2}^{m} 3^{m-i}\cdot 2^{d_1+\cdots+d_{i-1}}}{2^{d_1+\cdots+d_{m-1}+d'_m}}+(1+2\cdot a)\cdot 3^m=s^{m+d_1+\cdots+d_{m-1}+d'_m}(x)+(1+2\cdot a)\cdot 3^m\quad \text{odd}$$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(y)=\frac{3\cdot\dfrac{3^m\cdot y+3^{m-1}+\sum_{i=2}^{m} 3^{m-i}\cdot 2^{d_1+\cdots+d_{i-1}}}{2^{d_1+\cdots+d_{m-1}+d'_m}}+1}{2}=\frac{3^{m+1}\cdot y+3^m+2^{d_1+\cdots+d_{m-1}+d'_m}+\sum_{i=2}^{m} 3^{m+1-i}\cdot 2^{d_1+\cdots+d_{i-1}}}{2^{d_1+\cdots+d_{m-1}+d'_m+1}}$$

we obtain $r_{m+1}=3\cdot r_m+2^{(d_1+\cdots+d'_m)}=3^m+\sum_{i=2}^{m} 3^{m-i}\cdot 2^{d_1+\cdots+d_{i-1}}+2^{d_1+\cdots+d_{m-1}+d'_m}$ with $1\le d'_m<d_m$.

If $d'_m = d_m - 1$

if $s^{m+d_1+\cdots+d_{m-1}+d_m}(n) < n$ then $3^m < 2^{d_1+\cdots+d_{m-1}+d_m}$ and $\frac{3^{m+1}}{4} < 3^m < 2^{d_1+\cdots+d_{m-1}+d_m}$ then

$\frac{3^{m+1}}{8} < 3^{move} 2 < 2^{d_1+\cdots+d_{m-1}+d_m-1}$ if $\frac{3^m}{2^{d_1+\cdots+d_{m-1}+d_m}} < \frac{2}{3}$ then $\frac{3^{m+1}}{4} < 2^{d_1+\cdots+d_{m-1}+d_m-1} = 2^{d_1+\cdots+d_{m-1}+d'_m}$

$x = x' + a \cdot 2^{d_1+\cdots+d_{m-1}+d'_m}$ with $x' < 2^{d_1+\cdots+d_{m-1}+d'_m}$ and $a=0$ or $a=1$ then

$((x + 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\, 2^{d_1+\cdots+d_{m-1}+d'_m+1})) = x' + (1-a) \cdot 2^{d_1+\cdots+d_{m-1}+d'_m}$.

Case 1_I

$$s^{m+d_1+\cdots+d_{m-1}+d'_m}(x + 2^{d_1+\cdots+d_{m-1}+d'_m}) = \frac{3^m \cdot x + 3^{m-1} + \sum_{i=2}^{m} 3^{m-i} \cdot 2^{d_1+\cdots+d_{i-1}}}{2^{d_1+\cdots+d_{m-1}+d'_m}} + 3^m = 2 \cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x) + 3^m \text{ odd}$$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(x + 2^{d_1+\cdots+d_{m-1}+d'_m}) = \frac{3 \cdot (2 \cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x) + 3^m) + 1}{2} = 3 \cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x) + \frac{3^{m+1}+1}{2}$$

$s^{m+d_1+\cdots+d_{m-1}+d_m}(x)$ even and $m \equiv 0 (mod\, 2)$

$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(x + 2^{d_1+\cdots+d_{m-1}+d'_m})$ even

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x + 2^{d_1+\cdots+d_{m-1}+d'_m}) = \frac{3 \cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)}{2} + \frac{3^{m+1}+1}{4}$$

if $\frac{3^m}{2^{d_1+\cdots+d_{m-1}+d_m}} < \frac{2}{3}$ then $s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x + 2^{d_1+\cdots+d_{m-1}+d'_m}) < x + 2^{d_1+\cdots+d_{m-1}+d'_m}$ and $d'_{m+1} \geq 2$

else if $\frac{s^{m+d_1+\cdots+d_{m-1}+d_m}(x)}{2}$ odd then $s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x + 2^{d_1+\cdots+d_{m-1}+d'_m})$ even

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+3}(x + 2^{d_1+\cdots+d_{m-1}+d'_m}) = \frac{3 \cdot 2 \cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x) + 3^{m+1} + 1}{8}$$

$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+3}(x + 2^{d_1+\cdots+d_{m-1}+d'_m}) < x + 2^{d_1+\cdots+d_{m-1}+d'_m}$ and $d'_{m+1} \geq 3$

$y = (x + 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\, 2^{d_1+\cdots+d_{m-1}+d'_m+1})$

Case 1_II

$s^{m+d_1+\cdots+d_{m-1}+d_m}(x)$ even, $\frac{s^{m+d_1+\cdots+d_{m-1}+d_m}(x)}{2}$ even and $m \equiv 0 (mod\, 2)$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(x + 5 \cdot 2^{d_1+\cdots+d_{m-1}+d'_m}) = 3 \cdot (s^{m+d_1+\cdots+d_{m-1}+d_m}(x) + 2 \cdot 3^m) + \frac{3^{m+1}+1}{2} \text{ even}$$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x+5\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(\frac{s^{m+d_1+\cdots+d_{m-1}+d_m}(x)}{2}+3^m)+\frac{3^{m+1}+1}{4}$$ even

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+3}(x+5\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=\frac{3\cdot 2\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+5\cdot 3^{m+1}+1}{8}<x+5\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}$$

$d'_{m+1}\geq 3$ and $y=((x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\, 2^{d_1+\cdots+d_{m-1}+d'_m+1})+4\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\, 2^{d_1+\cdots+d_{m-1}+d'_m+3})$;

Case 1_III

$s^{m+d_1+\cdots+d_{m-1}+d_m}(x)$ odd, and $m\equiv 0(mod\, 2)$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(x+3\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3^m)+\frac{3^{m+1}+1}{2}$$ even

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x+3\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=\frac{3\cdot(s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3^m)}{2}+\frac{3^{m+1}+1}{4}$$

if $\frac{3^m}{2^{d_1+\cdots+d_{m-1}+d_m}}<\frac{2}{3}$ then a $d'_{m+1}\geq 2$ nd

$y=((x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\, 2^{d_1+\cdots+d_{m-1}+d'_m+1})+2\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\, 2^{d_1+\cdots+d_{m-1}+d'_m+2})$

else if $\frac{s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3^m}{2}$ odd then $s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x+3\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})$ even

$d'_{m+1}\geq 3$ and $y=((x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\, 2^{d_1+\cdots+d_{m-1}+d'_m+1})+2\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\, 2^{d_1+\cdots+d_{m-1}+d'_m+3})$;

Case 1_IV

$s^{m+d_1+\cdots+d_{m-1}+d_m}(x)$ odd, $\frac{s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3^m}{2}$ even and $m\equiv 0(mod\, 2)$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(x+7\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3\cdot 3^m)+\frac{3^{m+1}+1}{2}$$ even

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x+7\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(\frac{s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3^m}{2}+3^m)+\frac{3^{m+1}+1}{4}$$ even

$d'_{m+1}\geq 3$ and $y=((x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\, 2^{d_1+\cdots+d_{m-1}+d'_m+1})+6\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\, 2^{d_1+\cdots+d_{m-1}+d'_m+3})$;

Case 1_V

$s^{m+d_1+\cdots+d_{m-1}+d_m}(x)$ even and $m\equiv 1(mod\, 2)$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(x+3\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(s^{m+d_1+\cdots+d_{m-1}+d_m}(x))+\frac{3^{m+2}+1}{2}$$ even

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x+3\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(\frac{s^{m+d_1+\cdots+d_{m-1}+d_m}(x)}{2})+\frac{3^{m+2}+1}{4}$$

if $\frac{3^m}{2^{d_1+\cdots+d_{m-1}+d_m}}<\frac{2}{3}$ then a $d'_{m+1}\geq 2$ nd

$$y=((x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+1})+2\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+2})$$

else if $\frac{s^{m+d_1+\cdots+d_{m-1}+d_m}(x)}{2}$ odd then $s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x+3\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})$ even

$d'_{m+1}\geq 3$ and $y=((x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+1})+2\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+3})$;

Case 1_VI

$s^{m+d_1+\cdots+d_{m-1}+d_m}(x)$ even, $\frac{s^{m+d_1+\cdots+d_{m-1}+d_m}(x)}{2}$ even and $m\equiv 1(mod\,2)$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(x+7\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+2\cdot 3^m)+\frac{3^{m+2}+1}{2}\quad\text{even}$$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x+7\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(\frac{s^{m+d_1+\cdots+d_{m-1}+d_m}(x)}{2}+3^m)+\frac{3^{m+2}+1}{4}\quad\text{even}$$

$d'_{m+1}\geq 3$ and $y=((x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+1})+6\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+3})$;

Case 1_VII

$s^{m+d_1+\cdots+d_{m-1}+d_m}(x)$ odd, and $m\equiv 1(mod\,2)$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(x+5\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3^m)+\frac{3^{m+2}+1}{2}\quad\text{even}$$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(x+5\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=\frac{3\cdot(s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3^m)}{2}+\frac{3^{m+2}+1}{4}$$

if $\frac{3^m}{2^{d_1+\cdots+d_{m-1}+d_m}}<\frac{2}{3}$ then a $d'_{m+1}\geq 2$ nd

$$y=(x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+1})$$

else if $\frac{s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3^m}{2}$ odd then $s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x+5\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})$ even

$d'_{m+1}\geq 3$ and $y=((x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+1})+4\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+3})$;

Case 1_VIII

$$s^{m+d_1+\cdots+d_{m-1}+d'_m}\left(x+2^{d_1+\cdots+d_{m-1}+d'_m}\right)=\frac{3^m\cdot x+3^{m-1}+\sum_{i=2}^{m}3^{m-i}\cdot 2^{d_1+\cdots+d_{i-1}}}{2^{d_1+\cdots+d_{m-1}+d'_m}}+3^m=2\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3^m$$

$s^{m+d_1+\cdots+d_{m-1}+d_m}(x)$ odd, $\frac{s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3^m}{2}$ even and $m\equiv 1(mod\,2)$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}\left(x+2^{d_1+\cdots+d_{m-1}+d'_m}\right)=3\cdot\left(s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3^{m-1}\right)+\frac{3^m+1}{2}\quad \text{even}$$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}\left(x+2^{d_1+\cdots+d_{m-1}+d'_m}\right)=s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+\frac{s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3^m}{2}+\frac{3^m+1}{4}\quad \text{even}$$

$d'_{m+1}\geq 3$ and $y=\left(\left(x+2^{d_1+\cdots+d_{m-1}+d'_m}\right)\left(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+1}\right)\right)\left(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+2}\right)$;

in all cases examined for $n\equiv y\left(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+d'_{m+1}}\right)$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+d'_{m+1}}(n)=\frac{3^{m+1}\cdot n+3^m+2^{d_1+\cdots+d_{m-1}+d'_m}+\sum_{i=2}^{m}3^{m+1-i}\cdot 2^{d_1+\cdots+d_{i-1}}}{2^{d_1+\cdots+d_{m-1}+d'_m+d'_{m+1}}}<n\quad .$$

If $d_m-d'_m>1$ for $x+2^{d_1+\cdots+d_{m-1}+d'_m}+a\cdot 2^{d_1+\cdots+d_{m-1}+d'_m+1}$ and $b=1+2\cdot a$ with $a\geq 0$

$$s^{m+d_1+\cdots+d_{m-1}+d'_m}\left(x+b\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}\right)=\frac{3^m\cdot x+3^{m-1}+\sum_{i=2}^{m}3^{m-i}\cdot 2^{d_1+\cdots+d_{i-1}}}{2^{d_1+\cdots+d_{m-1}+d'_m}}+b\cdot 3^m=2^{(d_m-d'_m)}\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+b\cdot 3^m$$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}\left(x+b\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}\right)=3\cdot 2^{(d_m-d'_m-1)}\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+\frac{b\cdot 3^{m+1}+1}{2}\quad .$$

If $d_m-d'_m=2$

if $s^{m+d_1+\cdots+d_{m-1}+d_m}(n)<n$ then $3^m<2^{d_1+\cdots+d_{m-1}+d_m}$ and $\frac{3^{m+1}}{4}<3^m<2^{d_1+\cdots+d_{m-1}+d_m}$ then

$\frac{3^{m+1}}{16}<3^{move}\,4<2^{d_1+\cdots+d_{m-1}+d_m-2}$ if $\frac{3^m}{2^{d_1+\cdots+d_{m-1}+d_m}}<\frac{2}{3}$ then $\frac{3^{m+1}}{8}<2^{d_1+\cdots+d_{m-1}+d_m-2}=2^{d_1+\cdots+d_{m-1}+d'_m}$

$x=x'+a\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}$ with $x'<2^{d_1+\cdots+d_{m-1}+d'_m}$ and $0\leq a<4$ then

$$\left(\left(x+2^{d_1+\cdots+d_{m-1}+d'_m}\right)\left(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+1}\right)\right)=x'+\left(1-a\quad mod\quad 2\right)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}\quad .$$

Case 2_I

$s^{m+d_1+\cdots+d_{m-1}+d_m}(x)$ even and $m\equiv 0(mod\,2)$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(x+5\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(2\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+2\cdot 3^m)+\frac{3^{m+1}+1}{2}\quad \text{even}$$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x+5\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3^m)+\frac{3^{m+1}+1}{4}\quad \text{even}$$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+3}(x+5\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=\frac{3\cdot 4\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+5\cdot 3^{m+1}+1}{8}$$

if $\frac{3^m}{2^{d_1+\cdots+d_{m-1}+d_m}}<\frac{2}{3}$ then $d'_{m+1}\geq 3$ and

$$y=((x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+1})+4\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+3})$$

else if $s^{m+1+d_1+\cdots+d_{m-1}+d'_m+3}(x+5\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})$ even then $d'_{m+1}\geq 4$ and

$$y=((x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+1})+4\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+4})\quad ;$$

Case 2_II

$s^{m+d_1+\cdots+d_{m-1}+d_m}(x)$ even, $\frac{3\cdot 4\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+5\cdot 3^{m+1}+1}{8}$ odd and $m\equiv 0(mod\,2)$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(x+13\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(2\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+2\cdot 3^m)+\frac{3^{m+3}+1}{2}\quad \text{even}$$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x+13\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3^m)+\frac{3^{m+3}+1}{4}\quad \text{even}$$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+3}(x+13\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=\frac{3\cdot 4\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+5\cdot 3^{m+1}+1}{8}+3^{m+1}\quad \text{even}$$

$d'_{m+1}\geq 4$ and $y=((x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+1})+12\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+4})$;

Case 2_III

$s^{m+d_1+\cdots+d_{m-1}+d_m}(x)$ odd and $m\equiv 0(mod\,2)$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(x+2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(2\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x))+\frac{3^{m+1}+1}{2}\quad \text{even}$$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x+2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+\frac{3^{m+1}+1}{4}\quad \text{even}$$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+3}\left(x+2^{d_1+\cdots+d_{m-1}+d'_m}\right)=\frac{3\cdot 4\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3^{m+1}+1}{8}$$

if $\frac{3^m}{2^{d_1+\cdots+d_{m-1}+d_m}}<\frac{2}{3}$ then $d'_{m+1}\geq 3$

else if $s^{m+1+d_1+\cdots+d_{m-1}+d'_m+3}\left(x+2^{d_1+\cdots+d_{m-1}+d'_m}\right)$ even then $d'_{m+1}\geq 4$

$y=\left(x+2^{d_1+\cdots+d_{m-1}+d'_m}\right)\left(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+1}\right)$;

Case 2_IV

$s^{m+d_1+\cdots+d_{m-1}+d_m}(x)$ odd, $\frac{3\cdot 4\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3^{m+1}+1}{8}$ odd and $m\equiv 0(mod\,2)$

$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}\left(x+9\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}\right)=3\cdot\left(2\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)\right)+\frac{3^{m+3}+1}{2}$ even

$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}\left(x+9\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}\right)=3\cdot\left(s^{m+d_1+\cdots+d_{m-1}+d_m}(x)\right)+\frac{3^{m+3}+1}{4}$ even

$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+3}\left(x+9\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}\right)=\frac{3\cdot 4\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3^{m+1}+1}{8}+3^{m+1}$ even

$d'_{m+1}\geq 4$ and $y=\left(\left(x+2^{d_1+\cdots+d_{m-1}+d'_m}\right)\left(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+1}\right)+8\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}\right)\left(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+4}\right)$;

Case 2_V

$s^{m+d_1+\cdots+d_{m-1}+d_m}(x)$ even and $m\equiv 1(mod\,2)$

$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}\left(x+7\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}\right)=3\cdot\left(2\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+2\cdot 3^m\right)+\frac{3^{m+2}+1}{2}$ even

$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}\left(x+7\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}\right)=3\cdot\left(s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3^m\right)+\frac{3^{m+2}+1}{4}$ even

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+3}\left(x+7\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}\right)=\frac{3\cdot 4\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+7\cdot 3^{m+1}+1}{8}$$

if $\frac{3^m}{2^{d_1+\cdots+d_{m-1}+d_m}}<\frac{2}{3}$ then $d'_{m+1}\geq 3$

else if $s^{m+1+d_1+\cdots+d_{m-1}+d'_m+3}\left(x+7\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}\right)$ even then $d'_{m+1}\geq 4$

$y=\left(\left(x+2^{d_1+\cdots+d_{m-1}+d'_m}\right)\left(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+1}\right)+6\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}\right)\left(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+3}\right)$;

Case 2_VI

$s^{m+d_1+\cdots+d_{m-1}+d_m}(x)$ even, $\dfrac{3\cdot 4\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+7\cdot 3^{m+1}+1}{8}$ odd and $m\equiv 1(mod\,2)$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(x+15\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(2\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+6\cdot 3^m)+\frac{3^{m+2}+1}{2}\text{ even}$$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x+15\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3\cdot 3^m)+\frac{3^{m+2}+1}{4}\text{ even}$$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+3}(x+15\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=\frac{3\cdot 4\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+7\cdot 3^{m+1}+1}{8}+3^{m+1}\text{ even}$$

$d'_{m+1}\geq 4$ and $y=((x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+1})+14\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+4})$;

Case 2_VII

$s^{m+d_1+\cdots+d_{m-1}+d_m}(x)$ odd and $m\equiv 1(mod\,2)$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(x+3\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot 2\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+\frac{3^{m+2}+1}{2}\text{ even}$$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x+3\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+\frac{3^{m+2}+1}{4}\text{ even}$$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+3}(x+3\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=\frac{3\cdot 4\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3\cdot 3^{m+1}+1}{8}$$

if $\dfrac{3^m}{2^{d_1+\cdots+d_{m-1}+d_m}}<\dfrac{2}{3}$ then $d'_{m+1}\geq 3$

else if $s^{m+1+d_1+\cdots+d_{m-1}+d'_m+3}(x+7\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})$ even then $d'_{m+1}\geq 4$

$y=((x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+1})+2\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+3})$;

Case 2_VIII

$s^{m+d_1+\cdots+d_{m-1}+d_m}(x)$ odd, $\dfrac{3\cdot 4\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3\cdot 3^{m+1}+1}{8}$ odd and $m\equiv 1(mod\,2)$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(x+11\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(2\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+4\cdot 3^m)+\frac{3^{m+2}+1}{2}\text{ even}$$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x+11\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+2\cdot 3^m)+\frac{3^{m+2}+1}{4}\text{ even}$$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+3}(x+11\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=\frac{3\cdot 4\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+3\cdot 3^{m+1}+1}{8}+3^{m+1}$$ even

$d'_{m+1}\geq 4$ and $y=((x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\, 2^{d_1+\cdots+d_{m-1}+d'_m+1})+11\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\, 2^{d_1+\cdots+d_{m-1}+d'_m+4})$;

in all cases examined for $n\equiv y(mod\, 2^{d_1+\cdots+d_{m-1}+d'_m+d'_{m+1}})$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+d'_{m+1}}(n)=\frac{3^{m+1}\cdot n+3^m+2^{d_1+\cdots+d_{m-1}+d'_m}+\sum_{i=2}^{m}3^{m+1-i}\cdot 2^{d_1+\cdots+d_{i-1}}}{2^{d_1+\cdots+d_{m-1}+d'_m+d'_{m+1}}}<n$$.

If $d_m-d'_m>2$

if $s^{m+d_1+\cdots+d_{m-1}+d_m}(n)<n$ then $3^m<2^{d_1+\cdots+d_{m-1}+d_m}$ and $\frac{3^{m+1}}{4}<3^m<2^{d_1+\cdots+d_{m-1}+d_m}$ then

$\frac{3^{m+1}}{4\cdot 2^{d_m-d'_m}}<\frac{3^m}{2^{d_m-d'_m}}<2^{d_1+\cdots+d_{m-1}+d'_m}$ if $\frac{3^m}{2^{d_1+\cdots+d_{m-1}+d_m}}<\frac{2}{3}$ then. $\frac{3^{m+1}}{2\cdot 2^{d_m-d'_m}}<2^{d_1+\cdots+d_{m-1}+d'_m}$

$x=x'+a\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}$ with $x'<2^{d_1+\cdots+d_{m-1}+d'_m}$ and $0\leq a<2^{d_m-d'_m}$ then

$((x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\, 2^{d_1+\cdots+d_{m-1}+d'_m+1}))=x'+(1-a \;\; mod \;\; 2)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m}$.

Case 3_I

$m\equiv 0(mod\, 2)$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(x+(5+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(2^{(d_m-d'_m-1)}\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+2(1+2\cdot a)\cdot 3^m)+\frac{3^{m+1}+1}{2}$$

$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(x+(5+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})$ even

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x+(5+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(2^{(d_m-d'_m-2)}\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+(1+2\cdot a)\cdot 3^m)+\frac{3^{m+1}+1}{4}$$

$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x+(5+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})$ even

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+3}(x+(5+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=\frac{3\cdot 2^{(d_m-d'_m)}\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+(5+8\cdot a)\cdot 3^{m+1}+1}{8}$$

$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+3}(x+(5+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})$ even

…

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+(d_m-d'_m)+1}(x+(5+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=\frac{3\cdot 2^{(d_m-d'_m)}\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+(5+8\cdot a)\cdot 3^{m+1}+1}{2\cdot 2^{d_m-d'_m}}$$

if $\frac{3^m}{2^{d_1+\cdots+d_{m-1}+d_m}}<\frac{2}{3}$ then $d'_{m+1}\geq(d_m-d'_m)+1$

$$y=((x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+1})+(4+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+(d_m-d'_m)+1})$$

else if $s^{m+1+d_1+\cdots+d_{m-1}+d'_m+(d_m-d'_m)+1}(x+(5+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})$ even then $d'_{m+1}\geq(d_m-d'_m)+2$

$$y=((x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+1})+(4+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+(d_m-d'_m)+2})$$

for $n\equiv y(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+d'_{m+1}})$ then

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+d'_{m+1}}(n)=\frac{3^{m+1}\cdot n+3^m+2^{d_1+\cdots+d_{m-1}+d'_m}+\sum_{i=2}^{m}3^{m+1-i}\cdot 2^{d_1+\cdots+d_{i-1}}}{2^{d_1+\cdots+d_{m-1}+d'_m+d'_{m+1}}}<n \quad ;$$

Case 3_II

$m\equiv 1(mod\,2)$

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(x+(7+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(2^{(d_m-d'_m-1)}\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+2(1+2\cdot a)\cdot 3^m)+\frac{3^{m+2}+1}{2}$$

$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+1}(x+(7+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})$ even

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x+(7+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=3\cdot(2^{(d_m-d'_m-2)}\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+(1+2\cdot a)\cdot 3^m)+\frac{3^{m+2}+1}{4}$$

$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+2}(x+(7+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})$ even

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+3}(x+(7+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=\frac{3\cdot 2^{(d_m-d'_m)}\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+(7+8\cdot a)\cdot 3^{m+1}+1}{8}$$

$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+3}(x+(7+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})$ even

…

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+(d_m-d'_m)+1}(x+(7+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})=\frac{3\cdot 2^{(d_m-d'_m)}\cdot s^{m+d_1+\cdots+d_{m-1}+d_m}(x)+(7+8\cdot a)\cdot 3^{m+1}+1}{2\cdot 2^{d_m-d'_m}}$$

if $\frac{3^m}{2^{d_1+\cdots+d_{m-1}+d_m}}<\frac{2}{3}$ then $d'_{m+1}\geq(d_m-d'_m)+1$

$$y=((x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+1})+(6+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+(d_m-d'_m)+1})$$

else if $s^{m+1+d_1+\cdots+d_{m-1}+d'_m+(d_m-d'_m)+1}(x+(7+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})$ even then $d'_{m+1}\geq(d_m-d'_m)+2$

$$y=((x+2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+1})+(6+8\cdot a)\cdot 2^{d_1+\cdots+d_{m-1}+d'_m})(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+(d_m-d'_m)+2})$$

for $n\equiv y(mod\,2^{d_1+\cdots+d_{m-1}+d'_m+d'_{m+1}})$ then

$$s^{m+1+d_1+\cdots+d_{m-1}+d'_m+d'_{m+1}}(n)=\frac{3^{m+1}\cdot n+3^m+2^{d_1+\cdots+d_{m-1}+d'_m}+\sum_{i=2}^{m}3^{m+1-i}\cdot 2^{d_1+\cdots+d_{i-1}}}{2^{d_1+\cdots+d_{m-1}+d'_m+d'_{m+1}}}<n \quad .$$

Let's examine the relationship between k and m with $k-m=d_1+\cdots+d_m$:

if $n\equiv x(mod\,2^{d_1+\cdots+d_{m-1}+d_m})$ with x odd, $m>1$, $s^i(n)>n$ for $1\leq i<m+d_1+\cdots+d_{m-1}+d_m$

and $s^{m+d_1+\cdots+d_{m-1}+d_m}(n)=\frac{3^m\cdot n+3^{m-1}+\sum_{i=2}^{m}3^{m-i}\cdot 2^{d_1+\cdots+d_{i-1}}}{2^{d_1+\cdots+d_{m-1}+d_m}}<n$ then $\frac{3^m}{2^{d_1+\cdots+d_{m-1}+d_m}}<1$

let's consider the case where $d_1=\cdots=d_{m-1}=1$ and $d'_m=1$ then $d_1+\cdots+d_{m-1}+d'_m=m$ and $d_1+\cdots+d_{m-1}=m-1$, $k_m-m=m-1+d_m$ and $k_{m+1}-(m+1)=m+d'_{m+1}$

If $m\equiv 0(mod\,2)$ and $b=(5+8\cdot a)$ or if $m\equiv 1(mod\,2)$ and $b=(7+8\cdot a)$

as seen

if $\frac{3^m}{2^{d_1+\cdots+d_{m-1}+d_m}}=\frac{3^m}{2^{m-1+d_m}}<\frac{2}{3}$ then $d'_{m+1}\geq(d_m-1)+1$ so let's set $d'_{m+1}=d_m$ then

$$s^{m+1+m+(d_m-1)+1}(x+b\cdot 2^m)=\frac{3\cdot 2^{(d_m-1)}\cdot s^{m+m-1+d_m}(x)+b\cdot 3^{m+1}+1}{2\cdot 2^{d_m-1}}=\frac{3\cdot s^{m+m-1+d_m}(x)}{2}+\frac{b\cdot 3^{m+1}+1}{2\cdot 2^{d_m-1}}$$

$$s^{k_{m+1}}(x+b\cdot 2^m)=\frac{3\cdot s^{m+m-1+d_m}(x)}{2}+\frac{b\cdot 3^{m+1}+1}{2^{d_m}}<x+b\cdot 2^m \quad \text{with} \quad k_{m+1}=m+1+m+d_m=k_m+2$$

else $d'_{m+1}\geq(d_m-1)+2$ so let's set $d'_{m+1}=d_m+1$ then

$$s^{k_{m+1}}(x+b\cdot 2^m)=\frac{3\cdot 2^{(d_m-1)}\cdot s^{m+m-1+d_m}(x)+b\cdot 3^{m+1}+1}{4\cdot 2^{d_m-1}}<x+b\cdot 2^m \quad \text{with} \quad k_{m+1}=m+1+m+d_m+1=k_m+3 \quad .$$

For $\frac{3^m}{2^{d_1+\cdots+d_m}}=\frac{3^m}{2^{k_m-m}}<1$ then $2\cdot 3^m<2^{k_m-m+1}$

if $\frac{3^{m-1}}{2^{d_1+\cdots+d_{m-1}}}=\frac{3^{m-1}}{2^{k_{m-1}-(m-1)}}>\frac{2}{3}$ then $3^m>2^{k_{m-1}+2-m}=2^{k_m-1-m}$, $2\cdot 3^m>2^{k_m-m}$ and

$k_m-m = \lfloor\log_2(2\cdot 3^m)\rfloor = \lfloor 1+m\cdot\log_2(3)\rfloor$

else $k_m-m = k_{m-1}+2-(m-1)-1 = k_{m-1}-(m-1) + 1$.

Below is the algorithm code to generate the residue classes for $m \geq 1$:

```
get_dim(m)={my(log2_3=log(3)/log(2),X1=matrix(2,floor(1+m*log2_3)-m), d=0,dim=1);X1[1,1]=0;
if(m>=3,for(x=3,m,y=floor(1+(x-1)*log2_3);
for(i=1,y-x,for(j=i,y-x,if(i==1,X1[1+(d+1)%2,j]=1);X1[1+(d+1)%2,j]=X1[1+(d+1)%2,j]+X1[d+1,i]));
d=(d+1)%2);for(i=1,floor(1+m*log2_3)-m,dim=dim+X1[d+1,i]));dim;}
{m_max=9;dim_max=get_dim(m_max);N=vector(dim_max);R=matrix(2,dim_max);Dm=matrix(2,dim_max);
nr=0;m=1;log2_3=log(3)/log(2);kmm=floor(1+m*log2_3);c=1;N[1]=1;R[1,1]=1;Dm[1,1]=2;
print1("m = ",m,"    -    stopping time: ",kmm+m,"\nif n == ",N[1]," (mod 2^",kmm,")\ntotal residue classes:
",c,"\n\n");
for(m=1,m_max-1, kmm=floor(1+m*log2_3);mp1=m+1;kmmp1=floor(1+mp1*log2_3);
print1("m = ",mp1,"    -    stopping time: ",kmmp1+mp1,"\nif n == ");c1=0;
for(i=1,c,sdmm1=kmm-Dm[1+nr,i];for(j=1,Dm[1+nr,i]-1,rj=3*R[1+nr,i]+2^(sdmm1+j); c1=c1+1;
N[c1]=((2^kmmp1-rj)*lift(Mod(1/3^mp1,2^kmmp1)))%2^kmmp1;R[1+(1+nr)%2,c1]=rj;Dm[1+(1+nr)
%2,c1]=kmmp1-(sdmm1+j))); c=c1;N1=vector(c);N1=vecsort(N[1..c]);for(i1=1,c-1,print1(N1[i1],", "));
print1(N1[c]);nr=(1+nr)%2;print(" (mod 2^",kmmp1,")\ntotal residue classes:  ",c,"\n\n"))}
```

PARI/GP code of the algorithm

References

[1] L.E. Garner, On the Collatz 3n + 1 algorithm, *Proc. Amer. Math. Soc., 82 (1981), pp. 19-22*