A VERY ACCURATE APPROXIMATION FOR A NONTRIVIAL TRIGONOMETRIC FORMULA AND OTHER ADVANCED ANALYTIC CONSIDERATIONS.

ALEXIS ZAGANIDIS

Abstract. The first part of the present article is about advanced trigonometry formulas involving a rotating sphere and a source light infinitely far way. The second part of the present article is about two advanced conjectures on the global minimum of functions of class C^2 defined over the real coordinate space $\mathbb{R}^n$. The third part of the present article is about an advanced definition of the weak asymptomatic limit of functions of a single strictly positive real variable by applying recursively a cutoff function.

1. Advanced Trigonometry Formulas.

A very accurate approximation for the following nontrivial trigonometric formula :

$$\Delta(\omega,\phi) = |ArcSin\,(Sin\,(\omega)\,Cos\,(\phi)) - \omega\ Cos\,(\phi)\,| \leq f\,(\omega) \tag{1}$$

$$f\,(\omega) = \sqrt{\frac{\omega^2}{Sin^2\,(\omega)} - 1} - ArcSin\left(\sqrt{1 - \frac{Sin^2\,(\omega)}{\omega^2}}\right) \tag{2}$$

$$|ArcSin\,(Sin\,(23.44^\circ)\,Cos\,(\phi)) - 23.44^\circ \times Cos\,(\phi)\,| \leq f\,(23.44^\circ) = 0.255951^\circ \tag{3}$$

A very accurate approximation, that is simpler than the previous one, for the following nontrivial trigonometric formula over the smaller region $0 \leq \omega < \pi/3$:

$$\Delta(\omega,\phi) = |ArcSin\,(Sin\,(\omega)\,Cos\,(\phi)) - \omega\ Cos\,(\phi)\,| \leq f\,(\omega) \leq g\,(\omega) \tag{4}$$

$$g\,(\omega) = ArcSin\left(Sin\left(\frac{2\ \omega}{\sqrt{3}}\right)\right) - \frac{2\ \omega}{\sqrt{3}} \tag{5}$$

$$|ArcSin\,(Sin\,(23.44^\circ)\,Cos\,(\phi)) - 23.44^\circ \times Cos\,(\phi)\,| \leq g\,(23.44^\circ) = 0.277482^\circ \tag{6}$$

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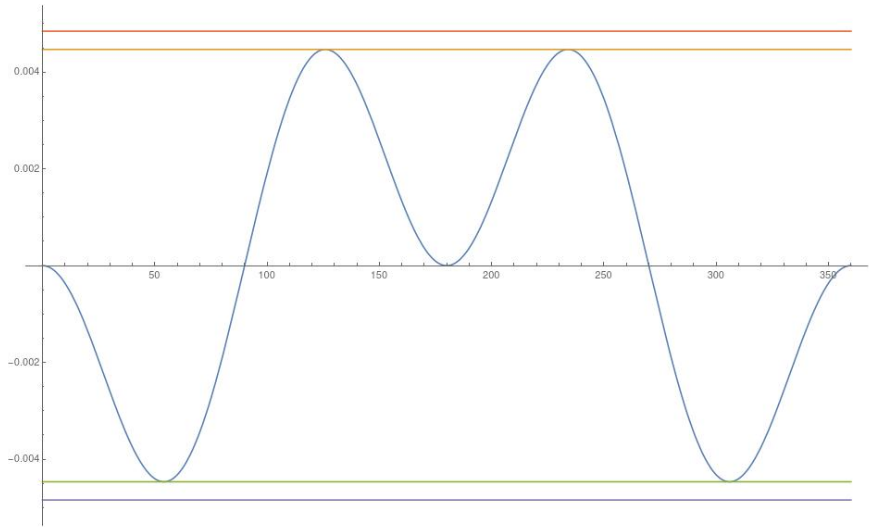

Figure 1. We have plotted the advanced trigonometric function $\Delta(23.44°, \phi)$, the upper bound $f(23.44°)$ and $g(23.44°)$, and the lower bound $-f(23.44°)$ and $-g(23.44°)$. We have plotted with the unit degree for the angle ϕ and with the unit degree for the angle output of the advanced trigonometric function $\Delta(23.44°, \phi)$.

Indeed, the angle α between the vectors $\vec{s} = (+\infty, 0, 0)$ (position of the source light infinitely far way) and the rotation vector $\vec{\omega} = (Cos(\phi) Sin(\omega), Sin(\phi) Sin(\omega), Cos(\omega))$ (the rotation vector of a sphere with the axial tilt angle ω and with the orbital angle ϕ) is the following :

$$\alpha = \frac{\pi}{2} - ArcSin(Sin(\omega) Cos(\phi)) \tag{7}$$

Let consider a source light infinitely far away from a sphere. The curved line parameterized with the variable θ on this sphere such that the light rays are tangential on that curved line, is the following :

$$(0, Sin(\theta), Cos(\theta)) \tag{8}$$

Let consider a rotation of the light rays with respect to this sphere along the y axis and with an upward angle $\alpha > 0$ (summer time):

(9)

$$RotationMatrix(\alpha, (0, -1, 0)) \cdot (0, Sin(\theta), Cos(\theta)) \xrightarrow{Spherical\ Coordinates}$$

(10)

$$\left(1, ArcTan\left(Cos(\alpha) Cos(\theta), \sqrt{Cos^2(\theta) Sin^2(\alpha) + Sin^2(\theta)}\right), -ArcTan(Sin(\alpha) Cos(\theta), Sin(\theta))\right)$$

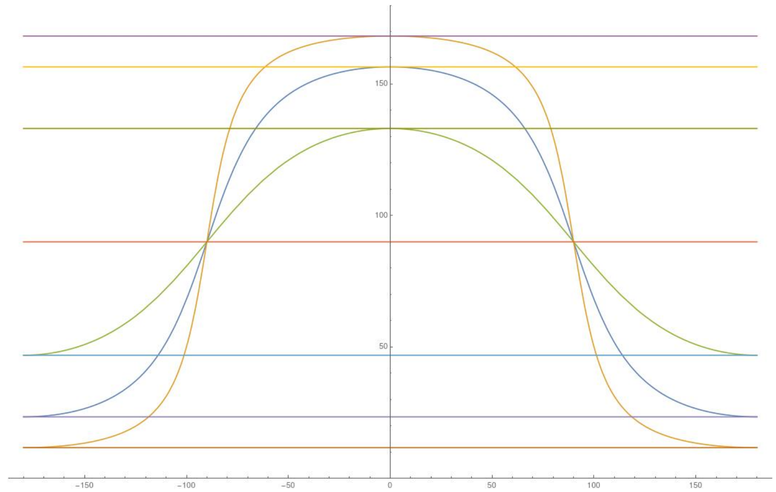

FIGURE 2. The latitude of the oblique light rays during the summer solstice that are tangential to the sphere for a given positive sunrise longitude and for a given negative sunset latitude. For simplicity, we define the zero latitude at the North Pole and we have defined the positive sunrise longitude as the exact opposite of the negative sunset longitude. We have plotted for several different numerical values of the axial title angle $\alpha = 0.5 \times 23.44°, 23.44°, 2 \times 23.44°$.

The longitude φ in polar units can be expressed with the latitude $\lambda = \pi/2 - \tilde{\lambda}$ in polar units :

$$\tilde{\lambda} = -ArcTan\left(Sin\left(\alpha\right)Cos\left(\theta\right), Sin\left(\theta\right)\right) \tag{11}$$

$$\theta = Arctan\left(Cos\left(\alpha\right)Tan\left(\lambda\right)\right) + \pi \; H\left(\pi/2 - |\tilde{\lambda}|\right) \tag{12}$$

$$\varphi = ArcTan\left(Cos\left(\alpha\right)Cos\left(\theta\right), \sqrt{Cos^2\left(\theta\right)Sin^2\left(\alpha\right) + Sin^2\left(\theta\right)}\right) \tag{13}$$

Let consider a rotation of the light rays with respect to this sphere along the y axis and with an downward angle $\alpha < 0$ (winter time):

(14)

$$RotationMatrix\left(\alpha, (0,1,0)\right) \cdot \left(0, Sin\left(\theta\right), Cos\left(\theta\right)\right) \xrightarrow{Spherical\ Coordinates}$$

(15)

$$\left(1, ArcTan\left(Cos\left(\alpha\right)Cos\left(\theta\right), \sqrt{Cos^2\left(\theta\right)Sin^2\left(\alpha\right) + Sin^2\left(\theta\right)}\right), ArcTan\left(Sin\left(\alpha\right)Cos\left(\theta\right), Sin\left(\theta\right)\right)\right)$$

The longitude φ in polar units can be expressed with the latitude $\lambda = \pi/2 - \tilde{\lambda}$ in polar units :

FIGURE 3. The latitude of the oblique light rays during the winter solstice that are tangential to the sphere for a given positive sunrise longitude and for a given negative sunset latitude. For simplicity, we define the zero latitude at the North Pole and we have defined the positive sunrise longitude as the exact opposite of the negative sunset longitude. We have plotted for several different numerical values of the axial title angle $\alpha = 0.5 \times 23.44°, 23.44°, 2 \times 23.44°$.

(16) $$\tilde{\lambda} = ArcTan\left(Sin\left(\alpha\right)Cos\left(\theta\right), Sin\left(\theta\right)\right)$$

(17) $$\theta = Arctan\left(Cos\left(\alpha\right)Tan\left(\lambda\right)\right) + \pi\ H\left(|\tilde{\lambda}| - \pi/2\right)$$

(18) $$\varphi = ArcTan\left(Cos\left(\alpha\right)Cos\left(\theta\right), \sqrt{Cos^2\left(\theta\right)Sin^2\left(\alpha\right) + Sin^2\left(\theta\right)}\right)$$

We consider the following 3D parametrization of the curved line on the sphere with the latitude coordinate $\lambda = \pi/2 - \tilde{\lambda}$ such that the oblique light rays (upward angle $\alpha > 0$ is summer time) are tangential on that curved line :

(19) $$\varphi\left(\tilde{\lambda}\right) = ArcSec\left(Cot\left(\alpha\right)Tan\left(\tilde{\lambda}\right)\right)$$

(20) $$\vec{r}\left(\tilde{\lambda}\right) = \left(-Cos\left(\varphi\left(\tilde{\lambda}\right)\right)Sin\left(\tilde{\lambda}\right), Sin\left(\varphi\left(\tilde{\lambda}\right)\right)Sin\left(\tilde{\lambda}\right), Cos\left(\tilde{\lambda}\right)\right)$$

We can deduce the following 3D parametrization of the tangential north pole vector $\vec{n}$ of the previous curved line with the latitude coordinate $\lambda = \pi/2 - \tilde{\lambda}$:

(21)
$$\vec{n}\left(\tilde{\lambda}\right) = RotationMatrix\left(\pi/2, \left(-Cos\left(\varphi\left(\tilde{\lambda}\right) - \pi/2\right), Sin\left(\varphi\left(\tilde{\lambda}\right) - \pi/2\right), 0\right)\right).\vec{r}\left(\tilde{\lambda}\right)$$

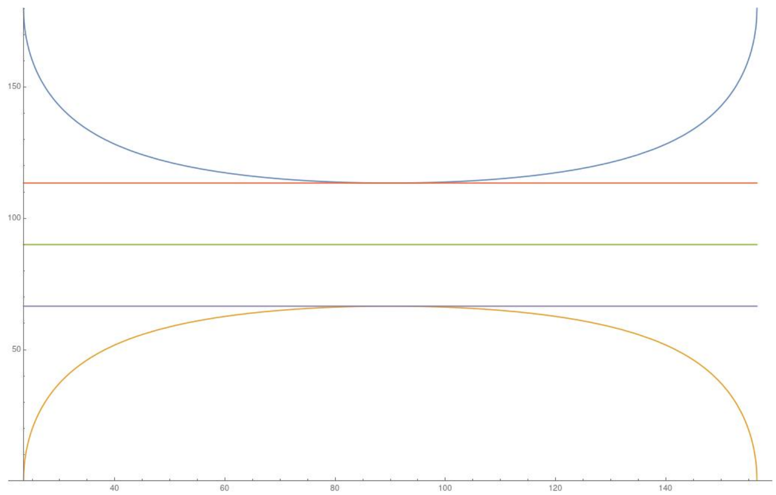

FIGURE 4. The angle η between the tangential north pole vector $\vec{n}$ and the oblique light rays that are also tangential for a given latitude. (bottom plot : $\alpha = 23.44°$ at the summer solstice; top plot : $\alpha = -23.44°$ at the winter solstice). This is approximately the angle between sunlight and the cardinal direction north at sunrise/sunset on the summer and winter solstice for a given latitude. For simplicity, we define the zero latitude at the North Pole.

We can deduce the following angle η between the tangential north pole vector $\vec{n}$ and the oblique light rays (upward angle $\alpha > 0$ is summer time) :

$$\eta\left(\tilde{\lambda}\right) = VectorAngle\left(\left(Cos\left(\alpha\right), 0, Sin\left(\alpha\right)\right), \vec{n}\left(\tilde{\lambda}\right)\right) \tag{22}$$

$$\left(Cos\left(\alpha\right), 0, Sin\left(\alpha\right)\right) = RotationMatrix\left(-\alpha, \left(0,1,0\right)\right).\left(1,0,0\right) \tag{23}$$

To generalize in other situations than the summer solstice and the winter solstice for a given orbital angle ϕ, the angle α between the position $\vec{s} = (+\infty, 0, 0)$ of the source light infinitely far way and the rotation axis $\vec{\omega}$ of the sphere is the following :

$$\alpha = \frac{\pi}{2} - ArcSin\left(Sin\left(\omega\right) Cos\left(\phi\right)\right) \tag{24}$$

The angle $\alpha = 90° - 23.44°$ with the orbital angle $\phi = 0°$ and the axial tilt angle $\omega = 23.44°$ correspond to the summer solstice, and the angle $\alpha = 90° - 23.44°$ with the orbital angle $\phi = 180°$ and the axial tilt angle $\omega = 23.44°$ correspond to the summer solstice.

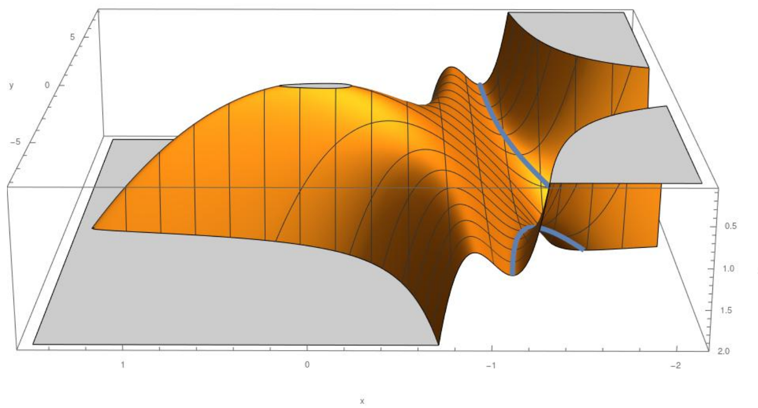

FIGURE 5. Illustration of a function $\tilde{f}\left(x,y\right) = x^2 + y^2\left(1+x\right)^3$ with a local minimum $\vec{p}_0 = (0,0)$ that is not a global minimum and with a vanish gradient at a single point $(0,0)$. However, the blue curved lines show an asymptotic vanishing of the gradient. The z-axis od the plot is reversed and the dimensions of the plot are : $\left(\left(-2.1, 1.5\right), \left(-8, 8\right), \left(0.05, 2.0\right)\right)$

2. Advanced conjectures about the global minimum of a function.

Let be a function f of class C^2 defined over the real coordinate space $\mathbb{R}^n$:

(25)
$$\exists ! \vec{y}_0 \left(\forall \vec{x}_n \left(\lim_{n\to+\infty} \nabla f\left(\vec{x}_n\right) = \vec{0} \to \lim_{n\to+\infty} \vec{x}_n = \vec{y}_0 \right) \wedge H\left(f\left(\vec{y}_0\right)\right) > 0 \Rightarrow \forall \vec{x}\left(f\left(\vec{x}\right) \geq f\left(\vec{y}_0\right)\right)\right)$$

The example function $\tilde{f}\left(x,y\right) = x^2 + y^2\left(1+x\right)^2$ has a single local minimum $\vec{p}_0 = (0,0)$ that is not a global minimum and with a vanish gradient at a single point $\vec{p}_0 = (0,0)$. Let consider the following curved line parameterized by the following equation :

$$\tilde{f}\left(x\left(y\right), y\right) = x\left(y\right)^2 + y^2\left(1 + x\left(y\right)\right)^3 = 0 \tag{26}$$

$$x\left(y\right) = -1 - \frac{1}{3y^2} \pm \sqrt{\frac{1+6y^2}{9y^4}} \tag{27}$$

$$\left(\vec{\nabla}\tilde{f}\right)\left(x\left(y\right), y\right) = \left(0, \frac{-2 \pm 2\left(1+6y^2\right)^{3/2}}{27y^5}\right) \tag{28}$$

$$\lim_{y \to +\infty} \left(\vec{\nabla}\tilde{f}\right)(x(y), y) = (0,0) \tag{29}$$

The gradient of the example function $\tilde{f}$ has indeed an asymptotic vanishing.

Sketch of the proof :

We do a coordinate transformation to transform $\mathbb{R}^n$ into an open ball V^n. We assume there is a point $\vec{p}_1 \neq \vec{p}_0$ where the function function f is smaller than the local minimum at the point $\vec{p}_0$. For any curved lines of class C^∞ starting at the point $\vec{p}_0$, finishing at the point $\vec{p}_1$ and parameterized by λ as $\vec{x}(\lambda)$, the function $f(\vec{x}(\lambda))$ reaches a local maximum $f(\vec{x}(\lambda_2))$ at λ_2 that is strictly higher than the local minimum. The gradient at the point $\vec{p}_2 = \vec{x}(\lambda_2)$ has a vanishing tangential projection with respect to the curved line $\vec{x}(\lambda)$. Therefore, the gradient at the point $\vec{p}_2 = \vec{x}(\lambda_2)$ has a none-vanishing perpendicular projection with respect to the curved line $\vec{x}(\lambda)$. It means we can find a another curved line of class C^∞ starting at the point $\vec{p}_0$, finishing at the point $\vec{p}_1$ and parameterized by λ as $\vec{x'}(\lambda)$ that is slightly longer and that reaches a local maximum $f\left(\vec{x'}(\lambda_3)\right)$ slightly smaller at λ_3 than the maximum $f(\vec{x}(\lambda_2))$ of the function $f(\vec{x}(\lambda))$ at λ_2. Therefore, the curved line of class C^∞ starting at the point $\vec{p}_0$, finishing at the point $\vec{p}_1$ and parameterized by λ as $\left(\vec{\tilde{x}}(\lambda)\right)$ has an arbitrary large length if the function $f\left(\vec{\tilde{x}}(\lambda)\right)$ reaches a local maximum with the lowest possible value. However, any curved lines of class C^∞ starting at the point $\vec{p}_0$, finishing at the point $\vec{p}_1$ and parameterized by λ as $\vec{x}(\lambda)$ with an arbitrary large length have some useless loops with respect to the lowest possible value of its local maximum. Absurd. CQFD.

Let be a function f of class C^2 defined over the real coordinate space $\mathbb{R}^n$:

$$\forall \vec{x}_n \left(\lim_{n \to +\infty} ||\vec{x}_n|| = +\infty \to \lim_{n \to +\infty} f(\vec{x}_n) = +\infty \right) \Rightarrow \exists \vec{y}_0 \forall \vec{x} \left(f(\vec{x}) \geq f(\vec{y}_0)\right) \tag{30}$$

Sketch of the proof :

We do a coordinate transformation into the spherical coordinates : $f(\vec{x}) \to f\left(r, \vec{\theta}\right)$. We define the function g as the following :

$$g(r) = \inf_{\vec{\theta}} \left(f\left(r, \vec{\theta}\right)\right) \tag{31}$$

$$g(0) = f\left(\vec{0}\right) \tag{32}$$

Trivially, we have the following limit :

$$\lim_{r \to +\infty} g(r) = +\infty \tag{33}$$

From elementary analysis, the function g has a global minimum which can be expresses as the following

$$g_{min} = \inf_{r} \left(g\left(r\right)\right) \tag{34}$$

$$= \inf_{r} \left(\inf_{\vec{\theta}} \left(f\left(r, \vec{\theta}\right) \right) \right) \tag{35}$$

$$= \inf_{r, \vec{\theta}} \left(f\left(r, \vec{\theta}\right) \right) \tag{36}$$

$$= f_{min} \tag{37}$$

Therefore, the global minimum of the function f exist since it is the global minimum of the function g. CQFD.

3. Advanced definition of a weak asymptomatic limit.

Let be the functions f and g defined on the real set $\mathbb{R}_{+}^{*} = (0, +\infty)$. Let consider the weak asymptomatic limit of the function f around a. By defining the function $g = f - a$, we consider the weak asymptomatic limit of g around 0. If the function g has a weak asymptomatic limit around 0, then the function f has a weak asymptomatic limit around 0.

Let define the following sequence of functions g_n by applying recursively a cutoff function h on a function g defined on $\mathbb{R}_{+}^{*} = (0, +\infty)$ in order to be able to define correctly the moving average of that function g :

$$g_0 = g \tag{38}$$

$$g_{n+1}\left(x\right) = \frac{\int_0^{+\infty} h\left(x, x'\right) g_n\left(x'\right) dx'}{\int_0^{+\infty} h\left(x, x'\right) dx'} \tag{39}$$

$$\forall x \left(x > 0 \rightarrow \int_0^{+\infty} h\left(x, x'\right) dx' < +\infty \right) \tag{40}$$

$$\forall x' \left(x' > 0 \rightarrow \lim_{x \rightarrow +\infty} h\left(x, x'\right) = 1 \right) \tag{41}$$

$$g_{n+1}\left(x\right) = \frac{1}{x} \int_0^{+\infty} e^{-\frac{x'}{x}} g_n\left(x'\right) dx' \tag{42}$$

$$h\left(x, x'\right) = e^{-\frac{x'}{x}} \tag{43}$$

$$g_{n+1}\left(x\right) = \frac{1}{x} \int_0^{+\infty} H\left(x - x'\right) g_n\left(x'\right) dx' \tag{44}$$

$$h\left(x, x'\right) = H\left(x - x'\right) \tag{45}$$

$$g_{n+1}\left(x\right) = \frac{2}{\sqrt{\pi} x} \int_0^{+\infty} e^{-\frac{x'^2}{x^2}} g_n\left(x'\right) dx' \tag{46}$$

$$h\left(x, x'\right) = e^{-\frac{x'^2}{x^2}} \tag{47}$$

(48)

The function g_n has an asymptomatic limit around 0 if and only if for any arbitrary small $\epsilon > 0$, it exists x_ϵ such that $x > x_\epsilon \to |g_n(x)| \leq \epsilon$. We can therefore deduce the following inequality for g_{n+1} :

$$x > x_\epsilon \to |g_{n+1}(x)| \leq \frac{1}{x}\int_0^{x_\epsilon} g_n(x')\,dx' + g_n(x_\epsilon)\,e^{-\frac{x_\epsilon}{x}} \tag{49}$$

Therefore, it exists $\tilde{x}_\epsilon$ such that :

$$x > \tilde{x}_\epsilon > x_\epsilon \to |g_{n+1}(x)| \leq \epsilon + g_n(x)(1+\epsilon) = \epsilon + \epsilon^2 \tag{50}$$

Therefore, if the function g_n has an asymptomatic limit around 0, then the function g_{n+1} has an asymptomatic limit around 0.

Therefore, we can define the following weak asymptomatic limit of the function f with the cutoff function h if and only if :

$$\lim_{n\to+\infty}\lim_{x\to+\infty}\inf_{x<x'}\left(|g_n(x')|\right) = 0 \tag{51}$$

With this definition, the functions $Sin(Log(x)), Sin(\sqrt{x}), Sin(x), Sin(x^2)$ have a weak asymptotic limit around 0.

With this definition, the functions $Log(x)\,Sin(\sqrt{x}), Log(x)\,Sin(x), Log(x)\,Sin(x^2)$ have also a weak asymptotic limit around 0 as well.

With this definition, the functions $\sqrt{x}\;Sin(\sqrt{x}), \sqrt{x}\;Sin(x), \sqrt{x}\;Sin(x^2)$ have also a weak asymptotic limit around 0 as well.

With this definition, the functions $x\;Sin(\sqrt{x}), x\;Sin(x), x\;Sin(x^2)$ have also a weak asymptotic limit around 0 as well.

With this definition, the functions $x^2Sin(\sqrt{x}), x^2Sin(x), x^2Sin(x^2)$ have also a weak asymptotic limit around 0 as well. It is particularly difficult to show analytically that the function $x^2Sin(\sqrt{x})$ has a weak asymptomatic limit around 0 with the cutoff function $h(x,x') = H(x-x')$ while it is much easier with the the cutoff function $h(x,x') = e^{-\frac{x'}{x}}$.

However the function $Log(x)\,Sin(Log(x)), \sqrt{x}\;Sin(Log(x)), x\;Sin(Log(x))$, $x^2\;Sin(Log(x))$ have not a weak asymptotic limit around 0.

Some important important limit properties are lost with respect to the classic definition of the asymptotic limit : the functions g^2 may not have a weak asymptomatic limit around 0 even if the function g has a weak asymptomatic limit around 0.

Open questions 01 : Does the function $g(\lceil x\rceil)$ and the function $g(\lfloor x\rfloor)$ have a weak asymptomatic limit around 0 if the function g has a weak asymptomatic limit around 0 ?

Open question 02 : It exists a bounded function f that has not a weak asymptomatic limit ?

Open question 03 : It exists a function f that has a weak asymptomatic limit around 0 with the cutoff function $h(x,x') = e^{-\frac{x'}{x}}$ but has not a weak asymptomatic limit around 0 with the cutoff function $h(x,x') = H(x-x')$?

Open question 04 : It exists a function f that has a weak asymptomatic limit around 0 with the cutoff function $h(x,x') = e^{-\frac{x'^2}{x^2}}$ but has not a weak asymptomatic limit around 0 with the cutoff function $h(x,x') = H(x-x')$?

Open question 05 : It exists a function f that has a weak asymptomatic limit around 0 with the cutoff function $h(x,x') = e^{-\frac{x'^2}{x^2}}$ but has not a weak asymptomatic limit around 0 with the cutoff function $h(x,x') = e^{-\frac{x'}{x}}$?

Open question 06 : Any functions f that has a weak asymptomatic limit around 0 with the cutoff function h_1 has also a weak asymptomatic limit around 0 with the cutoff function h_2 such that $x' < x \rightarrow h_2(x,x') \leq h_1(x,x')$ and $x < x' \rightarrow h_1(x,x') \leq h_2(x,x')$?

We can also compare the weak asymptomatic limit between two function f and $\tilde{f}$ around 0 with the same cutoff function h. There are two cases, the first case is the function f or the function $\tilde{f}$ that has the smallest n_0 such that :

$$\lim_{x\rightarrow+\infty} \inf_{x<x'} \left(|g_{n_0}(x')|\right) = 0 \tag{52}$$

or

$$\lim_{x\rightarrow+\infty} \inf_{x<x'} \left(|\tilde{g}_{n_0}(x')|\right) = 0 \tag{53}$$

The second case is performing the following ratio :

$$\lim_{n\rightarrow+\infty} \frac{\lim_{x\rightarrow+\infty} \inf_{x<x'} \left(|\tilde{g}n(x')|\right)}{\lim_{x\rightarrow+\infty} \inf_{x<x'} \left(|g_n(x')|\right)} = \begin{cases} <1, \\ =1, \\ >1. \end{cases} \tag{54}$$

Open question 07 : The comparison of the weak asymptomatic limit between two function f and $\tilde{f}$ around 0 with a cutoff function h_1 can be different than the the comparison of the weak asymptomatic limit between two function f and $\tilde{f}$ around 0 with a cutoff function h_2 ?
Open question 08 : The different cutoff functions give the same hierarchy on the set of functions of a single strictly positive real variable with respect to the weak asymptomatic limit ?

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