

Linear Combination of Real and Imaginary Parts of the Riemann ξ -Function

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Abstract

The Riemann hypothesis is proved true by finding two linear combinations of the real and imaginary parts of the Riemann ξ -function, that have simultaneously zeros only on the critical line according to the uncertainty principle.

Keywords: Variety, the Riemann hypothesis, the Riemann ξ -function, the Fourier transform, the uncertainty principle, the Riemann–Lebesgue lemma.

1 Introduction

As an engineer, I do not fully understand the importance of the Riemann hypothesis in pure mathematics and applied mathematics as well, but I know Riemann is one of the greatest mathematicians in the history of modern mathematics. It is the problem left unsolved by mathematicians such as Riemann, Hardy and Littlewood, and Selberg et al. in the last over 150 years that summons engineers' challenging spirit, as well. The Riemann hypothesis is about the distribution of zeros of the Riemann ζ -function which is a complex function. At zeros of a complex function, both the real part and the imaginary part are zero simultaneously. Finding the distribution of zeros means to solve the simultaneous equations of the real part and the imaginary part. Given the Abel-Ruffini theorem which states that there is impossibility to solve general polynomial equations of degree five or higher, we may conclude that the Riemann hypothesis is unsolvable at least by the simultaneous equations because the Riemann ζ -function is much more complex than polynomial equations. The Riemann hypothesis states that zeros of the Riemann ζ -function is located at a specific line. We are lucky that the zero distribution is not contracted in the linear combination of real and imaginary parts of the Riemann ζ -function. The proof of the Riemann hypothesis is then dependent on if we can find linear combinations of real and imaginary parts of the Riemann ζ -function that have simultaneously zeros on the critical line only. In the paper, we find the linear combinations. The publication of the work is the beginning of a new era in mathematics, however the proof is elementary without bringing out any mathematical advancement as expected.

2 Linear Combination of Functions

Theorem 2.1 (Variety of Linear Combination of Functions). *Let $f_1(x)$, and $f_2(x)$ be functions on the field x . If functions $g_1(x)$ and $g_2(x)$ have no poles on variety $V(f_1(x), f_2(x))$, then variety $V(f_1(x), f_2(x))$ is a subset of variety $V(g_1(x)f_1(x) + g_2(x)f_2(x))$.*

Proof. On $V(f_1(x), f_2(x))$, because $g_1(x)$, and $g_2(x)$ have no poles, we have

$$g_1(x)f_1(x) + g_2(x)f_2(x) = 0, \quad (1)$$

that is

$$V(f_1(x), f_2(x)) \subseteq V(g_1(x)f_1(x) + g_2(x)f_2(x)). \quad (2)$$

□

3 Riemann ξ -Function

The Riemann ζ -function is originally defined as

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} \quad s \in \mathbb{C}. \quad (3)$$

Now consider the following equation [1]

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = -\frac{1}{s(1-s)} + \int_1^{\infty} (t^{(1-s)/2} + t^{s/2}) \psi(t) \frac{dt}{t}, \quad t \in \mathbb{R} \quad (4)$$

which is rewritten as the Riemann ξ -function

$$\begin{aligned} \xi(s) &\equiv \pi^{-s/2} \Gamma(s/2) s(s-1) \zeta(s) = 1 + s(s-1) \int_1^{\infty} (t^{(1-s)/2} + t^{s/2}) \psi(t) \frac{dt}{t} \\ &= 1 + [(s-1/2)^2 - (1/2)^2] \int_1^{\infty} (t^{(1/2-s)/2} + t^{-(1/2-s)/2}) t^{1/2} \psi(t) \frac{dt}{t}, \end{aligned} \quad (5)$$

where

$$\psi(t) \equiv \sum_{n=1}^{\infty} e^{-n^2 \pi t}. \quad (6)$$

The nontrivial zeros of the Riemann ζ -function $\zeta(s)$ are identical with zeros of the Riemann ξ -function $\xi(s)$ [2]. G. H. Hardy [3] has proved that an infinite number of zeros lie on the critical line $s = 1/2$.

The Riemann ξ -function is further rewritten as follows

$$\begin{aligned} \xi(s) &= 1 + [(s-1/2)^2 - (1/2)^2] \int_1^{\infty} (t^{(1/2+1/2-s)/2} + t^{[1/2-(1/2-s)]/2}) \psi(t) \frac{dt}{t} \\ &= 1 + [w^2 - (1/2)^2] \int_1^{\infty} (t^{-w/2} + t^{w/2}) \psi(t) \frac{dt}{t^{1/2}} \\ &= 1 + [w^2 - (1/2)^2] \int_1^{\infty} [\exp(-\ln(t)w/2) + \exp(\ln(t)w/2)] \psi(t) \frac{dt}{t^{1/2}} \\ &= 1 + 2[w^2 - (1/2)^2] \int_1^{\infty} \cosh(\ln(t)w/2) \psi(t) \frac{dt}{t^{1/2}} \\ &= 1 + 2[w^2 - (1/2)^2] \int_1^{\infty} \cos(i \ln(t)w/2) \psi(t) \frac{dt}{t^{1/2}}, \end{aligned} \quad (7)$$

where

$$w = s - 1/2 = u + iv, \quad u, v \in \mathbb{R}. \quad (8)$$

Now let

$$z = iw = -v + iu. \quad (9)$$

Then, we have

$$\xi(s) \equiv \Xi(z) = 1 - 2 \left(z^2 + (1/2)^2 \right) \int_1^\infty \cos(z \ln(t)/2) \psi(t) \frac{dt}{t^{1/2}}. \quad (10)$$

Let $r = \ln(t)$, $r \in \mathbb{R}$, and notice the following identities

$$\begin{aligned} \cos(x + iy) &= \cos x \cosh y - i \sin x \sinh y, \\ \sin(x + iy) &= \sin x \cosh y + i \cos x \sinh y. \end{aligned} \quad (11)$$

Then $\Xi(z)$ is given as follows

$$\begin{aligned} \Xi(z) &= 1 - 2 \left(z^2 + (1/2)^2 \right) \int_0^\infty \cos(zr/2) \psi(e^r) e^{r/2} dr \\ &= 1 - 2 \left(v^2 - u^2 + (1/2)^2 - 2iuv \right) \\ &\quad \int_0^\infty [\cos(-vr/2) \cosh(ur/2) - i \sin(-vr/2) \sinh(ur/2)] \psi(e^r) e^{r/2} dr \\ &= 1 - 2 \left(v^2 - u^2 + (1/2)^2 \right) \int_0^\infty \cos(vr/2) \cosh(ur/2) \psi(e^r) e^{r/2} dr \\ &\quad - 2uv \int_0^\infty \sin(vr/2) \sinh(ur/2) \psi(e^r) e^{r/2} dr \\ &\quad + i \left[-2 \left(v^2 - u^2 + (1/2)^2 \right) \int_0^\infty \sin(vr/2) \sinh(ur/2) \psi(e^r) e^{r/2} dr \right. \\ &\quad \left. + 2uv \int_0^\infty \cos(vr/2) \cosh(ur/2) \psi(e^r) e^{r/2} dr \right]. \end{aligned} \quad (12)$$

The real and imaginary parts of $\Xi(z)$ are given as follows, respectively

$$\begin{aligned} \Re(\Xi(z)) &= 1 - 2 \left(v^2 - u^2 + (1/2)^2 \right) \int_0^\infty \cos(vr/2) \cosh(ur/2) \psi(e^r) e^{r/2} dr \\ &\quad - 2uv \int_0^\infty \sin(vr/2) \sinh(ur/2) \psi(e^r) e^{r/2} dr, \\ \Im(\Xi(z)) &= -2 \left(v^2 - u^2 + (1/2)^2 \right) \int_0^\infty \sin(vr/2) \sinh(ur/2) \psi(e^r) e^{r/2} dr \\ &\quad + 2uv \int_0^\infty \cos(vr/2) \cosh(ur/2) \psi(e^r) e^{r/2} dr. \end{aligned} \quad (13)$$

$\Xi(z) = 0$ is equivalent to the following equations,

$$\begin{aligned} \Re(\Xi(z)) &= 0, \\ \Im(\Xi(z)) &= 0. \end{aligned} \quad (14)$$

Now consider the following linear combination of $\mathfrak{R}(\Xi(z))$ and $\mathfrak{I}(\Xi(z))$

$$\begin{aligned}
L_s(u, v) &= uv\mathfrak{R}(\Xi(z)) + (v^2 - u^2 + (1/2)^2)\mathfrak{I}(\Xi(z)) \\
&= uv - \left[2(v^2 - u^2 + (1/2)^2)^2 + 2(uv)^2\right] \int_0^\infty \sin(vr/2) \sinh(ur/2) \psi(e^r) e^{r/2} dr \\
&= uv - 2 \left[(v^2 - u^2 + (1/2)^2)^2 + 4u^2(v/2)^2 \right] \int_0^\infty \cos(vr/2) \cosh(ur/2) \psi(e^r) e^{r/2} dr \\
&= 4u^2 \left\{ \frac{v}{4u} - 2 \left[\frac{(v^2 - u^2 + (1/2)^2)^2}{4u^2} + (v/2)^2 \right] \int_0^\infty \cos(vr/2) \cosh(ur/2) \psi(e^r) e^{r/2} dr \right\} \\
&= 4u^2 \left[\frac{(v^2 - u^2 + (1/2)^2)^2}{4u^2} + (v/2)^2 \right] \\
&\quad \left\{ \frac{1}{2u} \int_0^\infty \sin(vr/2) e^{-\frac{v^2 - u^2 + (1/2)^2}{2u} r} dr - 2 \int_0^\infty \sin(vr/2) \sinh(ur/2) \psi(e^r) e^{r/2} dr \right\} \\
&= \left[(v^2 - u^2 + (1/2)^2)^2 + (uv)^2 \right] \left\{ \int_0^\infty \sin(vr/2) \left[\frac{1}{2u} e^{-\frac{v^2 - u^2 + (1/2)^2}{2|u|} r} - 2 \sinh(ur/2) \psi(e^r) e^{r/2} \right] dr \right\} \\
&= \left[(v^2 - u^2 + (1/2)^2)^2 + (uv)^2 \right] F_s(u, v)
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
L_c(u, v) &= (v^2 - u^2 + (1/2)^2)\mathfrak{R}(\Xi(z)) - uv\mathfrak{I}(\Xi(z)) \\
&= (v^2 - u^2 + (1/2)^2) - \left[2(v^2 - u^2 + (1/2)^2)^2 + 2(uv)^2\right] \int_0^\infty \cos(vr/2) \cosh(ur/2) \psi(e^r) e^{r/2} dr \\
&= (v^2 - u^2 + (1/2)^2) - 2 \left[(v^2 - u^2 + (1/2)^2)^2 + 4u^2(v/2)^2 \right] \int_0^\infty \cos(vr/2) \cosh(ur/2) \psi(e^r) e^{r/2} dr \\
&= 4u^2 \left\{ \frac{(v^2 - u^2 + (1/2)^2)^2}{4u^2} - 2 \left[\frac{(v^2 - u^2 + (1/2)^2)^2}{4u^2} + (v/2)^2 \right] \int_0^\infty \cos(vr/2) \cosh(ur/2) \psi(e^r) e^{r/2} dr \right\} \\
&= 4u^2 \left[\frac{(v^2 - u^2 + (1/2)^2)^2}{4u^2} + (v/2)^2 \right] \\
&\quad \left\{ \frac{1}{2|u|} \int_0^\infty \cos(vr/2) e^{-\frac{v^2 - u^2 + (1/2)^2}{2|u|} r} dr - 2 \int_0^\infty \cos(vr/2) \cosh(ur/2) \psi(e^r) e^{r/2} dr \right\} \\
&= \left[(v^2 - u^2 + (1/2)^2)^2 + (uv)^2 \right] \left\{ \int_0^\infty \cos(vr/2) \left[\frac{1}{2|u|} e^{-\frac{v^2 - u^2 + (1/2)^2}{2|u|} r} - 2 \cosh(ur/2) \psi(e^r) e^{r/2} \right] dr \right\} \\
&= \left[(v^2 - u^2 + (1/2)^2)^2 + (uv)^2 \right] F_c(u, v),
\end{aligned} \tag{16}$$

where

$$F_s(u, v) = \int_0^\infty \sin(vr/2) \left[\frac{1}{2|u|} e^{-\frac{v^2 - u^2 + (1/2)^2}{2|u|} r} - 2 \sinh(ur/2) \psi(e^r) e^{r/2} \right] dr \tag{17}$$

and

$$F_c(u, v) = \int_0^\infty \cos(vr/2) \left[\frac{1}{2|u|} e^{-\frac{v^2-u^2+(1/2)^2}{2|u|}r} - 2 \cosh(ur/2) \psi(e^r) e^{r/2} \right] dr \quad (18)$$

and the following antiderivative was used

$$\begin{aligned} \int_0^\infty e^{-ax} \sin bxdx &= \frac{b}{a^2 + b^2}, \quad a > 0 \\ \int_0^\infty e^{-ax} \cos bxdx &= \frac{a}{a^2 + b^2}, \quad a > 0. \end{aligned} \quad (19)$$

Furthermore, F_s and F_c are written into the Laplace transformation as follows

$$\begin{aligned} F_s(u, v) &= \int_0^\infty \sin(vr/2) \left[\frac{1}{2|u|} e^{-\frac{v^2-u^2+(1/2)^2}{2|u|}r} - 2 \sinh(ur/2) \psi(e^r) e^{r/2} \right] dr \\ &= \Im(\mathcal{L}[f_s(r; u, v)H(r)](\mu)), \end{aligned} \quad (20)$$

and

$$\begin{aligned} F_c(u, v) &= \int_0^\infty \cos(vr/2) \left[\frac{1}{2|u|} e^{-\frac{v^2-u^2+(1/2)^2}{2|u|}r} - 2 \cosh(ur/2) \psi(e^r) e^{r/2} \right] dr \\ &= \Re(\mathcal{L}[f_c(r; u, v)H(r)](\mu)), \end{aligned} \quad (21)$$

where $H(r)$ is the Heaviside step function,

$$\mu = i\frac{v}{2} = \lambda + i\omega, \quad (22)$$

$$f_s(r; u, v) = \frac{1}{2|u|} e^{-\frac{v^2-u^2+(1/2)^2}{2|u|}r} - 2 \sinh(ur/2) \psi(e^r) e^{r/2}, \quad (23)$$

and

$$f_c(r; u, v) = \frac{1}{2|u|} e^{-\frac{v^2-u^2+(1/2)^2}{2|u|}r} - 2 \cosh(ur/2) \psi(e^r) e^{r/2}, \quad (24)$$

For F_s and F_c , we have

$$\begin{aligned} \lambda &= 0, \\ \omega &= \frac{v}{2}. \end{aligned} \quad (25)$$

The initial value theorem of the Laplace transformation shows that

$$\begin{aligned} F_s(0, v) &= \Re\left(\lim_{\mu \rightarrow \infty} \mathcal{L}[f_s(r; u, v)H(r)](\mu)\right) = f_s(0^-; u, v) \lim_{\mu \rightarrow \infty} \frac{1}{\mu} = 0, \\ F_c(0, v) &= \Im\left(\lim_{\mu \rightarrow \infty} \mathcal{L}[f_c(r; u, v)H(r)](\mu)\right) = f_c(0^-; u, v) \lim_{\mu \rightarrow \infty} \frac{1}{\mu} = 0. \end{aligned} \quad (26)$$

F_s and F_c are simultaneously zero for $u = 0$. Now let us consider the case of $u \neq 0$. To do this, further computation of F_s and F_c are necessary.

$$\begin{aligned} \mathcal{L}[f_s(r; u, v)H(r)](\mu) &= \mathcal{L}\left[\frac{1}{2|u|} e^{-\frac{v^2-u^2+(1/2)^2}{2|u|}r} H(r)\right](\mu) - 2\mathcal{L}\left[\sinh(ur/2) \psi(e^r) e^{r/2} H(r)\right](\mu) \\ &= \mathcal{L}\left[\frac{1}{2|u|} H(r)\right]\left(\mu + \frac{v^2 - u^2 + (1/2)^2}{2|u|}\right) - \mathcal{L}[\psi(e^r)H(r)](\mu - 1/2 - u/2) \\ &\quad + \mathcal{L}[\psi(e^r)H(r)](\mu - 1/2 + u/2), \end{aligned} \quad (27)$$

and

$$\begin{aligned}
\mathcal{L}[f_c(r; u, v)H(r)](\mu) &= \mathcal{L}\left[\frac{1}{2|u|}e^{-\frac{v^2-u^2+(1/2)^2}{2|u|}r}H(r)\right](\mu) - 2\mathcal{L}\left[\cosh(ur/2)\psi(e^r)e^{r/2}H(r)\right](\mu) \\
&= \mathcal{L}\left[\frac{1}{2|u|}H(r)\right]\left(\mu + \frac{v^2-u^2+(1/2)^2}{2|u|}\right) - \mathcal{L}[\psi(e^r)H(r)](\mu - 1/2 - u/2) \\
&\quad - \mathcal{L}[\psi(e^r)H(r)](\mu - 1/2 + u/2).
\end{aligned} \tag{28}$$

To calculate the Laplace transformation of $\psi(e^r)$, we expand $\psi(e^r)$ into the Taylor series as follows

$$\psi(e^r) = \sum_{n=1}^{\infty} e^{-n^2\pi e^r} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} (-n^2\pi e^r)^k = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} (-n^2\pi)^k e^{kr}. \tag{29}$$

Using the frequency shifting of the Laplace transformation, we obtain the following calculation.

$$\begin{aligned}
\mathcal{L}\left[\frac{1}{2|u|}H(r)\right]\left(\mu + \frac{v^2-u^2+(1/2)^2}{2|u|}\right) &= \frac{1}{2|u|} \frac{1}{\mu + \frac{v^2-u^2+(1/2)^2}{2|u|}} \\
&= \frac{1}{2|u|} \frac{\hat{\mu} + \frac{v^2-u^2+(1/2)^2}{2|u|}}{\left(\mu + \frac{v^2-u^2+(1/2)^2}{2|u|}\right)\left(\hat{\mu} + \frac{v^2-u^2+(1/2)^2}{2|u|}\right)} \\
&= \frac{1}{2|u|} \frac{\lambda + \frac{v^2-u^2+(1/2)^2}{2|u|} - i\omega}{\left(\mu + \frac{v^2-u^2+(1/2)^2}{2|u|}\right)\left(\hat{\mu} + \frac{v^2-u^2+(1/2)^2}{2|u|}\right)},
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
\mathcal{L}[\psi(r)H(r)](\mu) &= \mathcal{L}\left[\sum_{n=1}^{\infty} e^{-n^2\pi e^r} H(r)\right](\mu) = \sum_{n=1}^{\infty} \mathcal{L}\left[e^{-n^2\pi e^r} H(r)\right](\mu) \\
&= \sum_{n=1}^{\infty} \mathcal{L}\left[\sum_{k=0}^{\infty} \frac{1}{k!} (-n^2\pi e^r)^k H(r)\right](\mu) = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} (-n^2\pi)^k \mathcal{L}[H(r)](\mu - k) \\
&= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!(\mu - k)} (-n^2\pi)^k = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(\hat{\mu} - k)}{k!(\mu - k)(\hat{\mu} - k)} (-n^2\pi)^k \\
&= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(\lambda - k)}{k!(\mu - k)(\hat{\mu} - k)} (-n^2\pi)^k - i\omega \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!(\mu - k)(\hat{\mu} - k)} (-n^2\pi)^k \\
&= \lambda\Lambda(\mu) - \chi(\mu) - i\omega\Lambda(\mu),
\end{aligned} \tag{31}$$

where $\hat{\mu}$ is the complex conjugate of μ ,

$$\Lambda(\mu) = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!(\mu - k)(\hat{\mu} - k)} (-n^2\pi)^k = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(\mu - k)(\hat{\mu} - k)} \frac{(-n^2\pi)^k}{k!}, \tag{32}$$

and

$$\chi(\mu) = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{k}{k!(\mu - k)(\hat{\mu} - k)} (-n^2\pi)^k = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{k}{(\mu - k)(\hat{\mu} - k)} \frac{(-n^2\pi)^k}{k!}. \tag{33}$$

$\Lambda(\mu)$ and $\chi(\mu)$ are real valued functions and the exponential generating functions. Let the two exponential generating functions be

$$E_1(-n^2\pi; \mu) = \sum_{k=0}^{\infty} \frac{1}{(\mu - k)(\hat{\mu} - k)} \frac{(-n^2\pi)^k}{k!}, \quad (34)$$

and

$$E_2(-n^2\pi; \mu) = \sum_{k=0}^{\infty} \frac{k}{(\mu - k)(\hat{\mu} - k)} \frac{(-n^2\pi)^k}{k!}. \quad (35)$$

From the definition equations (34, 35), we have

$$E_2(-n^2\pi; \mu) = (-n^2\pi) \frac{dE_1(-n^2\pi; \mu)}{d(-n^2\pi)}. \quad (36)$$

Now, the functions $\Lambda(\mu)$ and $\chi(\mu)$ are rewritten as follows

$$\Lambda(\mu) = \sum_{n=1}^{\infty} E_1(-n^2\pi; \mu), \quad (37)$$

and

$$\chi(\mu) = \sum_{n=1}^{\infty} E_2(-n^2\pi; \mu) = \sum_{n=1}^{\infty} (-n^2\pi) \frac{dE_1(n; \mu)}{d(-n^2\pi)}. \quad (38)$$

Because

$$0 < E_1(-n^2\pi; \mu) = \sum_{k=0}^{\infty} \frac{1}{(\mu - k)(\hat{\mu} - k)} \frac{(-n^2\pi)^k}{k!} < \sum_{k=0}^{\infty} \frac{(-n^2\pi)^k}{k!} = e^{(-n^2\pi)}, \quad (39)$$

we have

$$0 < \Lambda(\mu) = \sum_{n=1}^{\infty} E_1(-n^2\pi; \mu) < \sum_{n=1}^{\infty} e^{(-n^2\pi)}. \quad (40)$$

Also

$$0 < \frac{dE_1(-n^2\pi; \mu)}{d(-n^2\pi)} = \sum_{k=1}^{\infty} \frac{1}{(\mu - k)(\hat{\mu} - k)} \frac{(-n^2\pi)^{k-1}}{(k-1)!} < \sum_{k=1}^{\infty} \frac{(-n^2\pi)^{k-1}}{(k-1)!} = e^{(-n^2\pi)}, \quad (41)$$

which gives

$$(-n^2\pi)e^{(-n^2\pi)} < (-n^2\pi) \frac{dE_1(-n^2\pi; \mu)}{d(-n^2\pi)} = \sum_{k=1}^{\infty} \frac{(-n^2\pi)}{(\mu - k)(\hat{\mu} - k)} \frac{(-n^2\pi)^{k-1}}{(k-1)!} < 0. \quad (42)$$

Summing equation (42) with respect to n gives

$$\sum_{n=1}^{\infty} (-n^2\pi)e^{(-n^2\pi)} < \chi(\mu) < 0. \quad (43)$$

Substituting equations (30) and (31) into equations (27) and (28) gives

$$\begin{aligned} \mathcal{L}[f_s(r; u, v)H(r)](\mu) &= \frac{1}{2|u|} \frac{\lambda + \frac{v^2 - u^2 + (1/2)^2}{2|u|} - i\omega}{\left(\mu + \frac{v^2 - u^2 + (1/2)^2}{2|u|}\right) \left(\hat{\mu} + \frac{v^2 - u^2 + (1/2)^2}{2|u|}\right)} \\ &\quad - (\lambda - 1/2 - u/2)\Lambda(\mu - 1/2 - u/2) - \chi(\mu - 1/2 - u/2) - i\omega\Lambda(\mu - 1/2 - u/2) \\ &\quad + (\lambda - 1/2 + u/2)\Lambda(\mu - 1/2 + u/2) - \chi(\mu - 1/2 + u/2) + i\omega\Lambda(\mu - 1/2 + u/2), \end{aligned} \quad (44)$$

and

$$\begin{aligned} \mathcal{L}[f_c(r; u, v)H(r)](\mu) &= \frac{1}{2|u|} \frac{\lambda + \frac{v^2 - u^2 + (1/2)^2}{2|u|} - i\omega}{\left(\mu + \frac{v^2 - u^2 + (1/2)^2}{2|u|}\right) \left(\hat{\mu} + \frac{v^2 - u^2 + (1/2)^2}{2|u|}\right)} \\ &\quad - (\lambda - 1/2 - u/2)\Lambda(\mu - 1/2 - u/2) - \chi(\mu - 1/2 - u/2) - i\omega\Lambda(\mu - 1/2 - u/2) \\ &\quad - (\lambda - 1/2 + u/2)\Lambda(\mu - 1/2 + u/2) - \chi(\mu - 1/2 + u/2) - i\omega\Lambda(\mu - 1/2 + u/2). \end{aligned} \quad (45)$$

Therefore

$$\begin{aligned} F_s = \Re(\mathcal{L}[f_s(r; u, v)H(r)](\mu)) &= \frac{1}{2|u|} \frac{\lambda + \frac{v^2 - u^2 + (1/2)^2}{2|u|}}{\left(\mu + \frac{v^2 - u^2 + (1/2)^2}{2|u|}\right) \left(\hat{\mu} + \frac{v^2 - u^2 + (1/2)^2}{2|u|}\right)} \\ &\quad - (\lambda - 1/2 - u/2)\Lambda(\mu - 1/2 - u/2) - \chi(\mu - 1/2 - u/2) \\ &\quad + (\lambda - 1/2 + u/2)\Lambda(\mu - 1/2 + u/2) - \chi(\mu - 1/2 + u/2), \end{aligned} \quad (46)$$

and

$$\begin{aligned} F_c = \Im(\mathcal{L}[f_c(r; u, v)H(r)](\mu)) &= \frac{1}{2|u|} \frac{-\omega}{\left(\mu + \frac{v^2 - u^2 + (1/2)^2}{2|u|}\right) \left(\hat{\mu} + \frac{v^2 - u^2 + (1/2)^2}{2|u|}\right)} \\ &\quad - \omega\Lambda(\mu - 1/2 - u/2) - \omega\Lambda(\mu - 1/2 + u/2). \end{aligned} \quad (47)$$

Because $\omega = v/2 \neq 0$, $F_c = 0$ only if

$$\frac{1}{2|u|} \frac{1}{\left(\mu + \frac{v^2 - u^2 + (1/2)^2}{2|u|}\right) \left(\hat{\mu} + \frac{v^2 - u^2 + (1/2)^2}{2|u|}\right)} + \Lambda(\mu - 1/2 - u/2) + \Lambda(\mu - 1/2 + u/2) = 0. \quad (48)$$

Substituting equation (48) into equation (46) for the fraction gives

$$\begin{aligned} F_s &= - \left[\lambda + \frac{v^2 - u^2 + (1/2)^2}{2|u|} \right] [\Lambda(\mu - 1/2 - u/2) + \Lambda(\mu - 1/2 + u/2)] \\ &\quad - (\lambda - 1/2 - u/2)\Lambda(\mu - 1/2 - u/2) - \chi(\mu - 1/2 - u/2) \\ &\quad + (\lambda - 1/2 + u/2)\Lambda(\mu - 1/2 + u/2) - \chi(\mu - 1/2 + u/2) \\ &= - \left[2\lambda - 1/2 - u/2 + \frac{v^2 - u^2 + (1/2)^2}{2|u|} \right] \Lambda(\mu - 1/2 - u/2) \\ &\quad - \left[1/2 - u/2 + \frac{v^2 - u^2 + (1/2)^2}{2|u|} \right] \Lambda(\mu - 1/2 + u/2) - \chi(\mu - 1/2 - u/2) - \chi(\mu - 1/2 + u/2). \end{aligned} \quad (49)$$

For F_s , $\lambda = 0$ and $\mu = iv/2$. Therefore

$$\begin{aligned}
F_s &= \left[1/2 + u/2 - \frac{v^2 - u^2 + (1/2)^2}{2|u|} \right] \Lambda(iv/2 - 1/2 - u/2) \\
&\quad + \left[-1/2 + u/2 - \frac{v^2 - u^2 + (1/2)^2}{2|u|} \right] \Lambda(iv/2 - 1/2 + u/2) \\
&\quad + \chi(iv/2 - 1/2 - u/2) + \chi(iv/2 - 1/2 + u/2) \\
&= 0,
\end{aligned} \tag{50}$$

that is

$$\begin{aligned}
&\left[\frac{v^2 - u^2 + (1/2)^2}{2|u|} - u/2 - 1/2 \right] \Lambda(iv/2 - 1/2 - u/2) \\
&\quad + \left[\frac{v^2 - u^2 + (1/2)^2}{2|u|} - u/2 + 1/2 \right] \Lambda(iv/2 - 1/2 + u/2) \\
&= \chi(iv/2 - 1/2 - u/2) + \chi(iv/2 - 1/2 + u/2).
\end{aligned} \tag{51}$$

Because $\chi(\mu) < 0$, we have

$$\begin{aligned}
&\left[\frac{v^2 - u^2 + (1/2)^2}{2|u|} - u/2 - 1/2 \right] \Lambda(iv/2 - 1/2 - u/2) \\
&\quad + \left[\frac{v^2 - u^2 + (1/2)^2}{2|u|} - u/2 + 1/2 \right] \Lambda(iv/2 - 1/2 + u/2) \\
&< 0.
\end{aligned} \tag{52}$$

Rearranging equation (52) gives

$$\begin{aligned}
&\left[\frac{v^2 - u^2 + (1/2)^2}{2|u|} - u/2 \right] [\Lambda(iv/2 - 1/2 - u/2) + \Lambda(iv/2 - 1/2 + u/2)] \\
&< \frac{1}{2} [\Lambda(iv/2 - 1/2 - u/2) - \Lambda(iv/2 - 1/2 + u/2)].
\end{aligned} \tag{53}$$

Therefore, because $\Lambda(\mu) > 0$, we have

$$\left[\frac{v^2 - u^2 + (1/2)^2}{2|u|} - u/2 \right] < \frac{1}{2}, \tag{54}$$

which is

$$\frac{v^2 - u^2 + (1/2)^2}{2|u|} - u/2 - 1/2 < 0, \tag{55}$$

that is to say

$$v^2 - u^2 + (1/2)^2 - u|u| - |u| < 0. \tag{56}$$

If $u > 0$, then we have

$$v^2 - 2u^2 + (1/2)^2 - u < 0, \tag{57}$$

which is equivalent to

$$v^2 < 2u^2 + u - (1/2)^2. \quad (58)$$

Otherwise, if $u < 0$, then we have

$$v^2 + (1/2)^2 + u < 0, \quad (59)$$

that is to say

$$v^2 < -u - (1/2)^2. \quad (60)$$

Summarizing the results of equations (26), (58) and (60), F_s and F_c are simultaneously zero only when $u = 0$. The Riemann hypothesis is true.

4 Conclusion

The Riemann hypothesis is proved true by choosing properly linear combinations of the real and imaginary parts of the Riemann ξ -function.

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