Iteration Orbit Control and Orbit Step Length Construction Method for the 3X+1 Conjecture

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Abstract: By setting the 3x + 1 iterative exponential orbit to construct and solve the initial number, and allowing the running orbit of this initial number to iterate under the control of the exponential orbit, this construction method proves that the step length of the 3x + 1 iterative orbit for specific types of numbers can be arbitrary or even infinitely long.
Keywords: Collatz conjecture; number theory; mathematical trajectory; cycle, combination

1.Introduction

I. Symbolic Meanings and basic definitions

Let $N^+ = (1,2,3...)$ denote the set of positive integers, $x, k, n_k, f^k(x) \in N^+$,

$x \equiv f^k(x) \equiv 1 (\mod 2)$,

$$f(x) = \frac{3x+1}{2^{n_1}}$$

$$f^2(x) = f(f(x)) = \frac{3f(x)+1}{2^{n_2}}$$

(1.0) $$f^k(x) = f(f^{k-1}(x)) = \frac{3f^{k-1}(x)+1}{2^{n_k}}$$

$f^k(x)$ is called the iterated function of $f(x)$, where

$$x_{0,} f^1(x_0), f^2(x_0), f^3(x_0), \ldots, \ f^k(x_0)$$

is called the iterated orbit of f(x)

Let $x_0, t, n_k, k \in N^+$, $x_0 > 1$, $(2, x_0 x_k) = 1$, x_0 be odd numbers, then there is

(1.1) $$2^{n_k} x_k = 3x_{k-1} + 1 \quad , \quad n_0 = 0$$

It is called the recurrence formula of the 3x + 1 conjecture; x_0 is called the

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initial number (hereinafter specified as an odd number unless otherwise stated), If in (1.0), we set $x = x_0$

$$x_0, f^1(x_0) = x_1, f^2(x_0) = x_2, f^3(x_0) = x_3, \ f^4(x_0) = x_4, ...$$

In this way, we have established a strict equivalence relationship between (1.0) and (1.1).

Thus, we establish a strict equality relationship between (1.0) and (1.1). From (1.0), we know that the operation carried out step by step starting from the initial number x_0 is an iterative operation; therefore, the (3x+1) operation is an iterative operation.This paper conducts analysis and proof based on (1.1), which is of great help in avoiding the use of heuristic or probabilistic methods to solve problems ,Please refer to [1] for the research background on the 3x+1 conjecture. facilitate subsequent proofs, we make the following definitions based on (1.1):

Definition 1: denote $T_t(x_0) = \{x_0, x_1, x_2, ..., x_t\}$, It is called the odd orbit iterated from the attain number x_0 to the cutoff number x_t, Call t the iteration step size

Definition 2: Denote $z_t(x_0) = \{n_1, \ n_2, ..., n_t\}$, which is called the even exponent orbit of iteration from the initial number x_0 to the cut-off number x_t (hereinafter referred to as the exponent orbit for short),If x_k is taken as the initial number, denote the exponent orbit as $z_t(x_k) = \{n_{k+1}, \ n_{k+2}, ..., n_t\}$.

Definition 3: denote $E_t(x_0) = n_1 + n_2 + ... + n_t$, $O_t(x_0) = t$, which are

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respectively called the total number of iterations over even numbers and the total number of iterations over odd numbers from the initial number x_0 up to the number x_t; denote $D_t(x_0)=E_t(x_0)+O_t(x_0)$, which is called the total number of iterations. If x_k is taken as the initial number, then $E_t(x_k)=n_{k+1}+n_{k+2}+...+n_{k+t}$, $O_t(x_k)=t$ etc.

Because it is often used in subsequent studies, we list here a few basic equations.Substituting (1.1) item by item, we have

$$x_k=\frac{1}{2^{n_k}}(3x_{n-1}+1)$$
$$=\frac{1}{2^{n_k}}\left(\frac{3}{2^{n_{k-1}}}\left(\frac{3}{2^{n_{k-2}}}\left(\cdots\left(\frac{3}{2^{n_1}}(3x_0+1)+1\right)\cdots+1\right)+1\right)+1\right)$$

To simplify and combine definitions 3, we have

(1.2) $$2^{E_k(x_0)}x_k=3^k x_0+\sum_{n=0}^{k-1}3^{k-n-1}\times 2^{E_n(x_0)},\quad E_0(x_0)=0$$

If x_k is taken as the initial number and x_{k+a} is taken as the terminal number), the above formula can be rewritten as

(1.3) $$2^{E_a(x_k)}x_{k+a}=3^a x_k+\sum_{n=0}^{a-1}3^{a-n-1}\times 2^{E_n(x_k)},\quad E_0(x_k)=0$$

Among them, $E_0(x_k)=0$ is mainly for the convenience of writing the equation and does not represent the actual value.

II. The 3x + 1 Conjecture （Collatz Conjecture） and problem-solving idea

1.Conjecture (3x + 1 or Collatz Conjecture) : Set $x_0\in N^+$, then x_0 has the total energy x_k=1 through finite step iteration

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2.There is an important conjecture in the 3x + 1 conjecture: the iterative step size of any number is finite .

3. Problem - solving Idea: The difficulty in proving the 3x + 1 conjecture lies in that the iterative orbit from the initial number x_0 to the terminal number x_t cannot be accurately predicted except through specific calculations. The problem - solving idea of this paper is that for a given arbitrary exponential orbit $z_t(x)=\{n_1, n_2,...,n_t\}$, solve for the initial number x and observe the iterative law of x. This is a brand - new problem - solving tool and provides a new idea for solving the 3x + 1 conjecture.

III. This paper will prove that:

Constructive theorem: set $h,k\in N^{+}$, $Z_k(x)$={1,1,1,....1}, where k is any positive integer, then it can always be constructed

(1.4) $$x=(2^{k+1}-1)+2^{k+1}h$$

Let the orbit $T_k(x)$ iterate under the control of the exponential orbit $Z_k(x)$ and have

(1.5) $$\lim_{k\to\infty} x_k = \infty$$

2. Lemma

Lemma 1: Set $a,b,k,m\in N^{+}$, $2^m a+b$, $b<2^m$, $(ab, 2)=1$, $k<m$, When performing k odd-numbered iterative operations on $2^m a+b$, b respectively, there is

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(2.1) $$E_k(b)=E_k(a.2^m+b),\quad E_k(b)<\mathrm{m}$$

Prove that by mathematical induction, when k =1, b, $a.2^{\mathrm{m}}+b$ is substituted into (1.1) and combined with the definition 3, we have

(2.2) $$2^{E_1(b)}x_1=3b+1$$

(2.3) $$2^{E_1\left(2^m a+b\right)}x_1^{/}=3\left(2^m a+b\right)+1$$

By sorting out (2.3), we have

(2.4) $$2^{E_1\left(a.2^m+b\right)}\left(x_1^{/}-3.a.2^{m-E_1\left(a.2^m+b\right)}\right)=3b+1$$

Compared with (2.2)(2.4), it is obvious that $E_1(b)=E_1\left(a.2^m+b\right),\ E_1(b)<m$, then the lemma is true.

When k=t, in terms of (1.2) associative definition 3, we have

(2.5) $$2^{E_t(b)}x_t=3^t b+\sum_{n=0}^{t\text{-}1}3^{t\text{-}n\text{-}1}\times 2^{E_n(b)}$$

(2.6) $$2^{E_t\left(2^m a+b\right)}\left(x_t^{/}\text{-}3^t.2^{m-E_t\left(2^m a+b\right)}.a\right)=3^t b+\sum_{n=0}^{t\text{-}1}3^{t\text{-}n\text{-}1}\times 2^{E_n\left(2^m a+b\right)}$$

Assuming that the left sides of equations (2.5) and (2.6) are equal, we have

(2.7) $$3^t b+\sum_{n=0}^{t\text{-}1}3^{t\text{-}n\text{-}1}\times 2^{E_n\left(2^m a+b\right)}\text{-}3^t b+\sum_{n=0}^{t\text{-}1}3^{t\text{-}n\text{-}1}\times 2^{E_n(b)}=0$$

Compare (2.5)(2.6) to the right, we have

(2.8) $$E_t(b)=E_t\left(a.2^m+b\right),\ E_t(b)<m$$

For k= t+1,According to (1.2) and definition 3 we have

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（2.9）
$$2^{E_{t+1}(b)}x_{t+1}=3^{t+1}b+\sum_{n=0}^{t}3^{t-n-1}\times 2^{E_n(b)}$$
$$=3^{t+1}b+2^{E_t(b)}+3\left(\sum_{n=0}^{t-1}3^{t-n-1}\times 2^{E_n(b)}\right)$$

（2.10）
$$2^{E_{t+1}(2^m a+b)}\left(x'_{t+1}-3^{t+1}.2^{m-E_{t+1}(2^m a+b)}.a\right)=3^{t+1}b+\sum_{n=0}^{t}3^{t-n-1}\times 2^{E_n(2^m a+b)}$$
$$=3^{t+1}b+2^{E_{t+1}(2^m a+b)}+3\sum_{n=0}^{t-1}3^{t-n-1}\times 2^{E_n(2^m a+b)}$$

(2.10)(2.9) minus the left hand side of the equation, combined with (2.7)(2.8) we have

（2.11）
$$\left(2^{E_t(2^m a+b)}-2^{E_t(b)}\right)+3\left(\sum_{n=0}^{t-1}3^{t-n-1}\times 2^{E_n(2^m a+b)}-\sum_{n=0}^{t-1}3^{t-n-1}\times 2^{E_n(b)}\right)=0$$

Compare (2.10)(2.9) to the right hand side of the equation

(2.12)
$$E_{t+1}(b)=E_{t+1}(a.2^m+b),\ E_{t+1}(b)<m$$

Then the lemma is true and the lemma is proved. For other types of proofs of this lemma, see[2].

Lemma 2: Set $x,h\in N^+$, $z_k(x)=\{n_1,\ n_2,\ldots,n_k\}$ be the exponent orbit of the iteration of any positive odd integer x . By choosing an appropriate h , we can always construct

（2.13）
$$x=r+2^{E_{k+1}(x_0)}h,\quad 0<r<2^{E_{k+1}(x_0)}$$

Which

（2.14）
$$-r_k=\sum_{n=0}^{k}3^{k-n}2^{E_k(x)},\quad E_0(x)=0$$

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such that $T_{k+1}(x)$ iterates under the control of $z_k(x)$.

Proof: Set $z_k(x)=\{n_1, n_2,...,n_k\}$,Construct the following equation based on Definition 3 and (2.14)

(2.15) $$R \equiv 3^{\varphi\left(2^{E_{k+1}(x)}\right)-k-1} r_k \equiv 3^{2^{E_{k+1}(x)-1}-k-1} r_k \left(\bmod 2^{E_{k+1}(x)}\right)$$

Where $\varphi\left(2^{E_{k+1}(x)}\right)=2^{E_{k+1}(x)-1}$ is an Euler function, $0<R<2^{E_{k+1}(x)}$.

Set

(2.16) $$r \equiv 2^{E_{k+1}(x)}+R\left(\bmod\ 2^{E_{k+1}(x)}\right)$$

Where $0<r<2^{E_{k+1}(x)}$, there is always an integer h, such that

(2.17) $$x=r+2^{E_{k+1}(x)}h$$

Obviously, as long as the appropriate value of h is chosen, the initial number x can be determined, such that $T_k(x)=\{x,x_1,..x_k\}$ iterates under the control of $z_k(x)$, according to the definition3, $E_{k+1}(x)=E_k(x)+n_{k+1}$, *where* $n_{k+1}\geq 1$, According to Lemma 3, $E_k(x)+m,$ for $m=1,2,3...$ and this can only be determined by substituting term by term and performing trial calculations. Let's illustrate with an example.

Let an arbitrary exponential orbit be $Z_6(x)=\{2,1,3,2,1,3\}$. Obviously, $E_1(x)=2, E_2(x)=2+1=3, E_3(x)=2+1+3=6,..., E_6(x)=12$. Substituting into (2.14), we have:

$$\begin{aligned}-r_6 &= \sum_{n=0}^{7-1} 3^{k-n-1} 2^{E_k(x)}\\ &=\left(3^6+3^5.2^2+3^4.2^3+3^3.2^6+3^2.2^8+3.2^9+2^{12}\right)\\ &=12013\end{aligned}$$

Substituting $E_7(x)=E_6(x)+1=13$ into (2.15) , we have

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$$R \equiv 3^{2^{E_6(x)}-7} r_6 \equiv 3^{4089} \times (-12013) \equiv -7527 \left(\bmod 2^{13}\right)$$

Substituting (2.16) we have

$$r \equiv 2^{E_6(x)+1} + R = 8192 - 7527 \equiv 665 \left(\bmod 2^{13}\right)$$

The x solution is:

(2.18) $$x_{01} = 665 + 2^{13} h_1$$

By the same token, substituting (2.15) for $E_7 = E_6(x)+2=14$, we have

$$R \equiv 3^{2^{E_6(x)+2}-7} r_6 \equiv -7527 \left(\bmod 2^{14}\right)$$

The x solution is:

(2.19) $$x_{02} = 8857 + 2^{14} h_2$$

Take h_1=0,1,h_2=0,1 to verify the following:

Table of Iteration Orbits of Odd Numbers Under the Control of Exponential Orbits

Tabel1:

		x	x_1	x_2	x_3	x_4	x_5	x_6
$T_6(x_{01})$	$h_1=0$	665	499	749	281	211	317	119
	$h_1=1$	8857	6643	9965	3737	2805	4205	1577
$Z_6(x_{01})$		0	2	1	3	2	1	3
$T_6(x_{02})$	$h_2=0$	8857	6643	9965	3737	2805	4205	1577
	$h_2=1$	41625	31219	46829	17561	13171	19757	7409
$Z_6(x_{02})$		0	2	1	3	2	1	3

In this example, when $E_7(x)=E_6(x)+1=13$ satisfies the iteration under the control of $z_6(x)$ orbit, so (2.18) is the minimum solution, and other solutions can be deduced from this minimum solution.

Lemma 4: Set $z_k(x)=\{n_1,n_2,...n_k\}$ be the exponential orbit of the iteration of any positive odd number x , and $T_k(x)=\{x,x_1,x_2,..x_k.\}|x_i\neq x_j,1\leq i,j\leq k$ be the odd - number orbit generated under the control of $z_k(x)$,If

(2.20) $$\frac{E_m(x)}{m}\leq\frac{E_{m\text{-}1}(x)}{m\text{-}1}<\frac{\log 3}{\log 2},\quad 2\leq m\leq k$$

Then there is always

(2.21) $$x<x_1<x_2,\ldots<x_k$$

Prove by mathematical induction: When k = 1 , $n_1=\frac{E_1(x)}{1}<\frac{\log 3}{\log 2}$, x is the initial number, Substitute into (1.1), we have

$$\frac{x_1}{x}=\frac{3}{2^{n_1}}\left(1+\frac{1}{3x}\right)>1$$

(2.22)

The lemma is true.

Suppose that the lemma is true when $k=m-1$, that is, $\frac{E_{\text{m-1}}(x)}{\text{m-1}}<\frac{\log 3}{\log 2}$, $n_{m-1}<\frac{\log 3}{\log 2}$ associated with defining 3, we have

(2.23) $$n_m=E_m(x)\text{-}E_{m-1}(x)=\frac{E_m(x)}{m}m-\frac{E_{m-1}(x)}{m-1}(m-1)$$
$$=\left[\frac{E_m(x)}{m}\text{-}\frac{E_{m-1}(x)}{m-1}\right]+\frac{E_{m-1}(x)}{m-1}$$

substitutes $\frac{E_m(x)}{m}\text{-}\frac{E_{m\text{-}1}(x)}{m\text{-}1}\leq 0$ into (2.23) , there is always

(2.24)
$$n_m<\frac{\log 3}{\log 2}$$

So, when $k=m$, the lemma is also true. According to (1.1) , we have

(2.25) $$\frac{x_m}{x_{m-1}}=\frac{3}{2^{n_m}}\left(1+\frac{1}{3x_{m-1}}\right)$$

Obviously, when (2.24) is substituted into (2.25) , there is

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always $3/2^{n_m} > 1$, we have

(2.26) $$x_{m-1} < x_m$$

Substituting (1.1) One by one, we have

(2.27) $$x < x_1 < x_2, \ldots < x_k$$

End of proof of lemma.

Corollary: Under the control of the exponential orbit $z_k(x) = \{1,1,\ldots 1\}$ for the odd - orbit $T_k(x)=\{x,x_1,x_2,..x_k\}$, $x_i \neq x_j, 1 \leq i,j \leq k$, there is always

(2.28) $$x < x_1 < x_2, \ldots < x_k$$

This corollary can be directly deduced from the lemma. Readers can prove it by themselves, and no further elaboration will be given here. Due to the importance of this corollary in the subsequent proof, we will conduct a calculation verification here. We take $z_9(x) = \{1,1,1,1,1,1,1,1,1\}$ and substitute it into (2.14). Then we have:

$$\begin{aligned} -r_9 &= \begin{bmatrix} 3^9 + 3^8.2 + 3^7.2^2 + 3^6.2^3 + 3^5.2^4 + 3^4.2^5 \\ +3^3.2^6 + 3^2.2^7 + 3\ .2^8 + 2^9 \end{bmatrix} \\ &= 58025 \end{aligned}$$

Substitute into (2.15):

$$R \equiv 3^{\varphi(\mathrm{E}_9(x)+1)-k} r_9 \equiv \text{-}58025 \times 3^{502} = \text{-}1 \left(\bmod 2^{10}\right)$$

Substitute into (2.16):

$$r \equiv 2^{10} - 1 \equiv 1023 \left(\bmod\ 2^{10}\right)$$

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Substitute into (2.17):

（2.29） $$x = 1023 + 2^{10} h$$

Taking h = 0,1,2 and substituting in (2.29) , we have $x = 1023,2047,3071$ and substituting in (1.1) , the calculations are validated as follows

Table of Iteration Orbits of Odd Numbers Under the Control of

Exponential Orbits

Tabel2:

	x	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
$T_9(x)$	1023	1353	2303	3455	5183	7775	11663	17495	26243	39365
	2047	3071	4607	6911	10367	15551	23327	34991	52487	78731
	3071	4067	6911	1032	15551	23327	34991	52487	78731	118097
$Z_9(x)$	0	1	1	1	1	1	1	1	1	1

It is proved that the exponential orbits of $T_9(x)$ are $z_9(x) = \{1,1,1,1,1,1,1,1,1\}$ and satisfy $x < x_1 < x_2 ,... < x_9$.

3. Proof of the theorem

Construction theorem: Set $h,k \in N^+$, where k is any positive integer,$Z_k(x)$={1,1,1,....1}, can always be constructed

（3.1） $$x = (2^{k+1} - 1) + 2^{k+1} h$$

Let the orbit $T_k(x)$ iterate under the control of the exponential orbit $Z_k(x)$, and there is

（3.2） $$\lim_{k \to \infty} x_k = \infty$$

Proof: in terms of $Z_k(x)$={1,1,1,....1}, in conjunction with Lemma2 (2.14) , always construct

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(3.3)
$$-r_k=\sum_{n=0}^{k}3^{k-n}2^{E_k(x)}=\frac{3^{k+1}-2^{k+1}}{3-2}$$

Constructed from (2.15) total energy

(3.4)
$$R\equiv 3^{\varphi(2^{k+1})-k-1}r_k\equiv 3^{2^k-k-1}r_k\equiv -1\left(\bmod 2^{k+1}\right)$$

There $\varphi\left(2^{k+1}\right)=2^k$ is an Euler function, $0<R<2^{k+1}$, constructed from (2.16) total energy

(3.5)
$$r\equiv 2^{k+1}-1\left(\bmod\ 2^{k+1}\right)$$

constructed from (2.17) total energy

(3.6)
$$x=\left(2^{k+1}-1\right)+2^{k+1}h$$

Obviously, as long as an appropriate value of h is chosen, the initial number x can be determined, so that $T_k(x)=\{x,x_1,...x_k\}$ iterates under the control of $z_k(x)$. According to Definition 3 $E_{k+1}(x)=E_k(x)+1$, $n_{k+1}=1$, and there is always

(3.7)
$$\frac{E_m(x)}{m}=\frac{E_{m-1}(x)}{m-1}=1\ \left|2\le m\le k\right.$$

According to Lemma 4 and the corollary, at this time, $x<x_1<x_2,...<x_k$, and obviously there is

(3.8)
$$\lim_{k\to\infty}x_k=\infty$$

Theorem proof, the theorem readers can combine (3.1) , lemma 4 inference selfcalculation verification.

4. Several Explanations

(1) The exponential orbit $Z_k(x)=\{1,1,1,....1\}$in the construction theorem is

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a form of digital characteristics. There are many such orbits. As long as they meet the conditions of Lemma 4, countless odd - number orbits $T_k(x)$ with arbitrary step lengths can be constructed under the control of the exponential $Z_k(x)$ orbit.

(2) According to (3.5), the odd iterative step size of an odd number $2^{k+1}-1$ is greater than k. Therefore, for a specific positive integer, regardless of the total iterative step size, the user-defined iterative step size k can be arbitrarily long or even infinitely large.

(3) Even if we can construct an initial number x with a very large step size, it is still unknown whether x_k=1 can necessarily be achieved through iterations with a finite step size. However, through the construction theorem, we have proved a conjecture in the 3x+1 conjecture, that is, the problem of "the iterative step size of any positive integer under 3x+1 iteration is always finite." Obviously, the answer is negative. For specific numbers, there exist many initial numbers with arbitrary step sizes or even infinitely large step sizes.

References

[1] Terence Tao, Almost all orbits of the Collatz map attain Almost bounded values, Forum of Mathematics, PI (2022) , Vol. 10: e121-56doi: 10.1017/FMP.2022.8.

[2] Jeffrey C. Lagarias The 3X + 1problem: An Overview, arXiv: 2111.02635v1[math. NT]4NOV 2021, Speculate C 2.

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[3] R. Terras, a stopping time problem on the positive integers, Acta Arith. 30(1976) , 241-252

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