Opening the Sieve of Eratosthenes, and Some Theorems on the Distribution of Primes

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Abstract

The Sieve of Eratosthenes is taken as a definition of primes and is examined in a way that "opens" it into an array of rows labeled as primes and columns labeled as numbers. Through the introduced concept of prime candidates, numbers in each row that have the potential of being declared primes in lower rows, the opened Sieve reveals repeating and inter-related patterns of these prime candidates as well as other defined entities. These allow development of a number of relations that prove useful in examining the distribution of primes, with some new theorems being proved. Included is a new and independent proof for Bertrand's postulate, two proofs for $\lim_{n\to\infty}\left((p_{n+1}-p_n)/p_n\right)=0$, and proofs of conjectures by Brocard, Legendre, Andrica, and Oppermann.

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1. INTRODUCTION

In the approach here to investigating the distribution of primes, new concepts, terminology, notation, and auxiliary functions are introduced, much of which is essential in allowing development and proof of theorems. The Sieve of Eratosthenes [1, 2] is taken as a definition of primes and is "opened up" to display patterns and relations. Many of these patterns are shown to be repetitive and are made expressible in finite closed form by means of generalizing the concept of repetends. The concept, notations, and theorems on generalized repetends are introduced in **Appendix B**. The expository part of this appendix and first two practical use examples there can be read in isolation of the rest of the paper, but must be read before continuing with **Section 4** and beyond where the notation and relations developed are used. The third practical use example can be read after the introduction of the function $S_i(N)$ in **Subsection 2.3**. Because there are a large number of new concepts and notations introduced in this paper, **Appendix A** summarizes these for quick reference. Not all the notations and relations developed are used towards the proofs in **Section 7**. None of **Section 3** is used for this. However, this avoids having to "re-invent these wheels" for any future use. **Appendix D** at the end holds the figures. These are used to assist intuition and provide empirical results. The overall style here is highly pedantic, in order to increase accessibility for novices in number theory.

2. THE OPENED SIEVE OF ERATOSTHENES

2.1 First Form of the Opened Sieve of Eratosthenes. For this paper, with the one exception N, which is non-negative real unless otherwise specified, all variables are non-negative integers, with ranges specified. To make the inner relations in the opened

Sieve of Eratosthenes overt and clearer visually, and therefore easier to discover, the display of the first form of the opened Sieve of Eratosthenes in **Figure 1** uses a minimum

Place Figure 1 here

number of mutually distinct distinguishing notations/symbols. Location is shown by a grid pattern of cells in rows and columns. Each column corresponds to a number j, where $j = 0,1,2,\ldots$. Each row i is also labeled p_i,[1] where $i = 0,1,2,\ldots$, and shows a dark cell where the respective prime $p_i > 1$ divides $j > 1$. If $j = 0$ or the number j in a row labeled p_i is divisible by a smaller prime, the cell is otherwise blank. The cell with $j = 1$ and $p_0 = 1$ is also dark. Otherwise, if the number j in the row labeled p_i remains a candidate for being a larger prime, the cell contains a 1. Let such values of j be called prime candidates, with $C_{i,n}$ the n^{th} prime candidate in the row labeled p_i, where $n = 1,2,\ldots$, and C_i an arbitrary prime candidate in this row. Therefore, a cell in row i with $p_i > 1$ contains a 1 if and only if the cell is not dark and the cell immediately above contains a 1.

The algorithm (algorithm step numbers begin with **A**) that results in the display of the first form of the opened Sieve of Eratosthenes is as follows:

A1. Start with a matrix array of blank cells $M(i, j)$, where $i, j = 0,1,2,\ldots$, out to any practical values desired. Additional labels for each row are to be determined by the rest of the algorithm following.

A2. In the first row, $i = 0$, darken the cell with $j = 1$, label row $i = 0$ with $p_0 = 1$, and place a 1 in each cell in this row from $j = 2$ onward. Therefore, $C_{0,n} = n + 1,\ \forall n \geq 1$.

A3. In the next row darken the cell immediately below the first cell in the row above having an entry 1. Call the value of j for this cell $j(\text{next } 1)$.[2] Label this next row i with $p_i = j(\text{next } 1)$. Row $i = 1$ is therefore labeled $p_1 = 2$.

A4. In this row, labeled $p_i = j(\text{next } 1)$, darken every $(p_i)^{\text{th}} = (j(\text{next } 1))^{\text{th}}$ cell after the first dark cell.

A5. Also, in this row labeled $p_i = j(\text{next } 1)$, for any cell that is not dark, insert an entry 1 if the cell immediately above contains a 1. Otherwise, leave the cell blank. Therefore, $C_{1,n} = 2n + 1,\ \forall n \geq 1$.

A6. Repeat steps **A3**, **A4**, **A5**, for $i = 2,3,4,\ldots$, out to any practical values i, j desired.

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[1] The value p_0 will be explained shortly. It is usual to denote primes by p_n, but we start with p_i to correspond better with our matrix notation $M(i, j)$ later in algorithm step **A1**.

[2] Using 1 as a symbol in and after algorithm step **A2** was not my first choice of symbol, but after formalizing step **A3** this became the most practical and simplest symbol to use.

The p_i that are generated by the above algorithm are, by definition, the primes, except that $p_0 = 1$, by usual convention, is not considered prime.[3] Since $p_0 = 1$ starts the algorithm, we call it an exceptional prime and the generated other p_i, regular primes.
This algorithm generates all primes (out to any practical value desired) by use of the opened Sieve of Eratosthenes and produces a display that allows analysis of the opened Sieve. From the above, we therefore have $C_{i,1} = p_{i+1}, \forall i \geq 0$.

In the remainder of this paper, the words cell, row and column mean, respectively, cell, row and column in the opened Sieve of Eratosthenes. Row i and the row labeled p_i are the same row. Also, the word Sieve means the Sieve of Eratosthenes.

The depiction in **Figure 1**, the first form of the opened Sieve of Eratosthenes for $i = 0,1,\ldots,16$ and $j = 0,1,\ldots,50$, shows explicitly where primes fall through and where other numbers get held up, and it admits to a number of relations/properties, most of which follow trivially from the algorithm.

Property 2.1.1. *In the row labeled* $p_i, i = 1,2,\ldots,$ *the dark cells are those with* j *divisible by* p_i. □

Property 2.1.2. *All* C_i *and (regular) primes greater than* $p_1 = 2$ *are odd.*

Proof: By **A3** and **A4** and the construct definition that only column numbers $j > 2$ with a 1 in a cell in the row labeled p_i can possibly be prime. □

Property 2.1.3. *No two adjacent cells in row* $i \geq 1$ *can both contain a* 1.

Proof: By **A3** and **A4**. □

Property 2.1.4. *In any row labeled* p_i, *where* $i \geq 1$, *cells with entry* 1 *must have column numbers* j *that differ by a multiple of* 2.

Proof: By **Property 2.1.2**. □

Theorem 2.1.5. *Primes* $p_i \geq 5$ *and prime candidates* C_i *in any row labeled* $p_i \geq 5$ *are all of the form* $6n \pm 1$, *where* $n \geq 1$.

Proof: This is true for $p_3 = 5$ and all regular prime candidates in the row labeled $p_2 = 3$ and therefore for all prime candidates in all lower rows since they inherit the underlying structure of the regular prime candidates in the row labeled p_2. Also, only

[3] It is usual to not consider $p_0 = 1$ a prime in order to preserve uniqueness of prime factorization in the Fundamental Theorem of Arithmetic.

prime candidates can be primes, so primes must exhibit the form of the prime candidates from which they derive. □

Property 2.1.6. *Once a cell is dark in **Figure** 1, no cell below in the same column can contain a* 1.

Proof: By **A5**. □

Theorem 2.1.7. *For any* $i = 1,2,3,...$, *the only cells in a row labeled* p_i *that are dark when no cells above in the same column are dark are those that have* $j = p_i^{1+A_i} p_{i+1}^{A_{i+1}} \cdots p_{i+n}^{A_{i+n}}$, *where* $n \geq 0$ *and* $A_i, A_{i+1}, \ldots, A_{i+n} \geq 0$.

Proof: For any $i = 1,2,3,...$, the only cells in a row labeled p_i that are dark when no cells above in the same column are dark are those that have j divisible by p_i but relatively prime to each of $p_1, p_2, ..., p_{i-1}$. Since all (regular) primes are relatively prime to each other, the Fundamental Theorem of Arithmetic means such j can only be of the form stated. □

For any $i \geq 0$, let $C_{i,0} = p_i$. Call the $C_{i,0}$ special prime candidates and the $C_{i,n}$ with $n \geq 1$ regular prime candidates. We will have occasion to use C_i as a non-specific prime candidate in row i (also labeled p_i). In summary, so far, we have

$$C_{i,0} = p_i, \ \forall i \geq 0, \tag{2.1.1}$$

$$C_{i+1,0} = C_{i,1} = p_{i+1}, \ \forall i \geq 0, \tag{2.1.2}$$

$$C_{0,n} = n+1, \ \forall n \geq 0, \tag{2.1.3}$$

$$C_{1,n} = 2n+1, \ \forall n \geq 1. \tag{2.1.4}$$

Theorem 2.1.8. *For* $i, n \geq 1$, a *number* $j \geq 0$ *is some regular prime candidate* $C_{i,n}$ *if and only if* $j \not\equiv 0 (\mathrm{mod}\, p_k), \ \forall k = 1,2,\ldots,i$.

Proof: By **A3**, **A4** and **A5**. □

As a corollary, we have the following:

Corollary 2.1.9. *For any* $i \geq 0$, *the product of any two regular* C_i *is a regular* C_i.

Proof: The statement is trivial for $i = 0$ and holds for $i \geq 1$ by **Theorem 2.1.8**. □

2.2 Second Form of the Opened Sieve of Eratosthenes. We now add a step in the algorithm, as follows:

A7. In such cells as discussed in **Theorem 2.1.7** insert an X. These are cells that are dark and have a cell with content 1 immediately above.

For $i=1$, all dark cells contain an X. For $i \geq 2$, there are dark cells with no X as well as dark cells with an X. For example, for any $(i \geq 2;\ n \geq 1)$ and $j = np_{i-1}p_i$, the cell in row i is dark with no X, while the cell in row i with $j = 3 + 6(n-1)$ is dark with an X. In row $i \geq 1$, the cell with $j = C_{i-1,1} = C_{i,0} = p_i$ is dark and contains an X. Let $X_{i,n}$ be the value of the n^{th} column number of a cell in row i (also labeled p_i) containing an X, where $i, n \geq 1$.[4] Let X_i be an arbitrary $X_{i,n}$ for given i. Explicit evaluation of the $X_{i,n}$ and enumeration in certain intervals of column numbers will be done later.

We will have occasion to use three general types of intervals, the first two of which will now be defined. For any $i, n \geq 0$, let $\mathrm{u}_{i,n}$ be the left-closed, right-open interval $\left[np_i,\ (n+1)p_i\right)$. We will also occasionally let $\mathrm{u}_{i,n}$ represent the cells in these intervals in row i, depending on context. Let u_i be an arbitrary $\mathrm{u}_{i,n}$ in row i. For any $(i \geq 0;\ n \geq 1)$, let $\mathrm{U}_{i,n}$ represent the left-closed, right-open interval $\left[p_i + (n-1)\prod_{k=0}^{i} p_k,\ p_i + n\prod_{k=0}^{i} p_k\right)$ or the cells in row i in that interval, depending on context.[5] Let U_i be an arbitrary $\mathrm{U}_{i,n}$ in row i. Using the standard notation $|\ |$ for magnitude or length, we have the following property:

Property 2.2.1. $\left|\mathrm{U}_{i+1}\right| = p_{i+1}\left|\mathrm{U}_i\right|,\ \forall i \geq 0$. □

In the first cell (start) of each $\mathrm{U}_{i,n}$ for $i \geq 1$, place an S. If an S and an X coincide in any cell, place SX in that cell. We do not consider cell content S as part of the algorithm since it is not used to find primes. It merely denotes the start of a $\mathrm{U}_{i,n}$, for convenience. **Figure 2a** depicts what we now call the second form of the opened Sieve for $i = 0,1,2,\ldots,16$ and $j = 0,1,2,\ldots,50$. **Figures 2b**, **2c**, **2d**, and **2e** depict the second form of the opened Sieve for the same values of i, but for $j = 50,51,\ldots,100$, $j = 100,101,\ldots,150$, $j = 150,151,\ldots,200$, and $j = 200,201,\ldots,250$, respectively.

Place Figures 2a-2e here

Figures 2a-2e are used to establish empirical results used for some theorems.

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[4] We can also define $X_{i,0} = 0,\ \forall i \geq 0$, although cells corresponding to these do not contain an X. We will not be using such.

[5] The choice of the symbols $\mathrm{u}_{i,n}$ and $\mathrm{U}_{i,n}$ was dictated by the first being small "unit" intervals and the latter, large intervals. The symbol $\mathrm{u}_{i,n}$ should not be confused with $u(i,n)$ that will be defined later in **Property 2.3.18**.

Theorem 2.2.2. *If* $j_1 \equiv j_2 (\text{mod}\, p_k),\ \forall k = 1, 2, ..., i$, *then the cells in the same row* $i \geq 1$ *and with column numbers* $j_1 \geq p_i$ *and* $j_2 \geq p_i$ *are otherwise identical*, *i.e.*, *they are both dark or non-dark and are both otherwise blank or have the same contents* 1, X, *or* SX.

Proof: By **A2**, **A3**, …, **A7**, and by definition of the placement of S. □

Theorem 2.2.3. *For all* $(i \geq 0;\ n \geq 1)$, *every* $\mathrm{U}_{i,n}$ *other than* $\mathrm{U}_{0,1}$ *has at least one regular* C_i, *so there are an infinite number of* C_i.

Proof: For $i = 0$, this follows by the definitions of $\mathrm{U}_{0,n}$ and C_0. For $i \geq 1$, this follows from the definition of $\mathrm{U}_{i,n}$, the infinity of the number of primes [3], and statement (2.1.2). □

In the above construct, for rows $i \geq 1$, all cells with an S also contain an X, but the converse is not true in general (apart from the start of each $\mathrm{U}_{i,n}$) for rows $i \geq 3$, *i.e.*, there are cells with an X occurring not at the start of such $\mathrm{U}_{i,n}$. This will be evident later once we establish the number of X in each $\mathrm{U}_{i,n}$.

The second form of the opened Sieve more readily admits to interesting and useful relations than does the first form.

Unlike the U_i, there are $\mathrm{u}_{i,n}$, with $i, n \geq 1$, that contain no C_i. For example, the interval $[13^3 - 13,\ 13^3 - 1)$, which is $\mathrm{u}_{6,13^2-1} = \mathrm{u}_{6,168}$, contains no C_i. This interval can also be written as $\mathrm{u}_{6,C_{5,35}} = [2184, 2197)$ and can be found empirically by extending **Figures 2a-2e** further, out to at least $j = 2197$. Another such interval is $\mathrm{u}_{9,59} = \mathrm{u}_{9,C_{8,9}} = [1334, 1357)$, found incidentally in the same empirical search. Empirically, except for $\mathrm{u}_{0,1}$, all $\mathrm{u}_{i,n}$ with $(0 \leq i \leq 5;\ n \geq 1)$ contain a regular C_i, as will become evident in **Appendix C**, using the later statement (4.2.2). It will be shown later that every $\mathrm{u}_{i,1}$ and then also every $\mathrm{u}_{i,n}$ with $(i \geq 1;\ 2 \leq n \leq C_{i,1} - 1)$ contains a regular C_i.[6]

Theorem 2.2.4. *Given any row* $i \geq 0$ *and any cell in that row with* $j \geq p_i$, *except for* $(i = 0;\ j = 1)$, *the cell at distance* $\prod_{k=0}^{i} p_k$ *cells to the right has the same appearances, i.e.*, *is also dark or non-dark and has the same contents* 1, X, SX, *or is otherwise blank, just as the given cell.*

Proof: The statement holds for $i = 0$, empirically. It remains to consider $i \geq 1$.

[6] It is the proof of $p_i < C_{i,1} < 2p_i,\ \forall i \geq 1$, later that will allow a new proof of Bertrand's postulate.

At distance $\prod_{k=0}^{i} p_k$ to the right from the cell with column number j the cell has column number $j_{new} = j + \prod_{k=0}^{i} p_k$, say. Therefore, for $i \geq 1$, we have

$$\begin{aligned} j_{new} &\equiv (j + \prod_{k=0}^{i} p_k)(\text{mod } p_r), \ \forall r \geq 1 \\ &\equiv j(\text{mod } p_r), \ \forall (r = 1, 2, ..., i). \end{aligned}$$

Therefore, by **Theorems 2.1.8** and **2.2.2**, and by the definition of placement of S, the statement also holds for $i \geq 1$. □

Corollary 2.2.5. *In any given row* $i \geq 0$, *the pattern of any number of consecutive cells starting with column number* $j \geq p_i$, *except for* $(i = 0; j = 1)$, *is repeated indefinitely at multiples of distance* $\prod_{k=0}^{i} p_k$. □[7]

Corollary 2.2.6. *Excluding the cell with* $(i = 0; j = 1)$, *in any number of combinations of rows from among those labeled* $p_0, p_1, \ldots, p_i$, *the pattern in any number of combinations of columns starting at* $j \geq p_i$ *is repeated indefinitely at multiples of distance* $\prod_{k=0}^{i} p_k$. □

For any $i \geq 1$, $\left(\prod_{k=0}^{i} p_k\right) - m$ is not divisible by any of $p_1, p_2, \ldots, p_i$ if $\left(\prod_{k=0}^{i} p_k\right) + m$ is not, and vice versa. Therefore, **Corollary 2.2.5** gives the following corollary:

Corollary 2.2.7. *For any* $(i \geq 0;\ m \geq 1)$, *there is symmetry of cell patterns in row* i (*excluding content* S) *about* $N_{i,m} = \frac{m}{2} \prod_{k=0}^{i} p_k$, *as long as the left side does not extend left beyond column* $j = p_i$. *Also, any* $m\left(\prod_{k=0}^{i} p_k\right) \pm 1 > p_i$ *are always regular* C_i, *and any* $\frac{m}{2}\left(\prod_{k=0}^{i} p_k\right) \pm 2 > p_i$ *for odd* m *are always regular* C_i. □

[7] This corollary shows there are an infinite number of endlessly repeating finite patterns in the opened Sieve of Eratosthenes, a fairly startling result for a generator of all primes. Apart from the Prime Number Theorem and a few other relations, prime numbers up to now have appeared to display what appears otherwise to be fairly random distribution. This repeating of patterns was the stimulus for development of the generalization of repetends, of which the notations and theorems are used explicitly or implicitly in the proofs of many relations in this paper. These notations and theorems also allow explicit evaluation in closed form of various functions within the opened Sieve of Eratosthenes.

The product $\prod_{k=0}^{i} p_k$ occurs frequently enough in this paper to make it often convenient to use the abbreviation

$$\Pi_i = \prod_{k=0}^{i} p_k,\ \forall i \ge 0.^{8} \qquad (2.2.1)$$

From now on, whenever we say a cell, u_i, U_i or any such related grouping of cells contains or has (a) prime candidate(s) (in words or symbols), we mean the cell(s) under consideration has/have column number(s) which is/are (a) prime candidate(s). Therefore, "a cell contains/has a 1" and "a cell contains/has a regular prime candidate" will mean the same thing, and these will mean the same as "there is a 1" or "there is a regular prime candidate" at a location. Similarly for X. Also, in order to avoid overly pedantic statements, from now on $X_{i,n}$ and $C_{i,n}$ and the cells they are associated with will be equated. For example, "there is an X between $C_{i,n}$ and $C_{i,n+m}$" will then mean that in the row labeled p_i, between the cells whose column numbers are $C_{i,n}$ and $C_{i,n+m}$, there is a cell containing an X.

It is only the cells containing an X that can prevent any cell containing a 1 immediately in the row above from having its column number j prime.[9] The distribution of cells containing an X is therefore of paramount importance in determining the distribution of primes.

Theorem 2.2.8. *For any* $(i \ge 0;\ n \ge 1)$, *given any regular* $C_{i,n}$ *in any given* U_i, there exists an $r \ge 0$ such that $C_{i,n} \pm r\Pi_i$ *lies immediately above a cell with an* X *in the row labeled* p_{i+1} *and in any given* U_{i+1} *to the right. Each cell with a regular* $C_{i,n}$ *in the given* U_i *is associated in a* $1{:}1$ *fashion with a cell containing an* X *in the given* U_{i+1} *by an* r *that is unique* $(\bmod\, p_{i+1})$.

Proof: Empirically, the statement holds for $i = 0$. It then remains to consider $i \ge 1$. U_{i+1} has p_{i+1} times as many cells as U_i, by **Property 2.2.1**, and the start of any U_i does not have the same column number as the start of any U_{i+1}. Therefore, there are $p_{i+1} - 1$ entire copies of U_i within the column numbers of this (or any other) U_{i+1}. For arbitrary $i \ge 1$, consider an arbitrary cell with a 1, and therefore column number $C_{i,n}$ for some $n \ge 1$, in a U_i with its column numbers entirely within those of $\mathrm{U}_{i+1,1}$. For all $r = 0,1,2,\ldots,(p_{i+1} - 1)$, we have $C_{i,n} + r\Pi_i$ is one of the column numbers of this or some other U_{i+1}.

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[8] This suits our purposes better than the standard primorial $p_n\# = \prod_{k=1}^{n} p_k,\ \forall n \ge 1$.

[9] In row $i \ge 1$, we have $X_{i,1} = p_i$, but no other X_i can be a prime.

In the row labeled p_{i+1}, every $(p_{i+1})^{\text{th}}$ cell from the first in any U_{i+1} onward is dark, by **A4**.

Since p_{i+1} is relatively prime to all other primes, $(C_{i,n} + r\Pi_i)(\text{mod}\, p_{i+1})$ forms a complete residue system $(\text{mod}\, p_{i+1})$ for $r = 0, 1, 2, ..., p_{i+1} - 1$. There are also no repeats in this residue system, since there are p_{i+1} terms. Therefore, since every $(p_{i+1})^{\text{th}}$ cell from the first in any U_{i+1} is darkened, the cell with column number $C_{i,n} + r\Pi_i$ in the row labeled p_i contains a 1, by **Theorem 2.1.8**, and finds itself immediately above a dark cell in some U_{i+1} for exactly one of these values of *r*. This dark cell therefore contains an X, by **A7**. This holds for each regular $C_{i,n}$ in the initial U_i. Also, each such X finds itself below only one such translated $C_{i,n}$, because of the uniqueness of each residue.

By **Corollary 2.2.5**, the same holds between any U_i and any U_{i+1} to the right. □

2.3 Introduction of Some New Notation and Functions. For any $i, n \geq 1$, we have $X_{i,n}$ is the column number of the cell in row i with the n^{th} X in that row. By the algorithm, we then have

$$X_{i,1} = C_{i,0} = p_i, \ \forall i \geq 1. \tag{2.3.1}$$

For the other X_i, we have the following theorem:

Theorem 2.3.1. $X_{i+1,n+1} = p_{i+1} C_{i,n}, \ \forall (i \geq 0; \ n \geq 1)$.

Proof: This follows by **Theorems 2.1.7** and **2.1.8**. □

This theorem is more insightful than **Theorem 2.2.8**. However, **Theorem 2.2.8** also admits to some insight, as will be shown in **Theorem 3.1.1**.

The notation $|\ |$ can be generalized beyond its standard meaning of magnitude. Let $|\text{U}_i|_1$ be the number of cells with a 1 in any U_i,[10] *i.e.*, let $|\text{U}_i|_1$ be the number of regular C_i in any U_i. More generally, in the row labeled p_i, we can use the notation $|N|_{1_i}$ for the number of cells with a 1 or the number of regular C_i less than or equal to N. Similarly, let $|\text{U}_i|_\text{X}$ be the number of cells with an X in any U_i and $|N|_{X_i}$ the number of X_i less than or equal to N. Later we will also generalize $\lfloor\ \rfloor$ and $\lceil\ \rceil$ when we define $\lfloor N \rfloor_\text{p}$, $\lfloor N \rfloor_{C_i}$, $\lfloor N \rfloor_{X_i}$, and $\lceil N \rceil^{C_i}$.

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[10] This notation is not entirely satisfactory since 1 is being used both as a number and a symbol. It might be better to use $|\text{U}_i|_{C_i / C_{i,0}}$, but this is awkward.

Theorem 2.3.2. $\left|\mathrm{U}_{i+1}\right|_{\mathrm{X}}=\left|\mathrm{U}_{i}\right|_{1},\ \forall i\geq 1$.

Proof: By **Theorem 2.2.8**. □

The third type of general interval we will be considering, and which has already been alluded to in **Corollaries 2.2.5** and **2.2.6**, is defined by

$$\mathrm{I}_{i,n_i}(N_i)=\left[N_i+(n_i-1)\Pi_i,\ N_i+n_i\Pi_i\right),\ \text{for any } (i\geq 0;\ n_i\geq 1;\ N_i\geq p_i).$$ [11]

We have **Theorem 2.3.2** and **Corollary 2.2.5** give

$$\left|\mathrm{I}_{i+1,n_{i+1}}(N_{i+1})\right|_{\mathrm{X}}=\left|\mathrm{I}_{i,n_i}(N_i)\right|_{1},\ \forall(i\geq 0;\ n_i,n_{i+1}\geq 1;\ N_i\geq p_i;\ N_{i+1}\geq p_{i+1}). \qquad (2.3.2)$$

Similar to $0!=1$, define

$$\prod_{k=1}^{0}(p_k-1)=1. \qquad (2.3.3)$$ [12]

We now have the following theorem:

Theorem 2.3.3. *Excluding* $\mathrm{U}_{0,1}$, *we have* $\left|\mathrm{U}_{i+1}\right|_{\mathrm{X}}=\left|\mathrm{U}_{i}\right|_{1}=\prod_{k=1}^{i}(p_k-1),\ \forall i\geq 0$. (2.3.3.0)

Proof: By induction on i:
(a) By inspection of **Figure 2a** and by **Corollary 2.2.5**, the statement holds for $i=0$.
(b) Therefore, at least for $r=0$ and excluding $\mathrm{U}_{0,1}$, we have

$$\left|\mathrm{U}_{r+1}\right|_{\mathrm{X}}=\left|\mathrm{U}_{r}\right|_{1}=\prod_{k=1}^{r}(p_k-1). \qquad \text{(b-1)}$$

(c) $\left|\mathrm{U}_{r+1}\right|_{1}=p_{r+1}\left|\mathrm{U}_{r}\right|_{1}-\left|\mathrm{U}_{r+1}\right|_{\mathrm{X}}$, by **Property 2.2.1** and the effect of X

$$=p_{r+1}\prod_{k=1}^{r}(p_k-1)-\prod_{k=1}^{r}(p_k-1),\ \text{by (b-1)}$$

$$=\prod_{k=1}^{r+1}(p_k-1).$$

Therefore, $\left|\mathrm{U}_{r+2}\right|_{\mathrm{X}}=\left|\mathrm{U}_{r+1}\right|_{1}=\prod_{k=1}^{r+1}(p_k-1)$, by **Theorem 2.3.2**.

Therefore, (2.3.3.0) also holds for $i=r+1$ and, by induction, for all $i\geq 0$. □

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[11] The index i in n_i and N_i is only needed for comparisons such as in (2.3.2).

[12] We might be tempted to use $\prod_{j=0}^{i}(p_j-1)$ from now on instead of $\prod_{j=1}^{i}(p_j-1)$, but $\prod_{j=0}^{i}(p_j-1)=0,\ \forall i\geq 0$, and so is not usable.

Using (2.3.2) and excluding any overlap with the interval $[1,2)$, we have **Corollary 2.2.5** and **Theorem 2.3.3** give

$$\left|\mathrm{I}_{i+1,n_{i+1}}(N_{i+1})\right|_{\mathrm{X}} = \left|\mathrm{I}_{i,n_i}(N_i)\right|_1 = \prod_{k=1}^{i}(p_k-1),\ \forall(i\ge 0;\ n_i,n_{i+1}\ge 1;\ N_i\ge p_i;\ N_{i+1}\ge p_{i+1}).$$

We now have the following corollary:

Corollary 2.3.4. *In any row* $i \ge 2$ *there are cells with an* X *and no* S.

Proof: Since there is only one S in any U_i, it suffices to prove $\left|\mathrm{U}_{i+1}\right|_{\mathrm{X}} \ge 2,\ \forall i \ge 2$. This follows from **Theorem 2.3.3**, since $\prod_{k=1}^{2}(p_k-1)=2$. □

The next theorem gives the main connection between C_i and regular primes:

Theorem 2.3.5. *For any* $i \ge 0$, *all regular* C_i *less than* $X_{i+1,2}$ *are regular primes*.

Proof: We have $C_{i,1} = p_{i+1},\ \forall i \ge 0$, by (2.1.2)
$= X_{i+1,1}$, by (2.3.1).

By the algorithm, for any $i \ge 0$, no other regular C_i less than $X_{i+1,2}$ is eliminated from being inherited in lower rows except as a regular prime. □

Consider real $N \ge 0$ and let $x_i(N)$ be the number of cells with an X in row $i \ge 1$ and with column number $j \le N$. Whenever we consider integral values of N, we will specify so explicitly. The function $x_i(N)$ has the following 3 properties:

Property 2.3.6. $x_i(N)=0,\ \forall(i\ge 1;\ 0\le N<p_i)$. □

Property 2.3.7. $x_i(N_2)\ge x_i(N_1),\ \forall(i\ge 1;\ N_2>N_1\ge 0)$. □

Property 2.3.8. $x_i(N)=n,\ \forall(i,n\ge 1;\ X_{i,n}\le N<X_{i,n+1})$.[13] □

Let $x_i(N_1,N_2)=x_i(N_2)-x_i(N_1),\ \forall(i\ge 1;\ N_1,N_2\ge 0)$, where order is important. Let $S_i(N)$ be the number of regular C_i in cells in row $i \ge 0$ and with column number $j \le N$. Also let $S_i(N_1,N_2)=S_i(N_2)-S_i(N_1),\ \forall(i\ge 0;\ N_1,N_2\ge 0)$, where order is important. By the definitions, we have the following 7 properties:

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[13] We could subsume **Property 2.3.6** into **Property 2.3.8** by allowing $n \ge 0$ and using $X_{i,0}=0,\ \forall i\ge 1$, but we will not be using $X_{i,0}$ in this paper.

Property 2.3.9. $S_i(N) \geq 0;\ \forall(i \geq 0;\ N \geq 0)$.[14] □

Property 2.3.10. $S_i(N_1, N_2) \geq 0,\ \forall(i \geq 0;\ N_2 > N_1 \geq 0$. □

Property 2.3.11. $S_i(N_2) \geq S_i(N_1),\ \forall(i \geq 0;\ N_2 > N_1 \geq 0)$. □

Property 2.3.12. $S_i(N) = 0,\ \forall(i \geq 0;\ 0 \leq N < C_{i,1} = p_{i+1})$. □

Property 2.3.13. $S_i(N) = n,\ \forall(i, n \geq 0;\ C_{i,n} \leq N < C_{i,n+1})$. □

Property 2.3.14. $S_i(p_i, N) = S_i(N),\ \forall(i \geq 0;\ N \geq 0)$. □

Property 2.3.15. $S_i(N_1, N_2) = S_i(N_1, N_3) + S_i(N_3, N_2),\ \forall(i \geq 0;\ N_1, N_2, N_3 \geq 0)$. □

By the algorithm, we have the following 2 properties:

Property 2.3.16. $S_{i+1}(N) = S_i(N) - x_{i+1}(N),\ \forall(i \geq 0;\ N \geq 0)$. □

Property 2.3.17. $S_{i+1}(N_1, N_2) = S_i(N_1, N_2) - x_i(N_1, N_2),\ \forall(i \geq 0;\ N_1, N_2 \geq 0)$. □

The main connection between C_i and C_{i+1} is given by the following property in which we define the function $u(i,n)$:

Property 2.3.18. $C_{i+1,n} = C_{i,n+u(i+1,n)},\ \forall i, n \geq 0$, *where* $u(i+1,n) = x_{i+1}(C_{i+1,n})$.[15] □

As a corollary, we have the following:

Corollary 2.3.19. *For any* $i, n \geq 0$, *there exists an* $m \geq 1$, *such that* $C_{i+1,n} = C_{i,m}$. *Specifically,* $m = n + u(i+1,n) = n + x_{i+1}(C_{i+1,n})$. □

If, for any $(i \geq 0;\ n \geq 1)$, there is no X immediately below $C_{i,n}$, then

$$C_{i,n} = C_{i+1,n-v}, \text{ where } v = x_{i+1}(C_{i,n}). \tag{2.3.4}$$

If, for any $(i \geq 0;\ n \geq 1)$, there is an X immediately below $C_{i,n}$, then

$$C_{i+1,n-v} < C_{i,n} < C_{i+1,n-v+1}, \text{ where } v = x_{i+1}(C_{i,n}). \tag{2.3.5}$$

Summarizing (2.3.4) and (2.3.5), we have the following:

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[14] We use $(i \geq 0;\ N \geq 0)$ here instead of $i, N \geq 0$ since i is integral and N is real.

[15] The function $u(i,n)$ simplifies notation when used as a subscript, and should not be confused with $\mathrm{u}_{i,n}$.

Property 2.3.20. $C_{i+1,n-v} \le C_{i,n} < C_{i+1,n-v+1}$, $\forall(i \ge 0;\ n \ge 1)$, *where* $v = x_{i+1}(C_{i,n})$ *and equality holds if and only if there is no* X *immediately below* $C_{i,n}$. □

The difference $p_{i+1} - p_i$ occurs frequently enough from now on that, for convenience, we define

$$\Delta_i = p_{i+1} - p_i,\ \forall i \ge 0. \tag{2.3.6}$$

By **Property 2.1.2**, we have

$$\Delta_i \ge 2,\ \forall i \ge 2. \tag{2.3.7}$$

We also define

$$\Delta_{i,n} = C_{i,n} - p_i,\ \forall i, n \ge 0,$$

which then gives

$$\Delta_{i,0} = 0,\ \forall i \ge 0,$$

and

$$\begin{aligned} \Delta_{i,1} &= C_{i,1} - p_i \\ &= p_{i+1} - p_i, \text{ by (2.1.2)} \\ &= \Delta_i, \text{ by (2.3.6).} \end{aligned}$$

The next theorem connects X_i and X_{i+1}:

Theorem 2.3.21. $X_{i+1,n+1} = X_{i,n+u(i,n)+1} + \Delta_i C_{i,n}$, $\forall i, n \ge 1$.

Proof: $X_{i+1,n+1} = p_{i+1} C_{i,n}$, $\forall(i \ge 0;\ n \ge 1)$, by **Theorem 2.3.1**
$= (p_i + \Delta_i) C_{i,n}$, by (2.3.6)
$= p_i C_{i-1,n+u(i,n)} + \Delta_i C_{i,n}, \forall i \ge 1$, by **Property 2.3.18**
$= X_{i,n+u(i,n)+1} + \Delta_i C_{i,n}$, by **Theorem 2.3.1**. □

The following theorem can be considered the x_i-counterpart of **Theorem 2.3.1** or the converse of **Property 2.3.16**:

Theorem 2.3.22. $x_{i+1}(N) = 1 + S_i\left(\dfrac{N}{p_{i+1}}\right)$, $\forall(i \ge 0;\ N \ge p_{i+1})$.

Proof: By **Theorem 2.3.1**, we have $X_{i+1,n+1} = p_{i+1} C_{i,n}$, $\forall(i \ge 0;\ n \ge 1)$.
For $i \ge 0$ and $X_{i+1,1} = p_{i+1} \le N < p_{i+1} C_{i,1} = X_{i+1,2}$, we have

$$1 \le \frac{N}{p_{i+1}} < C_{i,1} = p_{i+1},$$

$$x_{i+1}(N) = 1, \text{ by **Property 2.3.8**,}$$

$$\text{and} \quad S_i\left(\frac{N}{p_{i+1}}\right) = 0 \text{, by } \textbf{Property 2.3.12},$$

so that

$$x_{i+1}(N) = 1 + S_i\left(\frac{N}{p_{i+1}}\right), \ \forall (i \geq 0;\ X_{i+1,1} \leq N < X_{i+1,2}). \qquad (2.3.22.1)$$

For any $(i \geq 0;\ n \geq 1)$ and $X_{i+1,n+1} = p_{i+1}C_{i,n} \leq N < p_{i+1}C_{i,n+1} = X_{i+1,n+2}$, we have

$$C_{i,n} \leq \frac{N}{p_{i+1}} < C_{i,n+1},$$

$$x_{i+1}(N) = n+1 \text{, by } \textbf{Property 2.3.8},$$

$$\text{and} \quad S_i\left(\frac{N}{p_{i+1}}\right) = n \text{, by } \textbf{Property 2.3.13},$$

so that

$$x_{i+1}(N) = 1 + S_i\left(\frac{N}{p_{i+1}}\right), \ \forall (i \geq 0;\ n \geq 1;\ X_{i+1,n+1} \leq N < X_{i+1,2}). \qquad (2.3.22.2)$$

Combining (2.3.22.1) and (2.3.22.2) gives

$$x_{i+1}(N) = 1 + S_i\left(\frac{N}{p_{i+1}}\right), \ \forall (i \geq 0;\ N \geq p_{i+1}). \ \square$$

Since

$$x_i(N_1, N_2) = x_i(N_2) - x_i(N_1), \ \forall (i \geq 1;\ N_1, N_2 \geq 0),$$

$$\text{and } S_i(N_1, N_2) = S_i(N_2) - S_i(N_1), \ \forall (i \geq 0;\ N_1, N_2 \geq 0),$$

we have **Theorem 2.3.22** gives the following corollary:

Corollary 2.3.23. $x_{i+1}(N_1, N_2) = S_i\left(\frac{N_1}{p_{i+1}}, \frac{N_2}{p_{i+1}}\right), \ \forall (i \geq 0;\ N_1, N_2 \geq p_{i+1}). \ \square$

3. Some Information on Distances Between C_i

3.1 Maximum Distances Between C_i. By **Theorem 2.2.3**, each U_i other than $\mathrm{U}_{0,1}$ contains at least one regular C_i. Also, the U_i are finite and repeat indefinitely. Therefore, for any $(i \geq 0; j \geq 1)$ there exists an $n \geq 0$ such that there is a maximum distance $C_{i,n+j} - C_{i,n}$ achieved in each row, with $C_{i,n}$ in $\mathrm{U}_{i,1}$.[16] For any $(i \geq 0; j \geq 1)$, let the norm

$$d_{i,j} = \left\| C_{i,n+j} - C_{i,n} \right\| \qquad (3.1.1)$$

be the maximum such distance in row i. In general, we have

$$d_{i,j} < d_{i,j+k}, \ \forall (i \geq 0;\ j,k \geq 1). \qquad (3.1.2)$$

---

[16] We do not consider the possibility of $j = 0$ since $C_{i,n} - C_{i,n} = 0$ is too trivial to be of interest.

By (2.1.3), we have $C_{0,n} = n+1,\ \forall n \geq 0$, so that

$$d_{0,j} = \left\| C_{0,n+j} - C_{0,n} \right\|,\ \forall j \geq 1, \text{ by (3.1.1)}$$
$$= j. \qquad (3.1.3)$$

Theorem 3.1.1. $d_{i+1,j} \geq d_{i,j+1},\ \forall (i \geq 0;\ j \geq 1)$.

Proof: For given $(i \geq 0;\ j \geq 1)$, let $n \geq 0$ be an integer such that

$$d_{i,j} = C_{i,n+j} - C_{i,n}. \qquad (3.1.1.1)$$

By **Theorem 2.2.8** and **Corollary 2.2.5**, there is an $r \geq 1$ such that $C_{i,n+j-1} + r\Pi_i$ lies immediately above an X.

Therefore, for such r and by **Theorem 2.2.4**, we have $C_{i,n+j-1} + r\Pi_i = C_{i,t+j-1}$, for some $t > n$.

There is a $w \geq 0$ such that $C_{i+1,w+j}$ is the first C_{i+1} to lie to the right of column number $C_{i,t+j-1}$.

Therefore, by the algorithm, we have

$$C_{i+1,w+j} \geq C_{i,t+j},$$

with equality holding if and only if there are no additional cells with an X in row $i+1$ between $N = C_{i,t+j-1}$ (excl.) and $N = C_{i+1,w+j}$ (incl.) }.[17] (3.1.1.2)

Also, since we now have at least one X between $C_{i+1,w}$ (excl.) and $C_{i+1,w+j}$ (excl.), we then have

$$C_{i+1,w} \leq C_{i,t-1}, \qquad (3.1.1.3)$$

so that

$$C_{i+1,w+j} - C_{i+1,w} \geq C_{i,t+j} - C_{i,t-1}, \text{ by (3.1.1.2) and (3.1.1.3)}$$
$$> C_{i,t+j} - C_{i,t}$$
$$= C_{i,n+j} - C_{i,n}, \text{ by } \textbf{Corollary 2.2.5}$$
$$= d_{i,j}, \text{ by (3.1.1.1).} \qquad (3.1.1.4)$$

But, by definition of $d_{i+1,j}$, we have $d_{i+1,j} \geq C_{i+1,m+j} - C_{i+1,m},\ \forall m \geq 1$, and in particular for $m = w$.

Therefore, we have $d_{i+1,j} > d_{i,j}$, by (3.1.1.4), and so $d_{i+1,j} \geq d_{i,j+1}$, by (3.1.2).

Since given $i \geq 0$ and $j \geq 1$ were otherwise arbitrary, we then have

$$d_{i+1,j} \geq d_{i,j+1},\ \forall (i \geq 0;\ j \geq 1). \ \square$$

As a result of (3.1.2), we have **Theorem 3.1.1** immediately gives the following corollary:

Corollary 3.1.2. $d_{i+1,j} > d_{i,j},\ \forall (i \geq 0;\ j \geq 1)$. □

---

[17] We use (excl.) to mean exclusive of or excluding and (incl.) to mean inclusive of or including.

Lemma 3.1.3. *Let* $d_{i,j} = C_{i,n+j} - C_{i,n}$ *, for some* $i, j \geq 1$*. Then,* $n \geq 1$.

Proof: It suffices to prove $n \neq 0$. We do this by contradiction.
Suppose we have

$$d_{i,j} = C_{i,j} - C_{i,0}, \text{ for some } i, j \geq 1. \tag{3.1.3.1}$$

Then,

$$d_{i,j} = (C_{i,j} + r\Pi_i) - (C_{i,0} + r\Pi_i),\ \forall r \geq 1.$$

But, there is no C_i at $(C_{i,0} + r\Pi_i)$, for any $r \geq 1$, contradicting the definition of $d_{i,j}$ and **Corollary 2.2.5**.
Therefore, (3.1.3.1) is not possible.
Therefore, $n \neq 0$. □

Theorem 3.1.4. *For any* $i, j \geq 1$*, let* $n \geq 1$ *be an integer such that* $d_{i,j} = C_{i,n+j} - C_{i,n}$*. Then, there is at least one* X *between* $C_{i,n}$ *(excl.) and* $C_{i,n+j}$ *(excl.).*

Proof: By contradiction:
Suppose, for some $i, j, n \geq 1$, we have

$$d_{i,j} = C_{i,n+j} - C_{i,n} \text{ and no X between } C_{i,n} \text{ (excl.) and } C_{i,n+j} \text{ (excl.).} \tag{3.1.2.1}$$

Then, there is an $m \geq 1$ such that

$$C_{i,n} = C_{i-1,m}, \text{ by **Corollary 2.3.19**,} \tag{3.1.2.2}$$

and

$$C_{i,n+j} = C_{i-1,m+j}, \text{ by assumption (3.1.2.1) and the algorithm.} \tag{3.1.2.3}$$

Therefore,

$$\begin{aligned} d_{i,j} &= C_{i,n+j} - C_{i,n} \\ &= C_{i-1,m+j} - C_{i-1,m}, \text{ by (3.1.2.1) and (3.1.2.2).} \end{aligned} \tag{3.1.2.4}$$

But, by definition of $d_{i-1,j}$, we have $d_{i-1,j} \geq C_{i-1,k+j} - C_{i-1,k}$, $\forall k \geq 1$, and in particular for $k = m$.
Therefore, $d_{i-1,j} \geq d_{i,j}$, by (3.1.2.4), contradicting **Corollary 3.1.2**.
Therefore, assumption (3.1.2.1) is false. □

By (3.1.3), we have $d_{0,j} = j,\ \forall j \geq 1$. The next theorem provides a (lowest possible) lower bound for $d_{i,j}$, for all $i, j \geq 1$.

Theorem 3.1.5. $d_{i,j} = \left\| C_{i,n+j} - C_{i,n} \right\| \geq 2(i + j - 1)$, $\forall i, j \geq 1$. (3.1.5.0)

Proof: By induction on i:
(a) $C_{1,0} = 2, C_{1,1} = 3$, and $C_{1,n+j} = C_{1,n} + 2j$, $\forall j, n \geq 1$, by (2.1.4).
Therefore, the statement holds for $i = 1$.
(b) Therefore, at least for $k = 1$, we have

$$d_{k,j} = \left\| C_{k,m+j} - C_{k,m} \right\| \geq 2(k+j-1) \text{, } \forall j \geq 1 \text{.} \quad \text{(b-1)}$$

(c) Let

$$d_{k+1,j} = C_{k+1,n+j} - C_{k+1,n} \text{, for some } n \geq 1 \text{.} \quad \text{(c-1)}$$

Then, by **Theorem 3.1.4**, there is at least one X between $C_{k+1,n}$ (excl.) and $C_{k+1,n+j}$ (excl.).

Therefore, by **Corollary 2.3.19** and the algorithm, there is an $m \geq 1$ such that

$$C_{k+1,n} = C_{k,m} \text{ and } C_{k+1.n+j} \geq C_{k,m+j+1} \text{.} \quad \text{(c-2)}$$

Therefore,

$$\begin{aligned} d_{k+1,j} &= C_{k+1,n+j} - C_{k+1,n}, \text{ by (c-1)} \\ &\geq C_{k,m+j+1} - C_{k,m}, \text{ by (c-2)} \\ &\geq 2(k+j+1-1), \text{ by (b-1)} \\ &= 2\big((k+1)+j-1\big). \end{aligned}$$

Therefore, (3.1.5.0) also holds for $i = k+1$ and, by induction, for all $i \geq 1$. □

Theorem 3.1.6. *There is no upper bound on* $p_{n+1} - p_n$ *as* n *increases.*

Proof: Given any $i, j \geq 1$, there is an $m \geq 1$ such that $d_{i,j} = C_{i,m+j} - C_{i,m}$, with such a $C_{i,m}$ in each U_i.[18]

In particular, this holds for $j = 1$.

Since there can be no primes between $C_{i,m}$ (excl.) and $C_{i,m+1}$ (excl.), there must be successive primes p_n and p_{n+1}, with $p_n \leq C_{i,m}$ and $C_{i,m+1} \leq p_{n+1}$, so that

$$\begin{aligned} p_{n+1} - p_n &\geq d_{i,1} \\ &\geq 2i \text{, by **Theorem 3.1.5**.} \end{aligned}$$

Therefore, as i increases without bound, there must be successive primes p_n and p_{n+1} with $p_{n+1} - p_n$ increasing without bound. Also, n increases without bound to allow this, since p_{n+1}, and therefore $p_{n+1} - p_n$, remains finite otherwise. □

The usual proof that there can be any number of consecutive composite numbers considers the sequence $(N!+2), (N!+3), ..., (N!+N)$ for integer $N \geq 2$. All these are greater than but divisible by at least one of $2, 3, ..., N$, and so must form a sequence of $N-1$ consecutive composite numbers. The path to **Theorem 3.1.6** is more involved, but has at least three advantages. It shows where at least some of the strings of consecutive composites of ever increasing length are (at least two such strings of fixed length starting in each $\text{U}_{i,k}$ for i and k large enough, because of the symmetry in **Corollary 2.2.7**). It also adds coherence to the subject of distribution of primes by using the same information

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[18] Because of the symmetry in **Corollary 2.2.6**, there are actually at least 2 such $C_{i,m}$ in each $\text{U}_{i,k}$, for k large enough in comparison with j.

from the analysis of the opened Sieve of Eratosthenes that has been used here to derive other relations. Thirdly, its method is ultimately transparent, not relying on a "trick" method having only limited and superficial connection with the relations at hand. There is also the additional benefit of having another proof for a statement.

From here on, the notation and theorems in **Appendix B** concerning generalized repetends will be used. The notation {…} will denote exclusively a listing, as defined in **Appendix B**, and not a set. With use, the notation and basic theorems on generalized repetends quickly feel like a natural extension of arithmetic. Because of this, no formal referencing to the notation and used theorems in **Appendix B** will be made in the rest of this paper.

4. THE FUNCTIONS $t(i,n)$ AND $t(i,m,n)$

4.1 Slopes of $S_i(N)$ and $S_i(np_i)$ Over Intervals of Length Π_i. Excluding $(i=0;\ n=1;\ 1 \le N < 2)$, we have **Theorem 2.3.3** and **Corollary 2.2.5** give

$$\frac{S_i(N+n\Pi_i)-S_i\left(N+(n-1)\Pi_i\right)}{\Pi_i}=\frac{\prod_{j=1}^{i}(p_j-1)}{\prod_{j=0}^{i}p_j},\ \forall(i\ge 0;\ n\ge 1;\ N\ge p_i).$$ [19]

$$=\left\{1_{i=0},\left(\frac{1}{2}\right)_{i=1},\left(\frac{2}{6}\right)_{i=2},\left(\frac{8}{30}\right)_{i=3},\dots\right\}. \qquad (4.1.1)$$

Since $p_{i+1} \ge p_i+2,\ \forall i\ge 2$, the element values in (4.1.1) strictly decrease as a function of i for all $i \ge 0$, since $\frac{p_i-1}{p_i}<1,\ \forall i \ge 1$.

However,

$$\frac{S_i(N+n\Pi_i)-S_i\left(N+(n-1)\Pi_i\right)}{\Pi_{i-1}}=\frac{\prod_{j=1}^{i}(p_j-1)}{\prod_{j=0}^{i-1}p_j},\ \forall(i,n\ge 1;\ N\ge p_i)$$

$$=\left\{1_{i=1},1_{i=2},\left(\frac{8}{6}\right)_{i=3},\left(\frac{48}{30}\right)_{i=4},\dots\right\}, \qquad (4.1.2)$$

where the element values in (4.1.2) strictly increase as a function of i for all $i \ge 2$, since $\frac{p_i-1}{p_{i-1}}>1,\ \forall i\ge 3$. In other words, the slope of $S_i(N)$ between the ends of the interval

$$I_{i,n}(N)=\left[N+(n-1)\Pi_i,\ N+n\Pi_i\right),\text{ for any }(i\ge 0;\ n\ge 1;\ N\ge p_i), \qquad (4.1.3)$$

---

[19] In statement (2.3.3), we defined $\prod_{j=1}^{0}(p_j-1)=1$, similar to $0!=1$.

with respect to cell length is a strictly decreasing function of i for all $i \geq 0$. But with respect to the number of u_i in the interval this slope is a strictly increasing function of i for all $i \geq 2$. This property is one justification for defining the function $t(i,n)$, which we introduce next.

4.2 Introduction of the Functions $t(i,n)$ and $t(i,m,n)$. We have $p_0 = 1$. Also, (2.1.3) gives $C_{0,n} = n+1,\ \forall n \geq 0$. Therefore,

$$S_0(N) = \begin{Bmatrix} 0,\ \forall 0 \leq N < 1 \\ \lfloor N - p_0 \rfloor,\ \forall N \geq 1 \end{Bmatrix}$$
$$\geq \lfloor N - p_0 \rfloor,\ \forall N \geq 0.$$

Empirically,

$$S_1(N) = \begin{cases} 0,\ \forall 0 \leq N < 1 \\ \left\lfloor \dfrac{N - p_1 + 1}{p_1} \right\rfloor,\ \forall N \geq 1 \end{cases}$$
$$\geq \left\lfloor \frac{N - p_1 + 1}{p_1} \right\rfloor,\ \forall N \geq 0,$$

and $$S_2(N) \geq \left\lfloor \frac{N - p_2 + 1}{p_2} \right\rfloor,\ \forall N \geq 0.$$

Define $T_i(N)$, by

$$S_i(N) = \left\lfloor \frac{N - p_i + 1}{p_i} \right\rfloor + T_i(N),\ \forall (i \geq 1;\ N \geq 0). \tag{4.2.1}$$

Therefore,

$$S_i\big((n+1)p_i\big) = \left\lfloor \frac{(n+1)p_i - p_i + 1}{p_i} \right\rfloor + T_i\big((n+1)p_i\big),\ \forall (i \geq 1;\ n \geq 0)$$
$$= n + t(i,n),\ \text{where}\ t(i,n) = T_i\big((n+1)p_i\big).$$

This gives

$t(i,n+1) \geq t(i,n) - 1,\ \forall (i \geq 1;\ n \geq 0)$, with equality holding if and only if there is no regular C_i in $\mathrm{u}_{i,n+1}$, for any $(i \geq 1;\ n \geq 0)$. (4.2.2)

We can extend this definition of $t(i,n)$ to include $i = 0$ by defining

$$S_i\big((n+1)p_i\big) = n + t(i,n),\ \forall i,n \geq 0.\text{[20]} \tag{4.2.3}$$

Because of the properties of the slopes discussed for the intervals in (4.1.3), we have $S_{i+1}(np_{i+1}) > S_i(np_i),\ \forall i \geq 2$, and $t(i+1,n) > t(i,n),\ \forall i \geq 2$, at least from some value of n on.[21] Using **Figure 2a**, we have empirically that

$$t(0,n) = t(1,n) = t(2,n) = 0,\ \forall n \geq 0,$$

[20] Since there is no regular C_0 in $\mathrm{u}_{0,1}$, this changes the range of n in (4.2.2) for $i = 0$ to $n \geq 1$.

[21] The value of this n for given i will be determined later in **Theorem 5.1.10**.

and $t(3,n) = \{\ddot{0}_{n=0}, 0_1, 1_2, 2_3, 2_4, 2_5, \ddot{2}_6, \ldots\}$.

The values for $t(4,n)$ and $t(5,n)$ are given in **Appendix C**.

For any $(i \geq 0;\ N \geq p_i)$,

let $\lfloor N \rfloor_{C_i}$ be the greatest C_i, special or regular, less than or equal to N, (4.2.4)

and

let $\lceil N \rceil^{C_i}$ be the least C_i, special or regular, greater than or equal to N. (4.2.5)

Since, for any $(i, n \geq 0;\ N \geq p_i)$ we have **Property 2.3.13** gives

$$S_i(N) = n \text{ if and only if } C_{i,n} = \lfloor N \rfloor_{C_i}, \tag{4.2.6}$$

we then have (4.2.3) gives

$$C_{i,n+t(i,n)} = \lfloor (n+1)p_i \rfloor_{C_i},\ \forall i, n \geq 0, \tag{4.2.7}$$

or

$$C_{i,n+t(i,n)} \leq (n+1)p_i,\ \forall i, n \geq 0, \tag{4.2.8}$$

and

$$C_{i,n+t(i,n)+1} > (n+1)p_i,\ \forall i, n \geq 0. \tag{4.2.9}$$

Although this is analogous to the situation with integral floor functions, there is a difference. No regular C_i is a multiple of p_i, for any $i \geq 1$. Therefore, (4.2.8) and (4.2.9) give

$$C_{i,n+t(i,n)} < (n+1)p_i < C_{i,n+t(i,n)+1},\ \forall i, n \geq 1. \tag{4.2.10}$$

For any $i \geq 1$, equality in (4.2.8) then holds if and only if $n = 0$.

Setting $n = 1$ in (4.2.10) gives $C_{i.1+t(i,1)} < 2p_i < C_{i,2+t(i,1)},\ \forall i \geq 1$.

Since $2p_i < C_{i,0} = p_i$ is impossible, we must then have $t(i,1) \geq -1,\ \forall i \geq 1$. The value $t(i,1) = -1$ holds if and only if $C_{i,1} = p_{i+1}$ does not lie in the open interval $(p_i, 2p_i)$.

In **Theorem 5.2.1** we will prove $t(i,n) \geq 0,\ \forall i, n \geq 0$.

We can generalize definition (4.2.1) as follows:

$$S_i(C_{i,m}, N) = \left\lfloor \frac{N - C_{i,m} + 1}{C_{i,m}} \right\rfloor + T_i(C_{i,m}, N),\ \forall (i \geq 1;\ m \geq 0;\ N \geq 0), \tag{4.2.11}$$

so that

$$S_i\left(C_{i,m}, (n+1)C_{i,m}\right) = \left\lfloor \frac{(n+1)C_{i,m} - C_{i,m} + 1}{C_{i,m}} \right\rfloor + T_i\left(C_{i,m}, (n+1)C_{i,m}\right),\ \forall (i \geq 1;\ m, n \geq 0)$$

$$= n + t(i,m,n), \text{ where } t(i,m,n) = T_i\left(C_{i,m}, (n+1)C_{i,m}\right).^{22} \tag{4.2.12}$$

The variables m and n in $t(i,m,n)$ are not interchangeable.

As for $t(i,n)$ in (4.2.3), we can extend the definition of $t(i,m,n)$ in (4.2.12) to include $i = 0$ by defining

[22] By now it is more evident that the main reason for initially defining t is that it is much less cumbersome to use than T, especially as part of the second subscript index for C_i.

$$S_i\left(C_{i,m},(n+1)C_{i,m}\right)=n+t(i,m,n),\ \forall i,m,n\geq 0\,. \tag{4.2.13}$$

We also have

$$T_i(C_{i,0},N)=T_i(p_i,N),\ \forall(i\geq 1;\ N\geq 0)$$
$$=T_i(N)\,,$$
$$t(i,0,n)=t(i,n),\ \forall i,n\geq 0\,, \tag{4.2.14}$$
$$t(i,0)=0,\ \forall i\geq 0\,, \tag{4.2.15}$$
$$\text{and}\quad t(i,m,0)=0,\ \forall i,m\geq 0\,. \tag{4.2.16}$$

For any $(i,n\geq 1;\ m\geq 0)$, if there is no C_i between $nC_{i,m}$ (excl.)[23] and $(n+1)C_{i,m}$ (incl.)[24], we have $T_i\left(C_{i,m},(n+1)C_{i,m}\right)=T_i(C_{i,m},nC_{i,m})-1$. In such a situation we also have

$$t(i,m,n)=t(i,m,n-1)-1\,.$$

Empirically, $t(0,n)=0,\ \forall n\geq 0$, and all $\mathrm{u}_{0,n}$ with $n\geq 2$ have a regular C_0.[25]

For larger i, we already have (4.2.2) gives

$t(i,n+1)\geq t(i,n)-1,\ \forall(i\geq 1;\ n\geq 0)$, with equality holding if and only if there is no regular C_i in $\mathrm{u}_{i,n+1}$.

Also,

$S_i\left(C_{i,m},(n+1)C_{i,m}\right)=S_i\left((n+1)C_{i,m}\right)-S_i(C_{i,m}),\ \forall i,m,n\geq 0$, by definition

$=S_i\left((n+1)C_{i,m}\right)-m$, by **Property 2.3.13**,

so that (4.2.12) gives

$$S_i\left((n+1)C_{i,m}\right)=m+n+t(i,m,n),\ \forall i,m,n\geq 0\,.$$

Analogous to (4.2.5), for any $(i,m\geq 0;\ r\geq m;\ N\geq C_{i,m})$, we have

$S_i(C_{i,m},N)=S_i(N)-m=r$, if and only if $C_{i,m+r}=\lfloor N\rfloor_{C_i}$. (4.2.17)

Therefore,

$$C_{i,m+n+t(i,m,n)}=\left\lfloor (n+1)C_{i,m}\right\rfloor_{C_i},\ \forall i,m,n\geq 0\,, \tag{4.2.18}$$

or

$$C_{i,m+n+t(i,m,n)}\leq (n+1)C_{i,m},\ \forall i,m,n\geq 0\,, \tag{4.2.19}$$

and

$$C_{i,m+n+t(i,m,n)+1}>(n+1)C_{i,m},\ \forall i,m,n\geq 0\,. \tag{4.2.20}$$

Equality in (4.2.19) holds for $m\geq 1$ whenever $(n+1)$ is a regular C_i, say $C_{i,k}$, since $C_{i,m}$ and $C_{i,k}C_{i,m}$ are then both always regular C_i, by **Corollary 2.1.9**. Strict inequality holds for $i,m\geq 1$ in (4.2.19) whenever $(n+1)$ is not a regular C_i, since $(n+1)C_{i,m}$ can then not be a regular C_i. For $m=0$ we already have strict inequality holds in (4.2.19), for all $i,n\geq 1$, by (4.2.10).

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[23] The expression "(excl.)" stands for "(exclusive of)" or "(excluding)".
[24] The expression "(incl.)" stands for "(inclusive of)" or "(including)".
[25] Empirically, we have $t(0,n)=t(1,n)=t(2,n)=0,\ \forall n\geq 0$.

One of the ways the functions $t(i,n)$ and $t(i+1,n)$ are related is given in the following Theorem:

Theorem 4.2.1. $t(i+1,n) \geq t\left(i, n+\left\lfloor \frac{(n+1)\Delta_i}{p_i} \right\rfloor\right)+\left\lfloor \frac{(n+1)\Delta_i}{p_i} \right\rfloor - x_{i+1}\left((n+1)p_{i+1}\right),\ \forall i,n \geq 0$, *with strict equality holding for all* $\left\{\begin{matrix}(i=0;\ n\geq 0),\\ (i\geq 1;\ m\geq 1;\ n=mp_i-1)\end{matrix}\right\}$.

Proof: $S_{i+1}\left((n+1)p_{i+1}\right)=S_i\left((n+1)p_{i+1}\right)-x_{i+1}\left((n+1)p_{i+1}\right),\ \forall i,n\geq 0$,

by **Property 2.3.16**

$$= S_i\left((n+1)p_i+(n+1)\Delta_i\right)-x_{i+1}\left((n+1)p_{i+1}\right)$$

$$= S_i\left(\left(n+1+\frac{(n+1)\Delta_i}{p_i}\right)p_i\right)-x_{i+1}\left((n+1)p_{i+1}\right)$$

$$\geq S_i\left(\left(n+1+\left\lfloor\frac{(n+1)\Delta_i}{p_i}\right\rfloor\right)p_i\right)-x_{i+1}\left((n+1)p_{i+1}\right),$$

by **Property 2.3.11**

$$= n+\left\lfloor\frac{(n+1)\Delta_i}{p_i}\right\rfloor+t\left(i,n+\left\lfloor\frac{(n+1)\Delta_i}{p_i}\right\rfloor\right)-x_{i+1}\left((n+1)p_{i+1}\right),$$

by (4.2.3).

For $(i\geq 0;\ n=0)$, we have $S_{i+1}(p_{i+1})=0$, $S_i(p_{i+1})=S_i(C_{i,1})=1$, and $x_{i+1}(p_{i+1})=1$.

Since $\Delta_0=1=p_0$, we have $\frac{(n+1)\Delta_0}{p_0}=\left\lfloor\frac{(n+1)\Delta_0}{p_0}\right\rfloor,\ \forall n\geq 0$.

For all $i\geq 1$, we have $\frac{(n+1)\Delta_i}{p_i}=\left\lfloor\frac{(n+1)\Delta_i}{p_i}\right\rfloor,\ \forall(m\geq 1;\ n=mp_i-1)$.

By (4.2.3), we have

$$S_{i+1}\left((n+1)p_i\right)=n+t(i+1,n),\ \forall i,n\geq 0.$$

Therefore,

$$t(i+1,n)\geq t\left(i,n+\left\lfloor\frac{(n+1)\Delta_i}{p_i}\right\rfloor\right)+\left\lfloor\frac{(n+1)\Delta_i}{p_i}\right\rfloor-x_{i+1}\left((n+1)p_{i+1}\right),\ \forall i,n\geq 0,$$

with strict equality holding for all $\left\{\begin{matrix}(i=0;\ n\geq 0),\\ (i\geq 1;\ m\geq 1;\ n=mp_i-1)\end{matrix}\right\}$. □

The ranges $(i\geq 0;\ m\geq 1;\ n=mp_i-1)$ in **Theorem 4.2.1**, give the following:

Theorem 4.2.2. $x_{i+1}(mp_ip_{i+1})=m+t(i,m-1),\ \forall(i\geq 0;\ m\geq 1)$.

Proof: $x_{i+1}(N) = 1 + S_i\left(\dfrac{N}{p_{i+1}}\right),\ \forall(i \ge 0;\ N \ge p_{i+1})$, by **Theorem 2.3.22**.

Therefore,

$$x_{i+1}(mp_i p_{i+1}) = 1 + S_i(mp_i),\ \forall(i \ge 0;\ m \ge 1)$$
$$= m + t(i, m-1)\text{, by (4.2.3).} \ \square$$

As a result of **Theorems 4.2.1** and **4.2.2**, we have another connection between $t(i,n)$ and $t(i+1,n)$, given by the following corollary:

Corollary 4.2.3. $t(i+1, mp_i - 1) = m(\Delta_i - 1) + t(i, mp_{i+1} - 1) - t(i, m-1),\ \forall i, m \ge 1$. $\square$

Therefore,

$$t(i+1, p_i - 1) = t(i, p_{i+1} - 1) + \Delta_i - 1,\ \forall i \ge 1\text{, since } t(i,0) = 0,\ \forall i \ge 0\text{, by (4.2.15)}$$
$$> t(i, p_{i+1} - 1),\ \forall i \ge 2. \qquad (4.2.21)$$

Equivalent to the statement of **Theorem 4.2.1**, we can also write

$$t(i+1, n-1) \ge t\left(i, n-1+\left\lfloor \frac{n\Delta_i}{p_i} \right\rfloor\right) + \left\lfloor \frac{n\Delta_i + 1}{p_i} \right\rfloor - x_{i+1}(np_{i+1}),\ \forall(i \ge 0;\ n \ge 1).$$

Compartmentalizing n by $X_{i,r} \le n_{i,r} p_i < X_{i,r+1},\ \forall i, r \ge 1$, then gives

$$t(i+1, n_{i+1,r} - 1) \ge t\left(i, n_{i+1,r} - 1 + \left\lfloor \frac{n_{i+1,r}\Delta_i}{p_i} \right\rfloor\right) + \left\lfloor \frac{n_{i+1,r}\Delta_i + 1}{p_i} \right\rfloor - r,\ \forall(i \ge 0;\ r \ge 1).$$

Assuming $\lim_{i\to\infty} \dfrac{p_{i+1} - p_i}{p_i} = 0$ (which we prove later in **Subsection 7.3** and again in **Subsection 7.4**), and since $\Delta_i = p_{i+1} - p_i$, this will allow considering $1 \le n_{i+1,1} < \dfrac{p_i - 1}{\Delta_i}$,[26] for which range we have $t(i+1, n_{i+1,r} - 1) \ge t(i, n_{i+1,r} - 1) - 1$. There are cases in this range for which we have equality, *i.e.*, $t(i+1, n_{i+1,r} - 1) = t(i, n_{i+1,r} - 1) - 1$. Empirical examples of this are $t(17,1) = 12$, $t(18,1) = 11$ and $t(17,4) = 41$, $t(18,4) = 40$. Despite such, we will prove in **Theorem 5.2.1** that $t(i,n) \ge 0,\ \forall i, n \ge 0$.

5. MORE NEW NOTATIONS AND RELATIONS

5.1 Relating $u(i,n)$ and $u(i+1,n)$. In **Property 2.3.18**, we defined

$$u(i,n) = x_i(C_{i,n}),\ \forall(i \ge 1;\ n \ge 0). \qquad (5.1.1)$$

Using definitions (4.2.4) and (4.2.5), let

$$C_{i,(\underline{r})_i} = \lceil X_{i,r+1} \rceil^{C_i},\ \forall(i \ge 1;\ r \ge 0), \qquad (5.1.2)$$

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[26] Such $n_{i+1,1}$ exist whenever $\Delta_i \le (p_i - 1)/2$.

and

$$C_{i,(\overline{r})_i} = \lfloor X_{i,r+1} \rfloor_{C_i}, \ \forall (i \ge 1;\ r \ge 0). \qquad (5.1.3)$$

The symbols $(\underline{r})_i$ and $(\overline{r})_i$ represent variables whose values depend on i and r.[27]

By definition, we have $X_{i,1} = C_{i,0}, \ \forall i \ge 1$, so that by (5.1.2) and (5.1.3) we have

$$(\underline{0})_i = (\overline{0})_i = 0, \ \forall i \ge 1. \qquad (5.1.4)$$

Since no regular C_i is an X_i for any $i \ge 1$, we also have

$$(\underline{r})_i = (\overline{r})_i + 1, \ \forall i, r \ge 1. \qquad (5.1.5)$$

Lemma[28] 5.1.1. $u(i+1,n) = u(i,n) = 1, \ \forall \left(i \ge 1;\ 0 \le n \le (\overline{1})_i \right),$ (5.1.1.1)

and $u(i+1,n) < u(i,n), \ \forall \left(i \ge 1;\ n \ge (\underline{1})_i \right).$ (5.1.1.2)

Proof: We have

$$X_{i+1,r+1} = X_{i,r+u(i,r)+1} + \Delta_i C_{i,r}, \ \forall i, r \ge 1, \text{ by } \textbf{Theorem 2.3.21}. \qquad (5.1.1.3)$$

Therefore,

$$X_{i,1} < X_{i,2} < X_{i,3} < X_{i+1,2}, \ \forall i \ge 1, \qquad (5.1.1.4)$$

and

$$X_{i,r+2} \le X_{i,r+u(i,r)+1} < X_{i+1,r+1}, \ \forall i, r \ge 1. \qquad (5.1.1.5)$$

Since there are an infinite number of C_i for any $i \ge 0$, and since no regular C_i is an X_i for any $i \ge 1$, we then have

for any $i, n \ge 1$, there is an $r \ge 1$ such that $X_{i,r} < C_{i,n} < X_{i,r+1}$. (5.1.1.6)

By definition, we have $C_{i,0} = p_i, \ \forall i \ge 0$, and $C_{i,1} = p_{i+1} = X_{i+1,1}, \ \forall i \ge 0$.

We also have $u(i,0) = x_i(C_{i,0}) = 1, \ \forall i \ge 1$,

and $(\overline{1})_i \ge 0, \ \forall i \ge 1$, by (5.1.3).

We now proceed by considering all other possible locations for C_{i+1}, for given $i \ge 1$:

(i) If there are any $C_{i+1,n}$ in the range $X_{i+1,1} < C_{i+1,n} < X_{i+1,2}$, they have

$$n \ge 1$$

and $$u(i+1,n) = 1.$$

For any $(i \ge 1;\ n \ge 0)$, we also have $u(i,n) \ge 1$.

More precisely, if there are any $C_{i+1,n}$ in the range $X_{i+1,1} < C_{i+1,n} < X_{i,2}$, these have

$$u(i,n) = u(i+1,n) = 1.$$

Any $C_{i+1,n}$ in the range $X_{i,2} < C_{i+1,n} < X_{i+1,2}$ have

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[27] The reason for my choice of these symbols is that $C_{i,(\underline{r})_i}$ is the C_i that 'sits" atop $\lceil X_{i,r+1} \rceil^{C_i}$ and $C_{i,(\overline{r})_i}$ is the C_i that "butts its head" up against $\lfloor X_{i,r+1} \rfloor_{C_i}$.

[28] This is called a lemma instead of a theorem since it doesn't show that, for all $i, n \ge 1$, there is a $C_{i,n}$ satisfying $C_{i,n} < X_{i,n+1}$. This existence is proven later in **Corollary 5.1.3**.

$$C_{i+1,n} = C_{i,n+1} > C_{i,(\overline{1})_i}$$

and

$$u(i,n) > u(i+1,n).$$

(ii) There is a regular C_i above each X_{i+1}, for all $i \geq 0$, by construct, (5.1.1.7)
and no C_i at $N = X_{i,r+1}$, for any $i, r \geq 1$.
There may or may not be C_i and therefore C_{i+1} between $N = X_{i+1,r+1}$ (excl.) and $N = X_{i+1,r+k_{i+1,r}+1}$ (excl.) for any given $i, r \geq 0$ and some $k_{i+1,r} \geq 1$. But, because of (5.1.1.6), we may choose $k_{i+1,r}$ to be the smallest value that gives at least one C_i and therefore at least one C_{i+1} between $N = X_{i+1,r+k_{i+1,r}+1}$ (excl.) and $N = X_{i+1,r+k_{i+1,r}+2}$ (excl.). For such, we have

$$u\left(i+1,(\underline{r})_{i+1}\right) = r + k_{i+1,r} + 1. \qquad (5.1.1.8)$$

Therefore,

$$u\left(i,(\underline{r})_{i+1}\right) \geq u\left(i,\left(\underline{r+k_{i+1,r}+1}\right)_i\right),\ \forall r \geq 1, \text{ by (5.1.1.7) and (5.1.1.5)}$$

$$\geq r + k_{i+1,r} + 2, \text{ by (5.1.2) and (5.1.1.7)} \qquad (5.1.1.9)$$

$$> u\left(i+1,(\underline{r})_{i+1}\right), \text{ by (5.1.1.8).}$$

(iii) Now consider any $C_{i+1,n}$ in the range

$$X_{i+1,r+k_{i+1,r}+1} < C_{i+1,(\underline{r})_{i+1}} < C_{i+1,n} < X_{i+1,r+k_{i+1,r}+2}, \qquad (5.1.1.10)$$

should such exist for given $i, r \geq 1$.
These all have $u\left(i+1,n\right) = r + k_{i+1,r} + 1$. (5.1.1.11)
But, $u\left(i,n\right) \geq u\left(i,(\underline{r})_{i+1}\right)$, by **Property 2.3.7** and definition of u, since $n > (\underline{r})_{i+1}$

$$\geq u\left(i+1,(\underline{r})_{i+1}\right) + 1, \text{ by (5.1.1.5)}$$

$$\geq r + k_{i+1,r} + 2, \text{ by (5.1.2)}$$

$$> u\left(i+1,n\right), \text{ by (5.1.1.11).}$$

Since $i \geq 1$ was otherwise arbitrary, this exhausts all possible locations of $C_{i+1,n}$.
In summary, for any $i \geq 1$, we have $u(i+1,n) = 1 = u(i,n),\ \forall 0 \leq n \leq (\overline{1})_i$,
and $u(i+1,n) < u(i,n),\ \forall\left(i \geq 1;\ n \geq (\underline{1})_i\right)$. □

Empirically, we have $X_{1,n} < C_{1,n} < X_{1,n+1},\ \forall n \geq 1$. Therefore, straight-forward induction gives the following two corollaries to **Lemma 5.1.1**:

Corollary 5.1.2. $S_{i+1}(X_{i+1,n+1}) > S_i(X_{i,n+1}),\ \forall i, n \geq 1$. □

Corollary 5.1.3. $C_{i,n} < X_{i,n+1},\ \forall i, n \geq 1$. □

As a result of these last two corollaries, we immediately have the following two:

Corollary 5.1.4. $u(i+1,n) \le u(i,n-1),\ \forall i,n \ge 1$. □

Corollary 5.1.5. *If, for any* $i_0, n \ge 1$ *we have* $C_{i_0,m} < X_{i_0,n+1}$, for some $m \ge 1$, *then*

$$C_{i,m+i-i_0} < X_{i,n+1},\ \forall i \ge i_0 . \ \square$$

By (2.1.2), we have $C_{i,1} = C_{i+1,0},\ \forall i \ge 0$. **Corollary 5.1.3** now gives the following corollary:

Corollary 5.1.6. $C_{i,2} = C_{i+1,1},\ \forall i \ge 0$. □

Proof: Empirically, we have $C_{1,1} = 3 = C_{0,2}$.
For $i \ge 1$, we have $C_{i+1,1} = C_{i,1+u(i+1,1)}$, by **Property 2.3.18**
$= C_{i,2}$, by **Corollary 5.1.3**. □

As a corollary of **Corollary 5.1.6** we have the following corollary:

Corollary 5.1.7. $X_{i,3} = p_i p_{i+1},\ \forall i \ge 1$.

Proof: $X_{i,3} = p_i C_{i-1,2},\ \forall i \ge 1$, by **Theorem 2.3.1**
$= p_i C_{i,1}$, by **Corollary 5.1.6**
$= p_i p_{i+1}$, by (2.1.2). □

By **Corollary 5.1.5**, we can extend **Corollary 5.1.6** to give $C_{i,n+1} = C_{i+1,n}$ for larger values of n as i increases. For example, since $C_{2,2n} < X_{2,n+1},\ \forall n \ge 1$, empirically, we then have **Corollary 5.1.5** gives the following:

Corollary 5.1.7. $C_{i,2n} < X_{i,n+1},\ \forall (i \ge 2;\ n \ge 1)$. □

By **Theorem 2.3.5**, all regular C_i before $X_{i+1,2}$ are regular primes, for all $i \ge 0$. Also, $X_{i,2} < X_{i,3} < X_{i+1,2},\ \forall i \ge 1$, by (5.1.1.4) in **Lemma 5.1.1**. Therefore, **Corollary 5.1.2** gives the following corollary:

Corollary 5.1.8. *The number of* (*regular*) *primes before* $X_{i,2} = p_i^2$ *and also before* $X_{i,3} = p_i p_{i+1}$ *and* $X_{i+1,2} = p_{i+1}^2$ *is a strictly monotonically increasing function of* i, *for all* $i \ge 1$. □

By **Corollary 5.1.2**, we have

$$S_{i+1}(X_{i+1,n+1}) > S_i(X_{i,n+1}),\ \forall i,n \ge 1,$$

or $S_{i+1}(p_{i+1}C_{i,n}) > S_i(p_iC_{i-1,n}),\ \forall i,n \ge 1$, by **Theorem 2.3.1**.

We will strengthen this relation soon by proving

$$S_{i+1}(p_{i+1}C_{i,n}) > S_i(p_iC_{i,n}),\ \forall(i \ge 2;\ n \ge 0)\,. \qquad (5.1.6)$$

By **Property 2.3.18**, statement (5.1.6) is the same as

$$S_{i+1}(p_{i+1}C_{i,n}) > S_i(p_iC_{i-1,n+u(i,n)}),\ \forall(i \ge 2;\ n \ge 0)\,.$$

By **Theorem 2.3.1**, this gives

$$S_{i+1}(X_{i+1,n+1}) > S_i(X_{i,n+u(i,n)+1}),\ \forall(i \ge 2;\ n \ge 1)\,.$$

Empirically, we have $u(1,n) = \{1_{n=0}, \ddot{1}_1, \ddot{2}_2, \ldots\}$, so that

$$\begin{aligned} S_1(X_{1,n+u(1,n)+1}) &= S_1\left(X_{1,n+\{1_{n=0},\ddot{1}_1,\ddot{2}_2,\ldots\}+1}\right) \\ &= S_1\left(X_{1,\{2_{n=0},\ddot{3}_1,\ddot{5}_2,\ldots\}}\right) \\ &= \{1_{n=0}, \ddot{2}_1, \ddot{4}_2, \ldots\}\text{, empirically} \\ &= 2n,\ \forall n \ge 1 \\ &= S_2(X_{2,n+1}),\ \forall n \ge 1\text{, empirically.} \end{aligned}$$

The inequality in (5.1.6) therefore does not hold for $i = 1$.

In order to prove (5.1.6), we first review some relations. We have

$$S_{i+1}(N) = S_i(N) - x_{i+1}(N),\ \forall(i \ge 0;\ N \ge 0)\text{, by **Property 2.3.16**.}$$

Also,

$$S_i\left((n+1)p_i\right) = n + t(i,n),\ \forall i,n \ge 0\text{, by definition (4.2.3).}$$

Since there may be no regular C_i in a given $u_{i,n+1}$, we use (4.2.2), which states that

$$t(i,n+1) \ge t(i,n) - 1,\ \forall(i \ge 1;\ n \ge 0)\,. \qquad (5.1.7)$$

Empirically, we have $t(0,n) = t(1,n) = t(2,n) = 0,\ \forall n \ge 0$,

and

$$t(3,n) = \{\ddot{0}_{n=0}, 0_1, 1_{2=C_{2,0}-1}, 2_3, 2_4, 2_5, \ddot{2}_6, 2_7, 3_8, \ldots\}\,. \qquad (5.1.8)$$

Equivalently, we have

$$S_0(np_0) = S_1(np_1) = S_2(np_2) = n - 1,\ \forall n \ge 1\,, \qquad (5.1.9)$$

and

$$S_3(np_3) = \{\ddot{0}_{n=1}, 1_2, 3_{3=C_{2,0}}, 5_4, 6_5, 7_6, \ddot{8}_7, 9_8, 11_9, \ldots\}\,. \qquad (5.1.10)$$

We also have

$$C_{i,n} = p_i + \Delta_{i,n},\ \forall i,n \ge 0\text{, by definition,} \qquad (5.1.11)$$

and

$$C_{i+1,0} - C_{i,0} = \Delta_i,\ \forall i \ge 0\text{, by definition.} \qquad (5.1.12)$$

As a result, we have

$$\Delta_{i,0} = 0,\ \forall i \ge 0\,, \qquad (5.1.13)$$

$$\Delta_{i,1} = \Delta_i,\ \forall i \ge 0\,. \qquad (5.1.14)$$

Empirically,

$$\Delta_{2,n} = \{0_{n=0}, \ddot{2}_1, 4_2, \ddot{8}_3, \ldots\} \qquad (5.1.15)$$

and

$$\Delta_{3,n} = \{0_{n=0}, \ddot{2}_1, 6_2, 8_3, 12_4, 14_5, 18_6, 24_7, 26_8, (\ddot{32})_9, \ldots\}$$
$$\geq \{0_{n=0}, \ddot{2}_1, 6_2, \ddot{8}_3, \ldots\}. \qquad (5.1.16)$$

Since $C_{i+1,n} = C_{i,n+u(i,n)} \geq C_{i,n+1},\ \forall i,n \geq 0$, we then have (5.1.16) gives

$$\Delta_{i,n} \geq \{0_{n=0}, \ddot{2}_1, 6_2, \ddot{8}_3, \ldots\},\ \forall (i \geq 3;\ n \geq 0). \qquad (5.1.17)$$

We also have

$$x_{i+1}(p_{i+1}C_{i,n}) = x_{i+1}(X_{i+1,n+1}) = n+1,\ \forall i,n \geq 0.\text{[29]} \qquad (5.1.18)$$

We defined $\mathrm{u}_{i,n}$ as the left-closed, right-open intervals

$$\mathrm{u}_{i,n} = \left[np_i, (n+1)p_i\right),\ \forall i,n \geq 0.$$

As a result, for any $(i \geq 0;\ n \geq 1)$, the number of full u_i before $p_{i+1}C_{i,n}$ is

$$\left\lfloor \frac{(p_i+\Delta_i)(p_i+\Delta_{i,n})}{p_i} \right\rfloor = \left\lfloor \frac{p_i(p_i+\Delta_i+\Delta_{i,n})+\Delta_i\Delta_{i,n}}{p_i} \right\rfloor \geq (p_i+\Delta_i+\Delta_{i,n}). \qquad (5.1.19)$$

The last relation we need is the following:

$$\left.\begin{array}{l}\text{there is no } C_i \text{ above } N = p_{i+1}C_{i,0}, \text{ for any } i \geq 0, \text{ but there is}\\ \text{a (regular) } C_i \text{ above } X_{i+1,n+1} = p_{i+1}C_{i,n}, \text{ for all } (i \geq 0;\ n \geq 1).\end{array}\right\} \qquad (5.1.20)$$

Using the above relations, we now have the following theorem:[30]

Theorem 5.1.9. $S_{i+1}(p_{i+1}C_{i,n}) > S_i(p_iC_{i,n}),\ \forall (i \geq 2;\ n \geq 0)$,
and equivalently, $t(i+1, C_{i,n}-1) > t(i, C_{i,n}-1),\ \forall (i \geq 2;\ n \geq 0)$.

Proof: We have

$$S_{i+1}(p_{i+1}C_{i,n}) = S_i(p_{i+1}C_{i,n}) - x_{i+1}(p_{i+1}C_{i,n}),\ \forall i,n \geq 0, \text{ by } \textbf{Property 2.3.16}$$
$$\geq S_i\left(p_i(p_i+\Delta_i+\Delta_{i,n})\right) + \begin{Bmatrix} 0,\ n=0 \\ 1,\ n \geq 1 \end{Bmatrix} - x_{i+1}(p_{i+1}C_{i,n}), \text{ by (5.1.19), } \textbf{Property 2.3.11}, \text{ and (5.1.20)}$$
$$= \begin{Bmatrix} S_i\left(p_i(p_i+\Delta_i)\right) - 1,\ n=0 \\ S_i\left(p_i(C_{i,n}+\Delta_i+\Delta_{i,n}\right) + 1 - (n+1),\ \forall n \geq 1 \end{Bmatrix}, \text{ by (5.1.13) and (5.1.18)}$$
$$= \begin{Bmatrix} (C_{i,0}-1) + t(i, C_{i,0}-1) + (\Delta_i - 1),\ n=0 \\ (C_{i,n}+\Delta_i-1) + t(i, C_{i,n}+\Delta_i-1) + \Delta_{i,n} - n,\ \forall n \geq 1 \end{Bmatrix}, \text{ by (4.2.3)}$$
$$\geq \begin{Bmatrix} (C_{i,0}-1) + t(i, C_{i,0}-1) + (\Delta_i - 1),\ n=0 \\ (C_{i,n}+\Delta_i-1) + t(i, C_{i,n}-1) - \Delta_i + \Delta_{i,n} - n,\ \forall n \geq 1 \end{Bmatrix}, \text{ by (5.1.7)}$$

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[29] The proof of this for $n = 0$ is given in the start of the proof for **Theorem 5.1.10**.

[30] Although **Theorem 5.1.9** appears to be an unnecessary theorem at first sight in light of the following more general **Theorem 5.1.10**, **Theorem 5.1.9** is used in the proof of **Theorem 5.1.10**.

$$\ge \begin{Bmatrix} (C_{i,0}-1)+t(i,C_{i,0}-1)+(\Delta_i-1),\ n=0 \\ (C_{i,n}-1)+t(i,C_{i,n}-1)+\begin{Bmatrix}\{\ddot{2}_{n=1},4_2,\ddot{8}_3,\ldots\},\ i=2 \\ \{\ddot{2}_{n=1},6_2,\ddot{8}_3,\ldots\},\ \forall i\ge 3\end{Bmatrix}-n,\ n\ge 1 \end{Bmatrix},$$

by (5.1.15) and (5.1.17)

$$\ge S_i(p_iC_{i,n})+\begin{Bmatrix}(\Delta_i-1)_{n=0},\begin{Bmatrix}\{\ddot{1}_{n=1},2_2,\ddot{5}_3\ldots\},\ i=2 \\ \{\ddot{1}_{n=1},4_2,\ddot{5}_3,\ldots\},\ \forall i\ge 3\end{Bmatrix}\end{Bmatrix}, \text{ by (4.2.3)}$$

$$> S_i(p_iC_{i,n}),\ \forall(i\ge 2;\ n\ge 0).$$

Equivalently, $t(i+1,C_{i,n}-1)>t(i,C_{i,n}-1),\ \forall(i\ge 2;\ n\ge 0)$. □

We can strengthen **Theorem 5.1.9** by the following theorem:

Theorem 5.1.10. $S_{i+1}(np_{i+1})>S_i(np_i),\ \forall(i\ge 2;\ n\ge p_i)$,
and equivalently, $t(i+1,n)>t(i,n),\ \forall(i\ge 2;\ n\ge p_i-1)$.

Proof: For all $i\ge 0$, we have

$$X_{i+1,1}=p_{i+1}\le p_ip_{i+1}=p_{i+1}C_{i,0}<p_{i+1}^2=X_{i+1,2}, \tag{5.1.10.1}$$

so that

$$x_{i+1}(p_{i+1}C_{i,0})=1,\ \forall i\ge 0. \tag{5.1.10.2}$$

Also,

$$X_{i+1,r+1}=p_{i+1}C_{i,r},\ \forall(i\ge 0;\ r\ge 1), \text{ by \textbf{Theorem 2.3.1}}. \tag{5.1.10.3}$$

Since $C_{i,r}=C_{i-1,r+u(i,r)},\ \forall(i\ge 1;\ r\ge 0)$, by **Property 2.3.18**, we also have

$$X_{i,r+u(i,r)+1}=p_iC_{i,r},\ \forall(i\ge 1;\ r\ge 0). \tag{5.1.10.4}$$

Compartmentalize n by $X_{i,r}\le n_{i,r}p_i<X_{i,r+1},\ \forall i,r\ge 1$, so that (5.1.10.1), (5.1.10.2), and (5.1.10.3) give

$$n_{i,1}=\{1,2,\ldots,(C_{i-1,1}-1)\},\ \forall i\ge 1,$$

$$\text{and } n_{i,r}=\{C_{i-1,r-1},(C_{i-1,r-1}+1),\ldots,(C_{i-1,r}-1)\},\ \forall(i\ge 1;\ r\ge 2).$$

The listing $\{n_{i+1,r}\}_{r\ge 1}=\{n_{i+1,1},n_{i+1,2},n_{i+1,3},\ldots\}$ has no gaps from $n_{i+1,1}=1$ onwards.
Since $u(i,0)=x_i(C_{i,0})=x_i(X_{i,1})=1,\ \forall i\ge 1$, the listing $\{n_{i,r-1+u(i,r-1)}\}_{r\ge 1}$ also starts at $n_{i,r-1+u(i,r-1)}=1$. However, this listing $\{n_{i,r-1+u(i,r-1)}\}_{r\ge 1}$ does have gaps, say $\{G(i,r+1)\}$,[31] at the values of r at which $u(i,r-1)$ increases when r increases by 1. Whenever r increases by 1 but $u(i,r-1)$ does not increase, we can say the length of $\{G(i,r+1)\}$ is 0 and $\{G(i,r+1)\}$ is empty. Therefore,

$$\{n_{i+1,r}\}_{r\ge 1}=\{G(i,r+1)+n_{i,r-1+u(i,r-1)}\}_{r\ge 1}, \tag{5.1.10.5}$$

and there are no gaps on either side of this equation from the initial value 1. It is understood that equality in (5.1.10.5) is on an r-indexed element-by-element basis.
Using (5.1.10.2), (5.1.10.3), **Property 2.3.16**, and **Theorem 5.1.9**, we have

[31] $G(i,r+1)$ is a left-closed, right-open interval.

$$S_{i+1}(p_{i+1}C_{i,r-1}) = S_i(p_{i+1}C_{i,r-1}) - r > S_i(p_iC_{i,r-1}),\ \forall(i \ge 2;\ r \ge 1)\,. \tag{5.1.10.6}$$

For all $(r \ge 1;\ n_{i+1,r} \ge p_i)$, we have **Property 2.3.15** gives

$$S_{i+1}\left(\{n_{i+1,r}\}_{r\ge1}\,p_{i+1}\right) = S_{i+1}\left(\{C_{i,k=0}, C_{i,k=1}, C_{i,k=2}, \ldots, C_{i,k=r-1}\}_{r\ge1}\,p_{i+1}\right)$$
$$+S_{i+1}\left(\begin{Bmatrix}\{(C_{i,0}, C_{i,1})\}_{k=0}, \\ \{(C_{i,1}, C_{i,2})\}_{k=1}, \\ \vdots \\ \{(C_{i,r}, C_{i,r})\}_{k=r-1}\end{Bmatrix}_{r\ge1} p_{i+1}\right).^{32} \tag{5.1.10.7}$$

All regular C_i in each of the open intervals

$$\left(p_{i+1}C_{i,0}, p_{i+1}C_{i,1}\right), \left(p_{i+1}C_{i,1}, p_{i+1}C_{i,2}\right), \left(p_{i+1}C_{i,2}, p_{i+1}C_{i,3}\right), \ldots$$

are inherited as regular C_{i+1}, since there are no X_{i+1} within these open intervals to interfere.

Therefore, for all $(r \ge 1;\ n_{i+1,r} \ge p_i)$, and using k as a dummy variable, we have (5.1.10.1), (5.1.10.7), and **Property 2.3.16** give

$$S_{i+1}\left(\{n_{i+1,r}\}_{r\ge1}\,p_{i+1}\right) = S_i\left(\{C_{i,k=0}, C_{i,k=1}, C_{i,k=2}, \ldots, C_{i,k=r-1}\}_{r\ge1}\,p_{i+1}\right) - \{1_{k=0}, 2_{k=1}, 3_{k=2}, \ldots, r_{k=r-1}\}_{r\ge1}$$
$$+S_i\left(\begin{Bmatrix}\{(C_{i,0}, C_{i,1})\}_{k=0}, \\ \{(C_{i,1}, C_{i,2})\}_{k=1}, \\ \vdots \\ \{(C_{i,r}, C_{i,r})\}_{k=r-1}\end{Bmatrix}_{r\ge1} p_{i+1}\right),\ \forall i \ge 0$$

$$> S_i\left(\{C_{i,k=0}, C_{i,k=1}, C_{i,k=2}, \ldots, C_{i,k=r-1}\}_{r\ge1}\,p_i\right)$$
$$+S_i\left(\begin{Bmatrix}\{(C_{i,0}, C_{i,1})\}_{k=0}, \\ \{(C_{i,1}, C_{i,2})\}_{k=1}, \\ \vdots \\ \{(C_{i,r}, C_{i,r})\}_{k=r-1}\end{Bmatrix}_{r\ge1} p_{i+1}\right),\ \forall i \ge 2\text{, by (5.1.10.6)}$$

$$= S_i\left(\{C_{i,k=0}, C_{i,k=1}, C_{i,k=2}, \ldots, C_{i,k=r-1}\}_{r\ge1}\,p_i\right)$$
$$+S_i\left(\{G(i, r+1) + n_{i,r-1+u(i,r-1)}\}'_{r\ge1}\,p_i\right),$$

$\forall n_{i,r-1+u(i,r-1)} \ge p_i$, by (5.1.10.5), where $\{G(i, r+1) + n_{i,r-1+u(i,r-1)}\}'_{r\ge1}$ excludes $\{C_{i,r-1}\}_{r\ge1}$, and it is understood that comparison of the LHS and RHS of this equation is made on an element-by-element basis.

$$= S_i\left(\{G(i, r+1) + n_{i,r-1+u(i,r-1)}\}_{r\ge1}\,p_i\right)$$
$$= S_i(\{n_{i+1,r}\}_{r\ge1}\,p_i)\text{, by (5.1.10.5).}$$

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[32] As usual, we use $(\ldots,\ldots)$ to designate an open interval. The open intervals on the RHS are combined with the starting values indicated to give the intervals on the LHS.

Therefore,

$$S_{i+1}(np_{i+1}) > S_i(np_i),\ \forall (i \geq 2;\ n \geq p_i)\,,$$

and equivalently $t(i+1,n) > t(i,n),\ \forall (i \geq 2;\ n \geq p_i - 1)$. □

5.2 Proof of $t(i,n) \geq 0,\ \forall i,n \geq 0$, and $t(i,m,n) \geq 0,\ \forall i,m,n \geq 0$. Another consequence of **Lemma 5.1.1** is the following:

Theorem 5.2.1. $t(i,n) \geq 0,\ \forall i,n \geq 0$, *and* $t(i,m,n) \geq 0,\ \forall i,m,n \geq 0$. (5.2.1.0)

Proof: By (4.2.16), we have

$$C_{i,m+n+t(i,m,n)} = \lfloor (n+1)C_{i,m} \rfloor_{C_i}\,,\ \forall i,m,n \geq 0\,. \tag{5.2.1.1}$$

Empirically, we have

$$t(0,n) = t(1,n) = t(2,n) = 0,\ \forall n \geq 0\,. \tag{5.2.1.2}$$

Also,

$$\begin{aligned}(n+1)C_{0,m} &= (n+1)(m+1),\ \forall m,n \geq 0\text{, by (2.1.3)}\\ &= mn + m + n + 1\\ &= C_{0,m+n+mn}\text{, by (2.1.3).}\end{aligned}$$

Therefore, (5.2.1.1) gives

$$\begin{aligned}t(0,m,n) &= mn,\ \forall m,n \geq 0\\ &\geq 0\,. \end{aligned} \tag{t-3}$$

Similarly,

$$\begin{aligned}(n+1)C_{1,m} &= (n+1)\{2_{m=0}, \ddot{3}_1, \ddot{5}_2, \ldots\}\text{, empirically}\\ &= \left\{\begin{array}{l}\{2_{m=0}, \ddot{3}_1, \ddot{5}_2, \ldots \ddot{\}}_{n=0},\\ \{4_{m=0}, \ddot{6}_1, (\ddot{10})_2, \ldots \ddot{\}}_{n=1},\\ \{6_{m=0}, \ddot{9}_1, (\ddot{15})_2, \ldots\}_{n=2},\\ \vdots \end{array}\right\}\end{aligned}$$

and

$$C_{1,m+n} = \left\{\begin{array}{l}\{2_{m=0}, \ddot{3}_1, \ddot{5}_2, \ldots\}_{n=0},\\ \{\ddot{3}_{m=0}, \ddot{5}_1, 7_2, \ldots \ddot{\}}_{n=1},\\ \{\ddot{5}_{m=0}, \ddot{7}_1, 9_2, \ldots \ddot{\}}_{n=3},\\ \vdots \end{array}\right\}\text{, empirically,}$$

so that (5.2.1.1) gives

$$\begin{aligned}t(1,m,n) &= \left\{\begin{array}{l}\{\ddot{0}_{m=0}, \ddot{0}_1, 0_2, \ldots\}_{n=0},\\ \{0_{m=0}, \ddot{0}_1, \ddot{1}_2, \ldots \ddot{\}}_{n=1},\\ \{0_{m=0}, \ddot{1}_1, \ddot{3}_2, \ldots \ddot{\}}_{n=2},\\ \vdots \end{array}\right\}\\ &\geq 0,\ \forall m,n \geq 0\,. \end{aligned} \tag{5.2.1.4}$$

Empirically, we also have

$$(n+1)C_{2,m} = (n+1)\{3_{m=0}, \ddot{5}_1, 7_2, \ddot{(11)}_3, \ldots\}$$

$$= \left\{ \begin{array}{l} \{3_{m=0}, \ddot{5}_1, 7_2, \ddot{(11)}_3, \ldots \ddot{\}}_{n=0}, \\ \{6_{m=0}, \ddot{(10)}_1, 14_2, \ddot{(22)}_3, \ldots \ddot{\}}_{n=1}, \\ \{9_{m=0}, \ddot{(15)}_1, 21_2, \ddot{(33)}_3, \ldots\}_{n=2}, \\ \{12_{m=0}, \ddot{(20)}_1, 28_2, \ddot{(44)}_3, \ldots\}_{n=3}, \\ \vdots \end{array} \right\}$$

and

$$C_{2,m+n} = \left\{ \begin{array}{l} \{3_{m=0}, \ddot{5}_1, 7_2, \ddot{(11)}_3, \ldots\}_{n=0}, \\ \{\ddot{5}_{m=0}, 7_1, \ddot{(11)}_2, 13_3, \ldots \ddot{\}}_{n+1} \\ \{\ddot{7}_{m=0}, 11_1, \ddot{(13)}_2, 17_3, \ldots\}_{n=2}, \\ \{\ddot{(11)}_{m=0}, 13_1, \ddot{(17)}_2, 19_3, \ldots \ddot{\}}_{n=3}, \\ \vdots \end{array} \right\},$$

so that (5.2.1.1) gives

$$t(2,m,n) = \left\{ \begin{array}{l} \{\ddot{0}_{m=0}, \ddot{0}_1, 0_2, 0_3, \ldots\}_{n=0}, \\ \{0_{m=0}, \ddot{0}_1, 1_2, \ddot{2}_3, \ldots \ddot{\}}_{n=1}, \\ \{0_{m=0}, \ddot{1}_1, 2_2, \ddot{5}_3, \ldots\}_{n=2}, \\ \{0_{m=0}, \ddot{2}_1, 3_2, \ddot{8}_3, \ldots \ddot{\}}_{n=3}, \\ \vdots \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \{\ddot{0}_{m=0}, \ddot{0}_1, 0_2, 0_3, \ldots\}_{n=0}, \\ \{0_{m=0}, \ddot{0}_1, 1_2, \ddot{2}_3, \ldots \ddot{\}}_{n=1}, \\ \{0_{m=0}, \ddot{1}_1, 2_2, \ddot{5}_3, \ldots \ddot{\}}_{n=2}, \\ \vdots \end{array} \right\},$$ [33]

$$\geq 0, \ \forall m, n \geq 0. \qquad (5.2.1.5)$$

By (4.1.14) and (4.2.15), we have $t(i,0) = t(i,m,0) = 0, \ \forall i, m \geq 0$.

Therefore, it suffices now to consider $(i \geq 2;\ m \geq 0;\ n \geq 1)$ and proceed by induction on i:

(a) By (5.2.1.2) and (5.2.1.5), we have (5.2.1.0) holds for $(i = 2;\ m \geq 0;\ n \geq 1)$.

(b) Therefore, at least for $k = 2$, we have

$$t(k,n) \geq 0, \ \forall n \geq 1, \qquad \text{(b-1)}$$

and

$$t(k,m,n) \geq 0, \ \forall (m \geq 0;\ n \geq 1). \qquad \text{(b-2)}$$

(c) By (5.2.1.1), we have

$$C_{k+1, m+n+t(k+1,m,n)} \leq (n+1)C_{k+1,m}, \ \forall m, n \geq 0.$$ [34]

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[33] This is an example of the minimum n-repetend length being less than expected, by coincidence.

[34] Recall that, by (4.2.9), we have $C_{i,n} < (n+1)p_i, \ \forall i, n \geq 1$.

To prove $t(k+1,m,n)\ge 0,\ \forall(m\ge 0;\ n\ge 1)$, it then suffices to show

$$C_{k+1,m+n}\le(n+1)C_{k+1,m},\ \forall(m\ge 0;\ n\ge 1),$$

or $C_{k,m+n+u(k+1,m+n)}\le(n+1)C_{k,m+u(k+1,m)},\ \forall(m\ge 0;\ n\ge 1)$, by **Property 2.3.18**

or $C_{k,m+n+u(k+1,m+n)}<(n+1)\big((m+1+u(k+1,m)\big)p_k,\ \forall(m\ge 0;\ n\ge 1)$, by (4.2.10)

$$=\big(mn+m+n+1+(n+1)u(k+1,m)\big)p_k.$$

We have (4.2.10) also gives

$$C_{k,m+n+u(k+1,m+n)}<\big(m+n+1+u(k+1,m+n)\big)p_k,\ \forall m,n\ge 0.$$

Therefore, to prove $t(k+1,m,n)\ge 0,\ \forall(m\ge 0;\ n\ge 1)$, it now suffices to show

$$u(k+1,m+n)\le\big(mn+(n+1)u(k+1,m)\big),\ \forall(m\ge 0;\ n\ge 1).$$

A fortiori, since $u(i,m)\ge 1,\ \forall(i\ge 1;\ m\ge 0)$, it suffices to show

$$u(k+1,m+n)\le mn+(n+1),\ \forall(m\ge 0;\ n\ge 1). \qquad \text{(c-1)}$$

We have

$$u(1,n)=\left\{\begin{matrix}1,\ n=0\\ n,\ \forall n\ge 1\end{matrix}\right\}, \text{ empirically,}$$

and

$$u(i+1,n)\le u(i,n),\ \forall(i\ge 1;\ n\ge 0), \text{ by **Lemma 5.1.1**}. \qquad \text{(c-2)}$$

Empirically,

$$u(2,n)=\{1_{n=0},\ddot{1}_1,1_2,\ddot{2}_3,2_4,\ldots\}$$
$$\le n+1,\ \forall n\ge 0. \qquad \text{(c-3)}$$

Statements (c-2) and (c-3) give

$$u(k+1,m+n)\le m+n+1,\ \forall m,n\ge 0.$$

Therefore, (c-1) holds.

Therefore, including $t(k+1,m,0)=0,\ \forall m\ge 0$, we have

$$t(k+1,m,n)\ge 0,\ \forall m,n\ge 0.$$

Since $t(i,0,n)=t(i,n),\ \forall i,n\ge 0$, by (4.2.14), we then also have

$$t(k+1,n)\ge 0,\ \forall n\ge 0.$$

Therefore, (5.2.1.0) also holds for $i=k+1$.

By (5.2.1.2), (5.2.1.3), (5.2.1.4), (5.2.1.5), and induction, we have (5.2.1.0) then holds for all $i\ge 0$. □

As an immediate corollary to $C_{i,n+t(i,n)}<(n+1)p_i,\ \forall i,n\ge 1$, and $t(i,n)\ge 0,\ \forall i,n\ge 0$, we have the following, by setting $n=1$ and using $C_{i,1}=p_{i+1}\le C_{i,1+t(i,1)},\ \forall i\ge 0$:

Corollary 5.2.2. $p_{i+1}-p_i\le p_i-1,\ \forall i\ge 1$, *or* $p_{i+1}<2p_i,\ \forall i\ge 1$. □

5.3 Bertrand's Postulate. We can now give a proof of Bertrand's postulate [4] based on **Corollary 5.2.2**, by first proving the following more general theorem:

Theorem 5.3.1. *For any real* $N > \frac{3}{2}$, *there is at least one* (*regular*) *prime between* N (*excl.*) *and* $2N$ (*excl.*).

Proof: By **Corollary 5.2.2**, we have

$$p_{i+1} - p_i \le p_i - 1,\ \forall i \ge 1. \quad (5.3.1.1)$$

For $N \ge 2$, we may let $p_n = \lfloor N \rfloor_{\mathrm{p}}$ be the greatest (regular) prime[35] less than or equal to N, for some $n \ge 1$. Then,

$$p_n \le N, \quad (5.3.1.2)$$

$$p_n - 1 < N, \quad (5.3.1.3)$$

and

$$p_{n+1} > N. \quad (5.3.1.4)$$

But,

$$\begin{aligned} p_{n+1} - p_n &\le p_n - 1, \text{ by (5.3.1.1)} \\ &< N, \text{ by (5.3.1.2)}. \end{aligned} \quad (5.3.1.5)$$

Therefore,

$$\begin{aligned} p_{n+1} &< p_n + N, \text{ by (5.3.1.5)} \\ &\le N + N, \text{ by (5.3.1.2)} \\ &= 2N. \end{aligned} \quad (5.3.1.6)$$

Therefore $N < p_{n+1} < 2N$, by (5.3.1.4) and (5.3.1.6), so the statement holds for $N \ge 2$. Also, for $\frac{3}{2} < N < 2$, we have $N < 2 = p_1 < 2N$. □

The form of Bertrand's postulate that states there is at least one prime between N (excl.) and $2N$ (incl.), for all integral $N \ge 1$, therefore holds, *a fortiori*, by noting that the statement holds empirically for $N = 1$. This is a new proof, independent of others that exist, *e.g.*, those by Tchebychef [5], Ramanujan [6], Landau [7], and Erdős [8].

6. LOCATIONS OF SOME C_i

6.1 Locations of Some Regular C_i. In **Section 3** we introduced the maximum distance $d_{i,j} = \lVert C_{i,n+j} - C_{i,n} \rVert$ between C_i. We turn our attention now to the location of some C_i, specifically of some regular C_i.

The empirical results in **Appendix C** show that $t(i, n+1) \ge t(i, n),\ \forall (0 \le i \le 5;\ n \ge 0)$. As a result of (4.2.2) and Footnote 20, all $u_{0,n}$ with $n \ge 2$ have a regular C_0 and all $u_{i,n}$ with $(1 \le i \le 5;\ n \ge 1)$ have a regular C_i. However, the examples at the end of **Appendix C** show that there are $u_{i,n}$ with $i \ge 6$ and no regular C_i. Despite this, in **Theorem 6.1.1**

[35] More generally, we can allow $\lfloor N \rfloor_{\mathrm{p}}$ to be either an exceptional or regular prime for $N \ge 1$.

and **Corollary 6.1.2**, we will prove that at least all $\mathrm{u}_{i,n}$ with $(i \ge 6;\ \mathrm{u}_{i,n} < X_{i,3})$ have a regular C_i and therefore a regular prime.

By **Corollary 5.1.7**, we have $X_{i,3} = p_i p_{i+1},\ \forall i \ge 1$. Also,

$\mathrm{u}_{i,n} = \left[np_i, (n+1)p_i\right),\ \forall i, n \ge 0$, by definition.

Therefore, by **Theorem 2.1.8**, one way to prove there is a regular C_i in each $\mathrm{u}_{i,n}$ for $(i, n \ge 1;\ \mathrm{u}_{i,n} < X_{i,3})$ is to show that, for any $(i \ge 1;\ 1 \le n \le p_{i+1} - 1)$, we have $np_i + \{1_{k=1}, 2_2, \dots, (p_i - 1)_{p_i - 1}\} \not\equiv 0 \pmod{p_j},\ \forall j = 1, 2, \dots, i$, for at least one value of k.[36]
We do this in the next theorem:

Theorem 6.1.1. *For any* $\left(i \ge 1;\ n = 1, 2, \dots, (p_{i+1} - 1)\right)$, $\mathrm{u}_{i,n}$ *has a regular* C_i.

Proof: Because of the empirical results in **Appendix C**, it suffices to consider $(i \ge 6;\ n \ge 1)$.[37]
We have $X_{i,n+1} = p_i C_{i,n},\ \forall i, n \ge 1$, by **Theorem 2.3.1**.
Consider

$$np_i + j \equiv 0 \pmod{p_{\sigma(j)} C_{\sigma(j)-1, r(j)}},\ \forall j = 1, 2, \dots, (p_i - 1), \qquad (6.1.1.1)$$

where σ is a permutation on $\{1, 2, \dots, (p_i - 1)\}$, and where $r(j) \ge 1$.
In (6.1.1.1), since each modulus element in $\{1, 2, \dots, (p_i - 1)\} \pmod{p_i}$ corresponds to a unique modulus element in $-\{1, 2, \dots, (p_i - 1)\} \pmod{p_i}$, we may equivalently consider

$$np_i \equiv j \pmod{p_{\sigma(j)} C_{\sigma(j)-1, r(j)}},\ \forall j = 1, 2, \dots, (p_i - 1). \qquad (6.1.1.2)$$

Now consider

$$n_j p_i \equiv j \pmod{p_{\sigma(j)} C_{\sigma(j)-1, r(j)}},\ \forall j = 1, 2, \dots, (p_i - 1). \qquad (6.1.1.3)$$

For each j in (6.1.1.3), a solution n_j exists if and only if the greatest common divisor $\gcd(p_i, p_{\sigma(j)} C_{\sigma(j)-1, r(j)})$ divides j.
Since $(p_i, p_{\sigma(j)}) = 1,\ \forall j = 1, 2, \dots, (p_i - 1)$, we then have, for each j, a solution n_j exists if and only if $\gcd(p_i, C_{\sigma(j)-1, r(j)})$ divides j.
If $C_{\sigma(j)-1, r(j)} < p_i$, then $\gcd(p_i, C_{\sigma(j)-1, r(j)}) = 1$, and so $\gcd(p_i, C_{\sigma(j)-1, r(j)})$ divides j.
If $C_{\sigma(j)-1, r(j)} = p_i$, then $X_{\sigma(j), r(j)+1} = kp_i$, something that lies outside the range $j = 1, 2, \dots, (p_i - 1)$ and therefore does not concern us.
If $C_{\sigma(j)-1, r(j)} > p_i$, then $\gcd(p_i, C_{\sigma(j)-1, r(j)})$ can be 1 or p_i. Again, $\gcd(p_i, C_{\sigma(j)-1, r(j)}) = p_i$ is something that does not concern us.
This still leaves the question of whether such $C_{\sigma(j)-1, r(j)}$ exist for each $j = 1, 2, \dots, (p_i - 1)$.
If, for any given j, such a $C_{\sigma(j)-1, r(j)}$ exists, then (6.1.1.1) gives

[36] The index k is just a dummy variable.
[37] The proof given here works for any $i \ge 2$.

$$C_{\sigma(j)-1,r(j)} = \frac{n_j p_i + j}{p_{\sigma(j)}}. \tag{6.1.1.4}$$

We will show

$$n_j = n, \ \forall j = 1,2,\ldots,(p_i - 1), \text{ is impossible for } n < p_{i+1}. \tag{6.1.1.5}$$

By **Corollary 5.2.2**, this already holds for any $(i \geq 1;\ n = 1)$.
For $2 \leq n < p_{i+1}$, we have n can only be divisible by $p_1, p_2, \ldots,$ or p_i.
Also, since $C_{\sigma(j)-1,r(j)}$ is a regular $C_{\sigma(j)-1}$, we have **Theorem 2.1.8** gives

$$C_{\sigma(j)-1,r(j)} \text{ cannot be divisible by any of } p_1, p_2, \ldots, p_{\sigma(j)-1}. \tag{6.1.1.6}$$

But, for $n_j = n,\ \forall j = 1,2,\ldots,(p_i - 1)$, we have (6.1.1.4) gives $np_i + j$ must be divisible by $p_{\sigma(j)}$, for all $j = 1,2,\ldots,(p_i - 1)$. This contradicts (6.1.1.6).
Therefore, (6.1.1.5) is true.
Therefore, for any $\left(i \geq 1;\ n = 1,2,\ldots,(p_{i+1} - 1)\right)$, $\mathrm{u}_{i,n}$ has a regular C_i. □

By (5.1.1.4), we have $X_{i,3} < X_{i+1,2},\ \forall i \geq 1$. Also, by **Theorem 2.3.5**, all regular $C_i < X_{i+1,2}$, for $i \geq 0$, are (regular) primes. Therefore, **Theorem 6.1.1** immediately gives the following corollary:

Corollary 6.1.2. *All* $\mathrm{u}_{i,n}$ *with* $(i,n \geq 1;\ \mathrm{u}_{i,n} < X_{i,3})$ *contain a regular* prime. □

7. PROOF OF SOME EXTANT CONJECTURES ON PRIMES

7.1 Brocard's Conjecture. Brocard's conjecture [9] is that there are at least 4 (regular) primes between p_n^2 and p_{n+1}^2, for any $n \geq 2$. Empirically, the number of (regular) primes between p_n^2 and p_{n+1}^2 is

$$\{2_{n=0}, 2_1, 5_2, 6_3, 15_4, 9_5, 22_6, \ldots\}, \tag{7.1.1}$$

which appears to increase non-monotonically as a function of n for $n \geq 1$. Based on **Corollary 6.1.2**, we will prove the following theorem, which gives a result at least as strong as Brocard's conjecture:

Theorem 7.1.1. *For any* $n \geq 2$, *the number of primes between* p_n^2 *and* p_{n+1}^2 *is at least* $2\Delta_n \geq 4$.

Proof: We have $p_n p_{n+1} - p_n^2 = p_n(p_n + \Delta_n) - p_n^2 = \Delta_n p_n$.
Also, $p_{n+1}^2 - p_n p_{n+1} = p_{n+1}^2 - (p_{n+1} - \Delta_n) p_{n+1} = \Delta_n p_{n+1}$.
By **Corollary 6.1.2**, there are then at least Δ_n (regular) primes between p_n^2 and $X_{n,3} = p_n p_{n+1}$ and a further at least Δ_n (regular) primes in the last Δ_n u_{n+1} before $X_{n+1,2} = p_{n+1}^2$. The total number of primes between p_n^2 and p_{n+1}^2 is then at least

$2\Delta_n \geq 4,\ \forall n \geq 2$, by (2.3.7). □

7.2 Legendre's Conjecture. Legendre's conjecture [10] is that there is a (regular) prime between n^2 and $(n+1)^2$ for all $n \geq 1$. Empirically, the number of (regular) primes between n^2 and $(n+1)^2$ for $n \geq 1$ is $\{2_{n=1}, 2_2, 2_3, 3_4, 2_4, 4_5, \ldots\}$, which, like (7.1.1), also appears to increase erratically as a function of n. We will prove Legendre's conjecture using **Corollary 6.1.2** and the following lemma, whose proof is left to the reader:

Lemma 7.2.1. *For any $i \geq 0$, any $2p_i$ consecutive cells in the opened Sieve span at least one full* u_i. □

Proof of Legendre's Conjecture. For any $n \geq 1$, there is an $i \geq 0$ such that

$$p_i \leq n < (n+1) \leq p_{i+1}.$$

For these, we have

$$p_i^2 \leq n^2 < (n+1)^2 = n^2 + 2n + 1 \leq p_{i+1}^2,$$

so that the distance between these n^2 and $(n+1)^2$ is at least

$$2n+1 \geq 2p_i + 1.$$

Therefore, by **Lemma 7.2.1** and **Corollary 6.1.2** there is at least one (regular) prime between n^2 (excl.) and $(n+1)^2$ (excl.). □

7.3 The Conjecture that $\operatorname{Lim}_{n\to\infty}\left((p_{n+1} - p_n)/p_n\right) = 0$.

Proof: For given $i \geq 0$, let $p_{n(i)} = \lfloor p_i^2 \rfloor_{\mathrm{p}}$.

Then, **Corollary 6.1.2** and **Theorem 2.3.21** give

$$p_i^2 - p_i < p_{n(i)} < p_i^2 < p_{n(i)+1} < p_i^2 + p_i, \ \forall i \geq 1. \tag{7.3.1}$$

Since there are an infinite number of p_i, we may consider $\lim_{n(i)\to\infty}\left((p_{n(i)+1} - p_{n(i)})/p_{n(i)}\right)$.

Changing the variable $n(i)$ to simply n then gives

$$\begin{aligned}\lim_{n\to\infty}\frac{p_{n+1}-p_n}{p_n} &\leq \lim_{i\to\infty}\frac{(p_i^2+p_i)-(p_i^2-p_i)}{p_n}, \text{ by (7.3.1)}\\ &\leq \lim_{i\to\infty}\frac{2p_i}{p_i^2-p_i}, \text{ also by (7.3.1)}\\ &= 0. \ \square\end{aligned}$$

7.4 Andrica's Conjecture. Andrica's conjecture [11] is that

$$A_n = \sqrt{p_{n+1}} - \sqrt{p_n} < 1, \ \forall n \geq 1.$$

We will prove the following, again using **Corollary 6.1.2**:

Theorem 7.4.1. $\sqrt{p_{n+1}} - \sqrt{p_n} < 1, \ \forall n \geq 0$.

Proof: For $i \geq 2$, there are at least two u_i between p_i^2 and $X_{i,3} < X_{i+1,2} = p_{i+1}^2$.

Therefore, by **Corollary 6.1.2** and **Theorem 2.3.5**, there is then an n such that

$$p_i^2 < p_n < p_{n+1} < X_{i,3} < p_{i+1}^2 . \qquad (7.4.1.1)$$

Empirically, the smallest value here, $n = 5$, occurs for $i = 2$.
Therefore, for $n \geq 5$, there is an $i \geq 2$ such that

$$p_{n+1} - p_n < 2p_i, \text{ by } \textbf{Corollary 6.1.2}, \qquad (7.4.1.2)$$

and

$$\sqrt{p_{n+1}} + \sqrt{p_n} > 2p_i, \text{ by (7.4.1.1).} \qquad (7.4.1.3)$$

For any such p_n and p_{n+1}, we have

$$\begin{aligned} \sqrt{p_{n+1}} - \sqrt{p_n} &= \frac{p_{n+1} - p_n}{\sqrt{p_{n+1}} + \sqrt{p_n}} \\ &< \frac{2p_i}{2p_i}, \text{ by (7.4.1.2) and (7.4.1.3)} \\ &= 1. \end{aligned}$$

There are also exactly $\Delta_i \geq 2$ u_{i+1} between $X_{i,3} = p_i p_{i+1}$ and $X_{i+1,2} = p_{i+1}^2$, each with at least one prime, by **Corollary 6.1.2**. These are inherited from row i and include all primes between $X_{i,3}$ and p_{i+1}^2. For these, we have

$$\begin{aligned} \sqrt{p_{n+1}} - \sqrt{p_n} &= \frac{p_{n+1} - p_n}{\sqrt{p_{n+1}} + \sqrt{p_n}} \\ &< \frac{2p_{i+1}}{2p_{i+1}}, \text{ by (7.4.1.2) and (7.4.1.3)} \\ &= 1. \end{aligned}$$

As i increases from $i = 2$, the above includes all p_n with $n \geq 5$.
It remains to show that $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ also for $n = 0, 1, 2, 3, 4$. We have

$$\begin{aligned} \sqrt{p_1} - \sqrt{p_0} &= \sqrt{2} - 1 \\ &< 1.415 - 1 \\ &< 1, \end{aligned}$$

$$\begin{aligned} \sqrt{p_2} - \sqrt{p_1} &= \sqrt{3} - \sqrt{2} \\ &< 1.733 - 1.414 \\ &< 1, \end{aligned}$$

$$\begin{aligned} \sqrt{p_3} - \sqrt{p_2} &= \sqrt{5} - \sqrt{3} \\ &< 2.237 - 1.732 \\ &< 1, \end{aligned}$$

$$\begin{aligned} \sqrt{p_4} - \sqrt{p_3} &= \sqrt{7} - \sqrt{5} \\ &< 2.646 - 2.236 \\ &< 1, \end{aligned}$$

and

$$\begin{aligned} \sqrt{p_5} - \sqrt{p_4} &= \sqrt{11} - \sqrt{7} \\ &< 3.317 - 2.645 \\ &< 1. \end{aligned}$$

Therefore, $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all $n \geq 0$. □

Since, by **Theorem 3.1.6**, the gap between consecutive primes can be arbitrarily large, Andrica's conjecture, as stated, is unintuitive. However, the equivalent statement

$$p_{n+1} - p_n < \sqrt{p_{n+1}} + \sqrt{p_n},\ \forall n \geq 1,$$

is illuminating. In fact, this equivalent of Andrica's conjecture allows another proof for $\lim_{n\to\infty} \frac{p_{n+1} - p_n}{p_n} = 0$:

Proof: $p_{n+1} - p_n < \sqrt{p_{n+1}} + \sqrt{p_n},\ \forall n \geq 1$, gives

$$\frac{p_{n+1} - p_n}{p_n} < \frac{\sqrt{p_{n+1}} + \sqrt{p_n}}{p_n}$$

$$< \frac{\sqrt{2p_n} + \sqrt{p_n}}{p_n}, \text{ by } \textbf{Corollary 5.2.2}$$

$$= \frac{\sqrt{p_n}(\sqrt{2} + 1)}{p_n}$$

$$= \frac{(\sqrt{2} + 1)}{\sqrt{p_n}}.$$

Therefore, $\lim_{n\to\infty} \frac{p_{n+1} - p_n}{p_n} = 0$. □

7.5 **Oppermann's Conjecture.** Oppermann's conjecture [12, 13] is that for any integer $n \geq 2$ there is a (regular) prime between $n^2 - n$ (excl.) and n^2 (excl.), as well as between n^2 (excl.) and $n^2 + n$ (excl.).

Proof of Oppermann's Conjecture: For any $i \geq 1$, consider $p_i \leq n + \delta < p_{i+1}$, for integral δ.
For any $i, m \geq 0$, let $N = \partial \mathrm{u}_{i,m}$ denote the start of $\mathrm{u}_{i,m}$, *i.e.*, its first column number.
We have

$$n^2 = p_i^2 + 2\delta p_i + \delta^2 = (\partial \mathrm{u}_{i,p_i+2\delta}) + \delta^2,$$

$$n^2 - n = p_i^2 + (2\delta - 1)p_i + \left(\delta^2 - \delta\right) = (\partial \mathrm{u}_{i,p_i+2\delta-1}) + (\delta^2 - \delta),$$

and

$$n_i^2 + n_i = \left(p_i^2 + (2\delta + 1)p_i\right) + (\delta^2 + \delta) = (\partial \mathrm{u}_{i,p_i+(2\delta+1)}) + (\delta^2 + \delta).$$

Therefore, for any $i \geq 1$, between $n^2 - n$ and n^2, as well as between n^2 and $n^2 + n$, there is at least one full $\mathrm{u}_{i,m}$ with $m \geq 1$.
By **Corollary 6.1.2**, there is a (regular) prime in these u_i.
This holds for all $i \geq 1$ and, therefore, for all $n \geq 2$. □

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Appendix A: Summary of Notations and Their Definitions

Lower case Roman letters	Non-negative integers.
N	Non-negative real, unless otherwise indicated.
(excl.)	Exclusive of/excluding
(incl.)	Inclusive of/including
p_i	Primes, $p_0 = 1,\ p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ \ldots$, where $p_0 = 1$ is an exceptional prime and all p_i with $i \geq 1$ are regular primes.
Π_i	$\Pi_i = \prod_{j=0}^{i} p_j$, an abbreviated notation used for convenience.
$[\ldots,\ldots)$	The standard left-closed, right-open interval.
$\mathrm{U}_{i,n}$	$\mathrm{U}_{i,n} = \left[p_i + (n-1)\Pi_i, p_i + n\Pi_i\right),\ \forall(i \geq 0;\ n \geq 1)$.
U_i	An arbitrary $\mathrm{U}_{i,n}$ for given i.
$\mathrm{u}_{i,n}$	$\mathrm{u}_{i,n} = [np_i, (n+1)p_i),\ \forall i, n \geq 0$.
u_i	An arbitrary $\mathrm{u}_{i,n}$ for given i.
$\partial\mathrm{u}_{i,n}$	The start of $\mathrm{u}_{i,n}$, *i.e.*, its first column number.
$C_{i,n}$	The n^{th} prime candidate in row i of the opened Sieve. Only prime candidates can become primes in rows below. All $C_{i,0}$ are special prime candidates, and all $C_{i,n}$ with $n \geq 1$ are regular prime candidates.
$C_{i,0}$	$C_{i,0} = p_i$, for all $i \geq 0$, by definition. $C_{i,1} = C_{i+1,0} = p_{i+1}$, for all $i \geq 0$, by the algorithm that produces the opened Sieve, and by definition.
C_i	An arbitrary $C_{i,n}$ for given i.
Δ_i	$\Delta_i = p_{i+1} - p_i,\ \forall i \geq 0$
$\Delta_{i,n}$	$\Delta_{i,n} = C_{i,n} - p_i,\ \forall i, n \geq 0$

$X_{i,n}$ — The n^{th} value of the column number (where $n = 1, 2, 3, \dots$) in row i of the opened Sieve where there is an X, showing that a prime candidate in that column of the Sieve cannot be inherited below that row.
$X_{i,1} = p_i = C_{i,0}$, for all $i \ge 1$, by definition.
We can also define $X_{i,0} = 0,\ \forall i \ge 0$, even though there are no cells with an X at the column values $j = 0$.

X_i — An arbitrary $X_{i,n}$ for given i.

$S_i(N)$ — The number of regular C_i less than or equal to N.

$S_i(N_1, N_2)$ — $S_i(N_1, N_2) = S_i(N_2) - S_i(N_1)$.

$x_i(N)$ — The number of X_i less than or equal to N.

$x_i(N_1, N_2)$ — $x_i(N_1, N_2) = x_i(N_2) - x_i(N_1)$.

$u(i,n)$ — $u(i,n) = x_i(C_{i,n}),\ \forall (i \ge 1;\ n \ge 0)$. This change in notation facilitates use of $x_i(C_{i,n})$ as a subscript.

$T_i(N)$ — $S_i(N) = \left\lfloor \dfrac{N - p_i + 1}{p_i} \right\rfloor + T_i(N),\ \forall (i \ge 1;\ N \ge p_i)$.

$t(i,n)$ — $S_i(np_i) = \left\lfloor \dfrac{np_i - p_i + 1}{p_i} \right\rfloor + t(i, n-1),\ \forall i, n \ge 1$.
$t(i,n) = T\left(i, (n+1)p_i\right),\ \forall (i \ge 1;\ n \ge 0)$.
$S_i(np_i) = n - 1 + t(i, n-1),\ \forall (i \ge 0;\ n \ge 1)$, is an easier definition to work with and extends to $i = 0$.

$|\mathrm{U}_i|_1$ — The number of regular C_i in U_i. The 1 here is a symbol, not a number.

$|\mathrm{U}_i|_\mathrm{X}$ — The number of X in (an arbitrary) U_i.

$|N|_{1_i}$, $|N|_{C_i}$, $|N|_{X_i}$ — The number of regular C_i, special and regular C_i, and X_i, respectively, less than or equal to N. The subscript i is not needed if used inside $|\ |$, *e.g.*, $|\mathrm{U}_i|_\mathrm{X}$.

$\lfloor N \rfloor_\mathrm{p}$, $\lfloor N \rfloor_{C_i}$, $\lfloor N \rfloor_{X_i}$ — The greatest prime (exceptional or regular), C_i, and X_i, respectively, less than or equal to N. $\lfloor N \rfloor_{X_i}$ is not used in

	this paper.
$\lceil N \rceil^{p_i}$, $\lceil N \rceil^{C_i}$, $\lceil N \rceil^{X_i}$	The least p_i, C_i, and X_i respectively, greater than or equal to N. $\lceil N \rceil^{p_i}$ and $\lceil N \rceil^{X_i}$ are not used in this paper.
$C_{i,(\overline{n})_i}$	$C_{i,(\overline{n})_i} = \lfloor X_{i,n+1} \rfloor_{C_i}$, for any $(i \geq 1;\ n \geq 0)$.
$C_{i,(\underline{n})_i}$	$C_{i,(\underline{n})_i} = \lceil X_{i,n+1} \rceil^{C_i}$, for any $(i \geq 1;\ n \geq 0)$.
{...}	A list/listing of values, using ordered indices. See **Appendix B** on generalized repetends for specific definitions, notations, and theorems.
$\rfloor\!\lceil$ {...}	The element values in the list/listing are monotonically increasing as a function of their ordered indices.
$\prod_{j=1}^{0}(p_j - 1) = 1$	This is a definition by fiat, and is similar to $0! = 1$.

Appendix B: Generalizing the Concept of Repetends

In this appendix, we generalize the concept and notation of repetends, develop an algebra of rules for manipulation, and give three examples of how these can be used in mathematics.

In 1872, when Dedekind, Cantor, and Heine each gave their formal definitions of real numbers, it was already well-known that irrational numbers had non-repeating decimal digits. Two years later, in 1874, the word "repetend" appeared in literature to denote a repeated word, phrase, or sound[38]. The word "repetend" also began to be used for repeated decimal characters.[39]

However, there is presently no consistent notation for decimal repetends throughout the world. Depending on the country, a repetend is represented by parentheses $(a\ldots b)$, an overlying line (called a vinculum) $\overline{a\ldots b}$, an overlying arc $\overparen{a\ldots b}$, one or two overlying single dots $\dot{a}$ or $\dot{a}\ldots\dot{b}$, or an (ambiguous) ellipsis $a.bc\ldots$.[40]

One notation that suits our purposes is the single or one overdot[41], as follows:

$$a_1a_2\ldots a_n \cdot b_1b_2\ldots \dot{b}_m = a_1a_2\ldots a_n \cdot b_1b_2\ldots b_mb_mb_m\ldots$$

and $$a_1a_2\ldots a_n \cdot b_1b_2\ldots \dot{b}_m\ldots \dot{b}_{m+r} = a_1a_2\ldots a_n \cdot b_1b_2\ldots b_m\ldots b_{m+r}b_m\ldots b_{m+r}b_m\ldots b_{m+r}\ldots\ .$$

This notation can also be applied to strings or lists of characters outside its use with decimals. To do this, we first introduce the following concepts and notation:

Write $A=\{a_{n=1},a_2,\ldots\}$ to denote a list or listing of possible expressions[42] of A, as opposed to a set. We will deal here exclusively with lists.

We have/define

$$\{\{a_{n=1},a_2,\ldots\}\}=\{a_{n=1},a_2,\ldots\}, \tag{R-1}$$

a property that is not shared by sets. The list is ordered with respect to its indices. These indices will usually be non-negative integers, but need not be as long as they are ordered and we know what the index ordinal value is.[43] Unless otherwise indicated, we will assume the indices are the integral ordinal values. The first index will usually be shown as $n=$ to establish that we are working with a list/listing and not a set, as well as to identify the index as n.[44] We can also use $\{a_{n=1},a_{n=2},\ldots,a_{n=r},\ldots\}$ at any time. This helps avoid confusion, especially with more than one index variable. Any combination of notations that is not

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[38] According to the Merriam-Webster English dictionary.
[39] I have been unable so far to find out whether the literary or mathematical use came first.
[40] See Wikipedia: Repeating Decimals.
[41] We are going to use "overdot" instead of "overlying dot".
[42] We use "expressions" here in a generic sense, not a mathematical one. For precision in meaning, we could use "list" for the expression $\{a_{n=1},a_2,\ldots\}$ and the gerundive "listing" for the process of creating it, but this may be too pedantic.
[43] We must always be able to do a change of variables so we can list the elements using integral ordinal values when applying arithmetic rule (R-7) later. An index may then be negative, zero, or non-integral.
[44] This is not necessary and may be dispensed with if there is no ambiguity.

ambiguous is acceptable. For example,

$$\begin{Bmatrix} \{a_{n=1}, a_{n=2}, \ldots\}_{m=1}, \\ \{b_{n=1}, b_{n=2}, \ldots\}_{m=2}, \\ \vdots \end{Bmatrix} = \begin{Bmatrix} \{a_{n=1}, a_2, \ldots\}_{m=1}, \\ \{b_{n=1}, b_2, \ldots\}_{m=2}, \\ \vdots \end{Bmatrix}.$$

Call the expressions the elements of the list/listing. The elements $a_1, a_2, \ldots$ here may be anything, *e.g.*, characters, numerical values, mathematical formulae, images, etc. Unless otherwise indicated, it is understood that the listing $\{a_{n=m}, a_{m+1}, \ldots\}$ contains all the elements a_n with $n \geq m$, while $\{a_{n=m}, a_{m+1}, \ldots, a_{m+r}\}$ is a finite list of $r+1$ elements.

If f is a function[45] on $A = \{a_{n=1}, a_2, \ldots\}$, we define

$$f\left(\{a_{n=1}, a_2, \ldots\}\right) = \left\{\left(f(a_{n=1})\right)_{n=1}, \left(f(a_{n=2})\right)_{n=2}, \ldots\right\}. \qquad \text{(R-2)}$$

No matter what the value of $f(a_n)$ is, its ordinality in the list on the right is the same as that of a_n in the list on the left.

If $f(a_m) = a_m$, then $\left(f(a_m)\right)_m = (a_m)_m = a_m$, to agree with (R-2).

Now we can formally introduce our single or one overdot notation. For any $r \geq 1$, let

$$\begin{aligned} \{\dot{a}_{n=1}, a_2, \ldots, \dot{a}_r, \ldots\} &= \left\{\{a_{n=1}, a_2, \ldots, a_r\}_{m=1}, \{a_{n=1}, a_2, \ldots, a_r\}_{m=2}, \ldots\right\} \\ &= \left\{(a_1)_{k=1}, (a_2)_{k=2}, \ldots, (a_r)_{k=r}, (a_1)_{k=1+r}, \ldots, (a_n)_{k=n+(m-1)r}, \ldots\right\}, \end{aligned}$$

for all $m, n \geq 1$ such that $(a_n)_{k=n+(m-1)r}$ lies within $\{a_{n=1}, a_2, \ldots, a_r, \ldots\}$,

be a finite or infinite listing of a repeated list $\{a_{n=1}, a_2, \ldots, a_r\}$. This gives

$$\{\dot{a}_{n=1}, a_2, \ldots, \dot{a}_r\} = \left\{\{a_{n=1}, a_2, \ldots, a_r\}\right\} = \{a_{n=1}, a_2, \ldots, a_r\}.$$

In $\left\{\{a_{n=1}, a_2, \ldots, a_r\}_{m=1}, \{a_{n=1}, a_2, \ldots, a_r\}_{m=2}, \ldots\right\}$, we call $a_1, a_2, \ldots, a_r$ the repetend, or more specifically the n-repetend. We call r the repetend length, or more specifically the n-repetend length. For $r = 1$, we have

$$\{\dot{a}_1, \ldots\} = \left\{\{a_1\}_{m=1}, \{a_1\}_{m=2}, \ldots\right\} = \{(a_1)_{m=1}, (a_1)_{m=2}, \ldots\}.$$

From here on, we are concerned with elements that have quantifiable values[46]. We introduce double or two overdot notation in lists to denote the following property:

for any $r \geq 1$,

$$A = \{a_{n=1}, a_2, \ldots\} = \{\ddot{a}_{n=1}, \ldots, \ddot{a}_{1+r}, \ldots\},$$

if and only if $d = a_{k+r} - a_k$ is a constant, for all $k \geq 1$. (R-3)

For every $k \geq 1$, unless $d = 0$, it is not the elements $a_{n=k}, a_{k+1}, \ldots, a_{k+r-1}$ that are repeated here but their (arithmetic) pattern in a step-wise manner.

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[45] A function here need not produce numerical values.

[46] A function on these values, for our purposes now, must give numerical values, ones that can be used for numerical arithmetic.

For either single or double overdot notation the following hold:

(1) the elements may be real or complex;

(2) we will say that $A = \{a_{n=1}, a_2, \ldots\}$ is a repetend function of n, or an n-repetend function;

(3) for $A = \{a_{n=1}, a_2, \ldots\}$, we will call $a_k, \ldots, a_{k+r-1}$ for each $k \geq 1$ a repetend, or more precisely an n-repetend;

(4) we will call r the repetend length, or more precisely an n-repetend length; and

(5) we will call d the repetend difference, or more precisely the n-repetend difference.

Using the terminology "n-repetend" emphasizes which index variable is involved and is helpful when more than one index variable is involved.

If we write $\{a_{n=1}, \ldots, \ddot{a}_m, \ldots, \ddot{a}_{m+r}, \ldots\}$, for any $(m \geq 2;\ r \geq 1)$, the elements $a_1, a_2, \ldots, a_{m-1}$ may not satisfy (R-3). We still have $d = a_{m+k+r} - a_{m+k}$ is a constant, for all $k \geq 0$, and the first n-repetend may be $a_m, a_{m+1}, \ldots, a_{m+r-1}$. We may therefore move the double overdots to the right as follows:

Proposition 1. $\{a_1, \ldots, \ddot{a}_m, \ldots, \ddot{a}_{m+r}, \ldots\} = \{a_1, , \ldots, \ddot{a}_{m+k}, \ldots, \ddot{a}_{m+k+r}, \ldots\}$, for all $m, k, r \geq 1$. □

However, the converse is not true in general, although there are exceptions[47]. If we are given $\{a_{n=1}, a_2, \ldots, \ddot{a}_m, \ldots, \ddot{a}_{m+r}, \ldots\}$ and the start of the first an n-repetend is at $n = m > 1$, we may not be able to move the double overdots to the left to get

$$\{a_1, \ldots, \ddot{a}_m, \ldots, \ddot{a}_{m+r}, \ldots\} = \{a_1, \ldots, \ddot{a}_{m-k}, \ldots, \ddot{a}_{m-k+r}, \ldots\}, \text{ for any } k \geq 1.$$

For any $n \geq 1$,[48] given a single overdot notation listing, the single overdot notation can always be converted to the double overdot notation, as follows:

$$\{\dot{a}_{n=1}, \ldots, \dot{a}_r, \ldots\} = \{\ddot{a}_{n=1}, \ldots, \ddot{a}_{r+1}, \ldots\}, \text{ with } d = a_{k+r+1} - a_{k+1} = 0, \text{ for all } k \geq 0. \qquad \text{(R-4)}$$

This allows us to call r in both the single and double overdot notations by the same name, n-repetend length.

Converting the double overdot notation to a single overdot notation is possible if and only if $d = a_{k+r+1} - a_{k+1} = 0,$ for all $k \geq 0$.

As with the double overdots, the single overdots may also always be moved any distance to the right, but not always to the left.

In general,

$$\text{the smallest repetend length possible is } r = 1. \qquad \text{(R-5)}$$

$$\text{the smallest absolute repetend difference possible is } d = 0. \qquad \text{(R-6)}$$

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[47] An obvious exception is one in which the overdots have already been moved to the right before a list is presented to you, but there are others. One that uses **Proposition 4** is given at the end of this appendix.

[48] The choice of $n \geq 1$ here is for convenience only. The indices may be any values that can be assigned ordinal values relative to each other.

Along with (R-2), we impose rules for arithmetic with lists. For elements in two lists with the same ordinal index values we use the following rule:

$$\{a_{n=1}, a_2, \ldots, a_m, \ldots\} \pm \{b_{n=1}, b_2, \ldots, b_m, \ldots\} = \{(a_{n=1} \pm b_{n=1})_{n=1}, (a_{n=2} \pm b_{n=2})_{n=2}, \ldots, (a_{n=m} \pm b_{n=m})_{n=m}, \ldots\}. \quad \text{(R-7)}$$[49]

For any c, we impose the following two rules:

$$c + \{a_{n=1}, a_2, \ldots, a_m, \ldots\} = \{(c + a_{n=1})_{n=1}, (c + a_2)_2, \ldots, (c + a_m)_m, \ldots\}, \quad \text{(R-8)}$$

$$c\{a_{n=1}, a_2, \ldots, a_m, \ldots\} = \{(ca_{n=1})_{n=1}, (ca_2)_2, \ldots, (ca_m)_m, \ldots\}. \quad \text{(R-9)}$$

We multiply a list by a list in the same way as we do a matrix by a matrix:

$$\{a_{i=1}, a_2, \ldots\}^{\mathrm{t}} \{b_{j=1}, b_2, \ldots\} = \{a_i b_j\} = \left\{ \begin{matrix} a_{i=1}\{b_{j=1}, b_2, \ldots\}, \\ a_{i=2}\{b_{j=1}, b_2, \ldots\}, \\ \vdots \end{matrix} \right\}, \quad \text{(R-10)}$$

where the superscript t indicates transpose: $\{a_{i=1}, a_2, \ldots\}^{\mathrm{t}} = \left\{ \begin{matrix} a_{i=1}, \\ a_{i=2}, \\ \vdots \end{matrix} \right\}$.

Each element of one list can be treated independently from the other elements of that list for addition and subtraction with the elements of another list as follows:

$$\{a_{i=1}, a_2, \ldots\}^{\mathrm{t}} + \{b_{j=1}, b_2, \ldots\} = \left\{ \begin{matrix} a_{i=1}, \\ a_{i=2}, \\ \vdots \end{matrix} \right\} + \{b_{j=1}, b_2, \ldots\} = \left\{ \begin{matrix} \{a_{i=1} + \{b_{j=1}, b_2, \ldots\}\}_{i=1}, \\ \{a_{i=2} + \{b_{j=1}, b_2, \ldots\}\}_{i=2}, \\ \vdots \end{matrix} \right\}. \quad \text{(R-11)}$$

In (R-7) it is the index ordinal values that determine which elements are added together or subtracted, not what the indices are labeled as or what the index values are. If we restrict ourselves to all the elements of the lists here being real, we must then also have c in (RT-8) and (R-9) being real.

If a repetend element a_m consists of more than one character, it is often convenient to write $\ddot{a}_m = (\ddot{a})_m$ or $\ddot{a}_m = (\ddot{a_m})$, whichever is appropriate under the circumstances.

When two index variables are involved, each independently associated with repetends, we can write

$$\left\{ \begin{matrix} \{\ddot{a}_{n=1}, \ldots, \ddot{a}_{1+r_n}, \ldots\ddot{\}}_{m=1}, \\ \vdots \\ \{\ddot{b}_{n=1}, \ldots, \ddot{b}_{1+r_n}, \ldots\ddot{\}}_{m=1+r_m}, \\ \vdots \end{matrix} \right\}.$$

[49] We will not concern ourselves here with one list containing index ordinal values that do not occur in the other list. In general, what is done in such situations depends on the context.

This can be extended to any number of index variables, similar to matrix notation. In such a case the double overdot relation (R-3) for any variable applies with the other variables being kept constant. When possible, different index variables can be displayed in different index rows associated with the elements.

We say $\{a_{n=1},\ldots,a_m\}\geq\{b_{n=1},\ldots,b_m\}$ if and only if the relation $a_n \geq b_n$ holds term by term, where m need not be finite, and similarly for $=,\leq,>,<$.

With these definitions, we can now start to do manipulations and calculations. If a proof is not given for any of the following relations, it is left to the reader.

Proposition 2. *If f is a linear function on $A=\{a_{n=1},a_2,\ldots\}$, then*

$$f\left(\{\ddot{a}_{n=1},a_2,\ldots,\ddot{a}_{1+r},\ldots\}\right)=\left\{\ddot{\left(f(a_{n=1})\right)}_{n=1},\left(f(a_2)\right)_2,\ldots,\ddot{\left(f(a_{1+r})\right)}_{1+r},\ldots\right\}.\text{[50]} \qquad \text{(P2-0)}$$

Proof: For some constants B and C, we have

$$f(a_n)=Ba_n+C.$$

By (R-3), we have $a_{k+r}-a_k=d_A$, for all $k\geq 1$, where d_A is a constant.
Therefore, for any $k\geq 1$, we have

$$\begin{aligned}\left(f(a_{k+r})\right)_{k+r}-\left(f(a_k)\right)_k &= (Ba_{k+r}+C)_{k+r}-(Ba_k+C)_k\\ &= B\left((a_{k+r})_{k+r}-(a_k)_k\right),\text{ by (R-9)}\\ &= B(a_{k+r}-a_k)\\ &= Bd_A,\text{ a constant.}\end{aligned}$$

Therefore, by (R-2) and (R-3), we have (P2-0) holds. □

Proposition 3. *If in $\{\ddot{a}_{n=1},\ldots,\ddot{a}_m,\ldots\}$ and $\{\ddot{b}_{n=1},\ldots,\ddot{b}_m,\ldots\}$ we have $\{a_{n=1},\ldots,a_m\}\geq\{b_{n=1},\ldots,b_m\}$ and the respective n-repetend differences $d_a\geq d_b$, then*

$$\{\ddot{a}_{n=1},\ldots,\ddot{a}_n,\ldots\}\geq\{\ddot{b}_{n=1},\ldots,\ddot{b}_n,\ldots\}.\ \square$$

The statements of **Proposition 3** for the relations $=,\leq,>,<$ are left to the reader.
A word of caution in **Proposition 3** is needed here. If we have lists with different repetend lengths, say $\{\ddot{a}_{n=1},\ldots,\ddot{a}_{1+r_a},\ldots\}$ and $\{\ddot{b}_{n=1},\ldots,\ddot{b}_{1+r_b},\ldots\}$, with $r_a\neq r_b$, then the inequality comparison must be shown out to index $1+\text{lcm}(r_a,r_b)$ [51] here, and we must have $\frac{d_a}{r_a}\geq\frac{d_b}{r_b}$ (see why in **Theorem 9**) before we can say

$$\{\ddot{a}_{n=1},\ldots,\ddot{a}_{1+r_a},\ldots\}\geq\{\ddot{b}_{n=1},\ldots,\ddot{b}_{1+r_b},\ldots\}.$$

[50] This is an example in which placing the index values of n outside the parentheses (on the right side of the equation here) would be completely wrong, since $f(a)$ is undefined.
[51] As usual, "lcm" stands for least common multiple.

Statements for similar situations for the other relations $=, \leq, >, <$ are left to the reader.

Proposition 4. *If, for a given double overdot notation listing,* r *is a repetend length and* d *the repetend difference for that* r*, then, for any integral* $k \geq 2$*, we have* kr *is also a repetend length, with corresponding repetend difference* kd. □

Proposition 5. *If, for a given double overdot notation listing, there are two different repetend lengths* r_1 *and* r_2 *with respective corresponding repetend differences* d_1 *and* d_2*, then* $r_2 d_1 = r_1 d_2$.

Proof: By **Proposition 1**, we may assume that there is a $k \geq 1$ such that

$$\{a_{n=1}, a_2, \ldots\} = \{a_{n=1}, \ldots, \ddot{a}_k, \ldots, \ddot{a}_{k+r_1}, \ldots\} = \{a_{n=1}, \ldots, \ddot{a}_k, \ldots, \ddot{a}_{k+r_2}, \ldots\}.$$

Therefore, by **Proposition 4**, we also have

$$\{a_{n=1}, \ldots, \ddot{a}_k, \ldots, \ddot{a}_{k+r_1}, \ldots\} = \{a_{n=1}, \ldots, \ddot{a}_k, \ldots, \ddot{a}_{k+r_1 r_2}, \ldots\} \quad \text{(P5-1)}$$

and $$\{a_{n=1}, \ldots, \ddot{a}_k, \ldots, \ddot{a}_{k+r_2}, \ldots\} = \{a_{n=1}, \ldots, \ddot{a}_k, \ldots, \ddot{a}_{k+r_2 r_1}, \ldots\}. \quad \text{(P5-2)}$$

Since the RHS[52] of both listings (P5-1) and (P5-2) are the same, we have the repetend differences in the RHS of (P5-1) and (P5-2) are the same, *i.e.*,

$r_2 d_1 = r_1 d_2$, by **Proposition 4**. □

The result in **Proposition 5** can also be stated $\frac{r_2}{r_1} = \frac{d_2}{d_1}$. As an immediate corollary we have the following:

Corollary 6. *If, for a given double overdot notation listing, there are two different repetend lengths* r_1 *and* r_2 *with respective repetend differences* $d_1 < d_2$*, then* $r_1 < r_2$*. Conversely, if* $r_1 < r_2$*, then* $d_1 < d_2$. □

We now have the following converse of **Proposition 4**:

Proposition 7. *If, for a given double overdot notation listing,* r *is a repetend length, then any lesser repetend length must be a factor*[53] *of* r. □

Proof: By *reductio ad absurdum*:
For a given listing with two different repetends, if the two different first repetends start with different indices, shift one set of the repetend overdots to the right so the first elements of the two repetends coincide. Then relabel the indices so the first index of each repetend is 1. Let $A = \{\ddot{a}_1, a_2, \ldots, \ddot{a}_{1+r_1}, \ldots\} = \{\ddot{a}_1, a_2, \ldots, \ddot{a}_{1+r_2}, \ldots\}$ be the resultant listings starting with index 1, where r_1, r_2 are the respective repetend lengths. Without loss of generality, we may let

[52] "RHS" stands for right hand side.
[53] Including the possibility 1.

$$r_1 < r_2 \,. \qquad \text{(P7-1)}$$

Suppose, to the contrary of the proposition statement, that

$$r_1 \nmid r_2 \,. \qquad \text{(P7-2)}$$

Let the respective minimum repetend differences be d_1, d_2.

By (P7-1) and **Corollary 6**, we have

$$d_1 < d_2 \,. \qquad \text{(P7-3)}$$

Then, by **Proposition 1**, we have

$$\begin{aligned}
\{\ddot{a}_1, a_2, \ldots, \ddot{a}_{1+r_1}, \ldots\} &= \{\ddot{a}_1, a_2, \ldots, \ddot{a}_{1+r_1}, a_{2+r_1}, a_{3+r_1}, \ldots, a_{1+r_2}, \ldots\} \\
&= \{a_1, a_2, \ldots, \ddot{a}_{1+r_2-r_1}, \ldots, \ddot{a}_{1+r_2}, \ldots\} \\
&= \{a_1, a_2, \ldots, a_{1+r_2-r_1}, \ddot{a}_{2+r_2-r_1} \ldots, a_{2+r_2}, \ddot{a}_{3+r_2}, \ldots\} \\
&\;\vdots \\
&= \{a_1, a_2, \ldots, \ddot{a}_{1+r_2}, \ldots, \ddot{a}_{1+r_1+r_2}, \ldots\} \\
&\;\vdots
\end{aligned}$$

$$\begin{array}{lcl}
a_{1+r_2} - a_{1+r_2-r_1} = d_1 & \text{and} & a_{1+r_2} - a_1 = d_2 \\
a_{2+r_2} - a_{2+r_2-r_1} = d_1 & & a_{2+r_2} - a_2 = d_2 \\
\vdots & & \vdots \\
a_{2+r_1+r_2} - a_{2+r_2} = d_1 & & a_{2+r_1+r_2} - a_{2+r_1} = d_2 \\
\vdots & & \vdots
\end{array}$$

as well as

$$\begin{aligned}
a_{1+r_2-r_1} - a_1 &= d_2 - d_1 \\
a_{2+r_2-r_1} - a_2 &= d_2 - d_1 \\
&\;\vdots \\
a_{2+r_2} - a_{2+r_1} &= d_2 - d_1 \\
&\;\vdots
\end{aligned}$$

Therefore, there is a third repetend difference $d_3 = d_2 - d_1$ with its first repetend starting at a_1 also and with corresponding repetend length

$$\begin{aligned}
r_3 &= r_2 - r_1 \\
&< r_2, \text{ by (P7-1).}
\end{aligned}$$

Since we have $d_1 < d_2$, by (P7-3), **Corollary 6** gives

$$d_3 < d_2 \,. \qquad \text{(P7-4)}$$

Suppose $d_2 = bd_1$, for some integer $b > 1$.

Then $\frac{r_2}{r_1} = \frac{d_2}{d_1} = b$, by **Proposition 5**, contradicting (P7-1).

Therefore, we also have

$$\begin{aligned}
d_1 &\nmid d_2 \\
d_1 &\nmid d_3 \\
r_1 &\nmid r_3 \\
\text{and} \quad r_3 &\nmid r_2 \,.
\end{aligned}$$

This now sets up an iteration creating infinitely nested listings, all starting with index 1 and having corresponding repetend lengths $r_3, r_4, r_5, \ldots$ satisfying the condition $r_2 > r_3 > r_4 > r_5 > \cdots > r_1$, an impossibility for finite integers.
Therefore, supposition (P7-2) is false and we can only have r_1 divides r_2. □

Proposition 8. *For any constants* B, C, D, *we have*

$$B\{\ddot{a}_{n=1},\ldots,\ddot{a}_{1+r},\ldots\}+Cn+D=\{(Ba_{n=1}+C+D\ddot{)}_{n=1},(Ba_2+2C+D)_2,\\ \ldots,(Ba_{1+r}+(1+r)C+D\ddot{)}_{1+r},\ldots\}. \ \square$$

More general than **Proposition 8**, we have the following theorem:

Theorem 9. *Let* f *and* g *each be functions of integral* $n \geq 1$, *with* n*-repetends and with* n*-repetend lengths* r_f *and* r_g, *respectively. Then any linear combination of* $f(n)$, $g(n)$, *and* n *has* n*-repetends and has* $r = \mathrm{lcm}(r_f, r_g)$ *as an* n*-repetend length.*

Proof: Consider first case 1 in which the first n-repetends start at $n=1$, as follows:

$$f(n)=\{\ddot{a}_{n=1},\ldots,\ddot{a}_{1+r_f},\ldots\}\text{, with } n\text{-repetend difference } d_f=a_{1+r_f}-a_1, \qquad \text{(T9-1)}$$

and

$$g(n)=\{\ddot{b}_{n=1},\ldots,\ddot{b}_{1+r_g},\ldots\}\text{, with } n\text{-repetend difference } d_g=b_{1+r_g}-b_1. \qquad \text{(T9-2)}$$

Then, for any constants A, B, C, E, we have

$$Af(n)+Bg(n)+Cn+E=\{(Aa_{n=1}+Bb_{n=1}+C+E)_{n=1},(Aa_2+Bb_2+2C+E)_2,\ldots\}, \qquad \text{(T9-3)}$$

by (R-2) and (R-7).

For any integral $k, r \geq 1$, let

$$D_{k,r}=\left(Aa_{k+r}+Bb_{k+r}+(k+r)C+E\right)-(Aa_k+Bb_k+kC+E)\\ =A(a_{k+r}-a_k)+B(b_{k+r}-b_k)+rC.$$

If $r = \mathrm{lcm}(r_f, r_g)$, we then have

$$D_{k,r}=A\left(a_{k+\mathrm{lcm}(r_f,r_g)}-a_k\right)+B\left(b_{k+\mathrm{lcm}(r_f,r_g)}-b_k\right)+C\left(\mathrm{lcm}(r_f,r_g)\right)\\ =A\left(\frac{\mathrm{lcm}(r_f,r_g)}{r_f}\right)d_f+B\left(\frac{\mathrm{lcm}(r_f,r_g)}{r_g}\right)d_g+C\left(\mathrm{lcm}(r_f,r_g)\right), \qquad \text{(T9-4)}$$

by (T9-1), (T9-2), and **Proposition 4**,

is a constant, independent of k. Therefore, by (R-3), we have (T8-3) can be written

$$Af(n)+Bg(n)+Cn+E=\{(Aa_{n=1}+Bb_{n=1}+C+E\ddot{)}_{n=1},(Aa_{n=2}+Bb_{n=2}+2C+E)_{n=2},\ldots,\\ (Aa_{1+\mathrm{lcm}(r_f,r_g)}+Bb_{1+\mathrm{lcm}(r_f,r_g)}+C(\mathrm{lcm}(r_f,r_g)+1)+E\ddot{)}_{n=1+\mathrm{lcm}(r_f,r_g)},\ldots\},$$

i.e., $Af(n)+Bg(n)+Cn+E$ is an n-repetend function and has $\mathrm{lcm}(r_f, r_g)$ as an n-repetend length.
Now consider case 2 in which either of $f(n)$ or $g(n)$ has the first n-repetend starting at some value $n = n_f > 1$ or $n = n_g > 1$, respectively. By **Proposition 1**, we can choose the

first n-repetends for both $f(n)$ and $g(n)$ to start at the same value of $n \geq \max(n_f, n_g)$, say n_0. Ignoring the terms $a_1, a_2, \ldots, a_{n_0-1}$ and $b_1, b_2, \ldots, b_{n_0-1}$ now allows using the same arguments as in case 1. This completes the proof. □

As a result of **Theorem 9**, if $\min(r_f)$ and $\min(r_g)$ are the minimum n-repetend lengths of $f(n)$ and $g(n)$, respectively, then **Proposition 7** dictates that the minimum n-repetend length of any linear combination of $f(n)$, $g(n)$, and n can only be 1 or a proper factor of $\mathrm{lcm}\big(\min(r_f), \min(r_g)\big)$. For example, let

$$f(n) = \{\ddot{1}_{n=1}, 1_2, \ddot{3}_3, \ldots\} \text{ and } g(n) = \{\ddot{0}_{n=1}, 1_2, \ddot{0}_3, \ldots\}.$$

Then $\min(r_f) = \min(r_g) = 2$ and $f(n) + g(n) = \{\ddot{1}_{n=1}, 2_2, \ddot{3}_3, \ldots\} = \{\dot{1}_{n=1}, \dot{2}_2, \ldots\}$, so that the minimum n-repetend length of $f(n) + g(n)$ here is 1. Such a fortuitous combination for the right values of n cannot happen if $\min(r_f)$ and $\min(r_g)$ are not equal or one is not a multiple of the other, since otherwise the "corrective/reductionist" combinations can only occur at n-positions $\mathrm{lcm}\big(\min(r_f), \min(r_g)\big)$ from each other. In such situations, with or without "corrective/reductionist" combinations, the minimum n-repetend length of the linear combination of $f(n)$, $g(n)$, and n is $\mathrm{lcm}\big(\min(r_f), \min(r_g)\big)$.

In **Theorem 9** we looked at linear combinations of repetend functions f and g. The next theorem looks at composition of repetend functions f and g.

Theorem 10. *Let $f(n)$ be an n-repetend function, with minimum n-repetend length r_f and corresponding minimum n-repetend difference d_f. Let $g(n)$ be an integral n-repetend function with values within the domain of f, minimum n-repetend length r_g, and corresponding minimum repetend n-difference d_g. Then,*

(*i*) *$f\big(g(n)\big)$ is an n-repetend function, with $r_{f(g)} = \mathrm{lcm}(r_f, r_g)$ as an n-repetend length*;

(*ii*) *if $d_g = 0$ or d_g is not divisible by r_f, then $r_{f(g)} = \mathrm{lcm}(r_f, r_g)$ is the minimum n-repetend length of $f\big(g(n)\big)$; and*

(*iii*) *if $d_g = 0$, then $d_{f(g)} = 0$.*

Proof: Without loss of generality, we may assume $n, d_f, d_g \geq 0$, since otherwise we can always reverse directions or change the variable n. For convenience, we may also assume the n-repetends for $f(n)$ and $g(n)$ start at $n = 0$. Otherwise, we complicate notation by taking into account the least values of n that can start an n-repetend for each function, something that does not change the arguments otherwise.

The value $r_{f(g)} = \mathrm{lcm}(r_f, r_g)$ gives

$$f\left(g\left(n+r_{f(g)}\right)\right)-f\left(g(n)\right)=f\left(g(n)+\mathrm{lcm}(r_f,r_g)\frac{d_g}{r_g}\right)-f\left(g(n)\right),\ \forall n\geq 0,$$

by **Proposition 4**.

If $d_g = 0$, then $f\left(g\left(n+r_{f(g)}\right)\right)-f\left(g(n)\right)=0$, so that $f\left(g(n)\right)$ is an n-repetend function with $d_{f(g)}=0$.

Now consider $r_{f(g)}=\mathrm{lcm}(r_f,r_g)$ and $d_g>0$.

Since $g(n)$ is an integral function, we have d_g is an integer, and $\frac{r_{f(g)}}{r_g}d_g=\frac{\mathrm{lcm}(r_f,r_g)}{r_g}d_g$ is a positive integer divisible by r_f, say $\frac{r_{f(g)}}{r_g}d_g=mr_f$, where $m\geq 1$.

Therefore, $f\left(g\left(n+r_{f(g)}\right)\right)-f\left(g(n)\right)=f\left(g(n)+mr_f\right)-f\left(g(n)\right),\ \forall n\geq 0$

$=f\left(g(n)\right)+md_f-f\left(g(n)\right)$, by **Proposition 4**

$=md_f$, a constant, independent of n.

Therefore, $f\left(g(n)\right)$ is an n-repetend function, by (R-3), with $r_{f(g)}=\mathrm{lcm}(r_f,r_g)$ as an n-repetend length.

Also, $r_{f(g)}=\mathrm{lcm}(r_f,r_g)$ is the smallest integer giving these results if d_g is not divisible by r_f, since an integer smaller than $\frac{r_{f(g)}}{r_g}d_g=\frac{\mathrm{lcm}(r_f,r_g)}{r_g}d_g$ cannot then be divisible by r_f without violating **Proposition 4** and r_f being the minimum n-repetend length of $f(n)$.

We have now shown all of (*i*), (*ii*), and (*iii*) hold. □

By **Theorem 10** , (R-9), and **Proposition 4**, we immediately have the following corollary:

Corollary 11. *Let $f(n)$ be an n-repetend function and let a and b be any positive*[54] *integers. Then, $f(an+b)$ is an n-repetend function with the same minimum n-repetend length as $f(n)$.* □

As the first example of the usefulness of our repetend notation and relations, we use them to prove one of Ramanujan's [14] problems:

if n is any positive integer, prove that

$$\left\lfloor\frac{n}{3}\right\rfloor+\left\lfloor\frac{n+2}{6}\right\rfloor+\left\lfloor\frac{n+4}{6}\right\rfloor=\left\lfloor\frac{n}{2}\right\rfloor+\left\lfloor\frac{n+3}{6}\right\rfloor.$$

[54] There are many exceptions in which this corollary holds for either constant being negative.

Proof: $\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+2}{6} \right\rfloor + \left\lfloor \frac{n+4}{6} \right\rfloor = \{\ddot{0}_{n=1}, 0_2, 1_3, \ddot{1}_4, 1_5, 2_6, 2_7, \ldots\} + \{\ddot{0}_{n=1}, 0_2, 0_3, 1_4, 1_5, 1_6, \ddot{1}_7, \ldots\}$

$$+\{\ddot{0}_{n=1}, 1_2, 1_3, 1_4, 1_5, 1_6, \ddot{1}_7, \ldots\}$$

$$= \{\ddot{0}_{n=1}, 1_2, 2_3, 3_4, 3_5, 4_6, \ddot{4}_7, \ldots\}\,.$$

Similarly,

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+3}{6} \right\rfloor = \{\ddot{0}_{n=1}, 1_2, \ddot{1}_3, 2_4, 2_5, 3_6, 3_7, \ldots\} + \{\ddot{0}_{n=1}, 0_2, 1_3, 1_4, 1_5, 1_6, \ddot{1}_7, \ldots\}$$

$$= \{\ddot{0}_{n=1}, 1_2, 2_3, 3_4, 3_5, 4_6, \ddot{4}_7, \ldots\}\,. \ \square$$

The above proof does not actually require n to be an integer (although we would need to use a dummy integral index subscript) or even positive. A bit more involved problem, with similar proof, is showing that $\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+1}{5} \right\rfloor > \left\lfloor \frac{n+3}{7} \right\rfloor,\ \forall n \geq 3$.

Proof: $\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+1}{5} \right\rfloor = \{\ddot{1}_{n=3}, 1_4, 1_5, \ddot{2}_6, \ldots\} + \{\ddot{0}_3, 1_4, 1_5, 1_6, 1_7, \ddot{1}_8, \ldots\}$

$$= \{\ddot{1}_{n=3}, 1_4, 1_5, 2_6, 2_7, 2_8, 3_9, 3_{10}, 3_{11}, 4_{12}, 4_{13}, 4_{14}, 5_{15}, 5_{16}, 5_{17}, \ddot{6}_{18}, \ldots\}$$

$$+\{\ddot{0}_3, 1_4, 1_5, 1_6, 1_7, 1_8, 2_9, 2_{10}, 2_{11}, 2_{12}, 2_{13}, 3_{14}, 3_{15}, 3_{16}, 3_{17}, \ddot{3}_{18}, \ldots\}$$

$$= \{\ddot{1}_{n=3}, 2_4, 2_5, 3_6, 3_7, 3_8, 5_9, 5_{10}, 5_{11}, 6_{12}, 6_{13}, 7_{14}, 8_{15}, 8_{16}, 8_{17}, \ddot{9}_{18}, \ldots\} \quad \text{(A)}$$

$$\geq \{\ddot{1}_{n=3}, 1_4, 1_5, \ddot{2}_6, 2_7, 2_8, 3_9, 3_{10}, 3_{11}, 4_{12}, 4_{13}, 4_{14}, 5_{15}, 5_{16}, 5_{17}, 6_{18}, \ldots\}\,,$$ [55]

and $$\left\lfloor \frac{n+3}{7} \right\rfloor = \{\ddot{0}_{n=3}, 1_4, 1_5, 1_6, 1_7, 1_8, 1_9, \ddot{1}_{10}, \ldots\} \quad \text{(B)}$$

$$< \{\ddot{1}_{n=3}, 1_4, 1_5, \ddot{2}_6, 2_7, 2_8, 3_9, 3_{10}, \ldots\},\ \forall n \geq 6\,.$$

Comparing (A) and (B) for $n = 3, 4, 5$ completes the proof. $\square$

The method of proof used in this example shows a general technique for avoiding having to use the brute force technique of comparing repetends out to at least a least common multiple index, here of extending (A) and (B) out to $n = 3 + \mathrm{lcm}(3,5,7) = 108$.

As another example of the usefulness of generalized repetends, we look at a conjecture involving the opened Sieve of Eratosthenes.[56] Suppose we conjecture[57] that

$$S_3\left((n+1)p_3 + mC_{3,n}\right) \geq S_3\left((n+1)p_3\right) + m(n+1) - \begin{Bmatrix} 0,\ \forall (m \geq 0;\ n = 0) \\ 1,\ \forall (m \geq 0;\ n \geq 1) \end{Bmatrix}. \quad \text{(C)}$$

Empirically, we have

---

[55] This is an example in which the right overdots can be moved to the left, as shown.

[56] Generalized repetends were originally developed to help look for and analyze relations in the opened Sieve, because of **Corollary 2.2.5** there.

[57] The example following can be looked at once $S_i(N)$ and its properties are introduced in **Section 2.3**. This is a conjecture that I made, but found to be false, initially by trial and error.

$$
\begin{aligned}
S_3\left((n+1)p_3 + mC_{3,n}\right) &= S_3\big(5\{\ddot{1}_{n=0}, \ddot{2}_1, 3_2, 4_3, 5_4, 6_5, 7_6, 8_7, \ldots\} \\
&\qquad + m\{5_{n=0}, \ddot{7}_1, 11_2, 13_3, 17_4, 19_5, 23_6, \ddot{(29)}_7 \ldots\}\big) \\
&= S_3\big(5(m+1)_{n=0}, \ddot{(7m+10)}_1, (11m+15)_2, (13m+20)_3, \\
&\qquad (17m+25)_4, (19m+30)_5, (23m+35)_6, \ddot{(29m+40)}_7, \ldots\big) \\
&= \left\{\begin{array}{l}
\{0_{n=0},\ \ddot{1}_1,\ 3_2,\ 5_3,\ 6_4,\ 7_5,\ 8_6,\ \ddot{9}_7, \ldots \ddot{\}}_{m=0}, \\
\{1_{n=0},\ \ddot{4}_1,\ 6_2,\ 8_3, 10_4,\ 13_5,\ 14_6, \ddot{(17)}_7, \ldots\}_{m=1}, \\
\{3_{n=0},\ \ddot{6}_1,\ 9_2,\ 11_3, 13_4,\ 17_5, 21_6, \ddot{(25)}_7, \ldots\}_{m=2}, \\
\{5_{n=0},\ \ddot{8}_1,\ 12_2, 15_3, 19_4,\ 22_5, 27_6, \ddot{(33)}_7, \ldots\}_{m=3}, \\
\{6_{n=0},\ \ddot{9}_1,\ 15_2, 18_3, 24_4,\ 27_5, 33_6, \ddot{(40)}_7, \ldots\}_{m=4}, \\
\{7_{n=0}, \ddot{(11)}_1, 17_2, 22_3, 29_4, 32_5, 39_6, \ddot{(48)}_7, \ldots\}_{m=5}, \\
\{8_{n=0}, \ddot{(13)}_1, 21_2, 25_3, 33_4,\ 38_5, 46_6, \ddot{(56)}_7, \ldots \ddot{\}}_{m=6}, \\
\vdots
\end{array}\right\}
\end{aligned}
$$

and $$
\begin{aligned}
S_3\left((n+1)p_3\right) + m(n+1) &= S_3\left(5\{\ddot{1}_{n=0}, \ddot{2}_1, \ldots\}\right) + m\{\ddot{1}_{n=0}, \ddot{2}_1, \ldots\},\ \forall m \geq 0 \\
&= S_3(\{5_{n=0}, \ddot{(10)}_1, 15_2, 20_3, 25_4, 30_5, 35_6, \ddot{(40)}_7, \ldots\}) \\
&\qquad + m\{1_{n=0}, \ddot{2}_1, 3_2, 4_3, 5_4, 6_5, 7_6, \ddot{8}_7, \ldots\} \\
&= \{0_{n=0}, \ddot{1}_1, 3_2, 5_3, 6_4, 7_5, 8_6, \ddot{9}_7, \ldots\} \\
&\qquad + m\{1_{n=0}, \ddot{2}_1, 3_2, 4_3, 5_4, 6_5, 7_6, \ddot{8}_7, \ldots\} \\
&= \left\{\begin{array}{l}
\{0_{n=0},\ \ddot{1}_1,\ 3_2,\ 5_3,\ 6_4,\ 7_5,\ 8_6,\ \ddot{9}_7 \ldots \ddot{\}}_{m=0}, \\
\{1_{n=0},\ \ddot{3}_1,\ 6_2,\ 9_3,\ 11_4, 13_5, 15_6,\ \ddot{(17)}_7, \ldots\}_{m=1}, \\
\{2_{n=0},\ \ddot{5}_1,\ 9_2, 13_3, 16_4, 19_5, 22_6,\ \ddot{(25)}_7, \ldots\}_{m=2}, \\
\{3_{n=0},\ \ddot{7}_1,\ 12_2, 17_3, 21_4, 25_5, 29_6, \ddot{(33)}_7, \ldots\}_{m=3}, \\
\{4_{n=0},\ \ddot{9}_1,\ 15_2, 21_3, 26_4, 31_5, 36_6, \ddot{(41)}_7, \ldots\}_{m=4}, \\
\{5_{n=0}, \ddot{(11)}_1, 18_2, 25_3, 31_4, 37_5, 43_6, \ddot{(49)}_7, \ldots\}_{m=5}, \\
\{6_{n=0}, \ddot{(13)}_1, 21_2, 29_3, 36_4, 43_5, 50_6, \ddot{(57)}_7, \ldots \ddot{\}}_{m=6}, \\
\vdots
\end{array}\right\}.
\end{aligned}
$$ [58]

It is relatively quick and easy to see where conjecture (C) is false. The other major advantage of the above technique is that it reveals all the cases for which the conjecture is false, not just a single counter-example found by happenstance or arduous trial and error. Conceivably, this could allow us to see how to alter an opened Sieve conjecture, or any conjecture amenable to the use of generalized repetends, to make it true.

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[58] The minimum m-repetend length here is 1, starting from any value of $m \geq 0$, and so we can place the m-repetend double overdots at $m = 0$ and $m = 6$, as shown.

Appendix C: Empirical Values of $t(i,n)$ for $0 \le i \le 5$

To obtain the empirical values of $t(i,n)$ for any $i \ge 1$, the columns of the opened Sieve of Eratosthenes must be extended out to at least $j = p_i + \Pi_i$. The values for $i = 0,1,2,3,4$ are obtained from **Figures 2a-2e**. The extension of the Sieve for $i = 5$ can be done by writing a computer program based on the algorithm steps **A1** to **A7** or by using a spread sheet that allows at least $j = p_5 + \Pi_5 = 3321$ (I used Gnumeric Spreadsheet 1.10.16). In the following, the relative positions of the X_i are shown for $i = 3,4,5$. By $\rfloor\!\lceil\{\ldots\}$ we mean the list elements increase monotonically as a function of their ordered indices.

$$t(0,n) = t(1,n) = t(2,n) = 0, \ \forall n \ge 0\,;$$

$$t(3,n) = \{\underset{X_{3,1}}{\ddot{0}_{n=0}}, 0_1, 1_2, 2_3, \underset{X_{3,2}}{2_4}, 2_5, \underset{X_{3,3}}{\ddot{2}_6}, \ldots\}\,;$$

$$t(4,n) = \{\underset{X_{4,1}}{\ddot{0}_{n=0}}, 1_1, 2_2, 2_3, 3_4, 4_5, \underset{X_{4,2}}{5_6}, 5_7, 6_8, 6_9, \underset{X_{4,3}}{7_{10}}, 8_{11}, \underset{X_{4,4}}{8_{12}}, 8_{13}, 9_{14},$$
$$10_{15}, \underset{X_{4,5}}{10_{16}}, 10_{17}, \underset{X_{4,6}}{11_{18}}, 12_{19}, 12_{20}, 13_{21}, \underset{X_{4,7}}{13_{22}}, 14_{23}, 15_{24},$$
$$16_{25}, 16_{26}, 17_{27}, \underset{X_{4,8}}{18_{28}}, 18_{29}, \underset{X_{4,9}}{\ddot{(18)}_{30}}, \ldots\}\,;$$

$$t(5,n) = \{\underset{X_{5,1}}{\ddot{0}_{n=0}}, 2_1, 4_2, 6_3, 7_4, 8_5, 10_6, 11_7, 12_8, 15_9, \underset{X_{5,2}}{15_{10}}, 16_{11}, \underset{X_{5,3}}{17_{12}},$$
$$18_{13}, 19_{14}, 21_{15}, \underset{X_{5,4}}{22_{16}}, 24_{17}, \underset{X_{5,5}}{24_{18}}, 24_{19}, 27_{20}, 29_{21}, \underset{X_{5,6}}{30_{22}}, 31_{23}, 32_{24},$$
$$34_{25}, 35_{26}, 36_{27}, \underset{X_{5,7}}{38_{28}}, 38_{29}, \underset{X_{5,8}}{39_{30}}, 40_{31}, 42_{32}, 43_{33}, 45_{34}, 46_{35}, \underset{X_{5,9}}{48_{36}},$$
$$48_{37}, 49_{38}, 52_{39}, \underset{X_{5,10}}{53_{40}}, 54_{41}, \underset{X_{5,11}}{55_{42}}, 56_{43}, 58_{44}, 59_{45}, \underset{X_{5,12}}{59_{46}}, 61_{47}, 62_{48},$$
$$63_{49}, 65_{50}, 67_{51}, \underset{X_{5,13}}{67_{52}}, 69_{53}, 70_{54}, 72_{55}, 73_{56}, 74_{57}, \underset{X_{5,14}}{76_{58}}, 77_{59}, \underset{X_{5,15}}{78_{60}},$$
$$79_{61}, 81_{62}, 83_{63}, 84_{64}, 84_{65}, \underset{X_{5,16}}{86_{66}}, 87_{67}, 88_{68}, 90_{69}, \underset{X_{5,17}}{91_{70}}, 91_{71}, \underset{X_{5,18}}{93_{72}},$$
$$94_{73}, 96_{74}, 97_{75}, 98_{76}, 100_{77}, \underset{X_{5,19}}{101_{78}}, 102_{79}, 104_{80}, 106_{81}, \underset{X_{5,20}}{107_{82}}, 108_{83}, 108_{84},$$
$$110_{85}, 112_{86}, 113_{87}, \underset{X_{5,21}}{114_{88}}, 115_{89}, 116_{90}, 118_{91}, 120_{92}, 122_{93}, 123_{94}, 124_{95}, \underset{X_{5,22}}{125_{96}},$$
$$126_{97}, 128_{98}, 130_{99}, \underset{X_{5,23}}{131_{100}}, 132_{101}, \underset{X_{5,24}}{133_{102}}, 133_{103}, 135_{104}, 137_{105}, \underset{X_{5,25}}{137_{106}}, 138_{107}, \underset{X_{5,26}}{139_{108}},$$
$$140_{109}, 142_{110}, 144_{111}, \underset{X_{5,27}}{145_{112}}, 146_{113}, 147_{114}, 148_{115}, 150_{116}, 152_{117}, 154_{118}, 155_{119}, \underset{X_{5,28}}{156_{120}},$$
$$157_{121}, 158_{122}, 160_{123}, 162_{124}, 162_{125}, \underset{X_{5,29}}{163_{126}}, 164_{127}, 166_{128}, 168_{129}, \underset{X_{5,30}}{169_{130}}, 170_{131}, 172_{132},$$
$$173_{133}, 174_{134}, 176_{135}, \underset{X_{5,31}}{177_{136}}, 179_{137}, \underset{X_{5,32}}{179_{138}}, 180_{139}, 182_{140}, 183_{141}, \underset{X_{5,33}}{184_{142}}, 186_{143}, 186_{144},$$
$$187_{145}, 189_{146}, 191_{147}, \underset{X_{5,34}}{192_{148}}, 193_{149}, \underset{X_{5,35}}{194_{150}}, 196_{151}, 197_{152}, 198_{153}, 200_{154}, 201_{155}, \underset{X_{5,36}}{203_{156}},$$

$$203_{157},205_{158},207_{159},208_{160},209_{161},\underset{X_{5,37}}{211_{162}},211_{163},212_{164},214_{165},\underset{X_{5,38}}{215_{166}},216_{167},\underset{X_{5,39}}{217_{168}},$$

$$218_{169},221_{170},222_{171},\underset{X_{5,40}}{222_{172}},224_{173},225_{174},227_{175},228_{176},230_{177},\underset{X_{5,41}}{231_{178}},232_{179},\underset{X_{5,42}}{232_{180}},$$

$$234_{181},235_{182},236_{183},238_{184},239_{185},\underset{X_{5,43}}{240_{186}},241_{187},243_{188},246_{189},\underset{X_{5,44}}{246_{190}},246_{191},\underset{X_{5,45}}{248_{192}},$$

$$249_{193},251_{194},252_{195},\underset{X_{5,46}}{253_{196}},254_{197},\underset{X_{5,47}}{255_{198}},255_{199},258_{200},259_{201},260_{202},262_{203},263_{204},$$

$$264_{205},266_{206},268_{207},\underset{X_{5,48}}{270_{208}},270_{209},\underset{X_{5,49}}{\ddot{(270)}_{210}},\ldots\}$$

$$\geq \int\{\underset{X_{5,1}}{\ddot{0}_{n=0}},2_1,4_2,6_3,7_4,8_5,10_6,11_7,12_8,15_9,\underset{X_{5,2}}{15_{10}},\underset{X_{5,3}}{17_{12}},\underset{X_{5,4}}{22_{16}},\underset{X_{5,5}}{24_{18}},\underset{X_{5,6}}{30_{22}},$$

$$\underset{X_{5,7}}{38_{28}},\underset{X_{5,8}}{39_{30}},\underset{X_{5,9}}{48_{36}},\underset{X_{5,10}}{53_{40}},\underset{X_{5,11}}{55_{42}},\underset{X_{5,12}}{59_{46}},\underset{X_{5,13}}{67_{52}},\underset{X_{5,14}}{76_{58}},\underset{X_{5,15}}{78_{60}},\underset{X_{5,16}}{86_{66}},$$

$$\underset{X_{5,17}}{91_{70}},\underset{X_{5,18}}{93_{72}},\underset{X_{5,19}}{101_{78}},\underset{X_{5,20}}{107_{82}},\underset{X_{5,21}}{114_{88}},\underset{X_{5,22}}{125_{96}},\underset{X_{5,23}}{131_{100}},\underset{X_{5,24}}{133_{102}},\underset{X_{5,25}}{137_{106}},$$

$$\underset{X_{5,26}}{139_{108}},\underset{X_{5,27}}{145_{112}},\underset{X_{5,28}}{156_{120}},\underset{X_{5,29}}{163_{126}},\underset{X_{5,30}}{169_{130}},\underset{X_{5,31}}{177_{136}},\underset{X_{5,32}}{179_{138}},\underset{X_{5,33}}{184_{142}},\underset{X_{5,34}}{192_{148}},$$

$$\underset{X_{5,35}}{194_{150}},\underset{X_{5,36}}{203_{156}},\underset{X_{5,37}}{211_{162}},\underset{X_{5,38}}{215_{166}},\underset{X_{5,39}}{217_{168}},\underset{X_{5,40}}{222_{172}},\underset{X_{5,41}}{231_{178}},\underset{X_{5,42}}{232_{180}},\underset{X_{5,43}}{240_{186}},$$

$$\underset{X_{5,44}}{246_{190}},\underset{X_{5,45}}{248_{192}},\underset{X_{5,46}}{253_{196}},\underset{X_{5,47}}{255_{198}},\underset{X_{5,48}}{270_{208}},\underset{X_{5,49}}{\ddot{(270)}_{210}},\underset{X_{5,50}}{285_{220}},\ldots\}$$

$$= n + \{\underset{X_{5,r=1}}{0_{n=0}},1_1,2_2,3_3,3_4,3_5,4_6,4_7,4_8,6_9\}$$

$$+\int\{\ddot{5}_{\substack{n=10\\r=2}},5_{\substack{n=12\\r=3}},6_{\substack{n=16\\r=4}},6_{\substack{n=18\\r=5}},8_{\substack{n=22\\r=6}},10_{\substack{n=28\\r=7}},9_{\substack{n=30\\r=8}},12_{\substack{n=36\\r=9}},13_{\substack{n=40\\r=10}},13_{\substack{n=42\\r=11}},$$

$$13_{\substack{n=46\\r=12}},15_{\substack{n=52\\r=13}},18_{\substack{n=58\\r=14}},18_{\substack{n=60\\r=15}},20_{\substack{n=66\\r=16}},21_{\substack{n=70\\r=17}},21_{\substack{n=72\\r=18}},23_{\substack{n=78\\r=19}},25_{\substack{n=82\\r=20}},$$

$$26_{\substack{n=88\\r=21}},29_{\substack{n=98\\r=22}},31_{\substack{n=100\\r=23}},31_{\substack{n=102\\r=24}},31_{\substack{n=106\\r=25}},31_{\substack{n=108\\r=26}},33_{\substack{n=112\\r=27}},36_{\substack{n=120\\r=28}},$$

$$37_{\substack{n=126\\r=29}},39_{\substack{n=130\\r=30}},41_{\substack{n=136\\r=31}},41_{\substack{n=138\\r=32}},42_{\substack{n=142\\r=33}},44_{\substack{n=148\\r=34}},44_{\substack{n=150\\r=35}},47_{\substack{n=156\\r=36}},$$

$$49_{\substack{n=162\\r=37}},49_{\substack{n=166\\r=38}},49_{\substack{n=168\\r=39}},50_{\substack{n=172\\r=40}},53_{\substack{n=178\\r=41}},52_{\substack{n=180\\r=42}},54_{\substack{n=186\\r=43}},56_{\substack{n=190\\r=44}},$$

$$56_{\substack{n=192\\r=45}},57_{\substack{n=196\\r=46}},57_{\substack{n=198\\r=47}},62_{\substack{n=208\\r=48}},60_{\substack{n=210\\r=49}},\ddot{(65)}_{\substack{n=220\\r=50}},\ldots\}.$$

From the above, $t(i,n+1) \geq t(i,n),\ \forall(1 \leq i \leq 5;\ n \geq 0)$. Therefore, by (4.2.2), all $u_{0,n}$ with $n \geq 2$ and all $\mathrm{u}_{i,n}$ with $(1 \leq i \leq 5;\ n \geq 1)$ contain at least one regular C_i.

Two empty u_i are:

the last u_6 before $X_{6,36}$, *viz.*, $\mathrm{u}_{6,C_{5,35}} = [2184, 2197) = [p_1^3 p_2 p_4 p_6, 2197)$

($p_6 = 13$; with $C_{6,n} = 2183$, we have $C_{6,n+1} = C_{6,n} + 18$);

and the last u_9 before $X_{9,10}$, *viz.*, $\mathrm{u}_{9,C_{8,9}} = [1334, 1357) = [p_1 p_9 p_{10}, 1357)$

($p_9 = 23$; with $C_{9,n} = 1333$, we have $C_{9,n+1} = C_{9,n} + 28$).

Appendix D: Figures

column j →	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
P0=1			1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
P1=2				1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1		1	
P2=3						1		1				1		1				1		1				1		1				1		1				1		1				1		1				1		1	
P3=5								1				1		1				1		1				1						1		1						1				1		1				1		1	
P4=7												1		1				1		1				1						1		1						1				1		1				1			
P5=11														1				1		1				1						1		1						1				1		1				1			
P6=13																		1		1				1						1		1						1				1		1				1			
P7=17																				1				1						1		1						1				1		1				1			
P8=19																								1						1		1						1				1		1				1			
P9=23																														1		1						1				1		1				1			
P10=29																																1						1				1		1				1			
P11=31																																						1				1		1				1			
P12=37																																										1		1				1			
P13=41																																												1				1			
P14=43																																																1			
P15=47																																																			
P16=53																																																			

Figure 1. First form of the opened Sieve of Eratosthenes, from i=0 to i=16 and from j=0 to j=50.

column j →	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
P0=1			1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
P1=2			SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX
P2=3				SX		1		1		SX		1		1		SX		1		1		SX		1		1		SX		1		1		SX		1		1		SX		1		1		SX		1		1	
P3=5						SX		1				1		1				1		1				1		X				1		1				SX		1				1		1				1		1	
P4=7								SX				1		1				1		1				1						1		1						1				1		1				1		X	
P5=11												SX		1				1		1				1						1		1						1				1		1				1			
P6=13														SX				1		1				1						1		1						1				1		1				1			
P7=17																		SX		1				1						1		1						1				1		1				1			
P8=19																				SX				1						1		1						1				1		1				1			
P9=23																								SX						1		1						1				1		1				1			
P10=29																														SX		1						1				1		1				1			
P11=31																																SX						1				1		1				1			
P12=37																																						SX				1		1				1			
P13=41																																										SX		1				1			
P14=43																																												SX				1			
P15=47																																																SX			
P16=53																																																			

Figure 2a. Second form of the opened Sieve of Eratosthenes from i=0 to i=16 and from j=0 to j=50.

column j →	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
P0=1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
P1=2	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX
P2=3		SX		1		1		SX		1		1		SX		1		1		SX		1		1		SX		1		1		SX		1		1		SX		1		1		SX		1		1		SX	
P3=5				1		X				1		1				SX		1				1		1				1		1				1		X				1		1				SX		1			
P4=7				1						1		1						1				1		1				X		1				1						1		X						1			
P5=11				1						1		1						1				1		1						1				1						1								1			
P6=13				1						1		1						1				1		1						1				1						1								1			
P7=17				1						1		1						1				1		1						1				1						1								1			
P8=19				1						1		1						1				1		1						1				1						1								1			
P9=23				1						1		1						1				1		1						1				1						1								1			
P10=29				1						1		1						1				1		1						1				1						1								1			
P11=31				1						1		1						1				1		1						1				1						1								1			
P12=37				1						1		1						1				1		1						1				1						1								1			
P13=41				1						1		1						1				1		1						1				1						1								1			
P14=43				1						1		1						1				1		1						1				1						1								1			
P15=47				1						1		1						1				1		1						1				1						1								1			
P16=53				SX						1		1						1				1		1						1				1						1								1			

Figure 2b. Second form of the opened Sieve of Eratosthenes, from i=0 to i=16 and from j=50 to j=100.

column j →	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120	121	122	123	124	125	126	127	128	129	130	131	132	133	134	135	136	137	138	139	140	141	142	143	144	145	146	147	148	149	150
P0=1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
P1=2	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX
P2=3		1		1		SX		1		1		SX		1		1		SX		1		1		SX		1		1		SX		1		1		SX		1		1		SX		1		1		SX		1	
P3=5		1		1				1		1				1		X				1		1				SX		1				1		1				1		1				1		X				1	
P4=7		1		1				1		1				1						X		1						1				1		X				1		1				1						1	
P5=11		1		1				1		1				1								X						1				1						1		1				X						1	
P6=13		1		1				1		1				1														1				1						1		1										1	
P7=17		1		1				1		1				1														1				1						1		1										1	
P8=19		1		1				1		1				1														1				1						1		1										1	
P9=23		1		1				1		1				1														1				1						1		1										1	
P10=29		1		1				1		1				1														1				1						1		1										1	
P11=31		1		1				1		1				1														1				1						1		1										1	
P12=37		1		1				1		1				1														1				1						1		1										1	
P13=41		1		1				1		1				1														1				1						1		1										1	
P14=43		1		1				1		1				1														1				1						1		1										1	
P15=47		1		1				1		1				1														1				1						1		1										1	
P16=53		1		1				1		1				1														1				1						1		1										1	

Figure 2c. Second form of the opened Sieve of Eratosthenes, from i=0 to i=16 and from j=100 to j=150.

column j →	150	151	152	153	154	155	156	157	158	159	160	161	162	163	164	165	166	167	168	169	170	171	172	173	174	175	176	177	178	179	180	181	182	183	184	185	186	187	188	189	190	191	192	193	194	195	196	197	198	199	200
P0=1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
P1=2	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX
P2=3		1		SX		1		1		SX		1		1		SX		1		1		SX		1		1		SX		1		1		SX		1		1		SX		1		1		SX		1		1	
P3=5		1				SX		1				1		1				1		1				1		X				1		1				SX		1				1		1				1		1	
P4=7		1						1				X		1				1		1				1						1		1						1				1		1				1		1	
P5=11		1						1						1				1		1				1						1		1						X				1		1				1		1	
P6=13		1						1						1				1		X				1						1		1										1		1				1		1	
P7=17		1						1						1				1						1						1		1										1		1				1		1	
P8=19		1						1						1				1						1						1		1										1		1				1		1	
P9=23		1						1						1				1						1						1		1										1		1				1		1	
P10=29		1						1						1				1						1						1		1										1		1				1		1	
P11=31		1						1						1				1						1						1		1										1		1				1		1	
P12=37		1						1						1				1						1						1		1										1		1				1		1	
P13=41		1						1						1				1						1						1		1										1		1				1		1	
P14=43		1						1						1				1						1						1		1										1		1				1		1	
P15=47		1						1						1				1						1						1		1										1		1				1		1	
P16=53		1						1						1				1						1						1		1										1		1				1		1	

Figure 2d. Second form of the opened Sieve of Eratosthenes, from i=0 to i=16 and from j=150 to j=200.

column j →	200	201	202	203	204	205	206	207	208	209	210	211	212	213	214	215	216	217	218	219	220	221	222	223	224	225
P0=1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
P1=2	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1
P2=3		SX		1		1		SX		1		1		SX		1		1		SX		1		1		SX
P3=5				1		X				1		1				SX		1				1		1		
P4=7				X						1		1						SX				1		1		
P5=11										X		1										1		1		
P6=13												1										X		1		
P7=17												1												1		
P8=19												1												1		
P9=23												1												1		
P10=29												1												1		
P11=31												1												1		
P12=37												1												1		
P13=41												1												1		
P14=43												1												1		
P15=47												1												1		
P16=53												1												1		

column j →	226	227	228	229	230	231	232	233	234	235	236	237	238	239	240	241	242	243	244	245	246	247	248	249	250
P0=1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
P1=2	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX	1	SX
P2=3		1		1		SX		1		1		SX		1		1		SX		1		1		SX	
P3=5		1		1				1		X				1		1				SX		1			
P4=7		1		1				1						1		1						1			
P5=11		1		1				1						1		1						1			
P6=13		1		1				1						1		1						X			
P7=17		1		1				1						1		1									
P8=19		1		1				1						1		1									
P9=23		1		1				1						1		1									
P10=29		1		1				1						1		1									
P11=31		1		1				1						1		1									
P12=37		1		1				1						1		1									
P13=41		1		1				1						1		1									
P14=43		1		1				1						1		1									
P15=47		1		1				1						1		1									
P16=53		1		1				1						1		1									

Figure 2e. Second form of the opened Sieve of Eratosthenes, from i=0 to i=16 and from j=200 to j=250.