Title: **Collatz Conjecture Confirmed Through Connectivity of Odd and 8mod12 Positive Integers**

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Abstract: In this paper, I will prove all positive odd integers link incrementally through connected orbits to a value 2^c where $c \in \mathbb{N}$. These orbits connect because of a function I defined based on the connection between all odd and $8\text{mod}12$ positive integers. Applying this function infinitely many times to all positive odd integers always leads to a connection to power of 2 orbits. With these results, I'm able to prove the conjecture.

Keywords: Collatz function, orbit, trivial cycle, reverse Collatz relation, connecting orbits, 2-adic order, non-trivial cycle, bijection, cardinality, countably infinite, multiset

Introduction: The **Collatz function** is defined below where $n \in \mathbb{N}$:

$$C(n) = \left\{ \begin{array}{|c|} \hline \frac{n}{2}, n \equiv 0\text{mod}(2) \\ \hline 3n + 1, n \equiv 1\text{mod}(2) \\ \hline \end{array} \right\}.$$

The Collatz conjecture starts with substituting any positive integer n into the function above. If the value n is even, the result is $\frac{n}{2}$. If the value n is odd, the result is $3n + 1$. The conjecture says continuing this process for any positive integer n always leads to 1.

I define the **orbit** of a number n as the sequence $C^m(n)$ where $m \in \mathbb{N}_0$. Similarly, I define the set $R(n)$ by

$R_0(n) = C^0(n) = n$

$R_i(n) = C^i(n)$

where $i \in \mathbb{N}$, such that $R_a(n) \neq R_b(n)$ for $a, b \in \mathbb{N}$. For example, when $n = 5$, the orbit is shown below.

$C(5) = 5 \cdot 3 + 1 = 16$

$C(16) = \frac{16}{2} = 8$

$C(8) = \frac{8}{2} = 4$

$C(4) = \frac{4}{2} = 2$

$C(2) = \frac{2}{2} = 1$

$C(1) = 1 \cdot 3 + 1 = 4$

$C(4) = \frac{4}{2} = 2$

$C(2) = \frac{2}{2} = 1$

$C(1) = 1 \cdot 3 + 1 = 4$

This 4, 2, 1 **trivial cycle** continues to repeat, so $R(5) = \{5, 16, 8, 4, 2, 1\}$. Since $1 \in R(5)$, every orbit containing 5 also contains 1. Similarly, $R(n)$ contains 1 for any positive integer n that satisfies the conjecture. "As of 2020, the conjecture has been checked by computer for all starting values up to $2^{68} \approx 2.95 \times 10^{20}$."[1]

Another way to study the Collatz conjecture is to begin with 1 and follow orbits backwards. This **reverse Collatz relation**, where $n \in \mathbb{N}$, can be written as:

$$I(n) = \left\{ \begin{array}{|c|} \hline \{2n\}, n \equiv 0,1,2,3,5 \bmod(6) \\ \hline \{2n, \frac{n-1}{3}\}, n \equiv 4 \bmod(6) \\ \hline \end{array} \right\}.$$

For each case, I show $C(I(n)) = n$. $I(n) = 2n$ for all values of n, and $C(2n) = \frac{2n}{2} = n$ because the function divides its input by 2 if it is even. $I(n) = \frac{n-1}{3}$ for all $n \equiv 4 \bmod(6)$, because $\frac{n-1}{3} \equiv 1 \bmod(2)$ if and only if $3(\frac{n-1}{3}) + 1 = n \equiv 4 \bmod(6)$. This means that $C(I(n)) = C(\frac{n-1}{3}) = 3(\frac{n-1}{3}) + 1 = n$ because the function multiplies its input by 3 and adds 1 if it's odd.

For the remainder of this proof, iterations of the reverse Collatz relation will only use the $I(n) = 2n$ case, which is applicable to all integers.

After studying many Collatz orbits, certain connections became apparent. The most important was the connection between odd and $8 \bmod(12)$ positive integers. Two positive integers $q, s \in \mathbb{N}$ have **connecting orbits** if there exists at least one value $t \in \mathbb{N}$ such that $t \in R(q)$ and $t \in R(s)$.

Proposition: For $d, k \in \mathbb{N}_0$***, if*** $a = 2k + 1$, ***then*** $b \in R(a)$ ***and*** $b \in R(2^d)$ ***for some value*** $b \in \mathbb{N}$

Proof: Table 1 illustrates the odd and $8 \bmod(12)$ positive integer connection and how their orbits are structured going backwards in their orbits using only the $I(n) = 2n$ case of the reverse Collatz relation. Going forward in the orbits of $2k + 1$ and $12k + 8$ where $k \in \mathbb{N}_0$ gives $6k + 4$, which connects both orbits according to the definition of the Collatz function as seen in the equations below. Equation 1 shows how each positive odd integer must connect to a specific $8 \bmod(12)$.

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.	.	.
.	.	.
8k + 4	6(8k+4)+8=	48k + 32
4k + 2	6(4k+2)+4=	24k + 16
2k + 1	6(2k+1)+2=	12k + 8
6k + 4	=	6k + 4

Table 1: Connected Orbits

$C(2k + 1) = C(12k + 8)$

We know that $2k + 1$ is odd, and $12k + 8$ is even, so applying the Collatz function once gives

$3(2k + 1) + 1 = \frac{12k+8}{2} = 6k + 4.$

Multiplying by 2 gives

$$6(2k + 1) + 2 = 12k + 8 \equiv 8 \bmod(12). \quad (1)$$

Since all odd integers $2k + 1$ connect to $6k + 4$ as seen in the equations above and all even integers can be obtained from iterating $I(n) = 2n$ an infinite amount of times going backwards on each odd, all even integers obtained from these iterations must also connect to $6k + 4$ because they would be contained in the same orbit as the odd they came from. After applying the reverse Collatz relation to $2k + 1$ and $12k + 8$ integers v times where $v \in \mathbb{N}_0$, the Collatz function must be applied $v + 1$ times to connect to $6k + 4$. These connections are seen in the equations below. Equations 2, 3, and 4 show how each integer obtained by iterating $I(n) = 2n$ on $2k + 1$ a certain number of times must connect to the integer obtained by iterating $I(n) = 2n$ on $12k + 8$ the same amount of times.

Connecting orbits starting with $I(2k + 1) = 2(2k + 1) = 4k + 2 \equiv 2 \bmod(4)$ and $I(12k + 8) = 2(12k + 8) = 24k + 16 \equiv 16 \bmod(24)$ gives the equations below.

$C^2(4k + 2) = C^2(24k + 16)$

Because $4k + 2$ and $24k + 16$ are both divisible by 2, iterating the Collatz function once gives

$C(\frac{4k+2}{2}) = C(\frac{24k+16}{2}).$

We know that $\frac{4k+2}{2} = 2k + 1$ is odd, and $\frac{24k+16}{2} = 12k + 8$ is even, so applying the Collatz function one more time on both orbits gives

$3(\frac{4k+2}{2}) + 1 = \frac{24k+16}{4} = 6k + 4,$

and multiplying by 4 gives

$$6(4k + 2) + 4 = 24k + 16 \equiv 16\text{mod}(24). \quad (2)$$

Starting with $I^2(2k + 1) = 2^2(2k + 1) = 8k + 4 \equiv 4\text{mod}(8)$ and $I^2(12k + 8) = 2^2(12k + 8) = 48k + 32 \equiv 32\text{mod}(48)$, it follows that

$C^3(8k + 4) = C^3(48k + 32).$

Because $8k + 4$ and $48k + 32$ are both divisible by 2^2, iterating the Collatz function twice gives

$C(\frac{8k+4}{4}) = C(\frac{48k+32}{4}).$

We know that $\frac{8k+4}{4} = 2k + 1$ is odd, and $\frac{48k+32}{4} = 12k + 8$ is even, so applying the Collatz function one more time on both orbits gives

$3(\frac{8k+4}{4}) + 1 = \frac{48k+32}{8} = 6k + 4,$

and multiplying by 8 gives

$$6(8k + 4) + 8 = 48k + 32 \equiv 32\text{mod}(48). \quad (3)$$

Since $2k + 1$ connects to $6k + 4$, every integer obtained from

$I^v(2k + 1) = 2^v(2k + 1) = 2^{v+1}k + 2^v \equiv 2^v\text{mod}(2^{v+1})$

and

$I^v(12k + 8) = 2^v(12k + 8) = 3k * 2^{v+2} + 2^{v+3} \equiv 2^{v+3}\text{mod}(3 * 2^{v+2})$

where $v \in \mathbb{N}_0$ must also connect to $6k + 4$. Connecting the values $2^{v+1}k + 2^v$ and $3k * 2^{v+2} + 2^{v+3}$ to $6k + 4$ gives the equations below.

$C^{v+1}(2^{v+1}k + 2^v) = C^{v+1}(3k * 2^{v+2} + 2^{v+3})$

Because $2^{v+1}k + 2^v$ and $3k * 2^{v+2} + 2^{v+3}$ are both divisible by 2^v, iterating the Collatz function v times gives

$C(\frac{2^{v+1}k+2^v}{2^v}) = C(\frac{3k*2^{v+2}+2^{v+3}}{2^v}).$

We know that $\frac{2^{v+1}k+2^v}{2^v} = 2k + 1$ is odd, and $\frac{3k*2^{v+2}+2^{v+3}}{2^v} = 12k + 8$ is even, so applying the Collatz function one more time on both orbits gives

$3(\frac{2^{v+1}k+2^v}{2^v}) + 1 = \frac{3k*2^{v+2}+2^{v+3}}{2^{v+1}} = 6k + 4$

Multiplying by 2^{v+1} gives

$$6(2^{v+1}k + 2^v) + 2^{v+1} = 3k * 2^{v+2} + 2^{v+3} \equiv 2^{v+3}\text{mod}(3 * 2^{v+2}). \quad (4)$$

According to the left side of Equation 4, the power of 2 added after multiplying by 6 would be 2^{v+1}, which is divisible by 2 one more time than $2^v(2k + 1) = 2^{v+1}k + 2^v$ is before reaching an odd. To calculate the number of times a positive integer is divisible by 2 before reaching an odd, denoted as the **2-adic order**, I can derive a function $f(x)$ for all $x \in \mathbb{N}$.

According to Legendre's formula, the number of times $x!$ is divisible by a prime p is:

$V_p(x!) = \sum_{n=1}^{\infty} \lfloor \frac{x}{p^n} \rfloor.$ [2]

To calculate the 2-adic order, the value of p is 2.

The expression $\lfloor \frac{x}{2^n} \rfloor$ counts the number of times x is divisible by 2^n without remainder, and $\frac{x}{2^n}$ counts the number of times x is divisible by 2^n with remainder.

Subtracting these expressions gives $\frac{x}{2^n} - \lfloor \frac{x}{2^n} \rfloor$, which will output 0 when divisible by 2^n and a number in $(0, 1)$ when not divisible by 2^n.

Take the ceiling function so there can only be 2 outputs, $\lceil \frac{x}{2^n} - \lfloor \frac{x}{2^n} \rfloor \rceil = 0$ when x is divisible by 2^n, and $\lceil \frac{x}{2^n} - \lfloor \frac{x}{2^n} \rfloor \rceil = 1$ when x is not divisible by 2^n.

To switch these values, take 1 minus the ceiling function, so that $1 - \lceil \frac{x}{2^n} - \lfloor \frac{x}{2^n} \rfloor \rceil = 1$ when x is divisible by 2^n and $1 - \lceil \frac{x}{2^n} - \lfloor \frac{x}{2^n} \rfloor \rceil = 0$ when x is not divisible by 2^n.

Take the summation for $n = 1$ to $\log_2(x)$, because any given value of x can be divisible by 2 at most $\log_2(x)$ times before reaching an odd. Now, I define the function $f(x)$ which gives the 2-adic order of x for all $x \in \mathbb{N}$ as shown below.

$f(x) = \sum_{n=1}^{\log_2 x} 1 - \lceil \frac{x}{2^n} - \lfloor \frac{x}{2^n} \rfloor \rceil$ [3]

When a positive integer is written in the form

$2^v(2k + 1) = 2^{v+1}k + 2^v \equiv 2^v \text{mod}(2^{v+1})$

where $k, v \in \mathbb{N}_0$, the 2-adic order of that number is v because it is divisible by 2 v times before reaching an odd. As seen in Equation 4, we multiply $2^{v+1}k + 2^v$ by 6 and add 2^{v+1} to connect this orbit to the orbit of $6(2^{v+1}k + 2^v) + 2^{v+1}$. I define this as the function g such that

$g(x) = 6x + 2^{f(x)+1}$ where $x \in \mathbb{N}$.

The connected orbits created by the function g are seen below for the odd starting value 7. Figure 1 shows the orbit of 7 and its connection to the first power of 2 orbit. The starting numbers of these connected orbits come from $g(7) = 44$, $g(44) = 272$, $g(272) = 1,664$, $g(1,664) = 10,240$, and $g(10,240) = 65,536$. In each consecutive connecting orbit, the function g is replacing an odd with an $8\text{mod}(12)$. The orbit of 7 has six odds and each consecutive connecting orbit has one fewer odd than the previous orbit which eventually leads to a connection to an orbit with only 1 odd or power of 2 orbit.

Figure 1: Orbit of 7 Connecting to One Power of 2 Orbit

7	44	272	1664	10240	65536
22	22	136	832	5120	32768
11	11	68	416	2560	16384
34	34	34	208	1280	8192
17	17	17	104	640	4096
52	52	52	52	320	2048
26	26	26	26	160	1024
13	13	13	13	80	512
40	40	40	40	40	256
20	20	20	20	20	128
10	10	10	10	10	64
5	5	5	5	5	32
16	16	16	16	16	16
8	8	8	8	8	8
4	4	4	4	4	4
2	2	2	2	2	2
1	1	1	1	1	1

Figure 2 shows how the connection to power of 2 orbits for the odd integer 7 could continue because the function g can be applied to the starting values of each connected orbit to infinity. Running the function g on 65,536 gives

$$g(65,536) = 6(65,536) + 2^{16+1} = 6(65,536) + 2^{17} = 524,288$$

which is the next power of 2 orbit that 7 is connected to. All previous connected orbits end with 1, but this orbit would end with 8 implying that it is an even only orbit. Since the previous orbit had 1 odd and the function g is replacing an odd with an $8\text{mod}12$ in each iteration, the next connecting orbit must have zero odds. Because orbits must have at least 1 odd, a trivial cycle is added to each connected orbit for each additional power of 2 orbit connected, so all connecting orbits will have at least one odd and all finish at 1. This is why the orbit of 7 has 2 additional trivial cycles added to compensate for the 2 additional power of 2 orbits connected. Since all starting values up to 2^{68} have been proven by computer to go to 1, they would all follow the same pattern as 7 and connect to an infinite amount of power of 2 orbits through infinite iterations of g.

Figure 2: Orbit of 7 Connecting to More Power of 2 Orbits

7	44	272	1664	10240	65536	524288	4194304
22	22	136	832	5120	32768	262144	2097152
11	11	68	416	2560	16384	131072	1048576
34	34	34	208	1280	8192	65536	524288
17	17	17	104	640	4096	32768	262144
52	52	52	52	320	2048	16384	131072
26	26	26	26	160	1024	8192	65536
13	13	13	13	80	512	4096	32768
40	40	40	40	40	256	2048	16384
20	20	20	20	20	128	1024	8192
10	10	10	10	10	64	512	4096
5	5	5	5	5	32	256	2048
16	16	16	16	16	16	128	1024
8	8	8	8	8	8	64	512
4	4	4	4	4	4	32	256
2	2	2	2	2	2	16	128
1	1	1	1	1	1	8	64
4	4	4	4	4	4	4	32
2	2	2	2	2	2	2	16
1	1	1	1	1	1	1	8
4	4	4	4	4	4	4	4
2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1

I will now summarize what the function g is doing in the connected orbit process. First, remember that the function g is only giving the initial values of each consecutive connecting orbit and the full orbits have been given for illustration purposes to show how each orbit is connected to the next and to show that all orbits would be the same length. This connecting orbit process would not be needed if we already had the full orbit of the first orbit beginning with an odd because we could just look and see if it goes to one. Since we always start this process with an odd orbit, the first odd orbit would contain x odd integers and the 8mod12 orbit would contain $x - 1$ odd integers.This will always be true because these orbits are identical except for the initial values.This idea is one of the first things that lead me to believe the odd and $8\text{mod}12$ connection was meant to prove the conjecture. This connection would not make a lot of sense if the odd orbit contained infinite odd integers because the $8\text{mod}12$ orbit would contain $\infty - 1$ odd integers. This connection only makes sense if all orbits contain a finite number of distinct odd integers which will be confirmed soon. Since the function g is using the odd and 8mod12 connection to obtain the initial values of each consecutive connecting orbit, each consecutive connecting orbit will contain $x - 2$, $x - 3$, $x - 4$, …odd integers. Because this connecting orbit process is connecting the first orbit to the second orbit, the second orbit to the third orbit, the third orbit to the fourth orbit, …, this process is replacing each odd contained in the first odd orbit with an $8\text{mod}12$ in each consecutive connecting orbit, so all connecting orbits will finish the same way. If a connection to a power of 2 orbit is established, the initial odd orbit must decrease to 1 like the power of 2 orbit it has been connected to. We know this process can be iterated to infinity because each initial value after the first odd integer will always be even. We know this because the second initial value will always be an $8\text{mod}12$, and each initial value after this is coming from multiplying the previous initial value by 6 and adding some power of 2 to obtain the next initial value. Since these values are always even, they will always divide to an odd integer that will connect to an $8\text{mod}12$ contained in the next connecting orbit. The function g gives the initial value of the next orbit containing the $8\text{mod}12$ used to replace the previous odd. Because this process can be iterated to infinity, an infinite number of odd integers can be replaced from any initial odd orbit in this connecting orbit process. The last important thing to notice about this connecting orbits process is that it also gives an actual reason for the existence of the repeating 4,2,1 trivial cycle as seen above in Figure 2.

Another easier way to see that this connected orbit process works on the orbit of any odd integer is to just follow the odd and $8\text{mod}12$ connection the entire way down. I can start with any odd integer which will connect to a specific $8\text{mod}12$. This $8\text{mod}12$ will divide to a different odd integer that will connect to a different $8\text{mod}12$, and this process would continue to infinity because each odd integer will always connect to a specific $8\text{mod}12$ and each $8\text{mod}12$ will always divide to a specific odd integer. Using this process on the connected orbits of 7 to illustrate this idea, 7 connects to 44, 44 connects to 11, 11 connects to 68, 68 connects to 17, 17 connects to 104, 104 connects to 13, 13 connects to 80, 80 connects to 5, 5 connects to 32, and 32 connects to 1. The function g is giving the initial values of the connecting orbits that would contain each of the odd and $8\text{mod}12$ integers in this process. Since an odd is being replaced with an $8\text{mod}12$ with each iteration either way you do it and both processes can be iterated to infinity, an infinite number of odds can be replaced from any orbit. You will soon see that using these processes on the orbit of any odd integer will always connect that orbit to a power of 2 orbit.

After seeing the function g applied to the orbit of 7, we know that all orbits with a finite number of distinct odds not containing a non-trivial cycle must connect to a power of 2 orbit. I will now focus on proving the non-trivial cycle and infinite increasing odd orbits could not exist. A **non-trivial cycle** is an orbit with a repeating cycle that does not contain 1. The infinite increasing odd orbit contains distinct odds that increase to infinity instead of decreasing to 1. To prove these orbits could not exist, the function g and a **bijection** are used. A bijection can only exist between two sets that are the same size if and only if each element from one set is paired with one element in the other set using a bijective function. The bijection is used to connect odds to $8\text{mod}12$ values, and the function g can be established as a bijective function by showing that it has an inverse g^{-1}, such that

$g^{-1}(x) = \frac{x-2}{6}$ where $x \equiv 8\text{mod}(12)$.

The bijection must also be injective and surjective. Injective means that no odd can be paired with more than one $8\text{mod}(12)$, and no $8\text{mod}(12)$ can be paired with more than one odd. Surjective means that every odd pairs with at least one $8\text{mod}(12)$, and every $8\text{mod}(12)$ pairs with at least one odd. Applying the function g to an odd $2j+1$ where $j \in \mathbb{N}_0$ gives

$$g(2j+1) = 6(2j+1) + 2^{f(2j+1)+1} = 6(2j+1) + 2 = 8 + 12j \equiv 8\text{mod}(12).$$

Likewise, applying g^{-1} to a value $8 + 12j$, where $j \in \mathbb{N}_0$ gives

$$g^{-1}(8+12j) = \frac{8+12j-2}{6} = 2j+1,$$

so it is surjective. Assume a and b are distinct odds. If they were paired with the same $8\text{mod}(12)$ value through the function g, $6a + 2 = 6b + 2$, which could only be true if $a = b$. Likewise, if $6a + 2$ and $6b + 2$ are distinct $8\text{mod}(12)$ values paired with the same odd through g^{-1}, $a = b$, so it is injective as well. The bijection is created by placing all odds from the orbit with an odd starting number in the first set and running the function g on each of these odds giving a set of $8\text{mod}(12)$ values in the second set. The purpose of the bijection is to show all odds must be replaced with $8\text{mod}(12)$s no matter how many odds are in the orbit. An example of the bijection can be seen below for the odd integer 7.

Odds From Orbit of 7: {7, 11, 17, 13, 5, 1}

8mod(12)s Connecting to Orbit of 7: {44, 68, 104, 80, 32, 8}

The orbit of 7 is connected to the orbit of

$$g(7) = 6(7) + 2^{0+1} = 6(7) + 2 = 44.$$

The orbit of 44 would contain 11 because $\frac{44}{2^2} = 11$, and 11 is connected to $11 * 6 + 2 = 68$ which is contained in the orbit of

$g(44) = 6(44) + 2^{2+1} = 6(44) + 2^3 = 272.$

The orbit of 272 would contain 17 because $\frac{272}{2^4} = 17$, and 17 is connected to $17 * 6 + 2 = 104$ which is contained in the orbit of

$g(272) = 6(272) + 2^{4+1} = 6(272) + 2^5 = 1,664.$

The orbit of 1,664 would contain 13 because $\frac{1664}{2^7} = 13$, and 13 is connected to $13 * 6 + 2 = 80$ which is contained in the orbit of

$g(1,664) = 6(1,664) + 2^{7+1} = 6(1,664) + 2^8 = 10,240.$

The orbit of 10,240 would contain 5 because $\frac{10240}{2^{11}} = 5$, and 5 is connected to $5 * 6 + 2 = 32$ which is contained in the orbit of

$g(10,240) = 6(10,240) + 2^{11+1} = 6(10,240) + 2^{12} = 65,536.$

The orbit of 65,536 would contain 1 because $\frac{65536}{2^{16}} = 1$, and 1 is connected to $1 * 6 + 2 = 8$ which is contained in the orbit of

$g(65,536) = 6(65,536) + 2^{16+1} = 6(65,536) + 2^{17} = 524,288.$

The set of odds and the set of connected $8\text{mod}(12)$s have the same **cardinality**, which is a measure of the number of elements of the set. Because both sets have a cardinality of 6, all odds have been replaced. Since all odds have been replaced with $8\text{mod}(12)$s, you end up with the last connecting orbit as an even only orbit ending in 8 until a trivial cycle has been added. This is the same situation explained earlier in the paper for the orbit of 7 connecting to more than one power of 2. Remember the function g makes the first connection to a power of 2 orbit when the last connecting orbit has 2 odds and replacing one of those gives an orbit with only 1 odd or power of 2 orbit. This happens in the fourth connecting orbit whose first odd is 5 which connects to 32. Continuing the connection process on a power of 2 orbit gives an even only orbit until a trivial cycle has been added for each new power of 2 connected.

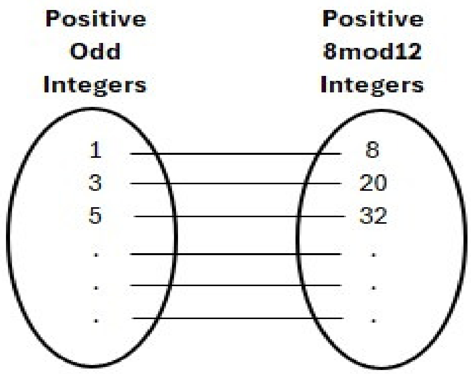
After seeing how the function g and bijection work on a finite orbit, they can now be applied to the infinite increasing odd counter example.

Odds From Orbit of Infinite Increasing Odds: $\{x_1, x_2, x_3, ...\}$

8mod(12)s Connecting to Orbit of Infinite Increasing Odds: $\{6x_1 + 2, 6x_2 + 2, 6x_3 + 2, ...\}$

The orbit of x_1 is connected to the orbit of $6x_1 + 2$. The orbit of $6x_1 + 2$ would contain x_2, and x_2 is connected to $6x_2 + 2$ which is contained in the orbit of $g(6x_1 + 2)$. The orbit of $g(6x_1 + 2)$ would contain x_3, and x_3 is connected to $6x_3 + 2$ which is contained in the orbit of $g^2(6x_1 + 2)$ and this process continues to infinity. The definition of cardinality used in the finite bijection can be extended to infinite sets as well. Since these infinite sets are mapped with bijective functions g and g^{-1}, they must have the same cardinality. These sets are also subsets of $\mathbb{N}$, and "any subset of the natural numbers is countable…A countable set that is not finite is said to be **countably infinite**."[4] Since both sets are the same size due to the bijection and are countably infinite, all odds must be replaced at infinity. This means that the orbit of x_1 must be connected to a power of 2 orbit at infinity. Since connected orbits must all end with the same values, the orbit of x_1 must end at 1 just like the power of 2 orbit would. This is confirmed by iterating the function g an infinite amount of times, which leads to a strictly decreasing connecting orbit at infinity that must be divisible by 2 an infinite amount of times. The only Collatz orbit that is strictly decreasing is a power of 2 orbit, so an infinite increasing odd orbit could not exist.

To see this another way, I will use the function g as a bijection on the entire set of odd integers as seen below.



Since every Collatz orbit would only contain a subset of the odd positive integers, and the bijection above clearly shows that all odd positive integers can be replaced with an $8\text{mod}12$, every odd integer in every orbit would be replaced with an $8\text{mod}12$ in the connecting orbit process.This means every Collatz orbit would connect to a power of 2 orbit at infinity and decrease to 1 like the power of 2 orbit they are connected to.

A third way to prove the infinite increasing odd counter example could not exist can be seen when looking back at Figures 1 and 2 that illustrate the connected orbits of the positive integer 7. The orbit of 7 has six distinct odd integers in Figure 1. In Figure 2 the orbit of 7 has eight indistinct odd integers because two additional trivial cycles were added to compensate for the two additional power of 2 orbits connected. The function g connects positive integers to power of 2 orbits based on the number of odds in the orbit regardless of whether they are distinct or not. This means that adding an infinite amount of trivial cycles to the orbits of all starting numbers up to 2^{68} would give these orbits the same number of odds as the infinite increasing odd orbit according to the function g. It follows that the infinite increasing odd orbit must connect to a power of 2 orbit using the same infinite iterations of g that would be applied to all numbers less than or equal to 2^{68} with infinite trivial cycles added. This again confirms that the infinite increasing odd counter example could not exist.

The function g and a bijection will now be used to disprove the non-trivial cycle counter example. When creating the set of odds for this counter example, we need to know how to handle the infinite amount of duplicates. There are two ways of dealing with repeated values entered into a set. The first involves only entering a repeated value once because the number of times an element appears does not matter. The other way involves entering all repeated values because the number of times an element appears does matter. This is called a **multiset**[5]. An example of a multiset would be a set containing the prime factors of an integer. The prime factorization of $36 = 2^2 * 3^2$ which gives the multiset {2,2,3,3}. Creating the set of odds for the non-trivial cycle counter example must have finite cardinality because the number of times these elements appear does not matter. For example, running an orbit on 7 finishes after one instance of the trivial cycle. Repeating the trivial cycle more than once does not change the orbit. As we saw earlier in this paper, the only time the trivial cycle needs repeated is when we are connecting orbits to additional power of 2 orbits as seen in Figure 2. In fact, a Mathematics Stack Exchange webpage mentions the idea of total stopping time, which measures the number of steps from the initial value to the first instance of 1 in a Collatz sequence[6]. This confirms that further repetitions of the trivial cycle do not matter and the non-trivial cycle would be treated the same way. Since I am only running the function g and bijection on one cycle of the non-trivial cycle, the set of odds would be finite, so the function g must connect the non-trivial cycle to a power of 2 orbit. Even if we consider infinite repetitions of the non-trivial cycle, iterating the function g an infinite amount of times would still connect it to a power of 2 orbit because all odds must be replaced at infinity since all odd integers would be replaced with $8\text{mod}12$ integers in each finite cycle. This proves that the trivial cycle is the only cycle that could exist which confirms that all odd integers must connect to a power of 2 orbit and decrease to 1. ☐

Conclusion: Since all odd positive integers connect to powers of 2 through orbits connected by the function g , all odd positive integers reach 1 when applying the Collatz function. Because all odd positive integers are contained in the orbits of all even positive integers, all even positive integers must reach 1 as well, which confirms the Collatz Conjecture is true.

The purpose of this paper was to prove all positive integers decrease to 1 when put into the Collatz function, but this connecting orbit process also works for all negative integers after making some minor changes to the Collatz and g functions. The modified Collatz and g functions are defined below for n and $x \in \mathbb{Z}^-$.

$$T(n) = \begin{cases} \frac{n}{2}, n \equiv 0 \bmod (2) \\ 3n - 1, n \equiv 1 \bmod (2) \end{cases} \qquad m(x) = 6x - 2^{f(x)+1}$$

The connected orbits created by the T and m functions are seen below for the odd starting value -7. Figures 1 and 3 are identical except for the negative signs proving all negative integers would connect to -1 using the modified functions for negative integers.

Figure 3: Orbit of -7 Connecting to One Negative Power of 2 Orbit

-7	-44	-272	-1664	-10240	-65536
-22	-22	-136	-832	-5120	-32768
-11	-11	-68	-416	-2560	-16384
-34	-34	-34	-208	-1280	-8192
-17	-17	-17	-104	-640	-4096
-52	-52	-52	-52	-320	-2048
-26	-26	-26	-26	-160	-1024
-13	-13	-13	-13	-80	-512
-40	-40	-40	-40	-40	-256
-20	-20	-20	-20	-20	-128
-10	-10	-10	-10	-10	-64
-5	-5	-5	-5	-5	-32
-16	-16	-16	-16	-16	-16
-8	-8	-8	-8	-8	-8
-4	-4	-4	-4	-4	-4
-2	-2	-2	-2	-2	-2
-1	-1	-1	-1	-1	-1

Figure 1: Orbit of 7 Connecting to One Power of 2 Orbit

7	44	272	1664	10240	65536
22	22	136	832	5120	32768
11	11	68	416	2560	16384
34	34	34	208	1280	8192
17	17	17	104	640	4096
52	52	52	52	320	2048
26	26	26	26	160	1024
13	13	13	13	80	512
40	40	40	40	40	256
20	20	20	20	20	128
10	10	10	10	10	64
5	5	5	5	5	32
16	16	16	16	16	16
8	8	8	8	8	8
4	4	4	4	4	4
2	2	2	2	2	2
1	1	1	1	1	1

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