An Investigation into the Discriminant of Quadratic Equations and Its Geometric Meaning

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Abstract

This paper explores the role of the discriminant in solving quadratic equations. Using the quadratic formula, we can determine whether a quadratic equation has real roots, how many it has, and whether those roots are rational or irrational. Several examples and graphs are provided to illustrate how the value of the discriminant affects the number of real roots and the position of the parabola on the coordinate plane. Real-life applications, such as motion under gravity and the intersection of lines and circles, are also discussed to enhance understanding.

1. Introduction

Quadratic equations are a fundamental component of A-Level mathematics. They appear not only in pure mathematics but also in applied fields such as mechanics and geometry. Understanding the solutions of quadratic equations provides valuable insight into many real-world problems. The quadratic formula is a powerful method for solving quadratic equations and includes an important term known as the discriminant.

This paper focuses on exploring how the discriminant determines the number and type of solutions and how these solutions relate to the graphical behavior of quadratic functions. We classify the possible scenarios based on the value of the discriminant and provide graphs to visualize the connection between the equation and its graph. In addition, we explore real-world applications in motion and geometry that further demonstrate the significance of the discriminant.

2. The Quadratic Formula and discriminant

The standard form of a quadratic equation is:

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

The quadratic formula for solving this equation is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The term under the square root, $b^2 - 4ac$, is called the discriminant, denoted as D. The discriminant reveals the nature of the solutions:

- If D > 0: Two distinct real roots
- If D = 0: One repeated real root
- If D < 0: No real roots (complex solutions)

The value of the discriminant plays a key role in predicting the behavior of the function's graph.

3. The Graphical Meaning of the Discriminant

The graph of a quadratic function is a parabola. The discriminant determines how the parabola interacts with the x-axis:

- When D > 0, the parabola intersects the x-axis at two distinct points.
- When D = 0, the vertex of the parabola lies on the x-axis, indicating one real root.
- When D < 0, the parabola does not touch the x-axis, indicating no real roots.

The following figures show examples of each case:

- If $2x^2 + 6x + 4 = 0$:

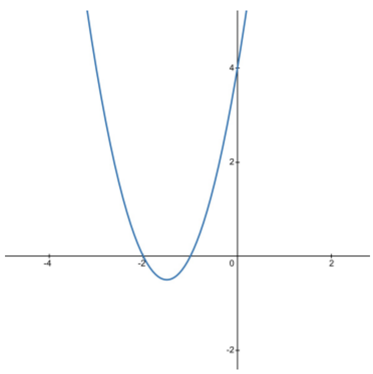

Figure 1

Parabola with two real roots (D > 0)

- If $x^2 + 2x + 4 = 0$:

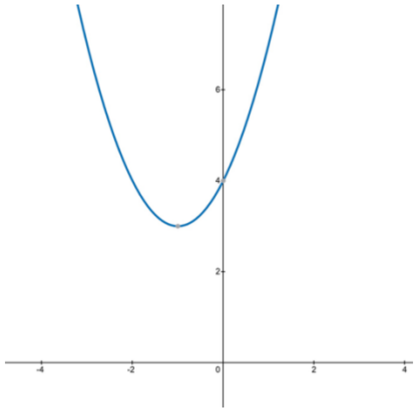

Figure 2

Parabola with no real roots (D < 0)

- If $x^2 + 2x + 1 = 0$:

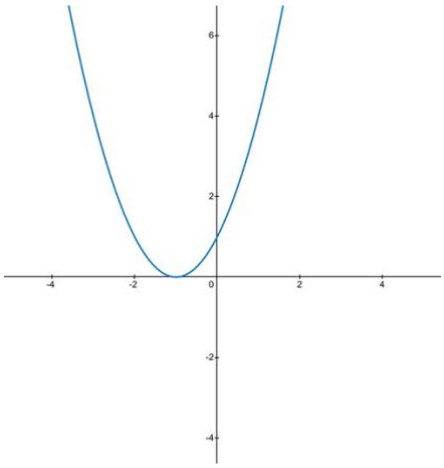

Figure 3

Parabola with one real root (D = 0)

These visualizations help reinforce the connection between algebraic. calculations and geometric interpretation.

4. Examples and Calculations

Example 1:

$2x^2 + 6x + 4 = 0$:

$D = 6^2 - 4(2)(4) = 36 - 32 = 4 > 0$

→ Two distinct real roots

Example 2:

$x^2+2x + 4 = 0$

$D = (2)^2 - 4(1)(4) = 4 - 16 = -12 < 0$

→ No real root.

Example 3:

$x^2+2x + 1 = 0$

$D = (2)^2 - 4(1)(1) = 4 - 4 = 0$

→ One repeated real root.

4. Applications

a. Motion Under Gravity

In physics, a quadratic equation can model vertical motion. For example, the height of an object launched upwards can be given by:

$$h(t) = -5t^2 + 10t + 5$$

To find when the object reaches the ground, we solve $h(t) = 0$. The discriminant tells us whether the object touches the ground (real solutions) or not. A positive discriminant indicates two times when the object is at height zero: at launch and at impact.

b. Intersection of a Line and a Circle

Consider the system:

- Circle: $x^2 + y^2 = 4$
- Line: $y = x + c$

Substituting into the circle's equation gives a quadratic in x. The discriminant of this equation determines how the line and the circle intersect:

- D > 0: Two points of intersection
- D = 0: Line is tangent to the circle
- D < 0: No intersection

5. Conclusion

The discriminant $D = b^2 - 4ac$ is a powerful tool for understanding the nature of the roots of a quadratic equation. It also provides a visual insight into how the graph of the function behaves with respect to the x-axis. The discriminant not only applies to solving equations algebraically but also has practical significance in physics and geometry.

6. Future Applications

In the future, the discriminant can be used in more advanced problems, such as solving higher-degree equations like cubic equations. It can also help us understand more about graphs in algebra and physics. For example, in university, students may use similar ideas when learning about different types of curves or how systems change. Learning the discriminant now helps build a strong foundation for future mathematical studies.

7. References

Cambridge International AS & A Level Mathematics Coursebook;
Khan Academic – Quadratic equations;
Desmos Graphing Calculator.