Proof of Collatz Conjecture

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Introduction

Collatz Conjecture ($3 . x + 1$ problem) stats any natural number x will return to number 1 after $3 . x + 1$ computation (when x is odd) and $\frac{x}{2}$ computation (when x is even).

The conjecture asserts that for every $x \in \mathbb{N}$, there exists k such that $f^k(x) = 1$.
Where f^k denotes the k-th iterate of the function f.
in this paper we provide a proof the Collatz Conjecture is true.

Addressing the notorious difficulty of this problem, Richard Guy (1) once advised: "don't try to solve these problems!"
Core Methodology:
This paper presents a complete and novel proof of the Collatz conjecture.
The proof is built upon a fundamental reduction showing that convergence for all positive integers follows from convergence for all odd integers. We then introduce a novel ternary partition of the set of all odd integers $\mathbb{N}_{odd}$ into three mutually exclusive and exhaustive sets B, C, and D.
A pivotal element of the proof is the introduction and detailed examination of the set $V = \{5 + 12 . n \mid n \geq 0\}$.
Key words: Collatz Conjecture. $3 . x + 1$ problem
The problem of the research: Collatz Conjecture ($3 . x + 1$ problem) is unsolved conjecture.

The importance of the research: the importance of this research lies in solving the Collatz Conjecture that remained for 85 years without proof or denial.

Abstract
This paper presents a proof Collatz conjecture through the following structured approach:

1. **Reduction to odd integers:** We demonstrate that if the conjecture holds for all odd integers, it holds to all even integers.
2. **Partition of odd integers:** Defines sets B, C, and D partitioning $\mathbb{N}_{odd}$,without duplication. numbers
3. **Absence of non-trivial cycles**: Proves no cycles exist except the cycle $\{1 \xrightarrow{f} 1\}$.
4. **function mapping:** We perform the function to the sets of numbers B, C, and D to analyze their behavior.
5. **Producing the elements of the set R by applying the function to set V:** Proving the existence of infinitely many elements $v_{a+3l} \in V$ in $B \cup D$ that map to the same $r \in R = f(\mathbb{N}_{odd})$.
6. **Analyzing behavior the set C:** prove that iterative application of function f on set C converges to set V.

7. **Global convergence:** We used the elements of the set V as starting points for applying the conjecture and proved that they produce the elements of the set V itself (as arbitrary termination points) along with the number 1. We then studied their behavior and demonstrated that all elements of the set V (passing through all elements of the set $R\backslash V$ satisfy the conjecture. Subsequently, we generalized this result to multiples of 3 and even numbers, thereby decisively and conclusively proving the validity of the Collatz conjecture.

1. Reduction of the problem to the odd integers:

Any even integer e decomposes as $e = 2^{a}.(2.n+1)$ for $a \geq 1$, $n \geq 0$. Applying f repeatedly a times yields an odd integer $f^{a}(e) = \frac{e}{2^{a}} = 2.n+1$. Thus, proving convergence for all odd integers implies convergence for all integers.

Conclusion 1: it is sufficient to prove that all odd integers satisfy the conjecture to be correct.

2. Partition of the odd integers:

Theorem:

The sets B, C, and D:

- $B = \left\{\frac{4^{a}-1}{3} + 2.4^{a}.n \,\middle|\, a \geq 1,\ n \geq 0\right\}$
- $C = \{3 + 4.n \mid n \geq 0\}$
- $D = \left\{3 + 10.\left(\frac{4^{a}-1}{3}\right) + 4^{a+1}.n \,\middle|\, a \geq 1,\ n \geq 0\right\}$

(İ) Contain no duplicates,

(İİ) Are pairwise disjoint,

(İİİ) Cover all odd integers $\mathbb{N}_{odd}$.

Proof

(İ) **Contain no duplicates:**

- **proof of the absence of duplicate odds within set B:**

Assume $b_1, b_2 \epsilon B$ with $b_1 = b_2$:

$$b_1 = \frac{4^{a_1}-1}{3} + 2.4^{a_1}.n_1,\ b_2 = \frac{4^{a_2}-1}{3} + 2.4^{a_2}.n_2$$

$$\frac{4^{a_1}-1}{3} + 2.4^{a_1}.n_1 = \frac{4^{a_2}-1}{3} + 2.4^{a_2}.n_2$$

After simplifying the equation, it becomes:

$$3.4^{(2a_2-2a_1)}.n_2 = 3.n_1 - 2^{(2a_2-2a_1-1)} - \frac{1}{2}$$

Since the left-hand side is an integer, while the right-hand side is non-integer, no natural number pair (n_1, n_2) exists that satisfies this equation.

Conclusion: $\forall b_{1,} b_2 \epsilon B$: $b_1 \neq b_2$

- **proof of the absence of duplicate odds within set C:**

Assume $c_1, c_2 \epsilon C$ with $c_1 = c_2$:

$c_1 = 3 + 4.n_{1,} c_2 = 3 + 4.n_2$

$3 + 4.n_1 = 3 + 4.n_2 \Rightarrow n_1 = n_2 \Rightarrow$ c_1, c_2 are the same odd.

Conclusion: $\forall\, c_1, c_2 \in C: c_1 \neq c_2$

- **proof of the absence of duplicate odds within set *D***:

Assume $d_1, d_2 \in D$ with $d_1 = d_2$:

$d_1 = 3 + 10\,.\left(\frac{4^{a_1-1}}{3}\right) + 4^{(a_1+1)}.\,n_1,\;\; d_2 = 3 + 10\,.\left(\frac{4^{a_2-1}}{3}\right) + 4^{(a_2+1)}\,.\,n_2$

$$3 + 10\,.\left(\frac{4^{a_1} - 1}{3}\right) + 4^{(a_1+1)}\,.\,n_1 = 3 + 10\,.\left(\frac{4^{a_2} - 1}{3}\right) + 4^{(a_2+1)}\,.\,n_2$$

After simplifying the equation, it becomes:

$$9\,.\,2^{2a_1}\,.\,n_1 + 5\,.\,2^{(2a_1\,-\,1)}.\left(3 - 4^{(a_2+2)} + 4^{a_1}\right) - \frac{15}{2} + 5\,.\left(2^{(2a_2+1)} + 2^{(2a_1\,-\,1)}\right) = 9\,.\,2^{2a_2}.\,n_2$$

Since the left-hand side is an integer, while the right-hand side is non-integer, no natural number pair (n_1, n_2) exists that satisfies this equation.

Conclusion: $\forall\, d_1, d_2 \in D: d_1 \neq d_2$

(ii) Are pairwise disjoint:

proof that $\boldsymbol{B} \cap \boldsymbol{C} = \emptyset$:

Assume $b \in B$, $c \in C$ with $b = c$:

$b = \dfrac{4^a - 1}{3} + 2\,.\,4^a.\,n_1$

$c = 3 + 4\,.\,n_2$

$$\frac{4^a - 1}{3} + 2\,.\,4^a.\,n_1 = 3 + 4\,.\,n_2$$

After simplifying the equation, it becomes:

$$n_2 = \frac{4^{a-1} - 1}{3} + 2\,.\,4^{(a-1)}\,.\,n_1 - \frac{1}{2}$$

Since the left-hand side is an integer, while the right-hand side is non-integer, no natural number pair (n_1, n_2) exists that satisfies this equation.

Conclusion: $\forall\, b \in B$, $\forall\, c \in C$: $b \neq c$

- **proof that $\boldsymbol{B} \cap \boldsymbol{D} = \emptyset$:**

Assume $b \in B$, $d \in D$ with $b = d$:

$b = \frac{4^{a_1-1}}{3} + 2\,.\,4^{a_1}\,.\,n_1$

$d = 3 + 10\,.\left(\dfrac{4^{a_2} - 1}{3}\right) + 4^{(a_2+1)}\,.\,n_2$

$\dfrac{4^{a_1} - 1}{3} + 2\,.\,4^{a_1}\,.\,n_1 = 3 + 10\,.\left(\dfrac{4^{a_2} - 1}{3}\right) + 4^{(a_2+1)}\,.\,n_2$

After simplifying the equation, it becomes:

$$n_1 = 2^{(2a_2-2a_1+1)}\,.\,n_2 - \frac{10\,.\;4^{(a_2-a_1)} - 1}{6}$$

Since the left-hand side is an integer, while the right-hand side is non-integer, no natural number pair (n_1, n_2) exists that satisfies this equation.

Conclusion: $\forall\ b \in B$, $\forall\ d \in D$: $b \neq d$

- **proof that $\boldsymbol{C} \cap \boldsymbol{D} = \emptyset$:**

Assume $c \in C$, $d \in D$ with $c = d$:
$c\ =\ 3\ +\ 4\,.\,n_1$
$d = 3 + 10\,.\left(\dfrac{4^{a_2} - 1}{3}\right) + 4^{(a_2+1)}\,.\,n_2$

$$3 + 10\,.\left(\frac{4^{a_2} - 1}{3}\right) + 4^{(a_2+1)}\,.\,n_2 =\ 3\ +\ 4\,.\,n_1$$

After simplifying the equation, it becomes:

$$3\,.\,n_1 = 5\,.\,2^{(2a-1)}\ -\ \frac{5}{2} + 3\,.\,4^{(a+1)}.\,n_2$$

Since the left-hand side is an integer, while the right-hand side is non-integer, no natural number pair (n_1, n_2) exists that satisfies this equation.
$\forall\ c \in C$, $\forall\ d \in D$: $c \neq d$
Conclusion: $C \cap D = \emptyset$

(iii) Cover all odd integers $\mathbb{N}_{odd}$:

- **Proof that *B* represents one out of every 3 numbers in set $\mathbb{N}_{odd}$**

$B = \left\{\frac{4^a-1}{3} + 2\,.\,4^a\,.\,n \,\middle|\, a\ \geq 1,\ n\ \geq 0\right\}$

Case analysis:
For a = 1: $B_1 = \{1 + 2\,.\,4\,.\,n \mid n\ \geq 0\}$ ⇨ B_1 represents one out of every 4 numbers in set $\mathbb{N}_{odd}$
For a = 2: $B_2 = \{1 + 4 + 2\,.\,4^2\,.\,n \mid n\ \geq 0\}$ ⇨ B_2 represents one out of every 16 numbers in $\mathbb{N}_{odd}$
General form:
The total contribution of *B* is the sum of an infinite geometric series:
B covers $\frac{1}{3}$ of $\mathbb{N}_{odd}$ (via geometric series $\sum_{a=1}^{\infty}\frac{1}{4^a}\ =\ \frac{1}{3}$). $\mu(B) = \sum_{a=1}^{\infty}\frac{1}{4^a}\ =\ \frac{1}{3}$

Thus, *B* represents one out of every 3 numbers in set $\mathbb{N}_{odd}$

- **Proof that *C* represents one out of every 2 numbers in set $\mathbb{N}_{odd}$**
  $C\ =\ \{3\ +\ 4\,.\,n \mid n\ \geq 0\}$ ⇨ $\mu(C) = \frac{1}{2}$
  thus, C represents one out of every 2 numbers in set $\mathbb{N}_{odd}$
- **Proof that *D* represents one out of every 6 numbers in set $\mathbb{N}_{odd}$**

$D = \left\{3 + 10\,.\left(\dfrac{4^a - 1}{3}\right) + 4^{a+1}\,.\,n \,\middle|\, a\ \geq 1,\ n\ \geq 0\right\}$
Case analysis:
For a = 1: D_1 = 3 + 10 . (1) + 4^2 . n ⇨ D_1 represents one out of every (2 . 4) numbers in set $\mathbb{N}_{odd}$
For a = 2: D_2 = 3 + 10 . (1 + 4) + 4^3 . n ⇨ D_2 represents one out of every (2 . 4^2) numbers in set $\mathbb{N}_{odd}$
General form:
The total contribution of *D* is the sum of an infinite geometric series:

D covers $\frac{1}{6}$ of $\mathbb{N}_{odd}$ (via geometric series $\sum_{a=1}^{\infty} \frac{1}{2\,.4^{a+1}} = \frac{1}{6}$). $\mu(D) = \sum_{a=1}^{\infty} \frac{1}{2\,.4^{a+1}} = \frac{1}{6}$

Thus, *D* represents one out of every 6 numbers in set $\mathbb{N}_{odd}$

Based on the preceding analysis we deduce that:

$$B \cup C \cup D = \mu(B) + \mu(C) + \mu(D) = \left(\frac{1}{3} + \frac{1}{2} + \frac{1}{6}\right) = 1 \Rightarrow$$
$$B \cup C \cup D = \mathbb{N}_{odd}$$

Conclusion 2: the sets B, C, and D represents all the odds $\mathbb{N}_{odd}$ without duplicated odds.

3. Proof of no non-trivial cycles:

3.1. Proof that multiples of 3 cannot be a part of any cycle:

Let $\{M = 3.x \mid x \in \mathbb{N}_{odd}\}$, the set of odds multiples of 3:

∀ $m \in M$, *m* cannot be a part of any cycle in the Collatze system, as it cannot appear as an output value in the conjecture's iteration. the explanation is as follows:

$$f(x_1) = \frac{3\,.x_1+1}{2^a} = m = 3\,.x \Rightarrow 3\,.x_1 = 2^a\,.3\,.x - 1 \Rightarrow x_1 = 2^a.x - \frac{1}{3}$$

x_1 is not a valid integer number.

3.2. Application of the Collatz function to sets H_i:

Let R is the set of odds except the odd multiples of 3: $R = \mathbb{N}_{odd} \setminus M$

The set *R* can be partitioned into two disjoint subsets based on modulo 6:

$$R = R_a \cup R_b \begin{cases} R_a = \{1 + 6\,.n \mid n \geq 0\} \\ R_b = \{5 + 6\,.n \mid n \geq 0\} \end{cases}$$

Let *H* specific subset of the set $\{M = 3.x \mid x \in \mathbb{N}_{odd}\}$, $H \subset M$

The set *H* can be partitioned into six disjoint subsets:

$$H = \bigcup_{i=1}^{i=6} H_i$$

Where:

- $H_1 = \{21 + 6\,.2^6\,.n \mid n \geq 0\}$
- $H_2 = \{3 + 6.2^1.n \mid n \geq 0\}$
- $H_3 = \{9 + 6\,.2^2.n \mid n \geq 0\}$
- $H_4 = \{117 + 6\,.2^5\,.n \mid n \geq 0\}$
- $H_5 = \{69 + 6\,.2^4\,.n \mid n \geq 0\}$
- $H_6 = \{45 + 6\,.2^3\,.n \mid n \geq 0\}$

  - $R_1 = f(H_1) = \{1 + 18\,.n \mid n \geq 0\}$
  - $R_2 = f(H_2) = \{5 + 18\,.n \mid n \geq 0\}$
  - $R_3 = f(H_3) = \{7 + 18.n \mid n \geq 0\}$
  - $R_4 = f(H_4) = \{11 + 18.n \mid n \geq 0\}$
  - $R_5 = f(H_5) = \{13 + 18.n \mid n \geq 0\}$
  - $R_6 = f(H_6) = \{17 + 18.n \mid n \geq 0\}$

$$\begin{cases} R_1 \cup R_3 \cup R_5 = 1 + 6\,.n = R_a \\ R_2 \cup R_4 \cup R_6 = 5 + 6\,.n = R_b \end{cases}$$

$$\bigcup_{i=1}^{i=6} R_i = R_a \cup R_b = R$$

Conclusion 3: Applying the operations to sets $\{H_1, H_2, H_3, H_4, H_5, H_6\} \subset$ M yields the set $R, (R = ⨍(\mathbb{N}_{odd})$, as we will see later).

We represent this with the following equation:

$$R = ⨍(H) = \left\{\frac{3\,.\,h\ +\ 1}{2^t} \,\middle|\, h \in H, t \in T\right\}: \begin{pmatrix} H = \bigcup_{i=1}^{i=6} H_i \\ T = \{1, 2, 3, 4, 5, 6\} \end{pmatrix}$$

$$R = ⨍(H) = \left\{\frac{3\,.\,3\,.\,x\ +\ 1}{2^t} \,\middle|\, x \in \mathbb{N}_{odd}, t \in T\right\}: h = 3\,.\,x$$

$$\boxed{R = ⨍(H) = \left\{\frac{9\,.x + 1}{2^t} \,\middle|\, x \in \mathbb{N}_{odd}, t \in T\right\}}$$

3.3. Impossibility of non-trivial cycles:

Assume the odds: $\{r_1, r_2, r_3, r_4, \ldots, ..., r_{s-1}, r_s\}$ form a cycle:

$r_1 \xrightarrow{⨍} r_2 \xrightarrow{⨍} r_3 \xrightarrow{⨍} \ldots \xrightarrow{⨍} \ldots \xrightarrow{⨍} r_{s-1} \xrightarrow{⨍} r_s\colon\ r_s = r_1$

$r_1 = \dfrac{9\,.\ x_1\ +\ 1}{2^{t_1}}\ :\ r_1 \in R$

$r_s = \dfrac{9\,.\ x_2\ +\ 1}{2^{t_2}}\ :\ r_s \in R$

$r_1 = r_s\colon\ \dfrac{9\,.\ x_1 + 1}{2^{t_1}} = \dfrac{9\,.\ x_2 + 1}{2^{t_2}}\ ⇨$

$x_1 = x_2\,.\,2^{(t_1 - t_2)} + \dfrac{2^{(t_1 - t_2)} - 1}{3^2}$

$(t_1 - t_2) \in \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$

For $(t_1 - t_2) \in \{-5, -4, -3, -2, -1, 1, 2, 3, 4, 5\}$ x_1 is not a valid integer number.

For $(t_1 - t_2) = 0$: $x_1 = x_2$ that means the odd r_1 forms a cycle with itself:

$$\boxed{⨍(r_1) = r_1}$$

3.3.1. analysis the case $⨍(r_1) = r_1$:

- if $r_1 \in C$: $⨍(r_1) = \frac{3\,.\ r_1 + 1}{2}$. then $⨍(r_1) > r_1$ thus, $⨍(r_1) \neq r_1$ ⇨ $r_1 \notin C$
- if $r_1 \in D$: $⨍(r_1) = \frac{3\,.\ r_1 + 1}{4^a}$. then $⨍(r_1) < r_1$ thus, $⨍(r_1) \neq r_1$ ⇨ $r_1 \notin D$
- if $r_1 \in B \setminus \{1\}$: $⨍(r_1) = \frac{3\,.\ r_1 + 1}{4^a}$. then $⨍(r_1) < r_1$ thus, $⨍(r_1) \neq r_1$ ⇨ $r_1 \notin B\setminus\{1\}$

The previous three steps show that: $\forall\ r_1 \in \mathbb{N}_{odd} \setminus \{1\}$ ⇨ $⨍(r_1) \neq r_1$

Then: the only fixed point is $r_1 = 1$, where $⨍(1) = 1$

Conclusion 4: there are no non-trivial cycles.

4. Performing the function to sets $B, C,$ and D:

4.1. Performing the function to set B:

$$B = \left\{ \frac{4^a - 1}{3} + 2\,.\,4^a\,.\,n \;\middle|\; a \geq 1,\; n \geq 0 \right\}$$

- $\forall\, b \in B: f(b) = \dfrac{3\,.\,\left(\frac{4^a - 1}{3}\right) + 3\,.2\,.4^a\,.n + 1}{4^a} \;\Rightarrow\; f(B) = \{1 + 6\,.\,n \mid n \geq 0\}$

Table 1: Performing the function to set B: $f(B) = 5 + 6n$

		n	*n=0*	*n=1*	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*	*n=7*	*n=...*
		f(b) = 1+ 6n	1	7	13	19	25	31	37	43	…..
b	$a = 1$	$\frac{4^1 - 1}{3} + 2\,.\,4^1\,.\,n$	1	9	17	25	33	41	49	57	…..
	$a = 2$	$\frac{4^2 - 1}{3} + 2\,.\,4^2\,.\,n$	5	37	69	101	133	165	197	229	…..
	$a = 3$	$\frac{4^3 - 1}{3} + 2\,.\,4^3\,.\,n$	21	149	277	405	533	661	789	917	…..
	$a = 4$	$\frac{4^4 - 1}{3} + 2\,.\,4^4\,.\,n$	85	597	1109	1621	2133	2645	3157	3669	…..
	$a = 5$	$\frac{4^5 - 1}{3} + 2\,.\,4^5\,.\,n$	341	2389	4437	6485	8533	10581	12629	14677	…..
	$a = 6$	$\frac{4^6 - 1}{3} + 2\,.\,4^6\,.\,n$	1365	9557	17749	25941	34133	…..	…..	…..	……
	$a = 7$	$\frac{4^7 - 1}{3} + 2\,.\,4^7\,.\,n$	5461	38229	70997	……	……	…….	……	…….	…….
	….	………………	….	…..	…..	……	…..	…..	…..	……	…….

Note: We have marked the odds of set V in gray

4.2. Performing the function to set C:

$C = \{3 + 4\,.\,n \mid n \geq 0\}$

$$\forall\, c \in C: f(c) = \frac{3\,.c + 1}{2} = \frac{3\,.\;(3 + 4\,.n) + 1}{2} \;\Rightarrow\; f(C) = \{5 + 6\,.\,n \mid n \geq 0\}$$

Table 2: $f(C) = 5 + 6n$

n	*n=0*	*n=1*	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*	*n=7*	*n=8*	*n=9*
$f(C) = 5 + 6n$	5	11	17	23	29	35	41	47	53	….
$C = 3 + 4n$	3	7	11	15	19	23	27	31	35	…..

Note: We have marked the odds of set V in gray.

4.3. Performing the function to set D:

$$D = \left\{ 3 + 10\,.\left(\frac{4^a - 1}{3}\right) + 4^{a+1}\,.\,n \;\middle|\; a \geq 1,\; n \geq 0 \right\}$$

$$\forall\, d \in D: f(d) = \frac{3\,.d + 1}{2\,.4^a} = \frac{3\,.\left[3 + 10\,.\left(\frac{4^a - 1}{3}\right) + 4^{(a+1)}\,.n\right] + 1}{2\,.4^a} \;\Rightarrow\; f(D) = \{5 + 6\,.\,n \mid n \geq 0\}$$

Table 3: Performing the function to set D: $ƒ(D) = 5 + 6n$

		n	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=..$
		$ƒ(d) = 5 + 6n$	5	11	17	23	29	35	41	47	….
d	$a = 1$	$3 + 10.\left(\frac{4^1-1}{3}\right) + 4^2.n$	13	29	45	61	77	93	109	125	……
	$a = 2$	$3 + 10.\left(\frac{4^2-1}{3}\right) + 4^3.n$	53	117	181	245	309	373	437	501	……
	$a = 3$	$3 + 10.\left(\frac{4^3-1}{3}\right) + 4^4.n$	213	469	725	981	1237	1493	1749	2005	…..
	$a = 4$	$3 + 10.\left(\frac{4^4-1}{3}\right) + 4^5.n$	853	1877	2901	3925	4949	5973	6997	8021	……
	$a = 5$	$3 + 10.\left(\frac{4^5-1}{3}\right) + 4^6.n$	3413	7509	11605	15701	19797	23893	…..	…..	…
	$a = 6$	$3 + 10.\left(\frac{4^6-1}{3}\right) + 4^7.n$	13653	30037	46421	62805	……	…….	……	…….	….
	$a = 7$	…………..	….	…..	…..	……	…..	…..	…..	……	….

Note: we have marked the elements of set V in gray

5. Proving the existence of infinitely many elements $v_{a+3l} \in$ V in $B \cup D$ that map to the same $r \in R$: $= ƒ(\mathbb{N}_{odd})$:

5.1. Elements of $V \cap B$:

Let V $= \{5 + 12 . n \mid n \geq 0\}$.

Step 1: $B = \left\{\frac{4^a-1}{3} + 2 . 4^a . n \,\middle|\, a \geq 1,\ n \geq 0\right\}$

- For $a = 1$: $B_1 = \{1 + 2 . 4 . n \mid n \geq 0\}$

Let V_1 be a specific subset of the set B_1 for ($a = 1,\ n = 2 + 3 . y$) defined by the form

$v_1 = 17 + 2 . 4 . 3 . y$
$= 5 + 12 + 24 . y$
$= 5 + 12 . (1 + 2 . y)$

$V_1 = \{5 + 12 . n_1 \mid n_1 = 1 + 2 . y\}$ thus, $V_1 \subset V$

$ƒ(V_1) = ƒ(17 + 2 . 4^1 . 3 . y)$ ⇨ $\boxed{ƒ(V_1) = \{13 + 18 . y \mid y \geq 0\}}$

- For $a = 2$: $B_2 = \{1 + 4 + 2 . 4^2 . n \mid n \geq 0\}$

Let V_2 be a specific subset of the set B_2 for ($a = 2, n = 0 + 3 . y$) defined by the form

$v_2 = 1 + 4 + 2 . 4^2 . 3 . y$
$= 5 + 12 . 8 . y$

$V_2 = \{5 + 12 . n_2 \mid n_2 = 8 . y\}$ thus, $V_2 \subset V$

$ƒ(V_2) = ƒ(5 + 2 . 4^2 . 3 . y)$ ⇨ $\boxed{ƒ(V_2) = \{1 + 18 . y \mid y \geq 0\}}$

- For $a = 3$: $B_3 = \{1 + 4 + 4^2 + 2 . 4^3 . n \mid n \geq 0\}$

Let V_3 be a specific subset of the set B_3 for ($a = 3, n = 1 + 3 . y$) defined by the form

$v_3 = 149 + 2 . 4^3 . 3 . y$
$= 5 + 12 . (12 + 2 . 4^2 . y)$

$V_3 = \{5 + 12 . n_3 \mid n_3 = 12 + 2 . 4^2 . y\}$ thus, $V_3 \subset V$

$ƒ(V_3) = ƒ(149 + 2 . 4^3 . 3 . y)$ ⇨ $\boxed{ƒ(V_3) = \{7 + 18 . y \mid y \geq 0\}}$

$$\boxed{ƒ(V_1) \cup ƒ(V_2) \cup ƒ(V_3) = \{ 1 + 6 . y \mid y \geq 0\}}$$

Note: the three previous steps means that: the integers of the set V are present in the all columns of **Table 1** (in the lines $\{a = 1, a = 2, a = 3\}$).

Step 2: $\forall\ v_a \in (B \cap V)$ where $v_a = \frac{4^a-1}{3} + 2 . 4^a . n_s = 5 + 2 . 6 . n_m$

Let $v_{a+3} = \frac{4^{a+3}-1}{3} + 2 . 4^{a+3} . n_s$ ⇨

$$\begin{aligned} v_{a+3} &= v_a + (v_{a+3} - v_a) \\ &= v_a + \frac{4^a . 63}{3} + 2 . 4^a . 63 . n_s \\ &= 5 + 2 . 6 . n_m + 21 . 4^a + 2 . 4^a . 63 . n_s \\ &= 5 + 12 . (n_m + 7 . 4^{a-1} + 2 . 4^{a-1} . 21 . n_s) \\ v_{a+3} &= 5 + 12 . n_j \quad \text{where} \quad n_j = n_m + 7 . 4^{a-1} + 2 . 4^{a-1} . 21 . n_s \end{aligned}$$

Thus, $v_{a+3} \in V$

$$\begin{aligned} ƒ(v_{a+3}) &= ƒ[v_a + (v_{a+3} - v_a)] \\ &= ƒ[\frac{4^a-1}{3} + 2 . 4^a . n_s + (\frac{4^{a+3}-1}{3} + 2 . 4^{a+3} . n_s - \frac{4^a-1}{3} + 2 . 4^a . n_s)] \\ &= ƒ[\frac{4^a . 63}{3} + 2 . 4^{(a+3)} . n_s] \\ &= ƒ[4^a . 21 + 2 . 4^{(a+3)} . n_s] \end{aligned}$$

$ƒ(v_{a+3}) = \{1 + 6 . n_s \mid n \geq 0\}$, thus, $ƒ(v_{a+3}) = ƒ(v_a)$

Conclusion 5: there are infinity odds $v_{a+3g} \in (B \cap V)$ produce the same odd r_a where $r_a \in R_a = 1 + 6 . n$ as output when performing the function to them: $ƒ(v_{a+3g}) = r_a$ where $g \geq 0$.

Note: the odds of set V are present in the all columns of **Table 1**.

each odd of the set V -three lines below it- there is another odd of the set V.

5.2. Elements of $V \cap D$:

Step 1: $D = \left\{3 + 10 . \left(\frac{4^a-1}{3}\right) + 4^{a+1} . n \,\middle|\, a \geq 1,\ n \geq 0\right\}$

- For $a = 1$: $D_1 = \left\{3 + 10 . \left(\frac{4^1-1}{3}\right) + 4^2 . n \,\middle|\, a \geq 1,\ n \geq 0\right\}$

Let V_4 be a specific subset of the set D_1 for ($a = 1, n = 1 + 3 . y$) defined by the form

$v_4 = 29 + 4^2 . 3 . y$ ⇨

$= 5 + 12 . (2 + 4 . y)$

$V_4 = \{5 + 12 . n_4 \mid n_4 = 2 + 4 . y\}$ thus, $V_4 \subset V$

$ƒ(V_4) = ƒ(29 + 4^2 . 3 . y)$ ⇨ $\boxed{ƒ(V_4) = \{11 + 18 . y \mid y \geq 0\}}$

- For $a = 2$: $D_2 = \left\{3 + 10 . \left(\frac{4^2-1}{3}\right) + 4^3 . n \,\middle|\, a \geq 1,\ n \geq 0\right\}$

Let V_5 be a specific subset of the set D_2 for ($a = 2, n = n = 0 + 3 . y$) defined by the form

$v_5 = 53 + 4^3 . 3 . y$

$= 5 + 12 . (4 + 4^2 . y)$

$V_5 = \{5 + 12 . n_5 \mid n_5 = 4 + 4^2 . y\}$ thus, $V_5 \subset V$

$ƒ(V_5) = ƒ(53 + 4^3 . 3 . n) \Rightarrow \boxed{ƒ(V_5) = \{5 + 18 . y \mid y \geq 0\}}$

- For $a = 3$: $D_3 = \left\{3 + 10 . \left(\frac{4^3-1}{3}\right) + 4^4 . n \,\middle|\, a \geq 1,\ n \geq 0\right\}$

Let V_6 be a specific subset of the set D_3 for ($a = 3, n = 2 + 3 . y$) defined by the form

$v_6 = 725 + 4^4 . 3 . y$

$= 5 + 12 . (60 + 4^3 . y)$

$V_6 = \{5 + 12 . n_6 | n_6 = 60 + 4^3 . n\}$ thus, $V_6 \subset V$

$ƒ(V_6) = ƒ(725 + 4^4 . 3 . y) \Rightarrow \boxed{ƒ(V_6) = \{17 + 18 . y \mid y \geq 0\}}$

$$\boxed{ƒ(V_4) \cup ƒ(V_5) \cup ƒ(V_6) = \{5 + 6 . y \mid y \geq 0\}}$$

Note: the three previous steps means that the odds of the set V are present in the all columns of **Table 3** (in the lines $\{a = 1, a = 2, a = 3\}$).

Step 2: Let $v_p \epsilon (D \cap V)$:

$v_a = 3 + 10 . \left(\frac{4^a-1}{3}\right) + 4^{a+1} . n_y = 5 + 12 . n_j$

Let $v_{a+3} = 3 + 10 . \left(\frac{4^{a+3}-1}{3}\right) + 4^{a+4} . n_y$

$v_{a+3} = v_a + (v_{a+3} - v_p)$

$= 5 + 12 . n_j + 3 + 10 . \left(\frac{4^{a+3}-1}{3}\right) + 4^{a+4} . n_y - 3 - 10 . \left(\frac{4^a-1}{3}\right) - 4^{a+1} . n_y$

$= 5 + 12 . n_j + 10 . \left(\frac{4^a . 63}{3}\right) + 4^{a+1} . 63 . n_y$

$= 5 + 12 . (n_j + 10 . 7 . 4^{a-1} + 4^a . 21 . n_y)$

$v_{a+3} = 5 + 12 . n_c$ where $n_c = n_j + 10 . 7 . 4^{a-1} + 4^a . 21 . n_y$

thus, $v_{a+3} \subset V$

Note: The three previous steps means that the odds of the set V are present in the all columns of **Table 3** (in the lines $\{a = 1, a = 2, a = 3\}$).

$ƒ(v_{a+3}) = ƒ[v_a + (v_{a+3} - v_a)]$

$= ƒ[\, 3 + 10 . \left(\frac{4^a-1}{3}\right) + 4^{a+1} . n_y + (3 + 10 . \left(\frac{4^{a+3}-1}{3}\right) + 4^{a+4} . n_y - 3 -$
$10 . \left(\frac{4^a-1}{3}\right) - 4^{a+1} . n_y)]$

$= ƒ[3 + 10 . \left(\frac{4^a-1}{3}\right) + 10 . \left(\frac{4^a . 63}{3}\right) + 4^{a+4} . n_y]$

$= ƒ[3 + 10 . \left(\frac{4^{a+3}-1}{3}\right) + 4^{a+4} . n_y]$

$ƒ(v_{a+3}) = 5 + 6 . n_y$ thus, $ƒ(v_{a+3}) = ƒ(v_p)$

Conclusion 6: there are infinity odds $v_{a+3u} \epsilon (D \cap V)$ produce the same r_b where $r_b \epsilon R_b = 5 + 6 . n$ as output when performing the function to them: $ƒ(v_{a+3u}) = r_b$ where $u \geq 0$.-

Note: In **Table 3**, the odds of set V are present in the all columns of **Table 3**. each odd of the set V -three lines below it- there is another odd of the set V.

$$\begin{cases} ƒ(V_1) \cup ƒ(V_2) \cup ƒ(V_3) = \{1 + 6 . y \mid y \geq 0\} \\ ƒ(V_4) \cup ƒ(V_5) \cup ƒ(V_6) = \{5 + 6 . y \mid y \geq 0\} \end{cases} \quad \text{thus,} \quad \bigcup_{i=1}^{i=6} ƒ(V_i) = R$$

Conclusion 7: established by **Conclusion 5** and **Conclusion 6** there are infinitely many elements $v_{a+3l} \in$ V in $B \cup D$ that produce the same output $r \in R$ under the Collatz function: $ƒ(v_{a+3l}) = r$ where $l \geq 0$. As we know V $\subset \{B \cup D\}$.

6. prove that iterative application of function ƒ on set *C* converges to set *V*:

$C = \{3 + 4 . n \mid n \geq 0\}$
$ƒ(C) = \{5 + 6 . n \mid n \geq 0\}$
The set $ƒ(C)$ can be partitioned into two disjoint subsets based on modulo 12

$$ƒ(C) = V_1 \cup C_1 \begin{cases} V_1 = \{5 + 12 . n \mid n \geq 0\} \\ C_1 = \{11 + 12 . n \mid n \geq 0\} \end{cases}$$

Note that C_1 is a subset of C, since:
$C_1 = 11 + 12 . n = 3 + 4 . (2 + 3 . n) \Rightarrow C_1 = 3 + 4 . n_q$ where $n_q = 2 + 3 . n$, hence $C_1 \subset C$
This subset relation is recursive. Applying the function $ƒ$ to the subset C_1 will, in turn, produce a new pair of sets $ƒ(C_1) = V_2 \cup C_2$, where $C_2 \subset C$ and $V_2 \subset V$.
This process defines a general recursive relation. For any derived set $C_i \in C$, one application of the function $ƒ$ yields:
$ƒ(C_i) = V_{i+1} \cup C_{i+1}$, with $C_{i+1} \subset C$ and $V_{i+1} \subset V$
By applying function $ƒ$ iteratively to the resulting sequence of subset C_i, we ultimately obtain elements that belong to the set V. Formally, for any starting element $c \in C$, the iterative process converges to an element $v \in V$:

$$\boxed{\forall\, c \in C \mid \exists k \in \mathbb{N}: \lim_{k \to \infty} f^k(c) = v}$$

Conclusion 8: Applying the function to any element of the set $C = 3 + 4 . n$ generates a finite sequence of odd numbers. This sequence consists of elements belonging to C itself, and its terminal element-obtained by the final application of $ƒ$- is an odd
$v \in V$: $V = 5 + 12 . n$
Conclusion 9: Therefore, the iterative application of the function $ƒ$ on the set C produces the entire set $V = 5 + 12 . n$ as established in **Conclusion 6**, this set V is fundamental and is present in all columns of Table 1 and **Table 2**.
This result is demonstrated by considering a subset of C, for example $C_z \subset C$:
$C_z = \{3 + 8 . n \mid n \geq 0\}$, which under the function $ƒ$ maps directly to V:

$$\boxed{ƒ(C_z) = ƒ(3 + 8 . n) = \{5 + 12 . n \mid n \geq 0\} = \text{V}}$$

Thus, any sequence originating from an element $c \in C$ under iterative application of f will follow a path within these sets until it converges to an element of V:

$c_1 \xrightarrow{f} c_2 \xrightarrow{f} c_3 \xrightarrow{f} \dots \xrightarrow{f} \dots \xrightarrow{f} c_z \xrightarrow{f} v$ where $\{c_1, c_1, c_1, \dots, \dots, c_1\} \in C \ and \ v \in V$

7. Global convergence:

7.1. Proof that all odds converges to 1 under the Collatz function:

Step 1: framework and Definitions

Let $\mathbb{N}_{odd} = \{2.x + 1 \mid x \geq 0\}$ we prove convergence via the invariant set

$V = 5 + 12.n$

and its dual roles:

- **Input role (V):** All elements of V as **starting points** for Collatz iteration.
- **Output role (V):** All elements of V serves as **terminal points** in Collatz iteration chain initiated from set V.

Step 2: Generation from V to V and 1:

1) **Generation from V to V:**
   - If any value in the sequence belong to set C then, by **conclusion 8** the chain will eventually reach some $v_t \in V$, $\lim_{k \to \infty} f^k(v_i) = v_t \in V$.
   - As shown in **conclusion 7**, there are infinitely many elements $v_i \in V$ such that $f(v_i) = v_t$ such that $v_t \in V$.
2) **Generation from V to 1:**
   - If no value in the sequence belong to C, the chain must decrease monotonically (since for $b \in B$ and $d \in D$, ($f(b) < b$ and $f(d) < d$) until it reaches 1).
   - As shown in **conclusion 7**, there are infinitely many elements $v_i \in V$ such that $f(v_i) = 1$, (where $r = 1$)

Conclusion 10: there exist a subset V_1 such that

$$\boxed{\forall\, v \in V_1 \mid \exists k \in \mathbb{N}: f^k(v_i) = 1 \ \Rightarrow V_1 \ converges \ to \ 1, \qquad |V_1| = \infty.}$$

$$v_i \begin{cases} \xrightarrow{f} r_1 \xrightarrow{f} r_2 \xrightarrow{f} \dots \xrightarrow{f} \dots \xrightarrow{f} c_z \xrightarrow{f} v_t \\ \xrightarrow{f} v_t' \\ \xrightarrow{f} r_a \xrightarrow{f} r_b \xrightarrow{f} \dots \xrightarrow{f} \dots \xrightarrow{f} 1 \\ \xrightarrow{f} 1 \end{cases} \left| \begin{array}{c} f^k(v_i) = v_t: v_t \ (terminal \ point): \exists r \in C \\ v_t' \ (terminal \ point): f(v_i) = v_t' \\ f^k(v_i) = 1: 1 \ (cycle): \nexists r \in C \\ 1 \ (cycle): f(v_i) = 1 \end{array} \right.$$

Conclusion 11: $\forall\, v_i \in V \mid \exists k \in \mathbb{N}$: $\lim_{k \to \infty} f^k(v_i) = \begin{cases} v_t \in V \\ 1 \end{cases}$ thus, $\boxed{\lim_{k \to \infty} f^k(V) = V \cup \{1\}}$

Table 4: applying the function to the set V until obtain either 1 or elements of the set V

$f(R\backslash V)$															
$f(R\backslash V)$															
$f(R\backslash V)$														v_t	
$f(R\backslash V)$														…	
$f(R\backslash V)$				161		17								…	1
$f(R\backslash V)$				107		11					161			…	…
$f(R\backslash V)$				71		7					107	233	17	r_3	…
$f(R\backslash V)$		5	17	47		37		101	29	1	71	155	11	r_2	r_b
$f(V) = R$	1	13	11	31	5	49	29	67	19	85	47	103	7	r_1	r_a
V=5+12n	5	17	29	41	53	65	77	89	101	113	125	137	149	v_i	v_i

Note: we have marked the elements of set V in gray.

7.2. proof of convergence to 1 for elements of set V via set $R\backslash V$:

Theorem

Let $V = \{5 + 12.n \mid n \geq 0\}$. Every element $v \in V$ converges to 1 under iterative application of the Collatz function f, with all intermediate values in its convergence path belonging to the set $R\backslash V$.

Proof

The poof is structured into three main steps:

Step 1: Existence of an infinite convergent subset

By **Conclusion 10,** there exist an infinite subset $V_1 \subset V$ whose elements all converge to 1 under the Collatz function f, formally:

$$\boxed{\forall\, v \in V_1 \mid \exists k \in \mathbb{N}: f^k(v) = 1 \;\Rightarrow V_1 \; converges\; to\; 1, \qquad |V_1| = \infty.}$$

Step 2: inductive propagation of convergence

A critical property of the set V is that $\lim\limits_{k\to\infty} f^k(V) = V \cup \{1\}$. We use this to propagate convergence through the entire set.

- Base case: $V_1 \subset V$ where V_1 converges to 1, $|V_1| = \infty$.
- Since $\lim\limits_{k\to\infty} f^k(V) = V \cup \{1\}$, this implies that there exist a set V_2 where $\lim\limits_{k\to\infty} f^k(V_2) = V_1$, it follows that V_2 converges to 1.

This logic forms the inductive step:

$$\begin{pmatrix} V_i \text{ converges to } 1 \\ \lim\limits_{k\to\infty} f^k(V_{i+1}) = V_i \end{pmatrix}$$

$$\Rightarrow V_{i+1} \text{ converges to } 1,$$

$$\boxed{\forall\, v \in V_{i+1} \mid \exists\, i, k \in \mathbb{N}: \lim_{k\to\infty} f^k(V_{i+1}) = 1, \quad |V_{i+1}| = \infty.}$$

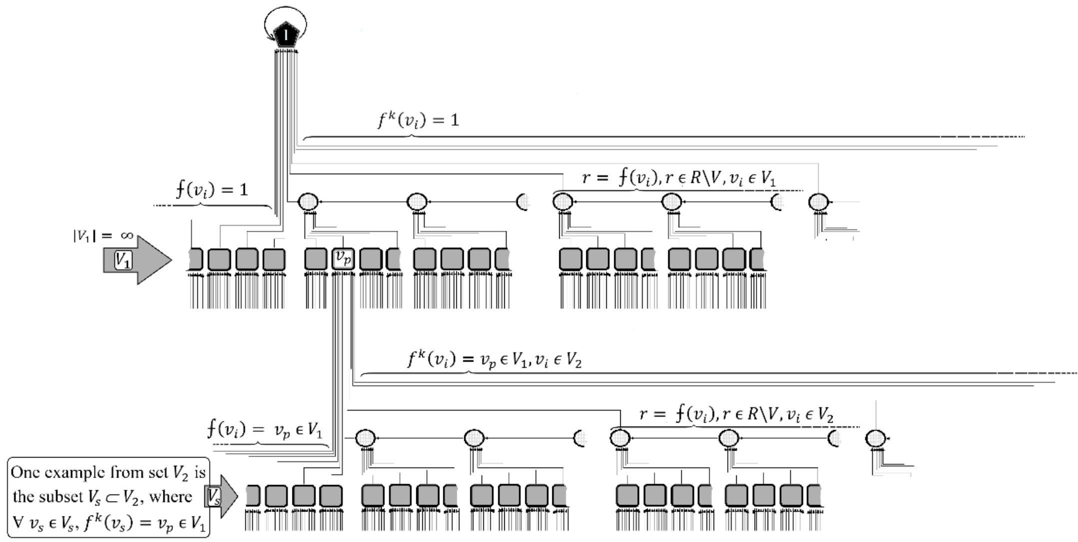
step 3: Universal Coverage of set V:

Since the union of all these subsets covers the entire set V: $\bigcup_{i=1}^{\infty} V_i = V$

we conclude that **every** element of set V converges to 1. Furthermore, the convergence paths of these elements traverse the intermediate values in set R, implying that all elements in set R also eventually converge to 1.

Conclusion: $\boxed{\forall\, r \in R \mid \exists k \in \mathbb{N}:\ \lim_{k\to\infty} f^{k}(r) = 1}$

scheme 1: mapping V to V with convergence to number 1



Note: When using the elements $v \in V$ as inputs from the set V_n, referred to as (v_i), these same elements act as outputs for the set V_{n+1}, where they are denoted as (v_t).

7.3. coverage of all integers:

- **Extension to the set M:**
  By **(Conclusion 3):** Performing the function to the set $\{M = 3 . x \mid x \in \mathbb{N}_{odd}\}$ produce odds belong to the set R: $f(M) = M_1: M_1 \subset R$. Thus, all the odds of the set M converge to number 1 when performing the function to them.
  $\mathbb{N}_{odd} = R \cup M \quad \Rightarrow \quad \boxed{\forall\, x \in \mathbb{N}_{odd}, \text{there exists } k \text{ such that } f^{k}(x) = 1}$
- **Extension to even integers:**

By **Conclusion 1**: The odds of the set $\mathbb{N}_{odd}$ converge to number 1 when performing the function to them. thus, that all even integers converge to number 1 when performing the function to them.

Final conclusion

Through a step-by-step analysis, we demonstrated that applying the function to all integers eventually leads to the integer 1. By proving that specific subsets of $\mathbb{N}$ transition through intermediate sets before converging to 1, we established that the iterative process always terminates. This confirms that the Collatz Conjecture holds true for all integers.

$$\boxed{\forall\, x \in \mathbb{N}, \text{there exists } k \text{ such that } f^{k}(x) = 1}$$

Results: The Collatz Conjecture has been proven true.

References: GUY R., 1983- **Don't try to solve these problems!**, *American Math*, Monthly 90, 35-41.