# The Instanton: a Conspicuous Case of Scale Transition

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**Abstract**. Light is the only physical phenomenon of our experience that transits all three scales of the world we inhabit: infrafinite, finite, as well as transfinite. The concept of instanton accomodates the transition of these scales in space and time. This fact indicates that the quantization might be in fact, the only true law of nature. In this respect the world we inhabit is unique: the Planck's quantization procedure asks for a special fundamental structure of the universe as an optical medium, which must be a Maxwell fish-eye. In order to apply the Planck's procedure of quantization to matter, one needs to extend the electromagnetic properties of the light fields to matter fields. At this juncture, the special relativity aroused a thesis which acts implicitly in all initiatives of theoretical physics: the *length* – which is a differentia of the concept of matter – is identical to the *distance*, which is a property of the vacuum concept, and can be revealed only by light. A proper usage of this thesis leads to the idea of Yang-Mills fields: the equivalents in matter of the electromagnetic fields from vacuum. The planetary atomic model, as the fundamental structure of the physical world is considered from this point of view. Consequences are suggested and/or described; some of them are pursued up to their conclusions, some remain at the level of logical speculations.

**Keywords**: Maxwell fish-eye, Bäcklund transformation, the instanton, Feynman's interpretation concept, de Broglie's light ray, holographic phenomenon, the eightfold way, quarks, protective measurements, Yang-Mills fields, gravity as a protective field, Bartolomé Coll's universal deformation

## A Profession of Faith

One can hear or, in the spirit of modern-time communications within the mankind, rather read from time to time, quite an alarming complaint, like the one once posted in cyberspace by theoretical physicist and philosopher Egbertus P. J. de Haas, in an old personal web page. As many others alike, this web page seems to have meanwhile vanished. However, by the courtesy of its author, we had the good chance of extracting the following words of warning and, fortunately, we were clever enough not to waste it. Quoting, therefore:

> The physicists who defend themselves by saying that quarks aren't real but *positivistic constructs of the mind*, with the sole intention to connect the observable, are forgetting the functioning of the human mind, *especially in the social context of education*. Pupils and students *start their lives in an Aristotelian-like realist philosophy* because that is the most common sense way to look at things and to interpret the stories of parents and schoolmasters. Adolescents learn to believe their teachers and to accept the reality of what is written in their schoolbooks. When we teach them in our science classes that stuff is made of molecules that molecules are made of atoms, atoms in their turn consist of electrons and nuclei, nuclei of protons and neutrons, protons and neutrons of quarks, then we imprint a view of what really is, in their absorptive minds. If education will continue to do so for decades, half the world will believe that quarks really exist. *Education can turn the biggest nonsense into common sense reality, by sheer force of authority and repetitive indoctrination. So education can make quarks as real for the one as it makes God real for the other. Once we start talking of quarks in our science classes, we make them part of reality, even if the positivist scientists who invented and used them claim a different view*. (E. P. J. de Haas, Philosophy, *our Italics*; *warmhearted thanks are due to Mr. Paul de Haas for kindly providing us once the old version of his web page containing this excerpt*)

We subscribe unreservedly to these conclusions. And with good reasons at that: while the scientific creation of a concept is the result, even the expression we should say, of that noble *fundamental Cartesian doubt*, once it reaches the public – if it ever does, in the first place – that concept appears as a dreadful irrefutable truth, mostly because it is endorsed by the "sheer force of authority", to use Paul de Haas' own words. The primary reason for this situation is that the public at large *has no access* to science, least of all to physics, so that, simply put, they are *incapable of doubting*. That is, the layman takes physics for granted, just the way it is served, with all the necessary cutlery at that, by some 'authority'. Paul de Haas seems to imply that the culprit is, by and large, to be found in the fact that society lives in that Aristotelian environment whose spirit is implanted in man's mind by *education*. Again, this is true, indeed! And unfortunate, we should add, especially for the theoretical physics. However, no one seems to realize it, for not too many people see it that way…

In fact, one may utter, in the 'layman's style', as it were: so what? what is so alarming about this?! The answer is easy for *all* to see and understand, from the above excerpt: the Aristotelian environment, *just like religion* for that matter, severs any *consideration of ethics*, to which an individual has access only by a *higher education*. This kind of education is, in the realm of science, the only social environment which gives the man a *slight chance* to become capable of doubting in a Cartesian acceptance. The higher education, therefore, is socially speaking that 'environment', if we may be allowed to say so in order to be in line with Paul de Haas' expression, which would allow one to openly and conscientiously accept that old… *dubito ergo cogito… ergo sum*. Having no higher education is not alarming by itself, indeed, but has harsh consequences on the very human condition, since the man emits judgments and, worse even, *claims their indisputability*, just because he has the chance to speak, as it were, especially in the present social environment.

Even though the higher education is not a sufficient condition by itself – as we have already mentioned, the chance it gives to a man is quite thin, since, according to Gustave Le Bon, the layman, regardless of his/her degree of education, closely reproduces the psychology of the crowds (Le Bon, 1906) – it is at least necessary in the formation of a *free individual*: as long as the individual *remains* in the Aristotelian environment, whereby, in the modern times especially, the *education* is regularly identified with a *training for social purposes*, he is obviously not free. There is no better expression than the one once articulated by Steven Weinberg, *may he rest in peace*!, with reference to religion. Quoting:

> … *with or without religion*, good people would tend to behave well and bad people would do evil things, but the *peculiar contribution* of religion throughout history has been *to allow good people do evil things*. One of the great achievements of science has been, not to make it impossible for intelligent people to be religious … but at least to make it possible for them *not to be religious*. *We should not retreat from this accomplishment*. [(Weinberg, 2001); *our emphasis*, *n/a*]

With due diligence on defining what, in our case here, 'good' and 'bad' may mean, a point upon which the religion has no issues at all, one can say, with Paul de Haas, about science in an Aristotelian context exactly what Steven Weinberg said in this excerpt about religion. And, when it comes to the very human roots of this situation, the problem with religion is exactly the same as that with science. For, the *evil things always reside in man's mistaking the products of own imagination for reality*, and this kind of reality is the only one that crowds can handle, to a certain extent, of course. This fact is independent of the social condition of the man, good and bad whatsoever, highly educated or not at all: it is our innate, *original sin*, as it were, and symptomatically, it is the only lever to move the crowds.

In science, though, unlike the case of religion, this condition appears to have become critical lately and, as the excerpt above of Paul de Haas shows, the situation is generated mainly by *confounding the mind creations with reality*. However, in the case of true science – that science that serves *the man* on his way to freedom, not *the social individual* who, in fact, assumes a fundamental chain of training in order to be enslaved into serving the society – there is a better chance of a 'right management', if we may say so, by mathematics. Among many quite popular quotes of good humor of the great mathematician Gregory Moisil – *may he rest in peace*! – there is one quite… serious, to be reckoned with:

> All that exists as *correct thinking* in this world *is either mathematics* or *liable to assume a mathematical form* (*our emphasis, n/a*)

Perhaps the original saying does not sound, *verbatim*, quite like that: on one hand it is quoted from the memory of an old – and only occasional, we have to admit it frankly – student of the great teacher. However, also adding to distortion may be the English rendition intended to put forward the gist of the saying: it may have contributed too, for a little departure from the original wording. In any case, the truth of this utterance is beyond any doubt: this is what the history of natural philosophy, and especially the modern physics – as the message of Paul de Haas shows – plainly confirm. To wit: no one questions, for instance, the reality of a *material point*, even though such a thing does not exist anywhere in our daily experience. However, everybody would agree that such a concept is only a *fiction* helping us in understanding the world we inhabit, and that this thinking is correct from a mathematical point of view…

… Which, unfortunately, is not the case of *quarks*, for instance. The reason for this is that none of the properties of the quarks *belong to our daily experience* or can be extrapolated starting thereof: they appear to have the same right to claim a real existence as, *e.g*. the scenario of a dream of Chimpden Earwicker, before he woke up to inspire James Joyce, or any one of the fantasies of E. T. A. Hoffman, for that matter. Yet, the great physicist Murray Gell-Mann, *may he rest in peace*!, the one who coined the term «quarks», had no problem whatsoever in declaring openly that "… the number three fitted perfectly *the way quarks occur in nature*" (*The Quark and the Jaguar*, W. H. Freeman & Company, New York, p. 181).

These last observations suggest a key towards understanding the problem and, in our opinion, even to offer a solution, by and large *socially accessible*, which, therefore can be taken as part of an Aristotelian environment: all mind creations of the modern physics are controlled by mathematics. After all, the quarks themselves are an excellent example such a mind creation. However, this very mathematics needs to be itself controlled by a sound natural philosophy derived from our experience, which is almost totally… absent today, to say the least. An excuse may be invoked here, in order to assume that such a control is impossible, and thus to repudiate the idea as belonging to utopia: the quarks' reality is beyond the reach of our senses, the only criterion that provides the layman with a slim chance of rightly judging a reality. Nonetheless, there is also a good chance that this criterion is misplaced.

Indeed, in keeping with the example of the material point above, one can figure out that everybody can imagine, even in an Aristotelian 'environment', that a material point can, indeed, adequately portray an isolated body. Perhaps many of us can go, without any mathematics whatsoever, even as far as connecting this concept to a sound reality: the more distant is a body, the more *realistic* is the image of material point we make out of it. The hard part would be to realize that the quarks are… material points. As we said, not too many among us seem to have realized this, since the modern mathematics does not appear to have any channel of communication with experience *at all levels*. Like, for instance, the old natural philosophy, which was closer to an Aristotelian environment, due to its closeness to our common experience ... The 'occurrence of quarks in nature', for example, as a *sure* phenomenon, is just a mathematical fact, and not too many among us have the chance to see in it the presence of a classical material point…

… and thus, it occured to us, that in presenting the modern physics to the public, no one has ever undertaken the burden of doing this from an appropriate natural philosophical point of view. To wit: from the very same point of view of that classical Aristotelian environment, whereby the image of the world is implanted into our minds by the 'sheer force of authority and repetitive indoctrination', indeed, but which, unlike quarks, we *can* 'absorb' in our minds due to the daily experience. More to the point, the physics is basically presented nowadays as part of technology. Along our studying experience in physics, we grew gradually aware that a concern of presenting physics within an Aristotelian atmosphere is, indeed, dearly missing, especially in the theoretical physics. And not just for pure educational purposes, but even for a *proper understanding* of the physics at large… So, we took advantage of the occasion presented by the *concept of instanton*, in order to show – and not just 'laterally', as it were, but by historically significant examples of the kind that made the modern physics – that there may be a way to scientifically fill in after all, even for such a crucial demand of our times!

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## Chapter 1 Under Louis de Broglie's Guidance

The key points of the idea of interpretation in wave mechanics is that, in addressing the concept of material particle – the necessary concept serving for interpretation according to wave-mechanical precepts – we have to use the square of a wave function. The routine opinion, made over time in practicing the classical physics, strongly indicates that this square represents a density. However, when carrying out an interpretation procedure, as its concept stands today, this density is referring to many, quite different things. We have, for instance, the ensembles of particles, as in the case of a Madelung-type interpretation of the wave function, in which case the square of the wave function is the *number density of particles*. At the other end of the possibilities of interpretation we have the ensembles of some representatives of particles, as in the case of Born-type interpretation, whereby the square of the wave function is referring to the *density of probability of presence of the particle* in a location at a time. And we have a host of cases in between, whereby the wave function can be even multidimensional, and therefore the density is actually a *quadratic form* involving some representative fields. In any of these cases the density serving the concept of interpretation of the wave mechanics by a physical procedure, has hardly anything to do with the classical Newtonian concept of density: that is, with *that characteristic describing the manner in which the matter fills the space at its disposal*.

This problem – that is, the discrepancy between the Newtonian density and the interpretative densities of modern physics – is not acknowledged in physics as such, and the fact that it was even addressed at a time by Louis de Broglie is a wonder by itself. Regardless of its academic value, when the issue has been documented, one could see at once that it contains a lot of references to one of the most important problems of the last century: *that of the scale transitions*. Louis de Broglie was the only one we are aware of, who ever tried to prove the assertion regarding the square of a genuine wave function of the physical optics in classical terms (de Broglie, 1926b,c). Along his path to proof, de Broglie followed the idea of a theory of ray optics in a remarkable analogy involving a special case of fluids. To wit: *a light ray to him is the analogous of a capillary tube*, whereby the wave surface is the analogous of a portion of the physical surface of the fluid confined within the tube [see for details (Mazilu, 2020); especially the Chapter 2 of that work].

In the framework of a modern interpretation concept, the fluid in a capillary tube is, physically speaking the only one moving 'by itself', as it were, under the *surface tension*. Of course, the analogy between a ray of light and the capillary fluid works as long as we are able to find a mandatory *wave surface*, playing the part of the *fluid surface* that moves inside the capillary tube. In short, Louis de Broglie has shown that the physical optics turns out to be an easy case of analogy here, inasmuch as the mathematics allows us to say that the wave surface of light can play the part of the surface of a fluid of particles 'flowing' along the ray. The hard part of the problem of this analogy is finding the reason of *capillarity* of the wave, which in the classical mechanical case is provided by the properties of matter: the surface tension of the fluid, which acts along the wall of the tube. Anyway, one can grasp, from this quite succint presentation, the idea of density used by de Broglie in his presentation: *it is a number density of particles*.

Along an extensive study of ours, whose recent products are a work dedicated to quantization of matter in the de Broglie's take (Mazilu, 2020), and to the first application of quantization, in Planck's original acceptance, to matter (Mazilu, 2022), we came to the surprising conclusion that the whole physics is, in fact, a matter of a *grand analogy*, as we would like to call it, to which the concept of scale transition (Mazilu, Agop, & Mercheș, 2019, 2021) is the driving engine, so to speak. The present work is all about elaborating on the idea of such analogy, following the Louis de Broglie's line as a guide, however, with the Max Planck's steps towards quantization applied to the case of matter. To de Broglie, the steps to quantization of matter have, apparently, nothing in common with the original Planck's quantization except the Planck's constant, which is declared universal by default, as it were: it allows us to construct the quantum strictly based on the concept of frequency, as connected to energy. After all, de Broglie has set the grounds for what later became the *second quantization* procedure and, with this, Planck's procedure remained in the back, so to speak, as the *old quantization* procedure.

Any procedure of quantization after the times of Planck, just repeats the leitmotif of the second quantization: *the Planck's constant is universal*. Whence a host of works, theoretical as well as experimental, trying to justify and even give interpretation to this constant. Even after the year 1967, when Harold Ralph Lewis has discovered an *invariant action* (Lewis, 1967), destined to be a 'generalization of the Planck's constant', as one often hears ever since, this constant has not lost its initially assumed position of universality, in any quantization regarding the matter. In this respect, an observation can be made (Mazilu, 2020), for guidance and encouragement of our appraisal of the concept of quantization: the class of invariants discovered by Harold Lewis, serving for 'generalization', *includes the classical Newtonian forces*, which, therefore, may appear as consequence of 'quantization law'. This further suggests that the quantization is a fundamental law of nature, of importance going beyond any axiomatics. A genuine 'quantum' different from Planck's can also be found among these invariants: it is what we have called the *Procopiu quantum* (Mazilu, 2022). The Stefan Procopiu's quantization procedure (Procopiu, 1913) is the only procedure following *faithfully*, for the case of matter, the Planck's procedure of quantization from the case of light. And the Procopiu's quantum is an exact Lewis invariant, just like any Newtonian force. This occurrence, we think, obligates us to a review of the idea of quantum of action, and an extension of it by the concept of invariant.

Since the physical optics was the realm of the first quantization and, as such, this kind of optics was the preferred field of reference of Louis de Broglie himself, it is only natural that we should start with it in a time history of incentives leading to the wave mechanics. However, we have to admit that the grand analogy we were talking about before, was implicitly present in physics from its very beginning. One can say, for instance, that from this point of view, the fact that de Broglie found a way of quantization in matter only guided by the special relativity is not quite fortuitous. To wit: as we shall show here, the whole relativity, as a standing physical theory is, indeed, a matter of analogy involving the idea of surface as it was conceived by Louis de Broglie in his construction of the concept of physical ray. This idea of surface, taken under the suggestive name of *instanton*, for reasons of scale transition to become clearer as we go on with this work [see (Mazilu, 2022) for our definition of the concept], is a key point of the modern theoretical physics. And we are set here to show the mathematical construction of a few geometrical models necessary in understanding the concept of instanton, and settle their place within the modern theoretical physics. This first chapter of the present work is dedicated to a brief update concerning the incentives we just mentioned above, that would once lead to the modern wave mechanics.

### 1.1 The Light Along the Line of Max Planck

In order to build the first quantum theory of light, Max Planck was compelled to invention, among a few others, of a *concept of resonator* [(Planck, 1900); see also (Planck, 1914), especially Part III, Chapter III, and Parts IV & V of the book]. It is important to insist right here on the Planck's reasoning, inasmuch as this reasoning is addressed to those *properties of matter* that, during the process of quantization of light, needed to be taken in consideration, explicitly we should say. And when we say 'explicitly' here, this qualification should not be taken as a pleonasm: after all, can one say, something cannot be done otherwise but explicitly. It should be taken under the meaning: *a physical structure*, *producing and absorbing light*. Fact is, and we insist on it here, that such a consideration was cut short 'halfway to completion', as it were, by the laws of Kirchhoff referring to light as a thermodynamic phenomenon, those very laws that laid grounds for the light quantization process. And, as the course of modern physics plainly shows, we need to reevaluate that old concept.

To start with, the resonator was conceived by Planck as an oscillator in ether. We need to lay stress on this: the *quantization of light* had always in the background an idea of *quantization of matter*. It was just natural, inasmuch as the light confined to a Wien-Lummer enclosure – the usual experimental device serving to study the thermodynamics of light – behaves indeed very much in the way described by the Kirchhoff's law of thermal equilibrium between radiation and matter. And along his study of the radiation problem, Planck felt compelled to make the statement of this law even more precise. Quoting:

> … This law states that *a vacuum* completely enclosed by reflecting walls, in which *any* emitting and absorbing bodies are scattered in *any* arrangement whatever, *assumes in the course of time the stationary state of black radiation*, which is *completely determined* by *one parameter* only, namely, *the temperature*, and in particular does not depend on *the number*, the *nature*, and the *arrangement* of the material bodies present. Hence, for the investigation of the properties of the state of black radiation *the nature of the bodies which are assumed to be in the vacuum is perfectly immaterial*. In fact, it does not even matter *whether such bodies really exist somewhere in nature*, provided their existence and their properties are consistent with the laws of thermodynamics and electrodynamics. If, for *any special arbitrary assumption* regarding the nature and arrangement of emitting and absorbing systems, we can find a state of radiation in the surrounding vacuum *which is distinguished by absolute stability*, *this state can be no other than that of black radiation*. [(Planck, 1914), pp. 135 – 136; *emphasis added*, *n/a*]

Notice the definition of *vacuum*: here it is identified with the one supporting light, more precisely, the thermodynamic light. The phrase: 'bodies which are assumed to be in the vacuum', leaves no doubt about that, but suggests more. We shall revisit this definition later, in matters regarding the quantization of the light itself, but in a larger theoretical environment, outside thermodynamics. Then the terms 'vacuum', 'light' and 'matter' will be delineated more precisely. The statement: 'temperature as the only parameter' did not raise then, and, actually, still does not raise nowadays any doubts of correctness within scientific community. And this, in spite of the fact that the *Wien's displacement law* – the physical law that 'ratifies' every theoretical spectral density of energy of radiation – explicitly shows that there are at least two more parameters to be considered in a radiation law: the *spectral density of radiation* and the *frequency* of that radiation. However, these parameters could not reach in physics the condition of absolute temperature: none of them was ever conceived as *statistic*, except perhaps in their quantitative evaluation, which in physics does not count as a valid definition of a concept. Let us go over to some preliminary details on this issue.

According to Wien's displacement law, any radiation law – under this last name one usually understands *a functional expression representing the spectral density of radiation* – is expressed as a function of only *two* variables: *frequency* itself, and the *ratio between frequency and temperature*. In its most general expression, the displacement law proclaims that such a function must have the general algebraic form:

$$E_\nu \propto \nu^3 \cdot g\left(\frac{\nu}{T}\right) \tag{1.1.1}$$

where $g(\ldots)$ is a universal function of its argument, $E_\nu$ is the spectral energy density of thermal radiation, corresponding to frequency $\nu$, and $T$ is the absolute temperature, as defined for a classical ideal gas, *i.e.*, *as a sufficient statistics*. Therefore the frequency can by no means be neglected as a parameter: it is at least for this reason that we cannot say 'temperature is the only parameter'. After all, it is its presence that compelled Planck to invent the resonators, and this is an undisputable fact!

However, the point at issue here is that one cannot see *how the frequency can be a statistic*, like the temperature from the case of classical ideal gas: by this the radiation fundamentally contrasts the classical ideal gas that serves for the definition of the absolute temperature. The temperature is associated with radiation based on the reason that in a Wien-Lummer enclosure 'the radiation reaches in time a stationary state', to put it in Planck's own words. The incidental gas' physical description *involves dynamics*, while the radiation only has a *purely energetic* description. There are notable consequences of this situation, two of which are worth mentioning right away, before we continue to explore Planck's own ideas on quantization.

Samuel Bruce McLaren, the one who, using the words of Harry Bateman, "heroically gave up his life (*in the WWI*, *a/n*) that others might live" (Bulletin AMS, Vol. 32, No. 2, 1926, p. 175), suggested a radical attitude. Namely, along the idea that in the Fresnel's physical theory of light a dynamical principle is only incidental, so that we need to give up a significant classical-mechanical approach. Quoting:

> To *save* the "æther" *it is necessary to give up the classical mechanics*. This paper shows that the theory of radiation can proceed *without the principle of minimum action*. A formula for the complete radiation naturally suggested is
>
> $$E_\lambda = 8\pi R\cdot\theta\cdot\lambda^{-4}\cdot\left\{1+\kappa_1\cdot\lambda\theta\cdot\left(e^{\frac{\kappa}{\lambda\theta}}-1\right)\right\}^{-1} \tag{1.1.2}$$
>
> $\kappa_1$ and $\kappa$ are arbitrary constants. This gives *a result similar to Rayleigh's* for large values of $\lambda\theta$, a result similar to Wien's for small values. [(McLaren, 1913); *emphasis added*, *a/n*]

Here $R$ is the ideal gas constant, $\theta$ is the temperature, and $\lambda$ is the wavelength of light, equivalent to frequency, by the usual formula $\nu = c/\lambda$. This formula satisfies all the classical criteria that made the case for Planck's formula. Incidentally, McLaren mentions especially the case of Rayleigh law of radiation, because this is the only case that can be theoretically validated based on the temperature of radiation as a statistic. In order that these statements may become more clear, we transcribe it in frequency, in order to be able to compare the result with the expression of Wien's displacement law (1.1.1). In such notations the above formula (1.1.2) reads:

$$E_\nu = \frac{8\pi R}{c^3}\nu^3 \cdot \frac{T}{\nu} \cdot \left\{1 + \kappa_1 \cdot \frac{T}{\nu} \cdot \left( e^{\kappa \frac{\nu}{T}} - 1 \right)\right\}^{-1} \tag{1.1.3}$$

For simplicity, the constant $\kappa$ here is supposed to have absorbed the light speed $c$ in its value. Then, obviously, this expression for spectral energy satisfies the basic requirement of the Wien's displacement law, because it is of the functional form given in equation (1.1.1), only with a special form of the function $g(\ldots)$ entering the algebraic expression in that equation. Again, obviously, the spectral density (1.1.3) satisfies the conditions of the two limiting cases taken by Planck as reference in obtaining his celebrated formula: Rayleigh-Jeans' for $\nu/T \to 0$

$$E_\nu \propto \nu^3 \cdot \left(\frac{\nu}{T}\right)^{-1} \tag{1.1.4}$$

and Wien's radiation law

$$E_\nu \propto \nu^3 \cdot e^{-\kappa \frac{\nu}{T}} \tag{1.1.5}$$

for $\nu/T \to \infty$ [see also (Mazilu, 2022), especially §2.1 of that work].

Symptomatically, in connection with this approach of the theory of equilibrium radiation, on the occasion of building his quantization procedure of light, Planck also had to invent *a new type of statistics*, which, in our opinion, was destined to overcome the sufficiency of the absolute temperature as a statistic [(Mazilu, Agop, & Mercheș, 2021), Chapter 2; see also, in this respect, (Mazilu, 2022), Chapter 2, especially §2.3]. Now, in obtaining his formula Samuel McLaren uses exclusively statistical reasons that can be summarized by saying that the energy distribution of thermal radiation is an *exponential distribution*, with no equipartition of energy connected to it, though. However, neither is there a quantization in this case! This very fact may be taken as symptomatic too: apparently, the quantization is tied up with a statistic, and we found that the important thing in this connection may be just *the type of statistics*, which goes beyond the exponential characteristic. Specifically, we also found out that the statistics in quantization is based on the particular class of distribution functions having *quadratic variances* when these variances are expressed as functions of ensemble means. Moreover, while in the case of light the discrete distributions of this type are relevant, in the case of matter we can make the case for continuum distributions (Mazilu, 2010, 2022). It is worth noticing that, according to this idea, the classical Newtonian forces – that is, the central forces with magnitude varying inversely with the square of distance – must be, in fact, the expression of a quantization. In other words, the classical dynamics may be taken as a quantization method, *avant la lettre* as it were, *but in the case of matter*. In fact, this may count, as a reason for the natural-philosophical change in emphasis: it is not the dynamics that is needed in the case of light, but the quantization. Likewise, it is not the dynamics which is fundamental either, to the description of matter, but the quantization as well. Thus, bringing dynamics into question may actually be redundant: *the quantization should be the only law of nature*!

Anyway, although Samuel McLaren remains at the qualitative theoretical stage – we have not verified how ‘similar’ (1.1.3) is to Planck’s analogue, with the two limiting cases of Rayleigh-Jeans and Wien’s laws of radiation – his results are something to reckon with, especially when recalling the fact that the physical theory of Fresnel might have no mechanical interpretation. As we said, in the first place the relation (1.1.3) is correct according to the classical criterion represented in equation (1.1.1); secondly, the Planck’s theory cannot be called into question as a physical theory. At least not now, when so many years have passed since its creation, and it provides so many correct theoretical explanations. After all, it is based on a sound statistics that turned out to be a universal instrument of theoretical physics (Carruthers, 1991).

Moreover, this statistics can be applied in its detailed steps to the case of matter, for an arbitrary physical meaning of the quantization constant, not just action (Mazilu, 2022). This fact is essential for the modern theory of chaos (Gutzwiller, 1984, 1990), a theory that, in contemporary physics, fills in for the missing milestones along the path from the quantization in light towards the quantization in matter. In our opinion, the *idea of chaos* was present in physics even from its modern founding. As Martin Gutzwiller himself, *may he rest in peace*!, puts it:

> Astronomers became increasingly aware of this problem during the last 60 years, but physicists began to recognize it only some 20 years ago. The phenomenon, which now goes under the name *chaos* (*original emphasis*, *a*/*n*), has since become a very fashionable topic of investigation. Innocent onlookers might suspect one more passing fad. I do not think it will turn out that way, though. *Chaos is not only here to stay*, *but will challenge many of our assumptions about the typical behavior of dynamical systems*. Since mechanics underlies our view of nature, we *will probably have to modify some of our ideas* concerning the harmony and beauty of the universe [(Gutzwiller, 1990), p. 2; *our emphasis*, *except as mentioned*, *a*/*n*].

First we must straighten the great physicist on one point: if ‘astronomers became increasingly aware’ is because the chaos is a natural phenomenon. However, if ‘physicists began to recognize it only some 20 years ago’, is because the chaos theory was, in germ only, is true, at the very foundations of the dynamics by Newton: it is contained in the very definition of the Newtonian static forces [(Newton, 1974), Book I, Section II, Proposition VII, especially the Corollary III of that proposition]. After all, this is the very reason why it ‘will challenge many of assumptions about the behavior of dynamical systems’. The present work shows also, among others, what are those assumptions to be challenged, and how. In the end, this was the case of quintessential Bohr’s quantization for matter, that can be taken for guidance: it also asks for the quantization of the *kinetic momentum*, and a deeper consideration of classical dynamics from a modern point of view shows, as we said, that the invention of the classical forces can also be considered as an expression of the concept of quantization [(Mazilu, 2020), §§6.4 and 6.5; see also (Mazilu, 2022)].

Coming back to our main stream of arguments here, the only truly criticizable fact regarding the Planck’s theory remains the one that his statistical method is based on equations involving *the correlation* of two ‘sub-processes’, if we may say so, components of the *light as a stochastic process*, and this characterization asks that the law of radiation should also be a *probability law*. The two ‘sub-processes’ are only inferred, so to speak, from the existence of Rayleigh-Jeans’ and, respectively, Wien’s experimental cases of thermal light, which do not involve the very universal function $g(\ldots)$, required by the Wien displacement law, but only special approximations of such a function. This fact, though, can only call into question the precepts of classical thermodynamics, not the physics itself. To wit, the Planck's correlation between the two cases is based on their *variances*: Planck's result is not referring to a density of probability *per se*, inasmuch as it involves just the variance of a statistical population characterized by that density of probability. However, it is only the density of probability corresponding to this variance that turns out to be universal (Carruthers, 1991). So, the question arises if the Planck's formula *itself* can be taken as referring to a probability, and the answer to this question appears to be positive, at least as we see an answer from the historical perspective.

Indeed, there is also, apparently, no quantization in a significant case which, in our opinion, deserves special attention, the one made a century ago by Irwin G. Priest. This case is not purely theoretical, like McLaren's, but simply regards the experimental data, like in the Planck's case (Priest, 1919); if it turns out to be theoretical, this fact can be assessed, in our opinion, only from the statistical perspective of the problem of thermal radiation. Priest's basic equation for spectral density of light is written in the old manner, also used by McLaren, *i.e.* using the wavelength instead of frequency:

$$E_{\lambda} = D_{1}T^{5}e^{-D_{2}\left(A^{-1/3}-(\lambda T)^{-1/3}\right)} \tag{1.1.6.}$$

or, dividing by the maximum spectral density at a given temperature – which is more convenient for fitting purposes – we have

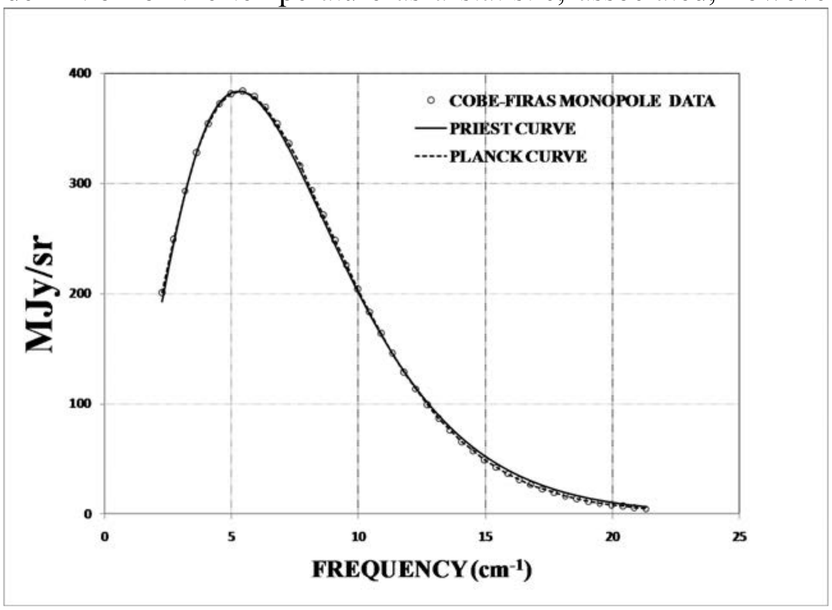
$$\frac{E_{\lambda}}{E_{\lambda m}} = e^{-D_{2}\left(A^{-1/3}-(\lambda T)^{-1/3}\right)} \tag{1.1.7}$$

Here $E_{\lambda m}$ is the maximum of spectral density, depending on the absolute temperature $T$; further on, $A$ and $D_{1,2}$ are constants. We have verified this formula on many cases among classical sets of data that led to Planck's conclusions. However, in order to bring a present-day example of the quality of this function in fitting, we have used the equation (1.1.7) in order to fit the COBE-FIRAS official data with it (Fixsen, Cheng, Gales, Mather, Shafer, & Wright, 1996). The data was first matched with a Gaussian, after a 'regularizing transformation', destined, according to statistical practices, to bring it to symmetry (Box & Cox, 1964):

$$\frac{\nu}{T} \rightarrow \sqrt[3]{\frac{\nu}{T}} \tag{1.1.8}$$

and then plotted against frequency, together with Planck's spectrum. The result is presented in the figure attached below, taken from (Mazilu, 2010). The two curves – Planck's and Priest's – are, at least visibly, not too far away from each other and from the experimental points. What we intend to sustain, is that a case can be made to the effect that the thermal light *can be described*, even in the Planck's manner, *by a probability density*, and that the primeval Einstein's interpretation of light (Einstein, 1905b) is just a particular case of this interpretation. To wit: Einstein used *only the Wien's limit* of the Planck's law of radiation, which is compatible with Maxwell's molecular statistics, and thus *a priori* uses a valid definition of temperature as a statistic. This incident may be taken as showing by itself a classical behavior of light, in case we are tempted to take it as an interpretation. Anyway, the case should raise a warning sign for any corpuscular theory of light, and so much the more for the quantization in the case of matter in general.

We take this juncture as an opportunity to stress an important idea arisen on the occasion of the researches for the laws of thermal radiation. Of course, both of the two limits of a radiation law – Rayleigh-Jeans' and Wien's radiation laws – use the temperature in constructing a thermodynamics of the thermal radiation, but there is a slight difference in the manner they are doing that. The Rayleigh-Jeans limit, of high temperature, is limited to the oscillator model for the waves of light, and uses the idea of the *equipartition of energy* for the ensemble of oscillators representing the light in a Wien-Lummer enclosure. Speculatively speaking it was objectionable – and Planck himself clearly saw this – on the grounds that it has not secured the temperature from the point of view of thermodynamical equilibrium required by Kirchhoff's laws: it simply took the temperature as given by that of an ensemble of oscillators. The Wien's limit of the radiation law, on the other hand, fills in for this omission, by considering the matter inside enclosure as an ideal gas. This is known to be the prototype thermodynamical system, allowing for a definition of the temperature as a statistic, associated, however, with the translational




degrees of freedom. Wily Wien figured out that the light, in its thermodynamical equilibrium with the gas, is just a physical expression of the possibility of transformation of the statistic on translational degrees of freedom of molecules into a statistic on vibrational degrees of freedom characterizing the light. Then the statistics just needs to be transferred directly from the classical one of Maxwell's referring to molecules, to the light waves referring to vibration, considering the thermodynamical equilibrium a fundamental physical process of realizing this transfer.

Obviously, the general traits of this transfer had to be taken in consideration, and they were summarized by Wien through two assumptions helping to solve the case. These assumptions, apparently inspired by the existing theories of the emission of electromagnetic field, were nevertheless directed to fill in for two necessary statistical properties, wanting at the time we are talking about. Quoting:

> 1° In a gas that radiates, *each molecule emits only radiations of a single wavelength*. The *frequency is a function only of the velocity of the molecule*;

> 2° The *intensity of radiation* limited by two very close *wavelengths*, *is proportional to the number of molecules* emitting the oscillations of that period. [(Wien, 1900); *our translation*, *emphasis added*, *a*/*n*]

An explanation is in order here: as long as the Rayleigh-Jeans case was taken as reference, it only proved that the classical equipartition energy simply does not work for an incidental statistical mechanics of radiation. The physics, in this instance, just missed a structural model of the molecule. However, it was almost sure that such a structure must involve electrical properties, for otherwise it would not be able to emit radiation of the electromagnetic nature. Thus, the first of the above assumptions is referring to the overall property allowing us to connect the two possible statistics involved in the Wien's displacement law: the speed of a molecule, known to be connected to the temperature as a statistic, and the frequency of radiation, which could not be seen in epoch as a statistic. The second of the above assumptions just fills in for the completely missing concept of frequency as a statistic analogous to temperature. Notice the idea of 'wavelength proportional to temperature', that proves, in our opinion, to be an essential theoretical concept in the problems of quantization.

Incidentally, let us also take notice of the significant circumstance that in the de Broglie's later theory of wave-particle duality, the fundamental hypothesis is based on relativistic assumptions forced upon monochromatic waves, whereby the frequency does not need to be a statistic: one wave corresponds to one frequency. Nevertheless, de Broglie encountered right away a contradiction between the definition of frequency and its relativistic use in connection with the Einstein's energy. Thus he was compelled to introduce the concept of group of waves in order to fill in for the missing statistic: the group of waves is limited to the waves in 'a very small interval of wavelengths', if it is to express the idea in the manner of Wien in the excerpt above. But, there is another face of this replacement, namely, that of the presence of such a statistic in the very classical idea of material point: the second assumption of Wien excerpted above admits that the intensity 'is proportional to the number of molecules emiting the light of the same period'. This assumption bluntly contradicts the Kirchhoff's laws, which, according to Planck show that the state of stationary radiation "does not depend on *the number*, the *nature*, and the *arrangement* of the material bodies present". It says that, on the contrary, we need to consider the structure of the bodies in the cavity and, when considering such bodies the state of radiation has a statistic associated with the ensemble of such bodies: the period of light.

As expected, there were objections raised on the occasion of the published work of Wien on the functional form of the law of radiation [for a detailed documentation see (Lummer, 1900) and the works cited there]. These objections would regard, indeed, the contradiction with Kirchhoff's law, but they can serve to make the point of departure from this law more precise. The moot point at the time was that, after all, the experimental data were not fully covered by Wien's radiation formula: it covered just 'the other end' of the spectrum radiation, as it were, complementary to the part covered by Rayleigh-Jeans radiation formula. However, Wien's answer to the critique by Lummer and Pringsheim contains a valid statistical-theoretical point, that needs to be considered as such and, moreover, needs to be transformed into a missing physical theory. Quoting:

> The hypotheses making the basis of this demonstration (*see the excerpt right above*, *a*/*n*) are, for the moment being, not confirmed by some other facts. But the objection raised by MM.

> Lummer and Pringsheim against my proof does not seem well-founded to me. MM. Lummer and Pringsheim state that the self-establishing radiation in a closed space *must be independent of the number of particles emitting radiations*, since *a single particle would suffice* (*according to Kirchhoff's laws*, *a/n*) *in establishing the regime of radiations*. This opinion is erroneous, for *setting up the equilibrium of radiations is a consequence of the second principle of Thermodynamics*, and *this principle is not valid but for a large number of molecules*. *It cannot be applied to a single molecule of gas*. [(Wien, 1900); *our translation*, *emphasis added*, *a*/*n*]

In other words, the problem of radiation is a clear place to exhibit the fact that the temperature is not a *sufficient statistic* even for the kinetic energy of molecules of the gas. The Maxwell demon, to mention just one renowned theoretical device based on sufficiency, is not a valid physically concept! According to its definition (Fisher, 1922, 1925), such a statistic – *i.e*. a sufficient statistic – *is independent of the size of a sample used in measuring it*. Assimilating the emission of light with a statistical measurement does not make sense for the temperature on a sample of *size one*. Whence the importance of the *Planck*'*s entropic approach* in the problem of thermal radiation!

Coming back to Planck, his statement to the effect that 'it does not matter whether such bodies really exist somewhere in nature', it is plainly contradicted today by the very progress of the modern theoretical physics. In order to ilustrate the issue we just remark here – of course, we shall revisit the subject later – that Planck invented the *concept of resonator* only in order to play the part of a 'material body present' in a Wien-Lummer cavity, capable to interact with the light. He felt himself under obligation *to realize* a kind of equilibrium *equivalent* with the thermodynamic equilibrium required by the Kirchhoff's law: without such an equilibrium, the absolute temperature would make no sense for light. In hindsight, this invention appears as a replica of another remarkable invention in the history of the natural philosophy: *Newton*'*s invention of the concept of forces*, whereby the equilibrium in question was an equilibrium of forces. And, even though the Kirchhoff's law stipulates that, from a thermodynamical point of view the matter's structure is inconsequential, Planck understands this as a *freedom to choose any structure one wants*. The reason is that, in order to define the temperature of radiation, one needs to have an equilibrium of some kind between matter and radiation *inside the Wien-Lummer enclosure*. Only then we may have a temperature of radiation, which can simply be taken as the temperature of thermodynamical equilibrium between matter and radiation: it is *the temperature of the matter in equilibrium with it*. However, from among the physical systems capable of interacting with the radiation, and also liable to have a definite temperature, *only the oscillators are convenient*. If the law of equipartition of energy is in force, they lead to the Rayleigh-Jeans radiation law. That is why the definition of Planck for these oscillators takes full advantage of the freedom offered by Kirchhoff's law, indeed, but also involves a vacuum structure that became conceptually obvious only in later times [see (Marciano, 1978)]. Quoting:

> Since, *according to this law*, (*Kirchhoff's*, *a/n*) *we are free to choose any system whatever*, we now select from all possible *emitting* and *absorbing* systems the simplest *conceivable* one, namely, one consisting of a large number N of similar *stationary oscillators*, each *consisting of two poles*, *charged with equal quantities of electricity of opposite sign*, which may move relatively to each other on a *fixed straight line*, *the axis of the oscillator*. [(Planck, 1914), p. 136; *emphasis added*, *n*/*a*]

Historically, the lines of theoretical development have been concerned only with the phrase 'according to this law we are free to choose'. However, the very Planck's choice shows, in fact, that *we are not quite so free to choose*: the choice must be made according to the laws of dynamics and electrodynamics. And, on one hand, with the choice of an oscillator, the temperature is not the only parameter: the frequency gets into play too, only the natural philosophy had, at that time, no means of dealing with its concept statistically, but only mathematically. On the other hand, the physical structure of that oscillator turned out to be an essential point of concern of the theoretical physics emerging from quantization. To wit: the idea of an oscillator along a 'fixed straight line' of Planck's, is in contradiction with the laws of dynamics. Going ahead of us: the Kepler motion asks for the Ampère current element model, as it is just natural. Quoting, again:

> It is true that it would be more general and *in closer accord with the conditions in nature* to assume the *vibrations* to be those of *an oscillator consisting of two poles*, each of which *has three degrees of freedom* of motion instead of one, i.e., to assume the vibrations *as taking place in space instead of a straight line only*. Nevertheless we may, *according to the fundamental principle stated above*, restrict ourselves from the beginning to the treatment of one single component, *without fear of any essential loss of generality of the conclusions we have in view*. [(Planck, 1914), p. 136; *emphasis added*, *n/a*]

The history of physics proved, quite contrary, that the generality was lost, and even in a fundamental way at that, but it may be preserved, from dynamical point of view, by a special formulation of dynamics. This formulation contains the equilibrium in a specific way according to the *idea of force characterizing a statics* (Wigner, 1954). However, in order to take heed of this idea, we need to learn some more lessons from the physics of light: specifically, we believe it worth coming back to the essentials of the theory of the most notable counterpart of Wien-Lummer cavity, involving the forces directly. On doing this we can discover quite a few points of theoretical interest (Mazilu, 2023a). First, and the foremost of them, is that, from the physical optics point of view represented by the Planck's choice of the structure of a resonator, the Planck's 'completely enclosed vacuum' must be actually a *Maxwell fish-eye optical medium* (Stavroudis, 1972) [see also (Buchdahl, 1978) and (Chen, 1978) on some essential theoretical-physical connections of the Maxwell fish-eye structure]. It is only this medium that allows for a *meaningful dipolar fundamental structure* accommodating the Planck's resonator. Secondly, the Katz's natural philosophy of charge, allows for a Planck-type quantization *in the matter*, as once realized by Stefan Procopiu [see (Mazilu, 2022) for a full documentation]. The Procopiu resonator is simply a *magnetic dipole*, but the geometrical structure of matter is just the same as that of the Planck's completely enclosed vacuum. For once this conclusion gives theoretical justification to Einstein's idea of application of the quantum statistics to the vibrations of solids (Einstein, 1907, 1912). But there is a more important natural-philosophical consequence that we extract from these facts, serving our present purposes.

Fact is that we need to make the ideea of *vacuum* more precise. The theoretical physics came to recognize the vacuum as a category in the sense of being a *predicate*: the vacuum is specific to any problem involving matter. It represents *the absence of that matter*. Of course, the vacuum is then a kind of *Kantian category*, once it represents the absence of matter. However, so is that matter, if present: a category. Furthers, as we said, Planck's vacuum is understood in the capacity of light, only accidentally containing some matter. Whence, the light must also be considered here as a category. The vacuum in general must be understood as containing matter and light as two opposed categories: the first one defined negatively – vacuum *is not* matter – the second one defined positively – the light *is* vacuum. The physics of last century proved that these two categories – light and matter – go into one another when disappearing, *not* into vacuum. Therefore the vacuum contains both of them, ranking equally, but can be characterized only by comparison with the matter: it is that category of matter missing the qualities of particles, therefore with no possibility of interpretation whatsoever. The category of light only offers the possibility of creating dipoles from vacuum, serving for fundamental structures. This should be, in our opinion, the moral of modern *vacuum tunneling* concept (Jackiw & Rebbi, 1976) connected with the idea of multiplicity of vacua.

### 1.2 The Optical Ray and the Planck's Resonator

The first problem to be solved in the de Broglie's order of things physical, is the construction of a ray in general [(de Broglie, 1926b,c); see also (Mazilu, 2020), *passim*]. Of course, such a problem involves some theoretical requirements for the physical description of a light ray. Optically speaking – the optics being the part of physics to be consulted in this instance – such a description means the knowledge of the equation of progression of the phenomenon of light along the ray. At this juncture the optical properties of the medium supporting light are essential, and among the natural-philosophical prerequisites on such a medium, the *refraction phenomenon* is essential: propagation of light can be mathematically described by the *Euler equation* corresponding to the extremum of *optical path*. This path is defined as an integral [see (Stavroudis, 1972, 2006), which are the works we follow closely for guidance]:

$$I \overset{def}{=} \int n(x,y,z)\cdot\sqrt{\langle dx|dx\rangle} \tag{1.2.1}$$

where $(ds)^2 \equiv \langle dx|dx\rangle$ is the square of arclength of the corresponding *geometrical path*, *i.e.* of the path in the empty space hosting the optical medium of *refraction index* $n(\boldsymbol{x})$. This background space is assumed here to be Euclidean. The variational problem associated with the integral from equation (1.2.1) – the Fermat's principle – provides the differential 'equation of motion along the ray', where the *geometrical path* length is playing the part of a 'time of motion'. Therefore, from this perspective, the motion represents a propagation, whose time is dictated by the geometrical path length from a position to another, as in the classical optics, where the light is assumed to propagate through free space. The differential equation in question is:

$$\frac{d}{ds}\left(n(\boldsymbol{x})\frac{d}{ds}|x\rangle\right) = \nabla n(\boldsymbol{x}) \tag{1.2.2}$$

In this kind of optical problems, the geometry is dealing with *coordinates of location* only. Thus, what is meant here by the symbol $|x\rangle$, as well as by $\boldsymbol{x}$, is a set of three coordinates in space, locating the positions in a Cartesian reference frame. Only, the Dirac's notation suggests an algebraic realization of the vector as a 3×1 matrix, *i.e.* a matrix with three lines and a column. In the cases where the reference frame is unique in space, like in path optics, there is no difference in meaning between the two notations. Such a difference may occur only when the reference frame changes along the path, and it is important for the global geometry involved in the optics of light, in that the geometrical quantities of physical interest are the *torsions* and not the *curvatures* (Cartan, 1931). We shall return on this topic, if needed. For now, coming back to our discussion path here regarding the classical optics, when algebraically expanded, the equation (1.2.2) gives

$$n(\boldsymbol{x})\left|x''\right\rangle+\left(\boldsymbol{x}'\cdot\nabla n(\boldsymbol{x})\right)\left|x'\right\rangle=\nabla n(\boldsymbol{x}) \tag{1.2.3}$$

where the diacritical mark means derivative on *s*, and a dot between vectors means dot product.

Everything depends here on the functional form of refraction index of the medium supporting the light. According to Planck's quantization procedure this optical medium needs to accommodate a fundamental material structure: *the electric dipole*. That is, we need to assume that in a Wien-Lummer enclosure, the thermodynamical equilibrium creates a medium whose *fundamental constitutive unit is a dipole*. This condition comes down to assuming a special functional form of the refraction index of the medium. To wit, this special functional form is:

$$n(\boldsymbol{x})=\left(1+\left\langle x|x\right\rangle\right)^{-1}\quad\therefore\quad\nabla n(\boldsymbol{x})=-2\left(1+\left\langle x|x\right\rangle\right)^{-2}\left|x\right\rangle \tag{1.2.4}$$

where $\langle x|x\rangle$ is the sum of squares of the coordinates along the path of light, taken as scaled with a gauge length, as it were, in order to be considered pure numbers. Understandably, in view of the previously presented idea of the quantization of light (see §1.1), the essential theme of the present work is justifying this choice for the refraction index of the optical medium. A 'choice' of the refraction index may seem as just a particular case of a medium, where the theoretical physics needs, in fact, the most general concept. However, let us not forget that the choice is only a mathematical fact, and the demand of generality is actually a matter of physics.

Start, for once, with the equation (1.2.3), which by using (1.2.4) can be rewritten as:

$$\left(1+\left\langle x|x\right\rangle\right)\left|x''\right\rangle-2\left\langle x|x'\right\rangle\left|x'\right\rangle+2\left|x\right\rangle=\left|0\right\rangle \tag{1.2.5}$$

Using the definition of the elementary arclength of the geometrical path, we have:

$$(ds)^2=\left\langle dx|dx\right\rangle=\left\langle x'|x'\right\rangle(ds)^2\quad\therefore\quad\left\langle x'|x'\right\rangle=1 \tag{1.2.6}$$

so that, in a Euclidean background, we can get the relations

$$\left\langle x'|x''\right\rangle=0,\quad\left\langle x''|x''\right\rangle+\left\langle x'|x'''\right\rangle=0 \tag{1.2.7}$$

and use them in order to conclude on the equation (1.2.5). First, by differentiating (1.2.5) itself, we get:

$$(1+\left\langle x|x\right\rangle)\left|x'''\right\rangle-2\left\langle x|x''\right\rangle\left|x'\right\rangle=\left|0\right\rangle \tag{1.2.8}$$

whence dot-multiplying this by $|x''\rangle$ and using the first relation from (1.2.7), we get:

$$\left\langle x''|x'''\right\rangle=0\quad\therefore\quad\left\langle x''|x''\right\rangle=const \tag{1.2.9}$$

Geometrically speaking, this last condition is referring to a curvature of the ray path through space: for this kind of continuum, the curvature of the path must be a constant. Returning then to equation (1.2.5) once again, however, this time only for dot-multiplying it by $|x''\rangle$ directly, and then using the first of equalities (1.2.7) and the result from the last of equations (1.2.9), gives the remarkable final equations to be used in adjusting the differential equation (1.2.8):

$$\frac{2\left\langle x|x''\right\rangle}{1+\left\langle x|x\right\rangle}=-\frac{1}{R^2},\quad\left\langle x''|x''\right\rangle\equiv\frac{1}{R^2} \tag{1.2.10}$$

Here $R$ is a non-dimensional constant, suggesting, again, the necessity of a gauge length to be introduced, this time, in a specific way imposed by the optics of media, *via* the curvature of the geometrical path. Inserting this result into equation (1.2.8) produces the final equation of the propagation along the ray:

$$\left|x'''\right\rangle + R^{-2}\left|x'\right\rangle = \left|0\right\rangle \tag{1.2.11}$$

This is the first occurrence, in the context of optics of material media, of such a third order linear differential equation which, as we will try to establish in the present work, is of a primary importance in theoretical physics, as long as this one is viewed from the perspective of a natural philosophy involving the scale transitions.

While this task will be gradually completed as we go along with our work, for now we have an observation that needs to be made in order to properly guide the work itself. Namely, the *optical medium* described by the refraction index from equation (1.2.4) should be considered a Riemannian manifold which turns out to be of *finite volume* and *positive curvature*. Thus this optical medium satisfies the basic geometrical requirements for playing the part of a Wien-Lummer cavity. Indeed, the elementary optical path of the medium is conformal Euclidean, assuming that $\langle dx|dx\rangle$ is Euclidean. In view of the idea of physical generality, it is perhaps the moment to insist for a little while on the equation (1.2.1) from this geometrical point of view. Take the elementary path

$$(ds)^2 = \frac{\langle dx|dx\rangle}{(1+\langle x|x\rangle)^2} \tag{1.2.12}$$

as the Riemannian metric of this optical realm. This metric is, in the views adopted here, a conformal Euclidean metric, with the conformality factor being spherically symmetric. Using the spherical symmetry, we have

$$(ds)^2 = \left(\frac{dr}{1+r^2}\right)^2 + \left(\frac{r}{1+r^2}\right)^2 (d\Omega)^2, \quad (d\Omega)^2 \overset{def}{=} (d\theta)^2 + \sin^2\theta\cdot(d\varphi)^2 \tag{1.2.13}$$

where $\theta$ and $\varphi$ are the spherical angles of colatitude and longitude, respectively, and $r^2 \equiv \langle x|x\rangle$. This form of the metric does not tell us very much about the Riemannian geometry of the optical realm, but it can tell us a lot if we go over to a meaningful variable. So, let us assume that $r$ takes values between two limits, as in the case of the radial coordinate in the classical Kepler problem. In this case it can be represented faithfully by formula:

$$r(\phi) = \tan(\phi/2) \tag{1.2.14}$$

so that the metric (1.2.13) can be written as

$$(ds)^2 = \frac{1}{4}\left\{(d\phi)^2 + \sin^2\phi\cdot(d\Omega)^2\right\} \tag{1.2.15}$$

This metric describes a realm spatially delimited by the magnitude $\tan\phi$ of the *eccentricity vector*, in the case of the classical Kepler motion, and with $\Omega$ representing the arclength, $\varphi$ say, of the geodesic on the unit sphere. By a change of variable:

$$z = i\frac{\cos(\phi/2) + ie^{-i\varphi}\sin(\phi/2)}{\cos(\phi/2) - ie^{-i\varphi}\sin(\phi/2)} \tag{1.2.16}$$

where $\varphi$ counts here as an arbitrary angle of longitude, the metric (1.2.15) can be reduced to the Beltrami-Poincaré form, characteristic to the hyperbolic plane:

$$(ds)^2 = -\frac{dz\cdot dz^*}{(z-z^*)^2} \tag{1.2.17}$$

We read this line of reasoning as telling us just what kind of Riemannian geometry governs the realm containing the center of force in the classical Kepler force. One thing is sure: this realm, containing, as the case may occur, the Earth, the Sun or the atomic nucleus, is clearly a Maxwell fish-eye! In this sense, the metric (1.2.12) is *universal*, indeed: it applies to any planetary system in the universe, at any space scale!

Let us, therefore, show what is its property that makes it so attractive, at least from our point of view, if nothing else. The geometry will be described here by introducing two geometrical parameters – which, therefore, are assumed to represent lengths – denoted $a$, $b$, and used in order to describe the Euclidean shape of the geodesics of the metric:

$$(ds)^2 = 4a^2b^2 \frac{\langle dx|dx\rangle}{(b^2 + \langle x|x\rangle)^2} \tag{1.2.18}$$

Here we have used the equation (1.2.12) for the elementary *optical path*. This is the metric of the realm called the Maxwell fish-eye, indeed, as it was described a long time ago by Constantin Carathéodory in his exquisite mathematical researches regarding the geometrical optics (Carathéodory, 1937). With this description we have our first and foremost of the incentives in choosing of the functional form (1.2.4) for the refraction index. Indeed, the Maxwell fish-eye is a perfect optical device whereby all light rays have the *properties of the lines of force of an electric dipole*: circles passing through two fixed points representing the locations of the two component charges (Stavroudis, 1972, 2006). In a Maxwell fish-eye, the light rays through any point in the space occupied by this medium, also pass through a point which is its inverse with respect to a given sphere. Should the necessity occur to operate an interpretation here, it obviously needs to be achieved by point particles having *charges*, and characterized by the Lorentz property: in order to acquire a charge of opposite sign, a position from such a medium needs to be *replicated by inversion* with respect to a certain, locally spherical surface.

Let us present some details of this statement, just because we need to be fairly familiar with the procedure in view of its application in the theory of embeddings, if for nothing else [we follow here (Carathéodory, 1937), §73]. These details involve the mathematical concepts connected to the three-dimensional space embedding into a four-dimensional Euclidean manifold: the first is the space form of an instanton, while the second involves the algebraic properties of the measured physical properties. For a good guidance on the topic, we recommend the exquisite work of Ruben Aldrovandi and José Geraldo Pereira on Geometrical Physics, especially the Chapter 23 (Aldrovandi & Pereira, 2017); so much the better as this guidance is offered in connection with classical non-Euclidean geometries. As, further on, the embedding procedure involves the stereographic projection, one may need a previous familiarization with this projection. A geometrically comprehensive presentation of the stereographic projection method is made in the booklet (Rosenfeld & Sergeeva, 1977), that we also recommend to the reader.

According to Constantin Carathéodory, the parameters $a$ and $b$ have the following geometrical meaning, for which they were specifically introduced in fact: the metric (1.2.18) is the metric in the three-dimensional boundary of a four-dimensional Euclidean sphere of radius $a$, projected stereographically on a three-dimensional Euclidean space at the distance $b$ from the center of the projection. This can be shown as follows: by analogy with the three-dimensional case, one takes the equation of a sphere in the four-dimensional case – a three-sphere, as it were, in view of the fact that a sphere in space is usually considered as a two-sphere – in Cartesian coordinates $\xi$, $\eta$, $\zeta$, $\tau$ in the form of quadratic equation:

$$\xi^2 + \eta^2 + \zeta^2 + \tau^2 = a^2 \tag{1.2.19}$$

The three-dimensional stereographic projection on an Euclidean tangent hyperplane, from a point located at the distance $b$ from it, is achieved by the formulas:

$$\frac{\xi}{x} = \frac{\eta}{y} = \frac{\zeta}{z} = \frac{\tau + a}{b} =: n \tag{1.2.20}$$

where $n$ is a parameter, playing the part of refraction index of the medium described by the metric (1.2.18). Introducing these coordinates in the equation (1.2.19), we get an equation that can be solved right away, giving two values of $n$:

$$n = 0, \quad n = \frac{2ab}{b^2 + r^2} \tag{1.2.21}$$

Here $r^2 \equiv \langle x|x\rangle$ is the Euclidean norm of the position vector of the projected point from the tangent hyperplane. The first one of these values corresponds the the 'south pole' of the hypersphere (1.2.19), $\tau = -a$ – the 'north pole', $\tau = a$, being the point where the hyperplane $(x, y, z)$ touches the hypersphere – where the correspondence realized by (1.2.20) is singular. On the other hand, the second one of these values corresponds to the projection of current point of coordinates $(\xi, \eta, \zeta, \tau)$, helping in representing it as a point in the tangent Euclidean space in coordinates $(x, y, z)$. According to equation (1.2.20), this representation is provided by the formulas:

$$\xi = \frac{2abx}{b^2 + r^2}, \quad \eta = \frac{2aby}{b^2 + r^2}, \quad \zeta = \frac{2abz}{b^2 + r^2}, \quad \tau = a\frac{b^2 - r^2}{b^2 + r^2} \tag{1.2.22}$$

which can be readily solved for $(x, y, z)$, in order to provide the Cartesian coordinates as:

$$x = b\frac{\xi}{a + \tau}, \quad y = b\frac{\eta}{a + \tau}, \quad z = b\frac{\zeta}{a + \tau}, \quad r^2 = b^2\frac{a - \tau}{a + \tau} \tag{1.2.23}$$

Now, using these last two equations, we can construct the four-dimensional Euclidean elementary distance:

$$(ds)^2 = (d\xi)^2 + (d\eta)^2 + (d\zeta)^2 + (d\tau)^2 \tag{1.2.24}$$

which turns out to be the metric (1.2.18).

Going a little bit ahead of us here, we see these results the following way: the Maxwell fish-eye is an optical medium describing the matter in a three-dimensional Euclidean space. The matter in this space is itself a Riemannian manifold, having the Euclidean metric (1.2.24), which is conformal with the Euclidean metric as in equation (1.2.18). The problem is not what the three-space represents – we know this from the daily experience of our life – but what the coordinates $(\xi, \eta, \zeta, \tau)$ are, and an answer presents itself right away: *they are charges*. This is a story first told to us by the geodesics of the conformal metric (1.2.18), which are lines characterizing the field of natural dipoles of the medium described by (1.2.18): either electric or magnetic.

On the other hand, any two of the four coordinates $(\xi, \eta, \zeta, \tau)$ can be associated in order to give either the square of an electric charge, or the square of a magnetic charge according to Katz's natural philosophy [see (Mazilu, 2020), §3.1]. The association is a stochastic process and, as we shall show here has everything in common with the stochastic type of processes once imagined by Carlton Frederick for the metric tensor of the spacetime (Frederick, 1976). According to this kind of view, the equation (1.2.19) would represent an electromagnetic continuum 'split into charges' by the procedure of embedding a three-dimensional Euclidean manifold. This, we know since Max Planck, is physically realized by a Wien-Lummer cavity enclosing light and matter in the form of some electric dipoles. Based on this, and using the analogy based on what we already know about quantization, and on the fact that matter and light are two inseparable categories, we can only guess that a similar procedure in the case of a matter continuu would be physically realized by a kind of Wien-Limmer cavity enclosing 'matter' and 'light', this one in the form of some magnetic dipoles. No matter what the case may turn out to be, from the physical optics point of view, which is the only *a priori* point of view to be used in this kind of problems, *the quantization procedure based solely on a Wien-Lummer cavity must necessarily be that of Max Planck*. We cannot speak of a direct second quantization, and the main reason for that is, in an expression of the divine Henri Poincaré, that we do not know in general, and expression for the energy of matter (Poincaré, 1897).

### 1.3 Portions of Surfaces and Electric Properties of Matter

Arthur Eddington has an important observation that, in our opinion, should guide the research of every physicist, especially if he/she wants to follow Louis de Broglie's path on building the theoretical physics. Indeed, it is significant that de Broglie followed an old natural-philosophical path, quite appropriate to the new steps in science at his time, that abides by the words of Eddington, written on the occasion of a proposal by Uhlenbeck and Goudsmit made just about the same historical time. That proposal regards the spin of the electron, when taken as part of the complex structures (Uhlenbeck & Goudsmit, 1926). One of the critiques raised against the proposal in question was that, in keeping with the classical meaning of the word 'spinning', the electron must be a spatially extended particle that may involve rotations with superluminal speeds. As we shall see here, such a logical inference may not be out of place for the instanton structure of an electron, but at the epoch we are talking about, this was hard to believe without an argument. And the complex factor $\sqrt{(-1)}$ came into argument, just like it did in the case of wave function of Schrödinger just about the same time [see (Mazilu, 2020), §1.1]. However, in the case of spin the occurrence of imaginary unit was intended to be taken as a prototype acceptance worth following mathematically. And the conclusion of this argument is just about the same in both cases, only Eddington expresses it more… academically, so to speak. Quoting:

> The mathematical definition of velocity ($dx/dt$) contains no special reference to motion in a dynamical sense; $x$ is merely the co-ordinate of a selected succession of world-points, and there is in the definition no guarantee that $dx$ is traversed by anything *except the thought of the mathematician*. In describing the electron as spinning, what happens is that, *faced with a hitherto unimagined structure*, *we make our thought skip faster than light round its boundary*, and by so doing succeed in seeing a correlation with a more familiar structure, namely, that of *an electron at rest*. The *correlating velocity has no more physical existence* than has the factor $\sqrt{-1}$ used to correlate the structure of the four-dimensional world to the more familiar structure of a four-dimensional Euclidean space. In a deeper analysis *we should not speak of a moving charge-element but of a charge-and-current vector*, motion being attributable only to *boundaries or analogous features of charge distribution* – not to *charge* (*original emphasis here*, *n/a*) but to *a charge* (*original emphasis here*, *n/a*). When in the cruder description the charge moves faster than light,

the charge-and-current vector $J^{\mu}$ becomes *space-like* (*original emphasis here*, *n/a*). [(Eddington, 1926); *our Italics*, *except as indicated*, *n/a*]

De Broglie clearly used the mathematics of relativity, whereby the idea of 'faster-than-light' particles originates, but on the occasion of a work on interpretation proper (de Broglie, 1926b,c) his reasoning took another turn, even though along the same general philosophical line. Going a little ahead of us, we can say that this line has strong historical roots in physics, as we shall see later along our story, but all of them respect the idea resuscitated by the words of Eddington excerpted above. In essence, we shall have to make our thought 'traverse a de Broglie tube' 'carried with a wave', as it were, and therefore we need to figure out how to describe *appropriately* a portion of surface delimited by a tube. The mathematical method styled briefly in this section is intended to support such a description.

We shall limit our considerations here to only the three-dimensional space, which we assume to be also the realm of daily events of our life. The vectors will be conceived either as entities defined by components in an Euclidean reference frame, or in the Dirac's matrix form. Thus, the position vector for instance, can be written in one of the following two forms:

$$\boldsymbol{x} = x^k \hat{\boldsymbol{e}}_k, \quad |x\rangle = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{1.3.1}$$

The first of these forms is the usual geometric script for the vectors, whereby the position vector is a linear combination of the unit vectors $\hat{\boldsymbol{e}}_k$ of the reference frame. The coefficients $x^k$ of this linear combination are the contravariant components of the position vector. The second writing – the matrix notation or, as we would like to call it in order to account for its origin, *the Dirac notation* – disregards the existence of the reference frame. It is appropriate in using for calculations in cases where the reference frame does not count: for instance, either in the cases of positions in the same reference frame, or in the cases where the reference frame is the same everywhere in space, as in the Cartan's approach of the Riemannian geometry (Cartan, 1931). However, there is a third case that seems to encompass these two: the case when the base vectors of the reference frame are constructed from coordinates, by the very same functional rule in any point in space. This is, for instance, the case of *a Beltrami reference frame* revealed by us in the case of Maxwellian approach to electricity [see (Mazilu, 2020), §6.1], and asks for a physically valid process of establishing the coordinates independently of the geometry.

In general, the reference frame is purely local: it can vary from point to point due to some physical reasons. Moreover, still due to some physical reasons, the reference frame may not be always orthogonal. In such cases, using the same general matrix notation as in equation (1.3.1), we write

$$\boldsymbol{x} = \langle x|\hat{\boldsymbol{e}}\rangle, \quad |\hat{\boldsymbol{e}}\rangle \equiv \begin{pmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{pmatrix}, \quad |\hat{\boldsymbol{e}}\rangle \cdot \langle \hat{\boldsymbol{e}}| = \boldsymbol{g}(\boldsymbol{x}) \tag{1.3.2}$$

Here $\boldsymbol{g}$ is the metric tensor, a matrix that, due to the fact that the reference frame is made out of vectors that are only normalized, not being orthogonal, has *1* as diagonal entries. If the metric tensor is the identity matrix, we have the usual Euclidean space, with the position expressed in Cartesian coordinates.

The basis of the differential geometry of space in the approach we use here, *i.e.* in the Élie Cartan's approach, is the observation that, physically speaking, an infinitesimal (or elementary) displacement involves both a variation in the position of a point *per se*, and a variation of the reference frame itself, according to a rule that may vary from one point to another:

$$d\boldsymbol{x} = dx^k \hat{\boldsymbol{e}}_k + x^k d\hat{\boldsymbol{e}}_k \tag{1.3.3}$$

Here the reference frame is understood as composed of a triad of unit vectors, having also a common origin. Thus, by the general geometrical rules, the elementary variations of the unit vectors of the reference frame can be expressed as linear combinations of these very vectors, with some differential coefficients that can be arranged as entries of a 3×3 matrix. Therefore, the evolution of the reference frame can be described by the so-called *Frenet-Serret equations*, written in the 'indicial form':

$$d\hat{\boldsymbol{e}}_k = \boldsymbol{\Omega}_k^j \hat{\boldsymbol{e}}_j \quad \therefore \quad \boldsymbol{\Omega}_k^j = d\hat{\boldsymbol{e}}_k \cdot \hat{\boldsymbol{e}}^j \tag{1.3.4}$$

Here, in order to use the summation rule over dummy indices, we introduced the *dual reference frame*, given by the unit vectors $\hat{\boldsymbol{e}}^k$. These are unit vectors that by their dot products give the *contravariant* metric tensor, which is the inverse matrix of the metric tensor defined by the usual reference frame. In general, the matrix $\boldsymbol{\Omega}$ has only zeros on the main diagonal if the reference frame is orthonormal. Indeed, by the virtue of definition of the metric tensor, we have

$$dg_i^j = \boldsymbol{\Omega}_i^j + \boldsymbol{\Omega}_j^i, \quad \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}^j = g_i^j \tag{1.3.5}$$

Thus, as just mentioned, we can make the properties of the matrix $\boldsymbol{\Omega}$ even more precise: it is always a skew-symmetric matrix, in the case of an orthonormal reference frame.

Now, with Frenet-Serret relations from equation (1.3.4), the equation (1.3.3) can be written as

$$d\boldsymbol{x} = s^k \hat{\boldsymbol{e}}_k, \quad s^k \equiv dx^k + x^j \boldsymbol{\Omega}_j^k \tag{1.3.6}$$

Obviously, both the components of $d\boldsymbol{x}$ as well as those of $d\hat{\boldsymbol{e}}_k$ must be exact differentials. In the framework of exterior calculus, this fact can be expressed by vanishing of the exterior differentials:

$$d \wedge d\boldsymbol{x} = \boldsymbol{0}, \quad d \wedge d\hat{\boldsymbol{e}}_k = \boldsymbol{0} \tag{1.3.7}$$

The whole geometrical construction of Élie Cartan is a mathematical consequence of these two equations, representing simple facts of differentiability. For once, by following the rules of working with exterior differential forms we can find, starting from (1.3.7), the following relations which connect the components of vector $d\boldsymbol{x}$ to the matrix $\boldsymbol{\Omega}$:

$$d \wedge s^k + \boldsymbol{\Omega}_j^k \wedge s^j = 0, \quad d \wedge \boldsymbol{\Omega}_j^k + \boldsymbol{\Omega}_m^k \wedge \boldsymbol{\Omega}_j^m = 0 \tag{1.3.8}$$

Here the Einstein's rule of sumation over dummy indices is observed, with the only difference that the monomials are defined by exterior multiplication, not by the usual numerical product, and the sign '∧' after differentiation symbol shows that it has to be an exterior differentiation. By obvious reasons, the first of equations (1.3.8) is usually called *compatibility equation*: it gives, indeed, the compatibility condition between the variation of the reference frame and the elementary displacements in space, as described within this reference frame. The second of the equations in (1.3.8) can be termed as the *Maurer-Cartan equation*, borrowing a name which describes the moving coframe of the Lie algebras.

Following the same rules of exterior multiplication and exterior differentiation, one can prove that the first equation (1.3.8) is a consequence of the second one, when combined with the definition of the differentials $s^k$ from equation (1.3.6). However, as they stand, the two equations (1.3.8) are the most general ones: they are valid regardless of the definition of $s^k$ provided by equation (1.3.6). Therefore the two equations (1.3.8) may very well not be obviously equivalent. This means that there are situations, essentially dictated by physical reasons, where we need to define the coordinate variations directly in terms of the variations of the reference frame, in which case $s^k$ do not have the simple structure given in (1.3.6), *i.e.* they cannot be neatly written as a sum of two differential components. In the like cases, one cannot say precisely how much from the infinitesimal variation of the position is pure displacement and how much of it is due to the contribution of the variation of reference frame. All one can say, in general, is that the components of the displacement vector are differential forms, and that they have to satisfy the general conditions from equation (1.3.8).

The mathematical method itself, for carrying out the task of introducing the physics into the natural philosophy, is based on some almost trivial statements regarding the foundations of the mathematics necessary in building a differential geometry. These statements emerged apparently largely unnoticed or, even if noticed, they have not been properly used to their full capacity, so to speak, at least not for physical purposes, anyway. In order to make this statement more obvious, we shall reproduce here two of the Élie Cartan's 'algebraical' theorems which are recognized to form the ground of his remarkable mathematical approach to differential geometry involving the so-called *moving frames* [for a clear presentation of the concept from the point of view we adopt here, see (Spivak, 1999), Volume II, Chapter 7]. Then, these theorems will be updated with a result of Yoshio Agaoka, used in a short description of the transport phenomena using Cartan's method for the classical case of the differential geometry of surfaces.

The Cartan's theorems in question are drawn here directly from one of Cartan's courses, published *via* the Russian geometrical school of S. P. Finikov [(Cartan, 2001); pp. 16 – 17, Theorems 7 & 9]. We appropriate them, for our purposes, under the name of *Cartan Lemmas 1 and 2*, only in order to be suitably used in making our point as explicit as possible. Here they are:

*Lemma 1*. Suppose that $s^1, s^2, \ldots, s^p$ is a set of linearly independent 1-forms. Then there exists a *convenient symmetric matrix*, $\boldsymbol{a}$ say, such that:

$$s^\alpha \wedge \phi_\alpha = 0 \quad \Leftrightarrow \quad \phi_\alpha = a_{\alpha\beta} s^\beta \quad with \quad a_{\alpha\beta} = a_{\beta\alpha} \tag{1.3.9}$$

where $\phi_1, \phi_2, \ldots, \phi_p$ is any other set of linearly independent 1-forms, and a summation over repeated indices is understood.

*Lemma 2*. Suppose the basic differential elements $du^1, du^2, \ldots, du^n$ are connected by a system of $p$ equations

$$\omega^1 = 0, \quad \omega^2 = 0, \quad \ldots \quad \omega^p = 0 \tag{1.3.10}$$

where $\omega^\alpha$, $\alpha = 1, 2, \ldots, p$ are linearly independent 1-forms. Then the 2-form $f$ constructed with the differentials $du^1, du^2, \ldots, du^n$ vanishes as a consequence of this system of equations if, and only if, $f$ can be written as the sum of exterior products

$$f = \omega^\alpha \wedge \phi_\alpha \tag{1.3.11}$$

where, again, summation over $\alpha$ is understood, and $\phi_\alpha$ are *p conveniently chosen* 1-forms.

The first one of these theorems is, by and large, known as *Cartan*'*s Lemma* proper in the specialty literature, and previously we designated it exactly that way [see *e.g*. (Mazilu, 2020); see §3.4]. As to the second one of the theorems, it carries no special name in the literature, being, in fact, used only occasionally. We intend to use it not just occasionally, but just as fundamentally as one uses the *Lemma 1*. Perhaps not in this very work, but this appears as a proper place to locate it anyway.

What seems to be essential in these lemmas, and is almost always stressed especially in some old treatises of geometry, but apparently forgotten lately – perhaps due only to the exclusive mathematical applications – is the fact that the symmetric matrix ***a*** from *Lemma 1*, as well as the 1-forms $\phi_\alpha$ from *Lemma 2*, are *things external to the geometrical problem at hand* and, moreover, *can be conveniently chosen*. We take these qualifications as meaning that they can be things geometrical, as originally intended, but for our purposes, we extend this meaning beyond the mathematical border: they can be *things physical* as well, *i.e*. things through which *the physics can be naturally introduced into geometrical theory* or *vice versa*. For instance, we need to introduce the physics in the theory of surfaces, in order to make it physical, thereby apt to serve the de Broglie's idea in constructing the light ray, or a ray in general for that matter. In concentrating on the local geometry in a position of a surface, without being interested of the global aspects of that surface, as it is almost always the case in physics, especially in the de Broglie's physical optics, this observation becomes essential. Consequently, we can use these two lemmas, primarily in order to choose some physical properties compatible with the geometrical ones. However, mention should be made, that there are a great many problems that the differential geometry allows us to solve using them.

With this task in mind, let us recall once again the convention referring to our use of numerical indices: insofar as either the space or the matter, contemplated as environments in the embedding problem necessary to physical interpretation, are apparently always three-dimensional, we reserve the Latin indices exclusively for this case. The Greek indices are used for any other dimension, as in the case above, but especially for dimension two, in the case of surfaces, dimension four in the case of the manifold of events, *viz*. the spacetime, or dimension five in the case of a Kaluza-Klein type theory.

While in a way we are compelled into recognizing that the differential geometry is preeminently well served by the Élie Cartan's method of construction of a geometry, at least from the physics' point of view, we have to recognize that the mathematical philosophical foundations of Cartan's method are also common facts of a purely linear algebraic nature. Two further ideas spring from this observation, namely, on one hand, that the Cartan's method can be extended as a general natural philosophical method into physics, as the previous section plainly shows. On the other hand, it occurred to us that this method can be used as the foundation of a theory of scale transition. These ideas, as well as the fact that we feel the urge of making the present text somewhat self-consistent, are the reason why in the present section we insist a little longer on the algebraical basis of the Cartan's method of construction of geometry. However, as this insistence is somehow out of the usual line, we first adduce an illustrating example from the physics of electricity, indicating the feasibility of the method in this branch of natural philosophy.

Take, for instance, the idea of a resonator: for once, it can be defined purely geometrically, as in the previous section. However, at some moment of using this definition for physical purposes, one has to take notice that it amounts to the idea that the motion of a reference frame in a charge sea *creates* pairs of charges, positive and negative, exactly as envisaged by Lorentz himself (Lorentz, 1892). From our perspective, the Lorentz contention means this: the matter is *incidentally* neutral from an electrical point of view – *i.e.* it is vacuum – inasmuch as this kind of neutrality is effective only on surfaces. Outside these surfaces, and in their immediate neighborhood, the charges are in a de Broglie region, 'moving at constant time', as de Broglie said, across the surface (de Broglie, 1926b). In order to illustrate our point of view, it is better to refer such a reasoning to the Lorentz's own works, and to the connection of his ideas with the classical works on electricity. In order to get a better grip on the subject, let us follow these ideas along with the classical problems of electricity.

The experience shows that the matter can carry charges, just the way it carries mass. However in such a case the image of ether, as interpreted by Samuel Earnshaw [see (Mazilu, Agop, & Mercheș, 2021), Chapter 1] seems impossible: in the ether we manifestly have waves not particles. Electromagnetic waves, it is true but, still, waves. It is on this occasion, that one can conclude, borrowing the later words of C. G. Darwin, that *the ether is a continuum which needs to be interpreted*. The hard part of this interpretation is that the ether appears as electrically neutral, and no one could see how the wave concept could be reconciled with the experimental idea of charge. It is at this point that Lorentz enters the stage with an idea of incidental electric neutrality. Quoting:

> If, after *arbitrary movements*, the matter is reduced to its *primary configuration*, and if, during these movements, *every element of a surface* which is *steadfastly attached to the matter* was crossed by *equal quantities of electricity in opposite directions*, all of the points of system will be found in their *primary positions* [(Lorentz, 1892), §57; *our translation and emphasis*]

Notice, first, that this hypothesis already assumes that an interpretation is in place, for otherwise one cannot describe a *surface* 'attached to *matter*': they need to have common *points*. On the other hand, if one takes the 'element of surface steadfastly attached to matter' as referring to an infinitesimal portion of a 'wave surface', the situation suggested by Lorentz in this excerpt is, indeed, the one envisioned by Louis de Broglie in his condition mentioned above, that we found 'strange' before [see (Mazilu, 2020), §2.1]. In view of this, we venture to assume that 'configuration' means here *an ensemble of classical material points*, so that when Lorentz says that an 'element of surface is attached to matter', we have o understand that this element of surface is determined by the positions of at least one material point, playing the part of *chosen positions* on describing a surface.

First, Lorentz finds that his assumption *is not always satisfied* – we should add: within the framework of Earnshaw interpretation – and by now we can even tell why, according to his own findings: it is a problem of *transport theory*. Indeed, there is a discrepancy between the time derivative, and substantial derivative involved in the transport of energy [see (Lorentz, 1892), pp. 423 – 424, §66]. However, Lorentz does not see in this a reason not to go any further with the model of matter, and this shows us just to what extent was he going with the fluid as a model in the interpretation problem: whatever cannot be conceived as valid for an ordinary fluid, cannot be applied to the ether, either. Quoting, indeed:

> If this hypothesis *cannot be admitted in the case of an ordinary fluid*, it *could not be applied to the electric fluid either*. However, this fact does not prevent our equations of motion from being accurate. Indeed, *the mass of this last fluid was supposed to be negligible*, and in calculating the variation $\delta T$ (*kinetic energy*, *n/a*) only that kinetic energy was considered *which is specific to the electromagnetic movements*; it will suffice therefore that the material points liable of these motions, and *which are not to be confused with the electricity itself*, enjoy the property of returning to the same positions *if for each surface element the algebraic sum of the quantities of electricity* by which it has been crossed, is *0*.
>
> Now, one is entirely free to try on the mechanism that produces the electromagnetic phenomena *any convenient assumption*, and while recognizing the difficulty of *imagining a mechanism that possesses the desired property*, it seems to me that we do not have the right to deny its possibility. [(Lorentz, 1892), §67; *our translation and Italics*]

Notice, incidentally, an observation intended to make us cautious, namely that the material points – the classical 'bodies' of dynamics – 'liable of motion' are 'not to be confused with electricity itself', a distinction which, we may say, brings forward the observation once made by Poincaré, about the impossibility of action upon ether (Poincaré, 1900). Also notice that the Lorentz matter thus interpreted, is the *counterpart of the physical universe at large*. Indeed, here we have to assume that 'the mass of *electric* fluid is supposed to be negligible', since the Coulomb forces dominate, while in a regular cosmology based on the general relativistic ideas, it is the charge that is 'supposed to be negligible', for the gravitation forces dominate.

The concept of Lorentz matter, therefore, speaks of a universe where the charge is *force-wise dominant*, for the mass, obviously, cannot be negligible in the sense that it is missing: the dynamics knows nothing of the concept of zero inertial mass. Fact is that we cannot 'dismiss the mass' in the construction of a physical theory of the universe, at least not the way we do it nowadays with the charge in the case of physical universe at large. As for the rest of the excerpt above, the most important thing, namely 'that mechanism… possessing the desired property' from the last sentence, was not to be 'assumed' anymore for, just about the period of time we are talking here, it was *physically accomplished in the form of the field generated via a periodic charge motion* by Heinrich Hertz [see, for instance, the English translations collected in (Hertz, 1893), for the fundamental works which instituted the modern theory of electromagnetic field].

Lorentz insisted at length in making the point clear that the interpretation of the electric matter is not a trivial thing. In order to clarify his essential idea, we think it is worth citing again the Lorentz's words: in characterizing the matter structure by an interpretation based upon the existence of electricity, these words constitute the crowning point of a long ascending path followed by electricity theory starting from the times of Ampère. Quoting, therefore:

> Here is now a system of hypotheses that give the value 0 for this variation (*of the kinetic energy*, *entering the extremum principle of mechanics*, *n*/*a*):
>
> a. There are *two systems of particles* participating in electromagnetic motions, systems that will be indicated by the letters N and N′.
>
> b. Any time *a certain particle* pertaining to one of these systems, *is to be found in the immediate vicinity of a particle of equal mass pertaining to the other system*.

c. The two systems always *have equal movements inversely oriented* or, stating it more exactly:

If two movements of the same duration start with the same initial positions and do not differ but by the sign of the components of the electric current, and if $P$ and $P'$ are points pertaining to systems N and N′ that coincide in the initial configuration, the point $P'$ will reach, in the second mouvement, the same final position the point $P$ reaches in the first movement.

This obviously implies that at the time of coincidence the points $P$ and $P'$ have equal and opposite velocities. Indeed, changing the signs (*of the components of current*, *a*/*n*) will reverse the velocity of the point $P$; but, according to the last hypothesis, this velocity must then become equal to that which the point $P'$ had previously.

Notice again that, in the course of a certain movement, a new particle $P'$ will coincide with a given particle $P$. Two juxtaposed wheels, having equal and opposed rotations of the same axis, may serve as an example. [(Lorentz, 1892), §69; *our translation and emphasis*, *a*/*n*]

We think that with these excerpts from Lorentz, the purpose is served in illustrating the role of the concept of surface in a comprehensive case of interpretation: the continuous Lorentz matter has all of the classically known physical qualities liable to generate forces, according to classical natural philosophy. These, as well known, are the gravitational mass and the charges, electric and magnetic. One can say that with the above considerations regarding the way of adding charges to the concept of interpretation, Lorentz has in store for us one of those physical instances that may have to be imagined by us, in order to make the Louis de Broglie's case: *following a ray in approaching a particle at constant time*.

It serves further our purpose here, recalling some steps of the historical path of electricity theory. This line of reasoning is destined to explain the electric neutrality of interacting material conductors of electricity. It started with Riemann, who realized Gauss' idea of interaction of currents (Gauss, 1833, 1845), by introducing what later came to be known as the Klein-Gordon equation (Riemann, 1858). Since Riemann used the concept of *retarded mass*, that perhaps appeared as highly speculative at the time, Enrico Betti stepped into argument, with an idea of *cycles of electricity* along a conductor traversed by a current (Betti, 1868). Betti's idea, apparently based on the concept of Fourier series, was criticized by Rudolf Clausius, on mathematical grounds (Clausius, 1868), and the Riemann's line of thought in electrodynamics remained at this level until Lorentz's work has emerged. Lorentz's ideas were undertaken by Einstein, however not along the Riemann-Betti line of reasoning, but along the Maxwell's line, thus leading to special relativity [see (Mazilu, 2020), Chapter 5, §§5.3 and 5.4]. Einstein's approach will be presented by us in detail in the next two chapters here. Within the present section we are interested in a mathematical issue connected with the Riemann's and Betti's ideas.

A conceptual problem occurs, regarding the definition of a surface without 'pegging it by points', as it were, but just involving considerations of continua. It is this issue that makes the difference between an Ampère element of current and the de Broglie's capillary tube model of the ray. To wit: in the case of de Broglie's ray, the wave surface is 'pegged' by an existing corpuscle that marks a position on the surface, and the corpuscle itself is followed, along the ray and 'at constant time' between *two imagined*, that is *non-marked surfaces*. Operationally, *i.e.*, kinematically, this kind of 'following' means moving with a speed higher than the speed of the particle itself. If we may use an analogy within the same spirit of de Broglie, we have here particles moving with the speeds of phases of the waves that get into compounding a group of waves: the speed of the particle itself is the group velocity, while the speeds of phase are the inverses of this group velocity with respect to the sphere represented by the propagation of light. On the other hand, in the case of an Ampère current element we have reverse situation: two marked surfaces, ‘pegged’ by static particles marking positions on them, with a non-marked surface between them, *that can only be imagined*, and along this surface the electricity should be inexistent, in order to assure neutrality, as in the case of Lorentz’s definition of electricity. As it turns out, such a construction is not entirely independent though, of the ‘pegged’ surfaces. Let us describe a mathematical possibility of such a construction.

First, let us make reference to an important concept of modern theoretical physics, in order to understand from the start what is the gist of the theory of surfaces we are seeking for here. Consider the surface as a *horizon* of the kind serving in the case for ‘membrane paradigm’ in the matters of black holes (Price & Thorne, 1988): we need to describe an *infinitesimal deformation* of surface, in order to accomplish through it the introduction of physics into the mathematical theory. Then, what one geometrically needs here, is the construction of a function $\boldsymbol{z}(u,v)$, describing the deformation of surface as a function of the coordinates $(u, v)$ on it, according to the vectorial equation (Guggenheimer, 1977)

$$\boldsymbol{r}(\varepsilon) = \boldsymbol{x} + \varepsilon \boldsymbol{z} \tag{1.3.12}$$

For the construction of $\boldsymbol{z}$, we use the metric form of the surface. In this case, the deformation is *infinitesimal* if:

$$\frac{ds^2(\boldsymbol{r},d\boldsymbol{r}) - ds^2(\boldsymbol{x},d\boldsymbol{x})}{\varepsilon} \xrightarrow[\varepsilon\to 0]{} 0 \tag{1.3.13}$$

where $\varepsilon$ is a parameter and $ds^2(\boldsymbol{x},d\boldsymbol{x})$ is the metric form, that is the first fundamental form of the surface, at position $\boldsymbol{x}$, calculated on the displacement $d\boldsymbol{x}$. According to (1.3.12, we can write the deformed metric as

$$ds^2(\boldsymbol{r},d\boldsymbol{r}) = ds^2(\boldsymbol{x},d\boldsymbol{x}) + 2\varepsilon(d\boldsymbol{x}\cdot d\boldsymbol{z}) + \varepsilon^2(d\boldsymbol{z}\cdot d\boldsymbol{z})$$

and then the deformation is infinitesimal in the sense of (1.3.13) if

$$d\boldsymbol{x}\cdot d\boldsymbol{z} = 0$$

Assuming an ‘Euclidean mentality’, there is always an arbitrary vector $\boldsymbol{q}$ of the ambient space, serving in writing $d\boldsymbol{z}$ in the form

$$d\boldsymbol{z} = \boldsymbol{q}\times d\boldsymbol{x} \tag{1.3.14}$$

The geometric arbitrariness of $\boldsymbol{q}$ can be significantly reduced, if we take notice that if $d\boldsymbol{z}$ is an exact differential vector, then we must have

$$d\wedge d\boldsymbol{z} = \boldsymbol{0} \quad \rightarrow \quad d\boldsymbol{q}\times^\wedge d\boldsymbol{x} = \boldsymbol{0} \tag{1.3.15}$$

Here ‘×^’ means that in the vector product, the usual multiplication needs to be replaced by an exterior multiplication of the differentials. Using the notation

$$d\boldsymbol{q} \overset{def}{=} j^k \hat{\boldsymbol{e}}_k \tag{1.3.16}$$

the condition (1.3.15) can be transcribed in the form

$$(-j^3\wedge s^2)\hat{\boldsymbol{e}}_1 + (j^3\wedge s^1)\hat{\boldsymbol{e}}_2 + (j^1\wedge s^2 - j^2\wedge s^1)\hat{\boldsymbol{e}}_3 = \boldsymbol{0}$$

which, in turn, comes down to the system of equations:

$$j^3\wedge s^1 = j^3\wedge s^2 = 0, \quad j^1\wedge s^2 - j^2\wedge s^1 = 0 \tag{1.3.17}$$

The first two of these equations show that $j^3 = 0$ on the surface, because $s^1$ and $s^2$ are independent in the geometry of a surface described by them. This means that the vector $d\boldsymbol{q}$ is situated in the tangent plane of the surface, *i.e.* it can be taken as *an intrinsic vector*. On the other hand, the last equation from (1.3.17) says something more. First, by the Cartan's *Lemma 1*, it can be transliterated into equation:

$$\begin{pmatrix} -j^2 \\ j^1 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \begin{pmatrix} s^1 \\ s^2 \end{pmatrix} \quad \therefore \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} j^1 \\ j^2 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \begin{pmatrix} s^1 \\ s^2 \end{pmatrix} \tag{1.3.18}$$

According to its 'intrinsic' property, the vector $\boldsymbol{dq}$ looks like a sort of 'complement' of the infinitesimal displacements $d\boldsymbol{x}$ on the surface. The similarity goes even deeper: in view of definition (1.3.16), the conditions for its integrability $d\wedge d\boldsymbol{q} = \boldsymbol{0}$ are

$$d\wedge j^1 + \Omega^1_2 \wedge j^2 = 0,\ d\wedge j^2 + \Omega^2_1 \wedge j^1 = 0,\ \Omega^3_1 \wedge j^1 + \Omega^3_2 \wedge j^2 = 0 \tag{1.3.19}$$

and, obviously, replicate the similar conditions for the components of $d\boldsymbol{x}$ given in equation (1.3.8) above. Indeed, representing the idea of 'pegged' surface by the condition $s^3 = 0$, the first set of equations from (1.3.8) can be written as

$$d\wedge s^1 + \boldsymbol{\Omega}^1_2 \wedge s^2 = 0, \quad d\wedge s^2 + \boldsymbol{\Omega}^2_1 \wedge s^1 = 0, \quad \boldsymbol{\Omega}^3_1 \wedge s^1 + \boldsymbol{\Omega}^3_2 \wedge s^2 = 0$$

Now, we start using the Cartan's Lemmas again, specifically with *Cartan's Lemma 1*. The entries $\Omega^3{}_1$ and $\Omega^3{}_2$ of the matrix $\boldsymbol{\Omega}$, should be the components of a ket vector $|\Omega^3\rangle$ representing the variation of the unit normal to surface at the given position, as in equation (1.3.4). Then, according to equation (1.3.9), the last of the relations above shows that the variation, by infinitesimal deformation, of the unit normal to surface, is an intrinsic vector that can be expressed *linearly* in terms of $s^1$ and $s^2$, *by a homogeneous relation, involving a conveniently chosen symmetric matrix*:

$$\boldsymbol{\Omega}^3_\alpha = b_{\alpha\beta} s^\beta, \quad b_{\alpha\beta} = b_{\beta\alpha}$$

Written symbolically, this means

$$\left|\Omega^3\right\rangle = \boldsymbol{b}\cdot|s\rangle, \quad \boldsymbol{b} = \boldsymbol{b}^t \tag{1.3.20}$$

where the upper index '$t$' stands for 'transposed'. Since, intuitively speaking, the variation of normal means a curvature of surface, the very matrix $\boldsymbol{b}$ should be related to the *curvature*, − we call it the *curvature matrix* − as in the classical theory of surfaces (Flanders, 1989).

Assuming that the curvature of surface is essential in its physics, especially in the physics of electricity, as the Lorentz theory implies, even if the surface is not marked by points *we choose* to read the third of the equations (1.3.19) as determining the ancillary vector $|j\rangle$ in terms of the curvature, according to the relation

$$|j\rangle = \boldsymbol{A}\cdot|\Omega^3\rangle \tag{1.3.21}$$

Here $\boldsymbol{A}$ is, again, a *convenient symmetric matrix*, introduced in order to satisfy *Cartan's Lemma 1*, representing the intuitive idea that the current generating the deformation is somehow related to the variation of curvature, as the experience instructs our intellect. Now, when we use (1.3.21), in conjunction with the geometrical definition of $|\Omega^3\rangle$ from equation (1.3.20) and equation (1.3.18), both written formally as:

$$\left|\Omega^3\right\rangle = \boldsymbol{b}\cdot|s\rangle, \quad \boldsymbol{i}\cdot|j\rangle = \boldsymbol{a}\cdot|s\rangle$$

we get from (1.3.21) the following *local* relation defining the matrix $\boldsymbol{A}$:

$$\boldsymbol{a} = \boldsymbol{i}\cdot\boldsymbol{A}\cdot\boldsymbol{b} \quad \therefore \quad \boldsymbol{A} = -\boldsymbol{i}\cdot\boldsymbol{a}\cdot\boldsymbol{b}^{-1} \tag{1.3.22}$$

Here $\boldsymbol{i}$ is the 2×2 fundamental skew-symmetric matrix from the second equality of equation (1.3.18): the notation is intended to suggest the obvious fact that $\boldsymbol{i}$ is the matrix replica of the imaginary unit from the case of complex numbers: its square is negative identity matrix.

Now, we have the possibility of characterizing a portion of surface by its normal: in a de Broglie tube, that normal should be the general direction of motion of the current of particles along the tube. Indeed, the relation (1.3.22) is not universally independent of the portion of surface around a certain position. However, it is locally useful, if we are able to identify the possible mechanisms of changing the surface profile, like the electromagnetic field in the genuine membrane paradigm from the theoretical case of the black holes. A first idea would be therefore characterization by a given normal: a portion of surface *per se* is the *ensemble of positions having the same normal*. This means that $|\Omega^3\rangle$ *must be a constant vector*, a condition that can be expressed in a differential form, taking advantage of equation (1.3.20). Let us insist upon this method, by using the previous results.

### 1.4 Surface Gauging by Curvature Change

Assume, first, that the *height* of the reference 'pegged' surface is defined, according to the rules of the classical differential geometry [see, for instance, (Struik, 1988), Chapter 2, §§2–5, 2–6 and 2–7] by a *quadratic form* in a point of this surface. The commonest idea is that this quadratic form represents the second fundamental form of the surface. However, in general, it can be taken as the *support function* of the surface, which is what we shall systematically do in our present argument. So, we take that the *height of a surface* over the reference surface is a quadratic form:

$$h \equiv \langle x|\boldsymbol{h}|x\rangle \tag{1.4.1}$$

Here the symbol $\boldsymbol{h}$ for the tensor of this quadratic form, as well as the symbol $h$ for the very value of the quadratic form itself, suggest the idea of *height*, and $|x\rangle$ is a position on the reference surface, around the point where we calculate the quadratic form. Assume, further, that the variation of this quadratic form can be calculated as in the classical theory of differentials, that is, by the rules of ordinary differentiation. Start, therefore, with the basic equation representing the differential of height according to the classical rules of differentiation, whereby the algebraic expression of the differential involves three distinct terms:

$$dh = \langle x|d\boldsymbol{h}|x\rangle + \langle dx|\boldsymbol{h}|x\rangle + \langle x|\boldsymbol{h}|dx\rangle \tag{1.4.2}$$

Assuming, further, that $|dx\rangle$ is defined by a *gauging* in the form

$$|dx\rangle = \boldsymbol{a}\cdot|x\rangle \tag{1.4.3}$$

where $\boldsymbol{a}$ is a matrix with differential entries, the equation (1.4.2) becomes

$$dh = \langle x|(d\boldsymbol{h} + \boldsymbol{a}^t\boldsymbol{h} + \boldsymbol{h}\boldsymbol{a})|x\rangle \tag{1.4.4}$$

Notice the specific definition (1.4.3) of this idea of gauging: it realizes a connection between the finite and infrafinite measures of positions on the surface. First of all, by such a definition the emphasis is placed on the scale transitions: from finite to infinitesimal measures. Secondly, the emphasis is shifted upon the differentiability of the matrices that characterizes this transition. So the matrices should have physical meaning too in this geometry, just like the coordinates.

With this observations in mind, we go for a few mathematical details regarding the result contained in equation (1.4.4). Notice, in the first place, that for a skew-symmetric matrix product $\boldsymbol{h}\cdot\boldsymbol{a}$ this variation (1.4.4) of the quadratic form of height is strictly defined by the variation of the height tensor. Indeed, if $\boldsymbol{h}\cdot\boldsymbol{a}$ is a skew-symmetric matrix, we have $(\boldsymbol{h}\cdot\boldsymbol{a})^t \equiv \boldsymbol{a}^t\cdot\boldsymbol{h} = -\,\boldsymbol{h}\cdot\boldsymbol{a}$, and the sum of the two matrix products in equation (1.4.4) is zero, and thus the equation for the variation of height reduces to:

$$dh = \langle x|d\boldsymbol{h}|x\rangle \tag{1.4.5}$$

If we know the symmetric matrix of height $\boldsymbol{h}$, this condition helps in constructing the very matrix serving for gauging $\boldsymbol{a}$. In other words, the variation of the height in a neighborhood of a certain point is controlled by the gauging equation (1.4.3), with the matrix $\boldsymbol{a}$ given by an equation of the form:

$$\boldsymbol{a}^t\cdot\boldsymbol{h} = (da)\boldsymbol{i},\quad \boldsymbol{i}\equiv\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.4.6}$$

Here $da$ is a differential factor, and the skew-symmetric matrix $\boldsymbol{i}$ has the property:

$$\boldsymbol{i}^2 = -\boldsymbol{1} \tag{1.4.7}$$

where $\boldsymbol{1}$ is the identity matrix. Solving (1.4.6) for $\boldsymbol{a}$ produces the matrix:

$$\boldsymbol{a} = -(da)\boldsymbol{h}^{-1}\boldsymbol{i} \quad \therefore \quad \boldsymbol{a}\equiv\frac{da}{\Delta}\begin{pmatrix} \beta & \gamma \\ -\alpha & -\beta \end{pmatrix} \tag{1.4.8}$$

where $\alpha$, $\beta$, $\gamma$ are the entries of the symmetric matrix $\boldsymbol{h}$:

$$\boldsymbol{h}\equiv\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \tag{1.4.9}$$

and $\Delta \equiv \det(\boldsymbol{h})$. The matrix $\boldsymbol{a}$ thus defined is, of course, only dependent on the height of the portion of surface considered, and replicates for this portion the property (1.4.7) of the matrix $\boldsymbol{i}$ from the case of complex plane. Specifically we have, by a simple calculation or, even simpler, using the Hamilton-Cayley equation:

$$\boldsymbol{a}^2 = -(da)^2\boldsymbol{1} \tag{1.4.10}$$

The theory can be constructed based on the same principles as those of construction of the second fundamental form, as long as the height of the surface – *i.e.*, the support function of the surface – is a quadratic form, which for physics's purposes, is a quite a general case.

Now, a few more words on the gauging condition of the surface, contained in its definition by equation (1.4.3). That equation associates the differentials of the coordinates around a point on surface, with the differential properties contained in the structure of the matrix $\boldsymbol{a}$. Fittingly, there are in this particular construction – that is, particular because it is based on differentials as in the classical geometry – still other forms of the matrix $\boldsymbol{a}$, that allow for a definition of the variation of the height of surface by the variation of its associated tensor alone. Indeed, the equation (1.4.4) shows that we can define the very variation of the height tensor $\boldsymbol{h}$ according to one of the two possibilities:

$$d\boldsymbol{h} + \boldsymbol{h}\boldsymbol{a} = \boldsymbol{0} \quad or \quad d\boldsymbol{h} + \boldsymbol{a}^t\boldsymbol{h} = \boldsymbol{0} \tag{1.4.11}$$

Actually, these conditions turn out to be completely equivalent, in view of the symmetry of the height tensor and the properties of the classical differential operation used here, whereby the matrix $d\boldsymbol{h}$ respects, in fact, the algebraical symmetry of the original matrix $\boldsymbol{h}$. Therefore, the whole argument can be taken as showing that we must have symmmetry of the product of matrices involved in the two expressions from the above equation (1.4.8), which means:

$$\boldsymbol{ha} = \boldsymbol{a}^t \boldsymbol{h} \tag{1.4.12}$$

Should a variation of the height tensor occur, whereby the symmetry of the height tensor is lost, this last condition is broken, and the matrix $\boldsymbol{a}$ can be defined in two different ways at will. As it stands now, however, the matrix $\boldsymbol{a}$ is uniquely defined by the first of the relations (1.4.11) as:

$$\boldsymbol{a} = -\boldsymbol{h}^{-1} d\boldsymbol{h} \quad \therefore \quad \boldsymbol{a} = -(dn)\boldsymbol{1} - \boldsymbol{E} \tag{1.4.13}$$

where the following notations have been used:

$$n \equiv \ln\sqrt{\Delta}, \quad \boldsymbol{E} \stackrel{def}{=} \begin{pmatrix} -(1/2)\omega^2 & -\omega^3 \\ \omega^1 & (1/2)\omega^2 \end{pmatrix} \tag{1.4.14}$$

and the matrix $\boldsymbol{E}$ has the differential forms of the $\mathfrak{sl}(2,R)$ coframe as entries:

$$\omega^1 = \frac{\alpha d\beta - \beta d\alpha}{\Delta}, \quad \omega^2 = \frac{\alpha d\gamma - \gamma d\alpha}{\Delta}, \quad \omega^3 = \frac{\beta d\gamma - \gamma d\beta}{\Delta} \tag{1.4.15}$$

which is to be met in the physical theory of the de Broglie waves [(Mazilu, 2020); see Chapter 3 there, especially equation (3.3.15); the whole §3.3 contains the essential algebraical properties of the $\mathfrak{sl}(2,R)$ manifold, based on these differentials]. The coframe (1.4.15) satisfies the Maurer-Cartan equations:

$$d \wedge \omega^k = C^k_{ij} \omega^i \wedge \omega^j \tag{1.4.16}$$

where the structure constants are taken as

$$C^1_{12} = 1, \quad C^3_{23} = 1, \quad C^2_{31} = -2 \tag{1.4.17}$$

the rest of them being zero. This algebraic structural arrangement for the $\mathfrak{sl}(2,R)$ algebra will be taken as standard in the present work, in matters of discussion of the essential properties of the Riemann manifolds of negative curvature in low dimensions. We have in mind typical example that will occur herein the dimensions 2, 3, and 4.

Now, just for a future theoretical benefit, we need to bring in an important observation: instead of equation (1.4.5), for this case we can have – by using for instance the first of the conditions (1.4.8) in equation (1.4.4), and then the second of (1.4.8) in the result thus obtained – the following condition:

$$dh = \langle x|\boldsymbol{a}^t \boldsymbol{h}|x\rangle \quad \therefore \quad dh = -\langle x|d\boldsymbol{h}|x\rangle \tag{1.4.18}$$

In other words, in this case, the variation of the height is the same as that from equation (1.4.5) but opposite in sign: *the* '*pegged*' *surface is deforming symmetrically*, which is a property that may be taken as characterizing the de Broglie region we once deemed as *strange*, inasmuch as the great physicist designated it as being a place of 'motion at constant time' *i.e. instantaneous motion* [see (Mazilu, 2020), §2.1]. Such a motion requires infinite velocity according to our experience, hence the term 'strange'.

Carrying now these observations back to the classical case of second fundamental form, as in the previous section, we can express the matrix $\boldsymbol{a}$ in terms of the matrix of curvatures $\boldsymbol{b}$ and its variation, as in equation (1.4.13). Then the matrix $\boldsymbol{A}$ itself, from equation (1.3.22) does not depend, indeed, but only on the curvature and its variation:

$$\boldsymbol{A} = \boldsymbol{i} \cdot \boldsymbol{b}^{-1} \cdot d\boldsymbol{b} \cdot \boldsymbol{b}^{-1} \quad \therefore \quad \boldsymbol{A} = \left\{ (dn)\boldsymbol{i} + \boldsymbol{i} \cdot \boldsymbol{E} \right\} \cdot \boldsymbol{b}^{-1} \tag{1.4.19}$$

so that the equation (1.3.21) becomes

$$\left| j \right\rangle \equiv \delta\boldsymbol{b} \cdot \left| s \right\rangle \quad where \quad \delta\boldsymbol{b} \equiv -\begin{pmatrix} \omega^1 & \tfrac{1}{2}\omega^2 + dn \\ \tfrac{1}{2}\omega^2 - dn & \omega^3 \end{pmatrix} \tag{1.4.20}$$

In other words, by infinitesimal deformation as defined here, the curvature matrix gathers a skew-symmetric part in need to be interpreted, for the curvature matrix loses its symmetry. In a classical view (Lowe, 1980), suggested by the mechanical deformation of the thin plates, this property should be connected with the *torsion* of surfaces. This further suggests the practicality of using the affine theory of surfaces.

Limiting the mathematics to what we have now, in this gauging, the infinitesimal deformation adds to the second fundamental form of reference 'pegged' surface characterized by matrix $\boldsymbol{b}$, a quadratic form having the matrix $\delta\boldsymbol{b}$ given in equation (1.4.20). The result is a surface having the support function $h$ as a quadratic form which is the sum of two quadratic forms which are mutually harmonic, that is they have the roots in a harmonic range, with their characteristic cross-ratio having the value $-1$, or a value compatible to this one:

$$h \equiv \left\langle s \middle| \boldsymbol{b} + \delta\boldsymbol{b} \middle| s \right\rangle = \alpha(s^1)^2 + 2\beta s^1 s^2 + \gamma(s^2)^2 + \omega^1(s^1)^2 + \omega^2 s^1 s^2 + \omega^3(s^2)^2 \tag{1.4.21}$$

since the coefficients are satisfying the algebraic condition:

$$\gamma \cdot \omega^1 - \beta \cdot \omega^2 + \alpha \cdot \omega^3 = 0 \tag{1.4.22}$$

The point of this construction is that the surface characterized by the support function (1.4.21) *should not be necessarily 'pegged'*: it is just an imagined surface in a continuum, that may or may not contain particles, *i.e.* a surface of the kind of those we imagine as being created by the electromagnetic field in ether. Since the condition (1.4.22) is also satisfied for the differentials $d\alpha$, $d\beta$ and $d\gamma$, the 'pegged' surface described by the matrix $\boldsymbol{b} + d\boldsymbol{b}$, has it too as a deformed surface. One can say that the two pegged surfaces delimit an Ampère element, represented by the de Broglie capillary tube.

Obviously, the case may also be made for an Ampère element whose delimiting surfaces are not 'pegged'. In this case, a generalization of Cartan's *Lemma 1* is essential, which completes our list of theorems necessary for introducing the physics into geometry. This generalization must be able to allow us distinguish the presence of *fields in matter* and their correlation with the *fields in space*, for instance. It has already been around for a relatively long time now, in the mathematical literature, under the name of Yoshio Agaoka, through the following theorem:

> Assume that $\pi_\alpha$ are $r$ exterior differential 1-forms representing a coframe with respect to an $r$-dimensional manifold in space. If $r$ differential $p$-forms $\omega^\alpha$ satisfy to equation
>
> $$\omega^\alpha \wedge \pi_\alpha = 0 \tag{1.4.23}$$
>
> then the $(p-1)$-forms $\gamma^{\alpha\beta}$ exist such that
>
> $$\omega^\beta = \gamma^{\beta\alpha} \wedge \pi_\alpha; \quad \gamma^{\beta\alpha} = \gamma^{\alpha\beta} \tag{1.4.24}$$
>
> for $\alpha, \beta = 1, 2, \ldots, r$ (Agaoka, 1989).

The Cartan's *Lemma 1* can be derived from this theorem of Agaoka as a particular value $p = 1$, in which case $\gamma^{\alpha\beta}$ must be $0$-forms, *i.e.* simply functions. The case $p = 2$ is 'maximal', as it were, in three dimensions, for in that case there are no exterior differential forms of degree higher than three.

However, there are exterior differential forms of degree three. In order to be able to use this theorem we shall try to read it in the spirit of the Cartan's *Lemma 2*, according to equation (1.3.11): there are two exterior differential 3-forms $\tau^{\alpha}$ which are null whenever the two *Lorentz 1-forms* $\pi_{\alpha}$, $\alpha = 1, 2$ *representing the contrariwise charge displacements* are null. In this case, the 3-forms should be, necessarily, a sum of exterior products:

$$\tau^{\beta} = \omega^{\beta\alpha} \wedge \pi_{\alpha} \tag{1.4.25}$$

where $\omega^{\alpha\beta}$ are four 2-forms that can be arranged into a matrix *not necessarily symmetrical*: the 2-forms from this equation carry two indices, not just one. By this, we try to legiferate the idea that $\tau^{\alpha}$ are zero whenever $\pi_{\alpha}$ are zero, but not *vice versa*: according to *Cartan's Lemma 2*, any one of the two differential 3-forms $\tau^{\alpha}$, separately, is null whenever the 1-forms $\pi_{\alpha}$ are null. Then the equations $\tau^{\alpha} = 0$ are to be treated according to the manner the equation (1.4.23) is treated, *i.e.*:

$$\gamma^{\beta\alpha} \wedge \pi_{\alpha} = 0 \quad \therefore \quad \gamma^{\beta\alpha} = \Gamma^{\beta\alpha\nu} \wedge \pi_{\nu}, \quad \Gamma^{\beta\alpha\nu} = \Gamma^{\beta\nu\alpha} \tag{1.4.26}$$

In words: the realm defined by the equations $\tau^{\alpha} = 0$ but where $\pi_{\alpha} \neq 0$, is geometrically described by *six exterior differential 1-forms* denoted here by $(\Gamma^{1\alpha\beta}, \Gamma^{2\alpha\beta})$, and symmetric in the last two indices. This realm should be the de Broglie's region of a *Lorentz surface* serving to define the concept of electricity: the 1-forms $\pi_{\alpha}$ are the two differentials vanishing on it, so that, from an electric point of view it should be a neutral surface, as the Lorentz's hypothesis requests. In other words, the realm described by the exterior 3-forms from equation (1.4.25) is a de Broglie zone where, according to previous observations, the phenomenon of holography dominates.

Thus, each one of *the two exterior 3-forms* $\tau^{\alpha}$ from equation (1.4.25) represents an infinitesimal matter manifold within a sea of charge, where, according to classical ideas connected to the concept of Ampère elements periodic processes imagined by Riemann, Betti, and Lorentz can take place. And since a particle can present to the world two kinds of charges at once, we need for such an interpretation two exterior 3-forms. Agaoka's theorem, thus conceived, is a clear signal for adopting the thought that the *Cartan's Lemma 2* is not sufficiently used in physics. For once, it may be taken as generalizing the idea of Dirac's constraints, that often lead to contradictions [see (Pons, 2005); also (Pitts, 2014)]. But the most important part it plays is in elucidating the content of the concept of density, to which we shall come again a few times during this very work.

### 1.5 A Classical (Re)Assessment of the Lorentz Theory of Matter

With André-Marie Ampère, the concept of Newtonian forces – and with it, naturally, the concept of action at a distance, of course – came to a crossroad. In order to offer a better understanding of the issue at hand, we find it necessary to present this moment of knowledge not so much by quotations from Ampère himself, according to our custom so far, but mainly by a discussion of the concept of *central forces*. The properties of such forces are key points in the arguments that, during the 19th century, led to the ideas that lie today at the foundations of the field theory. And, in our opinion, we must agree with Poincaré who once expressed how much the central forces meant for the development of physics, as compared to the energetical doctrine. Quoting:

The hypothesis of central forces *contained all the principles*; it entailed them as necessary consequences; it demanded both the *conservation of energy* and *that of masses*, and the *equality of action and reaction*, and the *law of least action*, which appeared, it is true, not *as experimental truths, but as theorems*; and whose enunciation had, at the same time, *that I don't know what* of a more precise and less general than their present form. [(Poincaré, 1905), p. 196; *our rendition and emphasis*, *a/n*]

The moment Ampère of human knowledge is the occasion when the very central forces came under scrutiny, and we chose to detail it for the good reason that some essential ones among 'all the principles', took then a turn which, eliminating "that I don't know what" of the central forces, as Poincaré would say, brought them in "their present form", which is not the most appropriate in building, say, a physics of the brain, for instance. In short, this kind of physics needs now, as it always did in fact, quite an alternative turn, to be found only in that very moment of our knowledge, because, subsequently, it was obliterated by the development of main-stream physics.

Now, the first in the order of things to be done here, is to clarify what are those key points which enticed the discussions that helped create the electrodynamics along the lines initiated by Ampère, *i.e.* with no consideration of the Faraday's induction phenomenon whatsoever. The Newtonian forces of physics were then – and we have to recognize that they still are now, to a great extent – central forces. No wonder then, that Ampère would try to find a way to use this kind of forces, along the way initiated by his illustrious predecessor, this time, though, in electrodynamics, where the essential premise of their existence is quite problematic, to say the least. That way can be perceived from the fact that Ampère was fully aware that there is a clear difference between the genuine Newtonian forces describing the action of gravitation or that of static electricity, and the forces acting between two *currents* (Ampère, 1823). [Incidentally, all the essential reports of Ampère can be found in the collection cited by us here as (Ampère, 1990)]. In order to see the difference in question, let us present the issue from a definite perspective on the central forces.

Quite important, from the standpoint of the electrodynamics rising in the first half of 19$^{th}$ century, is the property that allowed Ampère's equation of forces between *elementary currents*: if the forces are conservative and central, as in the case of Newtonian forces, *their magnitude does not necessarily depend exclusively on the distance* between the two places of the action at distance. This can be seen right away, assuming a typical central conservative force with the magnitude depending on coordinates separately, *i.e.* a force that can be written as a vector in an arbitrary Cartesian reference frame, where we write it as: $\boldsymbol{f}(\boldsymbol{r}) \equiv f(x, y, z)\cdot\boldsymbol{r}$, in view of the centrality property. This field has to satisfy the *Helmholtz conditions*: a scalar one amounting to $\nabla\cdot\boldsymbol{f}(\boldsymbol{r}) = 0$ and a vectorial one, in the form $\nabla\times\boldsymbol{f}(\boldsymbol{r}) = \boldsymbol{0}$; these are automatically satisfied by the original Newtonian forces. From the second of these conditions we have, in detail:

$$\nabla f(x,y,z)\times\boldsymbol{r} = \boldsymbol{0} \quad \therefore \quad \nabla f(x,y,z) \propto \boldsymbol{r} \tag{1.5.1}$$

if the force is not to be a constant in the chosen reference frame. The first Helmholtz condition then becomes:

$$\boldsymbol{r}\cdot\nabla f(x,y,z) + 3f(x,y,z) = 0 \tag{1.5.2}$$

and shows that the magnitude of force must be a homogeneous function of degree $-3$ in the coordinates. It is only in the particular cases where this function is $r^{-3}$, that we get the Newtonian forces going inversely with the distance squared. Otherwise, the magnitude $f(x,y,z)$ can very well be the reciprocal of a third degree homogeneous polynomial in the three coordinates, or a *–3/2* power of a homogeneous quadratic form, as in fact happened in the original Newton's case. Actually, it can be any other form leading to a homogeneous function having the degree *–3*. Combining (1.5.1) and (1.5.2) we find that the most general force satisfying both Helmholtz conditions:

$$\boldsymbol{f}(\boldsymbol{r}) = -(\frac{r^2}{3}) \cdot h_{-5}(x,y,z) \cdot \boldsymbol{r} \tag{1.5.3}$$

where $h_{-5}$ must be a homogeneous function of degree *–5*, as indicated by its lower index. This expression of the vector force can be rearranged to appear as proportional to a Newtonian force:

$$\boldsymbol{f}(\boldsymbol{r}) = h_0(x,y,z) \cdot (\frac{\boldsymbol{r}}{r^3}), \quad h_0(x,y,z) \equiv -(\frac{r^5}{3}) \cdot h_{-5}(x,y,z) \tag{1.5.4}$$

with the coefficient $h_0$ – a function *homogeneous of degree zero in coordinates*.

In other words, the most general force field satisfying the two Helmholtz conditions concurrently – taken as essential properties of Newtonian field of forces, and extended as fundamental properties of any conceivable central force – must be proportional to a genuine Newtonian force field, with the proportionality described, in a system of Cartesian coordinates, by a factor which can be either *a function of the ratios of coordinates*, or *a constant*, as in the genuine Newton's case. In order to be homogeneous of degree zero, the factor of proportionality can depend on coordinates both through the Euclidean distance *r*, and through some trigonometrical functions, which may be arbitrary in principle. One can therefore say that equation (1.5.4) generalizes the classical Newtonian case, which is what André-Marie Ampère apparently upheld, both conceptually, by his explicitly stated task, and factually, by building his expression of the electrodynamic force.

This generalization, however, is not quite complete, for it is referring only to a *space geometry*, with no involvement of physics whatsoever. In the classical case of genuinely Newtonian forces, one would have the *masses*, or the *charges*, or even both in fact, located at the two positions involved in the interaction at distance: the forces are *bilinear* in those physical quantities. In keeping with the idea of continuity and space extension of the matter, we may think of some mass elements, or charge elements – Hertzian 'higher order infinitesimals' – located at the two positions involved in the action at distance, and entering the equation of force through bilinear expressions, *i.e.* by their product. This was the original Newton's case, and so the equation of force may appear in the case of currents. The problem remains, though, concerning the *measure* of the current elements: while in the case of genuinely Newtonian forces one can think of the differentials of mass or charge, in the case of currents issues of relative directions occur. The common view arising just about the Ampère's epoch was that a current element should be represented by the time rate of variation of the charge, $d\boldsymbol{q}$ in the notation of the previous section – known as the *intensity* of the current – multiplied by the element of 'wire', thought to be a *line* in space: $Id\boldsymbol{l}$.

And thus, the force between two elements of wire, $d\boldsymbol{l}$ and $d\boldsymbol{l}'$ carrying currents of intensities *I* and *I'*, as provided by André-Marie Ampère in his pioneering work on electrodynamics (Ampère, 1990), can be written, in a form using modern notations, as [see (Assis, 1994), Chapter 4, Equations (4.14 – 15); see also (Darrigol, 2002), Appendix A, equation (A.3)]:

$$d^2\boldsymbol{f}(\boldsymbol{r}) = -I \cdot I' \cdot \left[ d\boldsymbol{l} \cdot d\boldsymbol{l}' - \frac{3}{2}(\hat{\boldsymbol{r}} \cdot d\boldsymbol{l})(\hat{\boldsymbol{r}} \cdot d\boldsymbol{l}') \right] \cdot \frac{\boldsymbol{r}}{r^3} \tag{1.5.5}$$

Here $\hat{\boldsymbol{r}}$ is the unit vector of $\boldsymbol{r}$, joining the *midpoints* of the two elements of current. This expression of the force is a *bilinear symmetric form* in the two current elements involved in the action at distance, so that the equation (1.5.5) can, indeed, be considered as a natural generalization of the classical central force, for the specific case of the current elements. It is in this circumstance that the vector $\hat{\boldsymbol{r}}$ becomes a field to be geometrically considered by itself, but this raises the problem of reading the formula of forces thus provided. Notice, however, a point to be necessarily considered: if we take the Helmholtz conditions as fundamental in defining the general Newtonian forces – and, again, we should mention the obvious implicit clause, that the forces are in vacuum – then the equation (1.5.5) needs to be compared with the first of the equations (1.5.4). Therefore, according to this conclusion, the Ampère coefficient from equation (1.5.5), if expressed in Cartesian coordinates, must be a homogeneous function of degree zero:

$$-I \cdot I' \cdot \left[ \boldsymbol{dl} \cdot \boldsymbol{dl'} - \frac{3}{2}(\hat{\boldsymbol{r}} \cdot \boldsymbol{dl}) \cdot (\hat{\boldsymbol{r}} \cdot \boldsymbol{dl'}) \right] \overset{def}{=} f_0(x,y,z) \tag{1.5.6}$$

The unfolding of physics since Ampère's epoch has imperatively asked for deciding on the physical structure of the elementary 'lengths' $\boldsymbol{dl}$ and $\boldsymbol{dl'}$ and, in our opinion, the equation (1.5.6) is the only one guiding our possible decisions. To justify the need for such a structure, it suffices to cite the 'excuse' made by Planck in the quote we excerpted by us in §1.1 for the choice of a special kind of dipole: the *concept* of a dipole needed charges with three degrees of freedom in motion at each one of its ends. However, the Kirchhoff's principles allowed him to choose only the simplest case of one degree of freedom along the direction of dipole, for they assured him that there should be no "fear of any essential loss of generality of the conclusions". With this observation in mind, let us continue our journey for an assessment of the Lorentz definition of the matter.

For once, we shall take the equation (1.5.5) just as it has been taken starting with Ampère himself, and as is suggested by Darrigol's notation: the key of reading is the general vector relation $\boldsymbol{r}^2 \equiv (\boldsymbol{l} - \boldsymbol{l'})^2$, showing what should be meant by the relative position of the two elements of current. Namely, $\boldsymbol{l}$ and $\boldsymbol{l'}$ are the *position vectors* of the two locations in the chosen reference frame, of the two current elements $I\boldsymbol{dl}$ and $I'\boldsymbol{dl'}$ involved in the action at distance. Then, according to Edmund Whittaker, the expression from equation (1.5.5) can be taken as the right expression incorporating the observation that the action of a closed circuit on an elementary current is perpendicular to this current [(Whittaker, 1910), pp. 89 *ff*]. Be this reading as it may, a hint of the depth of Ampère's inovation in using the central forces is apparent and, from our point of view, needs to be stated explicitly right away. It concerns the answer to the natural question: why would anyone need the force between two current *elements*, while the specific experience in constructing this physics is always referring to *finite* currents or *finite parts* of currents?

The answer is, in our opinion, obvious: *there is nothing*, *in this specific area of our experience*, *analogous to the Kepler motion*, to sustain the idea of a central force for the case of finite currents, as that motion once did for Newton. So, if one would want to continue Newton's tradition on forces, as Ampère did – according to his own statement from the very beginning of his extensive work – one would need to extract from experience those situations equivalent to the one faced by Newton himself. This operation cannot be accomplished but only conceptually, for it obviously involves matter formations without space extension – ideally differentials, if it is to give them a measure at any rate, that is the 'highest order infinitesimals' of Heinrich Hertz – in which case one can indeed operate with central forces, in the way suggested above. But there is a subtle catch here: if we are to exact a conceptual logic, then it does not make any sense to talk of the 'midpoints' of the current elements, since one cannot talk of the midpoint of a differential element.

However, the formula given in equation (1.5.5) is *abstracted* by Ampère from a whole set of experiments with *finite currents*, divided into four classes, destined to cover all typical situations of the relative positions of the two *finite* conductors involved in the action at distance [(Maxwell, 1892), Volume II, Part IV, Chapter II; see also (Whittaker, 1910), and (Assis & Chaib, 2015)]. Based on these facts of experience we conclude that, after such a careful classification and conclusions concerning the finite currents, Ampère just made a *conceptual transition of scale* from *finite* to *infrafinite* currents – the highest order of infinitesimals in a Hertzian natural philosophy – in order to be able to use the theory of Newtonian forces exactly as it was designed by its illustrious creator. Only, then, Ampère was induced into calculating the forces at a finite scale by a procedure of integration, so as to be allowed to use the experience in constructing the physics of currents. Indeed, any specific experiment with a current, necessary involves a closed circuit: the expression of force needs to satisfy this requirement. This explains the differential notation of the force – we first met it to Assis and then to Darrigol (*loc. cit.*), for no one of authors, at least not those in our study, seems to be concerned with its use – from the left hand side of equation (1.5.5): the *whole* force needs a *double integration* along the finite paths followed by each wire in turn. Not only this, but, noticing such detail, some other ones start showing up: an element of *real* current, for instance, may have a direction entirely independent of the path followed by its designated position. For, in reality, the wires are not lines, but three-dimensional structures, having space extension, and in such a space – a proper coordinate space, we should say – the currents can go in many different directions, not just one. Therefore, as the illustrious Gauss would say, this is precisely the place to observe the difference between the 'geometria situs' and the 'geometria magnitudinis', and to adapt the mathematics as a consequence of this observation (Gauss, 1833).

Regarding this adaptation, the great mathematician Joseph Liouville was the first one among scientists to draw attention to a specific issue connected with the force expression (1.5.5) even from the times of Ampère (Liouville, 1831): the possible noncentrality of the forces in electrodynamics. At that time he was just 20 years old, a student of Ampère, helping in editing the course notes and some of his Master's scientific communications. It is on this occasion that he learned about the newly raising science of electrodynamics, and even tried to publish his first article ever, that was originally addressed – critically we should say, however within a true scientific attitude – to electrodynamics. Unfortunately, he was pushing his production through the Academy, and 'les immortels' (Ampère, Arago, Maurice) had a limited possibility of understanding the *new mathematical principles* of the natural philosophy he used in electrodynamics. So that the Academy magnanimously rejected the publication of the work, giving the "young mathematician" an avuncular encouragement though, to insist on his path, for he showed "sagacity". We would have never been able to find what that work of youth of Liouville was about, had he not find some other ways of making himself known. Incidentally, we were able to appreciate just how 'new' are the 'mathematical principles' involved in the natural philosophy of electrodynamics of Ampère, from the nice work that laid the foundation of the modern theory of *fractional calculus* (Liouville, 1832). For once, though, after that 'kind' rejection, we should be thankful that Liouville agreed to publish what the immortals found worth publishing from that first production of him. First, in the Bulletin de Férussac a note appeared (Liouville, 1829), probably with the assistance of his friend Charles Sturm; this was the first publication ever of Liouville. Then, a short summary of his natural philosophy, where he made obvious what is essential in the electrodynamics based upon Ampère's experiments, has also appeared (Liouville, 1831), probably with Arago's assistance. The whole story of this turmoil of the young Liouville is vividly presented in (Lützen, 1990), Chapter VII. We are interested here only in that natural philosophy of electrodynamics, extracted by Liouville from the phenomenology built by Ampère himself, and used in foundation of electrodynamics.

Exactly to what the Liouville's note just cited is referring, is quite an interesting fact by itself, from a purely natural-philosophical point of view. Quoting:

> One obvious observation is immediate, *concerning the nature of the source of current*. If, in practice, two currents are set in the presence of one another, *the mobile current may be arbitrary*, but *the acting current is always closed*, or *liable to be considered closed*. This is a strict condition; for, the *electricity must move without interruption from one pole to the other*, of an arbitrary number of current sources.
>
> Since the acting current is closed, it follows that if there are forces from one element to another, the integral of which disappears when one of the currents is continuous, *our experiments will never be able to show these forces*. As for the practical results, they will be the same as in the case when these forces are zero.
>
> The existence of such forces is a priori not meaningless. It is known that the action of two magnetized molecules offers a very simple example. Indeed, a closed magnet has no action whatsoever on an arbitrary body, no matter of the degree of magnetization: for us it appears as non-magnetic; and yet, *when broken*, its *various portions can produce considerable effects*.
>
> Therefore, rigorously, one could assume that two voltaic elements act upon each other by *a compound action* as follows: 1° *a force* directed along the straight line passing through their middles and represented by Mr. Ampère's formula; 2° *four other forces* similar to those coming from *two magnetic molecules*, forces whose resultant *is not generally along the line joining* the two bodies [(Liouville, 1831); *our translation and emphasis*, *a*/*n*].

This excerpt, even if not conceptually complete or, more to the point, even if not exact, gives one a taste of the difference between the *action at distance* and *the force*: the action at distance may be realized not only by a single force, as in the case of central forces, but by *a set of forces*, so that such an action may appear as noncentral. This is the general idea of the new philosophy of forces initiated by Ampère in order to complete the classical Newtonian theory of forces. In a classical expression, it may even be admitted that the action at a distance may be realized even by *noncentral forces*, as in the later case of J. J. Thomson. While, for more details we recommend the Jesper Lützen's monograph on the life and works of Joseph Liouville (Lützen, 1990), for now we are in debt with an essential explanation on the kind of forces considered by Liouville in the above excerpt.

These forces are described only in the fragment published in the Bulletin de Férussac from the year 1829, especially the Figures 11 and 12 from page 449 of that tome (Liouville, 1829). Reading this first published work of the 'young mathematician', we are under impression that, with Liouville, we have a moment of the theory of electricity analog to that of Thomas Hobbes in the case of light: *the Liouville*'*s elements of current* are the analogues of the *lines of light* from the Hobbes' model of light, that later became *orbicular pulses* for the first physical model of the light ray, constructed by Robert Hooke [see (Mazilu, 2020), §2.1, and the works cited there, for a detailed story of the coming to being of the concept of light ray]. Right now, we are just following a suggestion contained in the analysis of Liouville: the noncentrality of electrodynamic forces is an effect of the process of association of charges into the natural neutrality characterized by the zero overall charge from the case of a dipole. And this process must be a stochastic process. As far as we can see, this was the very path taken by the physics itself, in the problem of the dynamics of currents.

Indeed, the program followed along the path taken by the mainstream physics is best illustrated through the words of Gauss from an 1845 letter to Wilhelm Weber. Quoting, again:

> Perhaps I will be able to delve a little bit more into these things, from which I had strayed away until you have pleased me with a visit at the end of April or the beginning of May, for you gave me hope. No doubt, I would have made my investigations public long ago, had it not been for the fact that, at the time I interrupted my preoccupation on them I was missing what I considered to be the actual keystone, namely *the derivation of the additional forces* (which are to be added to the mutual effect of *stationary parts of electricity in case they are in relative motion*) due to *the non-instantaneous, but time-propagating* (in a similar way to *light*) effect…
>
> *Nil actum reputans si quid superesset agendum*
>
> I did want to succeed in this at that time; but, so much as I remember, I left the study, not without the hope that I might succeed later, although – if I recall it correctly – with the personal conviction that it was first necessary *to have some practical idea on how the propagation occurs* [(Gauss, 1845); *our rendering and Italics*, *a*/*n*].

The Latin adage – in a suggestive translation: *Nothing has been done*, *if something remains yet to be done* – explaines that remarkable feature of the Gauss' scientific ethics, from the species of which we can hardly see something today: in any *public* productions one should be as thorough as it gets, and if it is not possible a right conclusion then better not make anything public. For, inherently, such productions are never *absolutely* thorough.

Notice the phrase that can be taken as an important conclusion regarding the object of physics involved in the electrodynamics of Ampère's times: 'stationary parts of electricity'. The whole physics of the parts of electricity was actually described in those times by the principles involved in the *statics of forces* between what is referred to by Gauss as 'stationary parts of electricity'. The 'additional forces' are then referring to such parts but in *motion*, and this is the way taken then by the future physics. This physics was, indeed, built towards the end of the 19th century and the beginning of the 20th century, on the particular foundations of electrodynamics, that suggestively took the form of 'electrodynamics of *moving bodies*', which culminated with the special relativity.

Another point here is the realization – which, according to Gauss' words in the above excerpt, may have been inspired by Wilhelm Weber – that the forces must have those *additions* due to the *relative* motion of the current elements. In other words, the kinematic forces are entirely different from their statical counterparts. The term 'relative motion', was understood by Gauss, and is still understood today – in spite of the discovery of electromagnetic waves – as a clear consequence of the *identification* of *current elements* with the *currents of particles in motion*, describing these currents inside the wires as a kind of fluxes of charge. Again, this very identification or, more to the point, the lack of its *proper recognition* by the natural philosophy, marks the whole physics, even as we have it today. We, however, *recognize* it: let us see where it leads, and what the consequences may be for what we think is a *proper* physics.

Our reason to consider Bernhard Riemann as a protagonist in constructing the electrodynamics is quite simple: while his teacher and friend, the great Carl Friedrich Gauss, only mentioned that he stopped short of finding a 'practical idea of how the propagation occurs', Riemann realized one of the most remarkable mathematical descriptions of such a practical idea. It is so remarkable that even today it is considered as one fundamental idea in the physical theory of fields. Namely, we have to notice that the propagation of light, as well as the propagation of heat in solids are only mentioned by Gauss. With the work of Riemann, however, we have a first instance of what later came to be known as the Klein-Gordon equation in the theory of fields. More importantly for us, this equation was used by Louis de Broglie in order to characterize his concept of physical ray [see (Mazilu, 2020), §2.1, equation (2.1.4)]. Riemann's own words in introducing it are:

> I took the liberty to communicate to the Royal Society a remark which brings the theory of electricity and magnetism into a *close connection with the theories of light and radiating heat*. I have found that the electrodynamic effects of galvanic current can be explained if one assumes that the effect of an *electric mass* on other masses does not occur *instantaneously*, but *propagates to these with a constant speed* (equal, within observational errors, to the speed of light). The differential equation for the propagation of electric force becomes, by this assumption, *the same as the equation of propagation of light and radiating heat* [(Riemann, 1858a); *our rendering and Italics here, a/n*; see also (Riemann, 1858b) for translations of the whole article; also the monograph (Laugwitz, 2008), has a translation of this very excerpt, on pp. 261 – 262].

Riemann's approach has been criticized from different points of view, on which we have no room of insisting right now; the interested reader is referred to the trustworthy literature already cited above, for due details in case they are needed indeed. However, from our point of view expressed all along our present endeavor, Riemann's work from which we excerpted the fragment right above was – in spite of some forced mathematical technicalities, and perhaps some misplaced natural philosophical conclusions – right in the place where it should have been, since from a physics' of charges point of view, no other place seems more adequate for applying its content. Quoting again:

> According to the existing view about *electrostatic action*, the potential function $U$ of arbitrarily distributed electrical masses, when $\rho$ is their density at points $(x, y, z)$, is defined by the condition
> $$\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}+\frac{d^2U}{dz^2}-4\pi\rho=0 \tag{1.5.7}$$
> and by the condition that $U$ is continuous and constant at infinite distance from the *acting masses*. A particular integral of the equation
> $$\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}+\frac{d^2U}{dz^2}=0 \tag{1.5.8}$$
> continuous everywhere except the point $(x',y',z')$, is

$$\frac{f(t)}{r} \tag{1.5.9}$$

and this is the potential function generated by the point $(x',y',z')$, *if there is the mass* $-f(t)$ *at the time t* located in that point.

Instead of this, I now assume that *the potential function U is determined by the condition*

$$\frac{d^2U}{dt^2} - \alpha^2\left(\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}+\frac{d^2U}{dz^2}\right) + 4\pi\alpha^2\rho = 0 \tag{1.5.10}$$

so that the potential function generated by the point $(x',y',z')$, *if the mass* $-f(t)$ *is located in it at the time t*, becomes

$$\frac{f(t-\frac{r}{\alpha})}{r} \tag{1.5.11}$$

[(Riemann, 1858); *our rendering and Italics*; see also the English translations (Riemann, 1858b, 1985)].

The formula (1.5.11) is usually seen as the precursor of the idea of retarded potentials, of which a first specimen (Lorenz, 1867) occurred just about the time when Riemann's work reached the scientific literature, *i.e.* about a decade after it was first presented to the Academy. However, we did not bring here the above excerpt in order to discuss issues that may seem of priority, but in order to notice the procedure, which fits harmoniously in a general natural philosophy. Notice that the Riemann's idea was to use the classical theory of *static forces* between two charges in order to describe the *instantaneous interaction*: the particles at different times interact just as the particles at the same time, only with different active masses, which can, however, be described quantitatively by the equation of potentials as proposed by Riemann.

Indeed, it is quite clear that what Riemann had in mind, was not so much the mathematical procedure of solution, that came to be known later as 'retardation', as much as he wanted to characterize the 'acting mass' in connection with its location with respect to the point where the action is exerted. We have noticed before that the Ampère's era natural philosophy regarding the forces, stands précised in the words of Joseph Liouville, that can be simply summarized by the observation that the Newtonian forces must be, in general, non-central forces. One can say that the problem of Newtonian forces was condemned from the very beginning to an uncertainty, a fact best expressed toward the end of the 19th century by Henri Poincaré on the occasion of the analysis of Hertz's mechanics. To wit: after the analysis of the possibilities of defining the forces in connection with masses, which, as well known, was Newton's main point in defining his gravitation force between bodies, the great natural philosopher is obligated to conclude that…

We are left therefore with nothing, and our efforts were unfruitful; we are compelled to adopt the following definition, which is nothing else but a confession of incapability: *the masses are coefficients*, *convenient to introduce in calculations*.

We will be able to redo the whole Mechanics, by attributing to all the masses different values. This new Mechanics will not be in contradiction with experience nor will it be contradicting the general principles of Dynamics (the principle of inertia, the proportionality of the forces with masses and with accelerations, equality of action and reaction, rectilinear and uniform motion of the center of gravity, the principle of areas).

Only, the equations of this new Mechanics will be *less simple*. Let's understand this well: only the first terms will be simpler, i.e. the ones we already know from experience; it would be possible that, by altering the masses by small quantities, the complete equation neither gain nor drop anything from their simplicity.

I insisted on this discussion longer than Hertz himself; I meant to show, though, that Hertz didn't simply look for quarrel with Galilei and Newton; we must agree to the conclusion that in the framework of the classical system *it is impossible to give a satisfactory idea for force and mass*. [(Poincaré, 1897); *our translation, original emphasis*, *a*/*n*]

Riemann was apparently well aware of these problems, prompted to criticality during the Ampère moment of human knowledge, by the necessity of characterization of the concept of *acting masses* in the case of dynamic electricity. No better proof for this awareness can be offered, than the precautions taken by him in posing the problem to be solved by the calculation of those 'additions' to static forces, as mentioned by Gauss, his illustrious predecessor. One can say that on this issue, Riemann went back to Newton, and took the solution of the problem from the very point it was left by Newton himself. Quoting, again:

Let S and S′ be two conductors traversed by a constant voltaic currents but not moving towards each other; let $\epsilon$ be an *electric mass element* in the conductor S that at the time t is located in the point (x,y,z); let $\epsilon'$ be an electric mass element of S′ that at the time t′ is in the point (x′,y′,z′). As regards the motions of the electric mass elements, which in each *conductor element* are opposite for the negative and the positive electricity, I assume that at every time moment they are so distributed that the sums

$$\sum \epsilon f(x,y,z), \quad \sum \epsilon' f(x',y',z') \tag{1.5.12}$$

extended over all the electric mass elements in the conductor can be neglected as compared with the same sums extended only over the positively electrical, or only over the negatively electrical mass elements, *as long as the function f and its differential quotients are continuous*. [(Riemann, 1858); *our English rendering and emphasis*, *a*/*n*; *compare the other existing translations*]

To us it is quite clear that Riemann realized how important is to distinguish between 'electric mass elements' (*Massentheilchen*) and 'conductor elements' (*Leitertheilchen*). It is easy to see that in this excerpt he defines in any moment a static state of the elements involved in the expression of Ampère force, based on the freedom of flow of charges within the conductor through which the flow proceeds. Such a state is conditional on the neutrality of the conductor at every time moment, whose mark involves an indeterminate function in the mathematical expression of the force – therefore of the potential – which has to satisfy general requirement of continuity and differentiability. In the equation proposed by Riemann such a function is quite arbitrary but depends on time nevertheless, suggesting that the mass itself should depend on time. The position dependence is then eliminated, and remains an uncertainty in the theory. However, it can be brought to bear again by Riemann's prescription given in equation (1.5.11). This is why we take Riemann's prescription as a necessity of defining *simultaneous* *mass elements*, of the species that started being only lately noticed in theoretical physics, in connection with the relativity prescriptions of non-local character [see (Amelino-Camelia, Freidel, Kowalski-Glikman, & Smolin, 2011), especially their suggestive Figure 2].

One can see that, in order to realize Gauss' idea, Riemann was induced into recognizing, apparently for the first time in the natural philosophy, the independence of the electric fluid from the material conductor of electricity. One can even say that this moment of our knowledge marked the recognition of a necessity of conceptual dissociation in modeling, between the conductor and the electricity running through it, thus making out of them a system equivalent to the classical ray, of the kind referred to by Louis de Broglie. The Liouville moment of describing the forces would appear, in this analogy, as only a description of the possible forces between the lines of current – that we like to call *Liouville elements* – inside the material conductor, which can be seen as the capillary tube containing them. Liouville's geometry associates, according to the Ampère theory, the midpoint of these elements with the application points of the forces. Thus, one can see that the association of elements in order to realize the Riemann's condition of neutrality of the conductor, becomes now a random process, that cannot be regulated but by a Lorentz's kind of assumption: *there should be a neutral surface on which the charge is zero* along the conductor, and only this surface is the home of an Ampère force *between conductors*. In hindsight this is the whole moral of a physical theory of infinitesimal deformation, as presented by us above in §1.3. If we may be allowed a speculation, a theorist of Riemann's astuteness, might have sensed here a contradiction lurking, between such a concept of action and the Newtonian definition of force. After all, it may be this awareness that determined him to withdraw the article from the Academy, not just some mathematically quantitative mistakes. But, let us see what these mistakes stand for by themselves, since they too are today part and parcel of our process of knowledge.

By the unavoidable course of his life, Riemann came very close to an Italian school of mathematics, mainly represented, on the subject we are interested here, by two notable Italian mathematicians from the University of Pisa: Enrico Betti and Eugenio Beltrami. Their names appear alongside the name of James Clerk Maxwell, sometimes even in issues of priority based on a Riemann kind of modern electrodynamics, against Maxwell's, issues to which we have strong reasons *not* to subscribe. Fact is that Enrico Betti contributed to reinforcing the Riemann's electrodynamics based on the principle presented above, while Eugenio Beltrami addressed a few direct critiques to Maxwell's ideas which, in our opinion, are far from posing questions of priority. Quite the contrary, on a deeper analysis Beltrami's critiques simply show that Maxwell ideas are, in fact, destined to *sustain* Riemann's electrodynamics, or *vice versa*. But we postpone a specific discussion for a later occasion, while for now we focus only on one of Betti's contribution to electrodynamics, connected with Gauss' ideas as communicated in the letter to Weber from which we excerpted above. On this occasion, though, we intend to show, on one hand, how Riemann's concept of electrodynamics was usually perceived at the time it got through into the open, and, on the other hand, that its necessary assessment was by no means exhausted by critique. Quite contrary, it appears to us that a proper assessment remained, by and large, unfulfilled. Also, Betti brought in a tangible *idea of phase* in one of its modern connotations, namely the one that led to the *concept of frequency* of the physical optics in the times of Fresnel. Quoting, therefore:

> In 1858 Riemann presented a paper at the Academy of Science from Gottingen, published, after his death, in the number six of the Poggendorff's Annalen for 1867, where he deduced the potential of two constant *closed currents* acting upon each other, admitting that *the action of electricity propagates in space with a constant velocity equal to that of light*, and assuming that *the current consists of the motion of two electricities*, positive and negative, *travelling simultaneously through the wire in opposite directions* and, moreover, that the sums of the products of positive and negative electricities by a function of the coordinates of the points of wire, are negligible when compared with the sums of the positive electricity alone, or of the negative electricity alone, multiplied by the same function. This concept of *electric current*, completely ideal, *is hardly in agreement with what is known about it*, and it seems that Riemann himself was not satisfied, thus withdrawing the article from the Secretariat of Academy, and renouncing to publish it later on. In this context, it seems to me that it is not without importance to show how *the electrodynamic actions can be explained by means of their propagation in time*, considering that *the action of dynamic electricity takes indeed place according to Newton's law for static electricity*, without being based on that concept though, but assuming instead that *the current consists of a periodic polarization of the elements of the wire*, which is more in agreement with all known facts [(Betti, 1868), *our rendition and Italics*, *a*/*n*; *see also a previous English translation* in (Betti, 1985)].

By comparison with Riemann's initial view, referring to an instanstaneous statics, necessary to build a state of the current element, Betti shows that the idea of current as a flow is usually perceived as 'the motion of two electricities traveling simultaneously in opposite directions', in the manner conceived later by Lorentz (see §1.3 above). It is not quite obvious that he succeeded in distinguishing between the 'mass element' and 'conductor element' as clearly as Riemann did, but apparently he targets this last one, understood by him as 'element of wire', gaining periodic polarity during the transport of charge, "more in harmony with all known facts". From the development of the original work, however, it seems that Betti shared the contemporary idea of a Liouville current element, whereby the path of charge and the wire are identified with each other. But let us follow closely the Betti's own work (Betti, 1868).

He starts with the observation that the interaction potential of two closed conducting material loops carrying currents, has the known mathematical form used, indeed, by Riemann himself, which we reproduce here within our previous notations as

$$U = I \cdot I' \cdot \oint_{l} \oint_{l'} \frac{\cos(d\boldsymbol{l}, d\boldsymbol{l}')}{r} ds \cdot ds' \tag{1.5.13}$$

where $ds$ is the element of length of the path $\boldsymbol{l}(t)$, while $ds'$ is the elementary length of the path $\boldsymbol{l}'(t)$. Now Betti notices (*loc. cit. ante*, §II), that in view of the definition $r^2 \equiv (\boldsymbol{l} - \boldsymbol{l}')^2$ the expression (1.5.13) can be rewritten as

$$U = -I \cdot I' \cdot \oint_{l} \oint_{l'} \frac{d^2(r^2)}{dsds'} \frac{ds \cdot ds'}{r} \tag{1.5.14}$$

and by a few manipulations based on partial integration, using the property of cyclicity of the two currrent loops, can be brought to the Riemann's own form

$$U = -\frac{I \cdot I'}{2} \cdot \oint_{l} \oint_{l'} \left[ r^2 \cdot \frac{d^2}{dsds'} \left( \frac{1}{r} \right) \right] ds \cdot ds' \tag{1.5.15}$$

This much we can get using an obvious mathematics and the experiment. The problem was to explain this formula by the interaction of the current elements, and to this end Betti introduces the idea of what he calls – and we cannot but agree with the name today – a *phase*:

> The elements of the two curves *s* and *s'* are *periodically polarized*, that is they act on each other as if they were *magnetic elements* with the axes in the direction of the tangents to the curves, and they had the respective moments *m* and *m'* variable with the time, that is
>
> $$m = f(t)ds, \quad m' = F(t)ds' \tag{1.5.16}$$
>
> where *f*(*t*) and *F*(*t*) are *functions that take the same values on very small intervals of time* equal to $p$.
>
> Assuming that the action propagates in space with the speed *c*, the potential of a line upon the other, over a whole period, will be
>
> $$U = \int_0^p dt \int_s \int_{s'} f(t)F(t - \frac{r}{c}) \cdot \frac{d^2}{ds ds'}\left(\frac{1}{r}\right) \cdot ds \cdot ds' \tag{1.5.17}$$
>
> The moments of the currents have not only *the same period*, but also vary with the same law, *and can differ only in the phase*. Then we have
>
> $$f(t) = e\varphi(t), \quad F(t) = e'\varphi(t+\sigma) \tag{1.5.18}$$
>
> where $\sigma < p$. [(Betti, 1868); *our translation*, *emphasis added*, *a*/*n*]

Working in the manner of Riemann himself, but based upon these last two formulas, Betti recovers the equation (1.5.15). And just like Riemann, he based his result on an apparently unsecured series of mathematical evaluations (Clausius, 1868) that cast enough doubts on the method itself. Clausius even suggests that this might be the reason determining Riemann to withdraw his original contribution from the Academy in 1858. However, what in our opinion is valuable in Betti's approach to Riemann's idea, is that it contains this sound *concept of phase*: in order to interact, the current elements must be 'in phase' from a certain point of view. Later on, Lorentz considered (see §1.3 above) this point of view as connected to the *existence of a surface*, which is, of course a more realistic approach, asking subsequently for the concept of wave. The wave alone would then be able to explain the concept of phase as introduced by Enrico Betti, by the *phenomenon of holography*. And this phenomenon was added to the classical phenomenology of light *via* de Broglie's quantization idea, which thus appears in a logical order for the natural completion of that phenomenology (Mazilu, 2020).

Whatever the possible reason – of a 'teleological' appearance, as it were – for the state of the facts may be, for that moment in time the Betti's conclusion was simpler:

> Therefore, the electrodynamic actions can be explained, *assuming that they propagate in space with a speed equal to that of light*, that they are *exercised according to Newton's law like the electrostatic actions*, that the *currents consist of a kind of polarization of their elements*, periodically variable, that the *law of variation is the same in all currents*, and that *the duration of the period is small* even compared to the time it takes the action to propagate to the distance unit. [(Betti, 1868); *our translation and emphasis*, *a*/*n*]

The phenomena thus expressed are simple, but we need to notice their ultimate reason: in reality, *ds* and *ds′* are *metric elements in a three-dimensional space*, and, according to Riemann's idea they need to be associated in time with each other, from the point of view of an interaction. The regularization theory of the Kepler motion encourages us to think that the two metrics are not even Euclidean, but… Riemannian, ultimately. This is an interesting idea, so let us just point out the way it comes to being.

To start with, there are no closed orbits in the classical Kepler problem, except under restrictive conditions in the initial data of the motion. In actual quantities these conditions come down to delimiting the region occupied by the center of orbit around the center of force: this means that the geometry of that region is a Lobachevsky geometry. Now, assume the planetary model: the region of the center of force here is the region of nucleus proper of the model, that may be assumed uniformly charged according to our experience. Nothing, apparently, prevents us from thinking about the electron as being alike the nucleus: a region of homogeneous charge, whose geometry is a Lobachevsky geometry of the hyperbolic plane. We can assume that the points of action of the forces are randomly distributed in the two regions, with equal probability. The conditions are such that we can even define what equal probability means here, since, geometrically speaking, the hyperbolic plane is *a Riemannian measurable manifold*. So, the association of two points of action of the force is a stochastic process, of the type envisaged by Riemann. The geometrical image of the association would then be a congruence of lines, forming an *Ampère element* delimited by two surfaces of negative curvature as in the previous §1.3. However, the gist of this construction cannot be fully revealed but by understanding its connection with relativity, so we shall have to revisit this issue later in the present work.

## Chapter 2 Special Relativity: the Unrecognized Theory of Scale Transition

Our undertake of the task of assessing the Louis de Broglie's idea from a modern theoretical physics' point of view (Mazilu, 2020) was, initially at least, an upshot of the fact that, during the study of physics we grew gradually confident that the physics of relativity is essentially incomplete without the concept of wave. The pursuit of necessary studies connected to the task, within this state of awareness, unveiled the fact that the human spirit has, indeed, followed the path of completing the relativity almost exclusively along *this way*, that is, by always searching for the place of wave within the theory. The search was, as a rule, discernible only 'objectively', we have to admit, manifested in a way specific to a context, since most of the times it is not explicitly addressed to the concept of wave. The theory of special relativity, from which the de Broglie's idea sprung, is a striking example of implicitly addressing the concept of wave.

We can say that the present work is all about the fact that the very idea of relativity – as well as a great many other fundamental ones, actually – are a consequence of the circumstance that our knowledge at large stays under the spell of a general law: *the scale transition invariance*. The scale invariance in physics is the point of view that reveals the hidden position of wave as a natural concept. It should then be just as natural that the knowledge follows an objective path leading to its completion: like all things conceptual, and therefore created by man, our knowledge is submitted to evolution, as are, in fact, all things ever made. And regarding the theoretical physics, there are two stages to be considered in this respect, in order to draw what we contemplate as the right conclusions needed in constructing a physics within the Louis de Broglie's spirit. These stages are marked, if we may say so, in time by what we should like to call the *de Broglie moment of theoretical physics*: one of them predates the de Broglie moment of physics, the other is an after-effect of it, as it were.

Now, while we can safely say that the theoretical physics *after* Louis de Broglie moment of humanity is, either directly or implicitly, under the spell of the wave-particle duality, and therefore there is plenty of material containing the right ideas to be reckoned with, when it comes to the physics *before* de Broglie, we cannot relate but to the *Einsteinian relativity*. In a way, this may be an advantage: having scarce material to choose from, the chances of subjective interference are minimal, and a right analysis of relativity might lead to sound criteria of selection from among those right ideas. In order to ease our understanding into these matters, the present chapter of our work, and the next one in fact, describes the fundamental issues of the relativity – as an Einsteinian doctrine, of course – the way we understand them. The emphasis is placed here upon those among issues of physics that led Einstein into constructing the relativity, in both of its instalments – the *special* and the *general* relativity – the way he did. It is thus shown that Einstein was 'compelled', if we may say so, into proceeding as he did by the necessities of a scale transition in physics. And what he achieved in this direction, carries the mark of a scale transition invariance which, occasionally, flared up in his very own work. This observation places, almost explicitly we should say, the concept of wave at the foundation of relativity in its both instalments, which is a lesson we need to learn properly and, of course, apply.

Leaving aside the notion of wave in the Darwin's definition of *interpretation* (Darwin, 1927), this last concept can actually be considered the central concept of physics all along its history: it is the concept around which the physics has been erected in its different directions as we have them today [see *e.g*. (Mazilu, 2020), *passim*]. In order to realize this fact, we just have to recall that the central theme of physics in each and every one of its productions is, and always was as a matter of fact, the description of *motion*. When elaborating on motion, we unavoidably have to touch the classical concept of *material point*, which is actually a matter of interpretation by itself. Further on, along the way of developing physics, one had to introduce *ensembles* of material points. This is the place where a contradiction creeps in the concept of motion, and the idea of wave begins to disclose its objective necessity.

Indeed, in the definition of the *concept of motion*, considered as a physical attribute of the classical material point, the natural philosophy came to recognize two major differentiae gradually entering the concerns of physics. And these differentiae, to wit: the *equation of motion*, and the *trajectory of motion*, are closely associated with the concept of interpretation. They are apparently lost when the *space extension* of the material point gets into physical scenario, or even in case of no spatial extension at all, when more than one physical attributes of the classical material point are to be taken in consideration: mass, charge, color *etc*. However, speaking, just for the moment being, only of the motion of classical material points, with no reference whatsoever to any idea of space extension – that is, no other than the kind of space extension to be described by the concept of distance between material points – we are going to detach now the basic operational definitions of the two differentiae of motion, as they appear in the different instances of the contemporary physics, or of the natural philosophy at large.

The equation of motion provides an order of the positions along the trajectory of motion. It is here the place where confusion finds its entrance into reasoning, and starts showing up through it: in an attempt to undertake this differentia of the motion – *viz*. the equation of motion – in order to apply it into describing the motion of Hertz material particles that may serve for interpretation, we need to take notice of the fact that, by itself, the trajectory of motion – the other differentia of the motion concept – is only a *geometrical* concept. The essential connotation of this last statement is, simply, operational. Namely, from the point of view of interpretation *per se*, the trajectory has to be considered as just a locus, in the geometrical connotation of the word: a possibly *disorderly* ensemble of positions, only arranged into a space form, with respect to the 'outside world', as it were. The equation of motion, on the other hand, is the 'device' serving in bringing a certain order *within* this very ensemble, and this order involves the outside world through the concept of time. The problems of physics started at this very point, *i.e.* when the time order along the trajectory began to involve the outside world. For, in that case, other material points are involved, with different trajectories, and especially with different equations of motion ordering these trajectories. And so the physics came to realize that the description of this connection with the outside world offers only very limited possibilities of being properly understood. Finally, along this path of engagement of our knowledge, the scientific community realized that, in fact, the possibility of comprehension would need the concept of interpretation in its acceptance according to wave mechanics, which involves *the concept of surface*.

Now, even from the times before Einstein actually, the relativity started being connected with physics through motion, as we said, but only in those cases where both of the above differentiae of the concept of motion – the equation of motion and the trajectory of motion – were known and, moreover, were of a special kind. To wit: the motion should have been *uniform* and *rectilinear* on one hand, as it was to Galilei and Newton and, on the other hand, the motion should have been *geodesic* in general, a novelty brought about by Einstein himself. It was noticed, even from the times of Newton, that the classical differentiae of the motion themselves – that is, uniformity and rectilinearity – are references in the description of the general concept, but the case is a little more complicate, involving some other branches of the physical thinking, even the natural philosophy in general, for that matter. In fact, the relativity, in its two instalments emanating from Einstein, *viz*. special relativity and general relativity, is a clear example of physics built exclusively around the concept of motion. In what concerns us here, we take it even beyond this important methodological feature: the relativity is dependent, and in a fundamental way we should say, on the idea of *scale transition* in both space and time.

In short, *Einstein's presentation* of relativity started from an *interpretation* of the Lorentz transformation which, however, had apparently nothing to do with the concept of wave, as required for the necessities of the wave mechanics (Einstein, 1905a). From this perspective we can safely say that the merit of recognizing that the interpretation leading to relativity requires the concept of wave, belongs exclusively to Louis de Broglie. An explanation of this observation is perhaps the best occasion to give one more reason for the present work. Along the due efforts of understanding the physics of scale transition, we grew increasingly certain that such an understanding is not properly possible without the de Broglie's idea of associating a frequency to a material point (Mazilu, 2023a). For, there is an objective reason for this idea, and that reason emerges from relativity, as Laurent Nottale asserted for the first time ever (Nottale, 1992). It is only the fact that the relativity itself has never been considered from such an angle, that hinders a proper understanding of the very scale transition basis of theoretical physics, and the de Broglie's idea offers the best angle, as it were, for such consideration. To wit: it gives us the possibility to complete the natural philosophy specifically, in order to be afterwards able to finalize a physics of the scale transition.

Thus, limiting for the moment our discussion only to the pioneering merits of Einstein, we can say that he considered the spacetime transformations in connection with the uniform rectilinear motion, and *only then* he proved that the field equations describing the light – in its electromagnetic stance, of course – are covariant with respect to these transformations. In other words, for Einstein the intensities of field are not to be considered as fundamental in the description of matter: they are simply just vectors or tensors, like any other physical quantities defined in space and time. It is only later, as we shall see, and forced by a cosmological point of view, that Einstein came to recognize the overwhelming importance of the electromagnetic fields in the construction of the world we inhabit (Einstein, 1919). But then, it seems to have become gradually clear to him, that the Maxwellian description of those fields is not sufficient for the task.

Be it as it may, it is starting from just the point of view of transformations, which fits perfectly the Maxwellian view of electromagnetism, that Einstein came to recognize the importance of *events* as *points in a metric spacetime*, thus making out of them the fundamental elements of the ensembles serving for an interpretation in physics. Accordingly, along this path, he was able to further realize the necessity of describing the fields in a new, four-dimensional arena, where the events – taken as elements located geometrically by positions and time moments alongside and equivalently, in a four-dimensional manifold – form ensembles serving for interpretation.

Once at this point, Einstein proceeded to a generalization of the spacetime metric, in order to include the gravitation in the picture, based on what he called *the Mach's principle*. However, this principle carries in itself a hint of cosmology in any of its formulations, starting with the one which can be drawn from Ernst Mach himself [(Mach, 1919), see, for instance, Appendix XX, pp. 542 – 543 of the work]. It so happened that on the occasion of application of the general relativity to the cosmological problem of the day – which, by the way, we think is *the only reason of existence of a general relativity*, if only in view of the Mach's principle, allegedly staying at the foundation of the first one of them all – some shortcomings were revealed, even for the Einsteinian natural philosophy at large, as a manner of proceeding in physics. It is at this juncture that Einstein was forced to rethink the concept of spacetime in terms of the concepts of space and time separately, almost in Newtonian terms we might say, and thus he explicitly introduced *an idea of scale transition* invariance in a specific metric form, which, in our opinion, needs to be further promoted for the benefit of the theoretical physics at large. This is the form involving the Cayley-Klein geometry, as presented by us in §1.3 above, which, as we have seen, has quite strong ties with the optics of light.

Always guiding our study after the divine Voltaire's often-cited adage: «judge a man by the questions he asks, rather than by the answers he offers», we were intrigued by the reasons which Einstein offers in order to justify the privileged position of light in the general economy of relativity and, as a matter of fact, in the economy of the physics at large. The first fact that strikes, in the content of these reasons, is that the electrodynamics seems to be involved only 'second-hand', as it were, by the idea of propagation. Quoting:

> The theory of relativity is often criticized for giving, *without justification*, a central theoretical rôle to the propagation of light, in that it founds the concept of time upon the law of propagation of light. The situation, however, is somewhat as follows. In order to give physical significance to the concept of time, processes of some kind are required which enable relations to be established between different places. *It is immaterial what kind of processes one chooses for such a definition of time*. It is *advantageous*, however, for the theory, *to choose only those processes concerning which we know something certain*. This holds for the propagation of light in vacuo in a higher degree than for any other process which could be considered, thanks to the investigations of Maxwell and H. A. Lorentz. [(Einstein, 2004), p. 28; *our emphasis*, *a*/*n*]

Like Planck beforehand, in making the case of resonator (see §1.1 above), Einstein introduces here an element of subjectivity, by that freedom in the choice of the definition of simultaneity: «it is immaterial what kind of processes one chooses…». And just like in the Planck's case, the physics ever since proved that the choice of light has, in fact, a deeper reason than the mere chance: it is not just by chance that «we know something certain…» concerning the propagation of light. So, we have started a detailed study of the problem, adopting the point of view that there is an objective reason for relativity, and this concerns the physical connection between light and matter. This is the reason why the present work serve still another purpose: it is also intended as a systematization of that study.

This very chapter starts with a short story of the special relativity moment of physics, aiming to make obvious the fact that special relativity possesses the faculty of a scale transition theory. It is shown that, in fact, the special relativity *started from an idea of scale transition* regarding the intimate structure of the bodies in motion. Only then, since this intimate structure was always thought in terms of electric properties of matter, inherent at the finite scale of our experience, was it natural for the special relativity to come to being as associated with electrodynamics as it did (Einstein, 1905a). The relativistic point of view thus appears to be that the very *same laws of physics apply to our world no matter of space scale one considers this world*. As to the time scales, they need entirely special considerations of physics.

The *scales* of space and time we are using in stating our outcomes on the ideas of *scale transition* here, are three, according to the classification of Nicholas Georgescu-Roegen: first comes the inherent *finite* scale, which is arbitrarily assigned to reality as such by our experience in its most common of instances, namely the human life on Earth. Then come its associated *infrafinite* and *transfinite* scales (Georgescu-Roegen, 1971), where *our imagination* and, inevitably, the physical thinking, has an important part to play. The *mathematical representation* of these scales in physics is achieved, at least for the necessities of the present work, by measures of things spatial and temporal. In the finite ranges, we have the *coordinates* and *time moments*, as measured by lengths and durations, established *via* reference frames and clocks; in the infrafinite ranges we have, ever since the Newton's times, the *differentials* of these coordinates and time moments, while in the transfinite ranges we have the geometrical concept of *absolute*. Since the mathematics of physics is here essentially Newtonian in spirit, due to the use of differentials for quantities at the infrafinite scale, one can safely say that the physics itself, which they entice, is essentially Newtonian in character, though without necessarily being classical. The work with these measures will be obvious as we go on with the development of the physical theory, leading also to a corresponding completion of the natural philosophy. The special relativity is one essential example of mathematical handling of these notions, and this is, again, one of the reasons we consider it first in the present work.

### 2.1 The Scale Faculty of Special Relativity

In the beginning was, as we said, *the Lorentz transformation*. It is starting from it, and based upon what would seem some obvious truths of our experience – like the invariance of the speed of light, for instance, which, since the times of Maxwell became a physical constant independent of space and time, and the existence of the central Newtonian forces at the finite scale of our experience – that Einstein constructed a theory whereby the equations describing the electromagnetism are invariant. One can even say that this construction was a further point needed by the Maxwell's electromagnetic theory, in order to be a *complete* electromagnetic theory. It meant that the light *as a phenomenon*, not only the physical magnitude of its speed, has to be described in a manner *explicitly* involving the invariance with respect to moments of time and locations, since the light is the one phenomenon conspicuously transiting the scales of space. And if the matter is electrical by its nature – at the *finite scale* of our existence, the charge prevails *statically*, by its Newtonian force, over the Newtonian force of gravitation [(Mazilu, 2020); Chapter 3, §3.1] – then the idea was taken *a priori* that the matter exists in a background dominated by electromagnetic field: the electromagnetic ether.

To start with, in expounding the ideas connected with the concept of Lorentz transformation we take just the one-dimensional case. We, obviously, must have a reason for limiting our considerations to this apparently particular case, but that reason will surface as we go along with this work, towards its end: for the moment we just work *a priori* under this take. Using notations that lately became a standard in most of the specialty works, the Lorentz transformation brought into play by Einstein in elaborating the basics of special relativity is, in a matrix form (Einstein, 1905a):

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ t \end{pmatrix} \tag{2.1.1}$$

where by $t$ and $t'$ we understand the time of an event multiplied by the speed of light, $c$ say. This means that an event located as $(x, t)$ in a certain reference frame, will be located as $(x', t')$ in a reference frame moving uniformly with a velocity $v$ with respect to it. Here we have used the definitions that became almost secular for the case:

$$\gamma^2(1-\beta^2) = 1, \quad \beta \equiv v/c \tag{2.1.2}$$

Once again, equation (2.1.1) assumes that the symbols $t$ and $t'$ actually mean the *lengths* of the light paths (along the common coordinate line, of course) in the corresponding reference frames. For a later convenience, notice that the 2×2 matrix from equation (2.1.1) can be written in the form:

$$\begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.1.3}$$

*i.e.* as a linear combination involving the identity matrix and a special null-trace matrix. This last matrix represents, geometrically speaking, an *involution*: applying it twice in any of its actions – *linear*, in two dimensions, or *homographic*, in one dimension – amounts to the action of the identity matrix.

Now, the family of Lorentz matrices from equation (2.1.1) can be presented as a continuous Lie group with one parameter, namely the relative velocity of reference frames with respect to one another (Mandelshtam, 1933). Any two matrices of this family commute, as it can be easily verified. By an admissible change of parameter, such a family of matrices can be presented as a group of hyperbolic rotations. The 'admissibility' in question is referring to both the general mathematical aspect of the physical behavior of parameter, and the connectivity of the Lorentz group: the Lorentz transformation (2.1.1) is continuously connected with the identity transformation, and this connection is described in terms of parameter $\beta$. On the other hand, though, it would seem that the parameter $\beta$, as it is used in the equations (2.1.1) and (2.1.2), cannot assume but real values between $-1$ and $1$, in view of the physical fact that in our experience there are *not* known motions of material particles with velocities surpassing the value of the constant $c$, assigned by James Clerk Maxwell to the light. In this case, since $\beta$ is a continuum parameter, it should always be represented by a continuous function that assumes values only within that limited interval. And indeed, we can write the Lorentz matrix from (2.1.1) in the exponential form:

$$\exp\left\{-s\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\} \equiv \begin{pmatrix} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{pmatrix}, \quad \tanh s \equiv \beta \tag{2.1.4.}$$

generalizing (2.1.3), in the sense that this last one becomes a variant of the 'complete' exponential transformation (2.1.4.) of the connected Lorentz group. In the approximate cases where the first order in the parameter $s$ is valid and can be considered, *i.e.* for values of this parameter allowing the approximations: $\cosh s \cong 1$, $\sinh s \cong s$ and $\tanh s \cong s$, the matrix from equation (2.1.4.) goes into that from (2.1.1). Here $s$ is a new real parameter, and whatever this parameter may be, the values of its hyperbolic tangent cover naturally the interval of real numbers from $-1$ to $1$.

The above approximations are obviously valid for small values of the relative velocity *v*, vhen referred to the light speed, an observation that instated the idea that the finite scale of the world we inhabit is actually the world of 'small velocities' with respect to that of light, as represented by the constant *c*. It is this conclusion that has been carried over into the construction of a physics that can be characterized as *exclusively based upon the concept of motion*. Thus, the fact becomes manifest, that the Lorentz transformation can be transcribed as a hyperbolic rotation, as we already mentioned above:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{pmatrix} \cdot \begin{pmatrix} x \\ t \end{pmatrix} \tag{2.1.5}$$

Such a rotation, obviously, leaves the indefinite quadratic form:

$$(x')^2 - (t')^2 = x^2 - t^2 \tag{2.1.6}$$

unchanged, just as its approximate counterpart (2.1.3) does. This last invariance condition was taken initially, and is taken sometimes even today, as the basis of theory of special relativity, inasmuch as (2.1.6) is 'invariant' with respect to the values of the parameter *s*. Mention should be made, that the matrix realizing this 'complete' Lorentz transformation, belongs to the same class as the 'approximate' variant of the transformation given in (2.1.3), *i.e.* it is a linear combination of the same two basic 2×2 matrices used in equation (2.1.3):

$$\begin{pmatrix} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{pmatrix} = \cosh s \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \sinh s \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.1.7}$$

This is another form of the expression of connectivity *via* the exponential formula for such a particular group of transformations.

The group is, customarily, also applied as such, apparently without any *a priori* reason, in the infrafinite range of spacetime, *i.e.* for the differentials of space and time, in order to write the metric form of this continuum, interpreted by ensembles of events. As a consequence, this metric was *a priori* taken in a form that, in time, brought so much trouble within the Einsteinian natural philosophy, leading to apparently unnecessary paradoxes. This is the well-known quadratic differential form called Lorentz metric or, sometimes, Minkowski metric. The difference between these metrics is only of nuance, so to speak: while the first name is referring mostly to physics, being based on the idea of the group transformations, the second name assumes exclusively the formal geometrical properties of the continuum of events. According to this geometry, the difference between the square of the light path length and the square of any other path length, in a certain direction in space, represents a metric of the spacetime continuum, entirely analogous to the Euclidean metric of the space of our experience. Anyway, fact is that, in the spirit of such geometrical philosophy, as it were, the *infrafinite* variant of (2.1.6) is taken as formally the same:

$$(ds)^2 = (dx)^2 - (dt)^2 \tag{2.1.8}$$

and this, therefore, speaks of a *formal scale transition invariance* between finite and infrafinite. Such a mathematical philosophy is practiced by Einstein – and, implicitly, by anybody else working in theoretical physics along his natural-philosophical doctrine, of course – all along his works on relativity and, in fact, not only there. As we shall see here in due time, the quadratic form (2.1.8) is not necessarily 'natural' in a physical context, but can be derived in connection with a special interpretation of the Lorentz transformation, associated with an alowable change of parameter in its mathematical expression.

It is our opinion that there should be – at least mathematically, therefore from a conceptual point of view – a reason for the transition between the metrics (2.1.6) and (2.1.8), that defines the special relativity as a scale transition theory. In pursuing physics here, we need to notice that the classical incentive stays in the interpretation of light, whose remarkable expression is the association of the physical constant representing the ratio between electrostatic and electrodynamic units, with a velocity: *the light speed*. It is this association that brought into play the idea of involvement of wave in the process of interpretation, for once just by the Maxwell electrodynamics. In this process, the matter is somewhat overshadowed, thus making necessary, in a way, the Louis de Broglie's approach to the concept of wave.

**2.2 The Newtonian Motivations of Special Relativity**

As we already acknowledged above, the form (2.1.1) of the Lorentz transformation can be considered only as approximately valid, by comparison with the connected exponential group of transformations given by equation (2.1.5). Namely, we have in the equation (2.1.1) the case of small relative velocities corresponding to the transformation from equation (2.1.5). In fact, it is along this path of thinking that the name 'relativity' came to being. More to the point, the natural philosophy holds that at small velocities we should have to deal with the classical *Galilean relativity*, valid in the finite world we inhabit, and specifically connected to our only possibility of existence: the Earth surface and the field it reveals. As this kind of relativity involves just the motion, we would have to add the electromagnetic field to it, in order to compare it to the Einsteinian relativity. It is along this path of knowledge that we have learned, in time, of course, that the Einsteinian point of view here – manifested by the involvement of the electromagnetic field in the Galilean physics – asks for special transformations of those fields, which are the corresponding special cases of the relativistic transformations above [see (Le Bellac & Lévy-Leblond, 1973) for details]

The point is that, just by being 'relativity', the previous Einsteinian geometrical theory is, in fact, an *interpretation* (not quite *in the precise sense of Charles Galton Darwin*, involving the concept of wave, but an interpretation anyway!) *given by Einstein to the Lorentz transformation*. It is classical in character, inasmuch as its fundamental feature is that it takes after the example of classical Galilean kinematics, which describes the motion. However, it is well known that, on the contrary, the original Lorentz transformation is referring, in fact, to an invariance of *the field equations* involved in the Maxwellian electrodynamics (Lorentz, 1899, 1904). Thus, the motion – and the matter that necessarily comes with it, of course – would fall, in this approach, on a second level of importance, if we may say so, as indeed it did historically.

Less well known, though – in fact almost never acknowledged anyway, in any of the modern treatises on the subject – is the original purpose of this transformation. It is behind this *declared* purpose, that one can perceive, lurking in the background, an idea of scale transition, of which we are so fond. So it is worth insisting for a little while, on this beautiful piece of the history of modern physics. Actually, since we must expound here only a concise – but comprehensive, nevertheless, if we may! – story of some quite well-known historical facts, for the clarity of this presentation it is best to follow the exquisite summarizing once done by the distinguished natural philosopher who was, and still is in fact, Henri Poincaré, in the introduction to his celebrated *Dynamics of the Electron* from *Rendiconti di Palermo* (Poincaré, 1906). It is important we think, at this juncture, to mention that we particularly value the views of Poincaré on physics, due to the fact that, being mathematician by formation, he always approached physics with the Newtonian spirit of a natural philosopher. Not too many mathematicians can do this, mostly within today's conditions in mathematics, to say nothing about physics!

According to his account from the work just cited, by the end of the 19th century, notably after the times of Michelson-Morley experiment, it became clear that the *motion of Earth through ether* cannot be documented as such, no matter of the experimental point of view. It is, indeed, this last impossibility that has been proved, in a decisive manner, by the results of Michelson-Morley experiment. Again, this historical point was entirely analogous to the corresponding one of the Galilean relativity. To wit, there is, no doubt, a close analogy: the *Earth drifting* through ether is alike a *ship sailing* at constant speed on a quiet sea. There is no possibility to account for this motion by experiments done exclusively on the Earth, just as there is no possibility to account for the *uniform motion* of a ship by experiments done exclusively on it. This medium, that is the ether, was always conceived in connection with the light, and because the light came to be recognized as a phenomenon of electromagnetic nature – especially after the works of Maxwell on electrodynamics – one had to deal in physics with an *electromagnetic ether*, as mentioned above. Thus, the electric and magnetic properties of the matter came to be considered essential, and the negative result of the celebrated Michelson-Morley experiment had to be assessed accordingly. An assessment of our knowledge, necessary from at least two points of view: an *ontological* one, for it serves into making us understand the mechanism of the universe, and a *gnoseological* one, for it gives us the possibility to update the existing natural philosophy.

The widely recognized protagonist of this process of assessment was the illustrious Hendrik Antoon Lorentz, who saw in the negative result of that experiment an opportunity to characterize the *internal forces of matter*. This is, indeed, the initial reason, put forward by Lorentz himself, in constructing his renowned transformation, and we find the best concise expression of it to Poincaré. Quoting, therefore, from Poincaré, on this subject:

> The LORENTZ's idea can be summarized as follows: if one can impress to any system *a common translation* with no modification of any of the perceived phenomena, it is because *the equations of an electromagnetic medium are not altered by certain transformations*, which we call the LORENTZ transformations; *two systems*, *one immovable the other in translation*, thus become the *exact image* of one another [(Poincaré, 1906); *our translation and emphasis*, *n/a*]

We need some elucidations in connection with this excerpt. Obviously, in view of historical scientific environment, by *system* here one must understand the classical 'system of material points'. The Poincaré's specification of a 'common translation' leaves no doubt about this fact: it is obviously referring to an ensemble of such points, moving *collectively*. As usual in the classical physics, an interpretation is already in place here but, let us say it again, *without the concept of wave*. Then, if these fundamental components of the interpretation process – which, once again, are not yet 'wave phenomena', if it is to use an expression of Louis de Broglie – are carrying charges, they should repel each other, and they cannot constitute a physical structure. This is why Poincaré always searched for, and even found some stresses to keep them together: the *Poincaré stresses*. It is just natural, we have to admit: according to *Earnshaw's classical theorem* (Stratton, 1941) any configuration of such identical material particles is physically unstable, so that it cannot make up a physical structure. It is perhaps worth mentioning and keeping in mind as particularly significant, that such a possibility of construction exists, for instance in the form of the modern Bardeen-Cooper-Schrieffer theory of solids, but it requires special environments in the form of *coordinate spaces associated with particular collective motions*.

The most important observation in connection with the above excerpt from Poincaré, though, is referring to the target of the Lorentz's *original* transformation and, obviously, this target appears to be twofold. Typically, it is taken and, of course, it always has been taken indeed, as the transformation that 'does not alter the equations of an electromagnetic medium'. However, let us render due consideration to the second part of the above formulation of Poincaré, which says that by such a transformation 'two systems, one immovable, the other in translation', must become 'the exact image of each other'. According to this statement, the Lorentz's transformation also targets 'the systems', which are specifically altered by it in such a way that the behavior of ether remains formally unchanged, and thus unnoticeable (see §1.3 for details of Lorentz's original idea; notice that Lorentz alluded, for the first time ever, to the necessity of the *concept of surface* in the identification of a 'system'). And, as the 'alteration' involves the 'impressed translation', suggesting a classical intervention of force for 'alteration', one might think of the motion according to the classical standards, involving the intervention of man. This should be the reason why the motion of the Earth cannot be specified with respect to ether: with a consecrated expression, one cannot physically account for the *absolute motion* of the Earth.

However, according to Poincaré, the Lorentz's approach may be deemed, at some point or another, as a convenient hypothesis used to serve in putting things in order only momentarily. From this point of view it is, conceivably, carrying nothing fundamental within it, since it may appear as just a *panacea*, if we may be allowed this expression. This is the reason why Lorentz himself, being obviously convinced that the idea should be carrying a much heavier connotation for our knowledge, in general, than this appears at the first sight, undertook the task of clarifying and, possibly, simplifying the theory quite a few times in his own works. One of these attempts of Lorentz at refining the theory of the electric matter in order to satisfy the facts contained in the results of Michelson-Morley experiment, remained historically notable as an essential work at the foundations of physics. It is now the opportune time to quote from this fundamental work of theoretical physics:

> The experiments of which I have spoken (*the Trouton-Noble kind of experiments*, *n/a*) are not the only reason for which a new examination of the problems connected with the motion of the Earth is desirable. POINCARÉ has objected to the existing theory of electric and optical phenomena in moving bodies that, in order to explain MICHELSON's negative result, *the introduction of a new hypothesis has been required, and that the same necessity may occur each time new facts will be brought to light* (Poincaré, 1900). Surely, this course of inventing special hypotheses for each new experimental result is somewhat artificial. It would be more satisfactory, if it were possible to show, by means of certain fundamental assumption, and without neglecting terms of one order of magnitude or another, that many electromagnetic actions are entirely independent of the motion of the system. Some years ago, I have already sought to frame a theory of this kind (Lorentz, 1899). I believe now to be able to treat the subject with a better result. The only restriction as regards the velocity will be that *it be smaller than that of light*. [(Lorentz, 1904); *our emphasis*; *citations inserted*]

The general idea of approach can be assumed here as well known, and its description can be safely deferred to the original works. What we are interested in, for the moment, is that expression of Poincaré, regarding the fact that 'two systems, one immovable, the other in translation, become the exact image of one another'. Again, it can have a twofold meaning. First of all, the whole point of *any* physical theory, when compared to experiment, is that it involves a necessary hypothesis which is not commonly recognized as such by the natural philosophy. To wit: while it is always about a system *in motion*, the physical theory needs obviously the definition of its counterpart, the 'immovable system'. Indeed, considering the target of physics at the times of Lorentz and Poincaré – in fact, the physics' target of all times before and ever since – we need to recognize that, in all fairness, we cannot have in our knowledge any idea of what an 'immovable system' might be, simply because *we do not have* an 'immovable body' in our experience.

Everything is in motion around us: what we perceive as immovable is actually moving with the Earth, which is moving around the Sun, which is moving around the Milky Way, which is moving with the local group of galaxies, which… and the list of physically possible relative component motions of an apparently 'immovable system' can continue *ad infinitum*, as it were. All we certainly have at our disposal from experience – and therefore we can use in deciding the physical shape of the finite world we inhabit – is, on one hand, an invented medium, *viz*. the electromagnetic ether, within which the above quite amalgamated motion allegedly takes place. On the other hand, we also have a state of bodies of *relative immobility with respect to us*, the observers, from which we can infer the properties of some forces connected with the distinguished properties of the matter: the *electricity*, in the specific case of Lorentz and Poincaré.

Now, if this is all we have in our experience, then what we need to consider first is whether the forces involved in matter – which, by the way, were an *invention* too: Newton's invention, of course – can have an objective existence or rather not. Incidentally, in order to document the term 'invention' used by us here, we advise the reader to consult *Principia* in its original Latin edition, prepared by Roger Cotes, where the word *inventione* is used in connection to forces and even to orbits (see Book I, Section II). Some later renditions of *Principia* are preserving the Newton's Latin expression, some are not, converting it into *determination*. This, obviously, alters the original meaning, giving to forces that artificial physical objectivity we generally assume today. However, apparently it *does not* reflect the Newton's original intention; see also (Faraday, 1857). These forces are *scale-transitive* mind creations, *i.e*. they can be supposed to act the same way no matter of the space scale but, we have to say it, in a special definition of the forces, involving the *idea of gauging*, which we would like to call the *Berry-Klein gauging* procedure [(Berry & Klein, 1984); see also (Mazilu, 2020), Chapter 4, especially §4.2]. Only in this case can one say that the Newtonian forces are scale-transitive indeed, and this is just the way Newton himself endeavored to secure his invention from a natural-philosophical point of view. This statement needs itself a special assessment on different acounts, of which, we think, theoretical physics 'took already care', if we may say so, through an objective process of thinking, to be unveiled as we go along with this work.

The basis of our current understanding of this issue can be rationally realized, if we proceed along the following lines: describe the electric matter in terms of a *classical interpretation* – *i.e*. an interpretation not using any wave idea in connection with the concept of particle – by ensembles of Hertz material particles. These particles possess, besides gravitational mass, both electric and magnetic static charges. Then, taking after the classical routine, *we are free to assume* the existence of an ensemble of such Hertz material particles in equilibrium, under electric, magnetic and gravitational static forces. Indeed, any linear combination of such forces *between identical particles*, is liable to sustain a mechanical equilibrium, because the gravitational force is attractive, while the other two forces are repulsive for identical particles. However, such an ensemble of identical particles is *fictitious*, insofar as *in actuality* the different Newtonian forces thus assigned to a particle prevail upon each other at *different scales of space and time*. In fact, we can even say that it is this property the one that asks for a scale description of the physical world we inhabit. To wit, the gravitation prevails at the grand scale of the universe; on the other hand, the electric and magnetic forces prevail at the finite scale revealed to us by our daily experience and, apparently, they also prevail at the microscopic scale of the world. This means that, in actuality, we cannot have a *static ensemble of particles* at our disposal within the daily experience: *physics only allows us to think of it*, *in reality it does not even exist*!

It is the daily experience which further shows that the electric and magnetic *static* forces act in a 'tandem', so to speak, as a force whose expression is linear in the electric and magnetic fields, involving, still linearly, the two kinds of charges, electric and magnetic (Harrison, Krall, Eldridge, Fehsenfeld, Wade, & Teutsch, 1963):

$$\boldsymbol{F}_{st} = q_e \boldsymbol{e} + q_m \boldsymbol{b} \tag{2.2.1}$$

Here the vectors $\boldsymbol{e}$ and $\boldsymbol{b}$ play the part of the intensities of the field of forces *at the location where it acts*. These forces characterize a mechanical equilibrium whereby the particles possesing charges are in a *stationary state*. Then, assume, further, that a *state of motion* is described by a Lorentz-transformed force, involving the *static force* from equation (2.2.1) and *a rotated counterpart*, with *the rotation acting upon intensities of the field, and defined by the static charges*:

$$\boldsymbol{F} = q_e \boldsymbol{e} + q_m \boldsymbol{b} + \frac{1}{c}\boldsymbol{v} \times (q_e \boldsymbol{b} - q_m \boldsymbol{e}) \tag{2.2.2}$$

Now, if the equations describing the evolution of the intensities of fields are 'symmetric' *i.e.*, according to the prescriptions of Maxwell's electrodynamics, we have:

$$\begin{aligned} &\nabla \cdot \boldsymbol{e} = 4\pi q_e \rho, \quad \nabla \times \boldsymbol{e} = -\frac{1}{c}\frac{\partial \boldsymbol{b}}{\partial t} - \frac{4\pi}{c} q_m \boldsymbol{j} \\ &\nabla \cdot \boldsymbol{b} = 4\pi q_m \rho, \quad \nabla \times \boldsymbol{b} = \frac{1}{c}\frac{\partial \boldsymbol{e}}{\partial t} + \frac{4\pi}{c} q_e \boldsymbol{j} \end{aligned} \tag{2.2.3}$$

where $\rho$ is the numerical density of particles, while $\boldsymbol{j}$ is their current, interesting things start to show up. For once, these equations have the virtue of reducing themselves to the usual Maxwell equations for either $q_m = 0$ or $q_e = 0$. Notice, however, that with no such quantitative consideration on charges – which is quite particular and, therefore, from natural-philosophical point of view they should be, in a way, irrelevant – we can define two new field variables *via* the *genuine rotation* generated by the two charges:

$$e\boldsymbol{E} = q_e \boldsymbol{e} + q_m \boldsymbol{b}, \quad e\boldsymbol{B} = -q_m \boldsymbol{e} + q_e \boldsymbol{b}, \quad e^2 = q_e^2 + q_m^2 \tag{2.2.4}$$

after which the force (2.2.2) becomes *the Lorentz force* as we usually know it in our experience:

$$\boldsymbol{F} = e\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right) \tag{2.2.5}$$

while the Maxwell's equations (2.2.3) become naturally non-symmetrical, the way we know them from any of the textbooks summarizing that experience:

$$\nabla \cdot \boldsymbol{E} = 4\pi e\rho, \quad \nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}$$
$$\nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c} e \cdot \boldsymbol{j} \tag{2.2.6}$$

However, while in the first symmetric version, the rotation is determined by the ratio of charges, which in turn needs a special natural philosophy involving these charges [(Katz, 1965); see also (Mazilu, 2020), §3.1], in the Lorentz's version the theory is pending on a genuine *space rotation* that needs *central forces acting sideways*. This notion may seem contradictory, but we use it, nevertheless, in order to pinpoint a fact of which we need to account theoretically.

Namely, insofar as a force is created by a physical characteristic of a particle – gravitational mass, magnetic charge, electric charge, *etc* – it is, no doubt, central: the particle creating it is the obvious center of force. On the other hand, when it comes to *the action* of such a force, it can be twofold: the force can act along the direction to the particle that created it, or transversally to that direction, *i.e. sideways*, with an expression of J. J. Thomson. Besides the fact that, at the first sight, this concept of transversal action is strange by itself, from the point of view of motion it requires a special arena where *the forces have to be logarithmic*, although not central [(Mazilu, 2020); Chapter 6, §6.2]. This arena cannot be but the Louis de Broglie's region that we have found 'strange' (*loc. cit. ante*, Chapter 2, §2.1), which, as we have hinted previously, in the case of light is an expression of the holographic property of the universe. In the end, this requirement leads to the necessity of a wave image, as de Broglie's theory stipulates, but it turns out to be valid along with the Maxwellian electrodynamics, just as Lorentz intended to show in the first place.

Now, for a better understanding of the issue, it is perhaps worth presenting it from a Newtonian perspective, however involving plainly the name of Maxwell himself. The equation (2.2.2) presents the dynamical force as different from the statical one, including a sideways force, if it is to use again the term of J. J. Thomson. This extra term is constructed so as to make the formula of rotation of the force field a plain rotation as in equation (2.2.4). Along the idea of a grand analogy, this case reminds us of a similar one that occurred to Newton on the occasion of a cosmogonic explanation of the world, which he was asked to recommend for religious purposes. Quoting from a letter of Newton to Bishop Richard Bentley:

> To the last part of your letter, I answer, first, that if the earth (without the moon) were placed any where with its centre in the *orbis magnus*, and stood still there without any gravitation or projection, and there at once were infused into it both a gravitating energy towards the sun, and *a transverse impulse of a just quantity moving it directly in a tangent to the orbis magnus*; the *compounds of this attraction and projection would*, according to my notion, *cause a circular revolution of the earth about the sun*. But the transverse impulse must be a just quantity; for if it is too big or too little, it will cause the earth to move in some other line. Secondly, I do not know any power in nature which would cause this transverse motion without the divine arm. Blondel tells us somewhere in his book of Bombs, that Plato affirms, that the motion of the planets is such, as if they had all of them been created by God in some region very remote from our system, and left fall thence towards the sun, and *so soon as they arrived at their several orbs, their motion of falling turned aside into a transverse one*. And this is true, *supposing the gravitating power of the*

> *sun was double at the moment of time in which they all arrive at their several orbs*; but then the divine power is here required in a double respect, namely, to turn the descending motions of the falling planets into a side motion, and, at the same time, to double the attractive power of the sun. So, then, gravity may put the planets into motion, but, without the divine power, it could never put them into such a circulating motion as they have about the sun; *and therefore for this, as well as other reasons, I am compelled to ascribe the frame of this system to an intelligent Agent* [(Bentley, 1838), pp. 209 – 210, *our Italics*]

There is, therefore, *in the mind of Newton*, an event *imprinted* in the history of motion by the 'transverse' circular motion, that can be described in the following fashion: the initial motion of the planets is *radial* towards a center represented by the Sun, according to the action of static Newtonian forces; this radial motion is suddenly turned into *transverse motion* at the moment when 'they arrived at their respective orbs'. In hindsight, it is this transverse motion that makes the object of the classical dynamics when describing the Kepler motion as a dynamical problem. The 'transverse impulse of a just quantity' is then represented by some *initial velocities of the Kepler motion*, that Newton assigns to an 'intelligent Agent'. In keeping with the dreadful atheistic modern view of the world, we can tell nowadays that the 'intelligent Agent' acted in the spirit of J. J. Thomson or, better yet, in the spirit of a *Maxwell's demon* – an invention playing the part of such an Agent [(Maxwell, 1904), pp. 338–339] – acting by rotating the force as in equation (2.2.4), at the 'just moment' of time, and that was the moment when the planet, in its purely radial motion, 'reached its orb'. And, still continuing along this dreadful atheistic way, we dare to associate the rotation of force at the position of its action, with the action of a field of a statistic nature, posited by Maxwell himself as a matter of fact, as follows.

The classical Poisson equation was, from the very beginning (Poisson, 1812) taken to mean the *preponderance of matter over the field*: the density of matter determines the field of forces in matter. Both concepts – matter and field – were naturally brought to human awareness, with this apparently natural 'dominance', if we may say so, by the Newtonian theory of forces. There is, however, a renowned case whereby the Poisson's equation is 'rustled up', as it were, within the above-mentioned classical habit of *defining the field* when the *density of matter* is given, and used *into defining the density* when *the field is given*. This became one main point in the Louis de Broglie's theoretical doctrine [(de Broglie, 1935); see also (Mazilu, 2020), §2.4]. Contemplating this doctrine, we are compelled to find such an idea significant even from another point of view, namely because it is a construction that served to build an image of the electric ether based on *considerations of statics*, not dynamics.

Indeed, the essence of the problem of ether, is well represented by the so-called *Maxwell stress system*, described by Clerk Maxwell in the Chapters 4 and especially 5 of the first volume of his classical treatise (Maxwell, 1892). Even though this system is mostly cited as an example of the failure to describe the ether as what is classically conceived as an isotropic incompressible medium [see especially (Love, 1944), where the system of Maxwell stresses is presented from different mechanical perspectives in various places of the book], we think that it is still in position to straighten up some of our modern physical concepts, especially that of *static ensemble of equilibrium*, so necessary to any theory of interpretation. As we said, Maxwell himself did not seem to have used his system of static stresses very much. In hindsight, this appears to have happened mostly because he seems to have been carried away by the electromagnetic image of the light, whereby the dynamics appears to be the essential working ingredient. By the same token, however, the subsequent neglect of the Maxwell stress system in physics may have been due to a deeper, objective reason, that can be assigned to the necessity of interpretation in physics. We will turn back to this important issue later on.

Maxwell's problem was to find the stresses induced by the action of forces in ether, in order to explain the omnipresent gravitational and electric forces. The attraction was represented in those times, just as it is today, by the Newton's forces, which proved also to be valid for electricity, as Charles Coulomb would have long shown. Maxwell did not take into consideration this property directly, but first translated it into a problem *involving the continua*: finding the stresses *statically equivalent* with a system of forces in general. Notice that these stresses had also to face, later on, the fact that the matter does not seem to be dragged by ether, which was proved experimentally toward the end of the 19th century by the Michelson experiment. This circumstance too, may have participated to ignoring the case as inessential, inasmuch as neither the gravitation, for instance, nor the electric action could be consequently explained as drag forces. This conclusion was even reinforced by Henri Poincaré, as displayed in the excerpt above, who, moreover, specifically showed that the electric matter of Lorentz is in default with respect to classical dynamics, inasmuch as it does not obey the classical principle of action and reaction (Poincaré, 1900). He even pushed this property into describing the forces of gravitation, and the forces of cohesion of matter in general, thus inventing the so-called *Poincaré stresses* (Poincaré, 1906).

The mathematics of a force generated by matter was described in those times the way it is still described today, and this way is expressed by its essential mathematical concept, the Poisson equation, as we said. This will be rewritten here in the form:

$$\nabla^2 U(x,y,z) = 4\pi\rho(x,y,z) \tag{2.2.7}$$

In this equation $U(x, y, z)$ is the potential of the forces in a medium of density $\rho(x, y, z)$. The way this equation is constructed in modern times – that is, by the Gauss integral theorem – confers to the density a Newtonian quality, of a characteristic of matter describing the way it fills the space at its disposal. If this medium is electrically active, then $\rho$ is taken as the density of electricity and $U$ is an electric potential. Maxwell apparently took equation (2.2.7) as *defining the density* of the medium, rather than defining the potential, for the following good reason: he proved that the equation of equilibrium of a system of stresses is satisfied with the volumetric forces corresponding to a matter with density given by (2.2.7). Indeed, the equation of equilibrium of a continuous stress system in general, in its simplest form, asserts that the divergence of the second order stress tensor, $\boldsymbol{t}$ say, is given by the density of volume forces $\boldsymbol{f}$ (Love, 1944):

$$\nabla \cdot \boldsymbol{t} + \boldsymbol{f} = \boldsymbol{0} \tag{2.2.8}$$

When specifically applied to the stress tensor $\boldsymbol{t}$ defined by the matrix

$$t_{ij} = \frac{1}{4\pi}\left(\frac{\partial U}{\partial x^i}\frac{\partial U}{\partial x^j} - \frac{1}{2}\delta_{ij}(\nabla U)^2\right), \quad x^i = x,y,z \tag{2.2.9}$$

the equation (2.2.8) is identically satisfied for a force density $\boldsymbol{f}$ given by

$$\boldsymbol{f} \overset{def}{=} \frac{1}{4\pi}(\nabla^2 U)\cdot\nabla U \tag{2.2.10}$$

In other words, according to equation (2.2.8), the stress system (2.2.9) is *statically equivalent* with the system of volume forces (2.2.10) of the matter having a density given by Poisson's equation. Thus the gravitation, for instance, can be conceived as a tension due to these stresses through ether, and likewise the electric force. The Poincaré conclusion about Lorentz material system can be taken as showing that such a system of stresses is insufficient to do the job they are called for, no matter of the system of forces taken into consideration.

Now, if we replace the gradient components from the matrix (2.2.9) by the components of a logarithmic force, *i.e.* of the central force deriving from a logarithmic potential:

$$U(\boldsymbol{r}) = \kappa \cdot \ln r$$

where $\kappa$ is a constitutive constant, the Maxwell stress (2.2.10) generated by this field is statically equivalent with the system of volume forces

$$\boldsymbol{f}(\boldsymbol{r}) = -\frac{\kappa}{4\pi}\frac{1}{r^3}\hat{\boldsymbol{r}} \tag{2.2.11}$$

and correspond to a Maxwell tensor

$$t_{ij} = \frac{\kappa}{4\pi}\left(\frac{x^i x^j}{r^2} - \frac{1}{2}\delta_{ij}\right), \quad r^2 \equiv x^2 + y^2 + z^2 \tag{2.2.12}$$

These last two equations suggest unearthing an interesting story connected with the name of Eugenio Beltrami, in need to be unraveled at one point along our work (Beltrami, 1886). For now, though, we just concentrate upon the static force (2.2.11).

That force has the appearance of a central force, and has been, indeed, used as such by Joseph John Thomson who, in our opinion is its main promoter in matters physical (Thomson, 1910, 1913). However, at a point along the physical theory, J. J. Thomson was compelled to asume a ‘jump’, so to speak, in order to account for the fact of quantization. The mechanism assumes that the force (2.2.11) acts only outside the tubes of force, for *a quantum of matter* cannot be but inversely proportional with the square of the radial distance. The force acting ‘sideways’ can only be derived in connection with a logarithmic potential, as above (Mazilu, 2020). According to J. J. Thomson, the force (2.2.11) is the force exercised by a *dipole* along its own direction, but as one can see, it appears as a kind of statistic. It should be then logically inferred that the statistic in question is made possible by the existence of such dipoles, thus giving one more reason for the existence and necessity of Planck’s quantization procedure (see §1.1).

Going a little ahead of us, we think that the time is ripe in order to take due notice of the fact that the previous Maxwellian theory can be taken as a sound basis for another important concept, currently used, by and large, in the modern *theory of critical phenomena* [one can consult the work of Cyril Domb dedicated to subject (Domb, 1996), which makes the idea of critical point remarkably clear from all the pertinent differentiae of the concept]: that is the *concept of molecular field*, introduced to common awareness by Pierre Weiss, on the occasion of one of the first theoretical undertakes of the description of critical phenomena in the case of magnetism (Weiss, 1907). The first move would be a framing of the idea molecular field in the order of ‘things Fresnelian’, so to speak, for this is, indeed, the natural case. In order to do this, we quote the very Pierre Weiss’ words:

> I propose to myself to show here that a theory of ferromagnetism can be founded on an extremely simple hypothesis concerning *these mutual actions* (*of the magnetic molecules*, *a*/*n*). I assume that *every molecule experiences, from the part of the ensemble of the surrounding molecules*, an action equal to that of a *uniform* field (*original emphasis*, *a*/*n*) NI proportional to the

intensity of magnetization and of the same direction with it. One could give to NI the name of internal field in order to mark the *analogy with the internal pressure of van der Waals*. Indeed, this field, *adding itself to the exterior field*, *will account for the high intensity of magnetization of the ferromagnetic bodies*, by the laws of the paramagnetic bodies, in the same manner with the *internal pressure* which, *adding itself to the external pressure*, accounts for the *high density of the liquids by invoking the compressibility of the gases*. However, this expression is liable to lead to frequent confusions. I would prefer instead the name *molecular field* (*original emphasis*, *a*/*n*). We shall be led to consider, here as well as elsewhere, *a sphere of molecular activity*. [(Weiss, 1907); *our translation*; *emphasis added except as mentioned*, *a*/*n*]

How is this definition of the molecular field fitting in the present context? In order to see how, we refer the reader to some previous results [see (Mazilu, Agop, & Mercheș, 2021), Chapter 2]. The argument can be précised the following way: since Weiss is talking of the analogy involving a gas, we need to recall that, in the Clausius virial theorem for the gas, as it is used in the thermodynamics of the real gases in the van der Waals take, the temperature is no more a *sufficient statistic*. This means that it does not make any sense at the level of a molecule: the Maxwell demon cannot exist! So, necessarily, intermolecular central forces deriving from a logarithmic potential are involved in the calculations of the virial, as the forces above. Then the Maxwell-Beltrami stresses (2.2.12) should be, in fact, equivalent with the Thomson's inverse cubic forces, and this is the molecular field of Weiss. The concept is incomplete as yet, on a few accounts, but we shall become aware of its necessities of completion as we go with our developments. We are just aware of their logical existence, in order to be able to pick them up when the case may occur.

To continue our main streak of discussion here, the bottom line is that the relativity, as an expression of the necessity of interpretation, needed the concept of wave: the collective motion of an ensemble of particles, impossible to be considered as static, however still in need to be viewed as an ensemble of *simultaneous particles*. Otherwise, the interpretation itself, as a necessary step in the construction of a theory of physical structures, could not be a full concept. Starting from this point of view on the necessities of the theoretical physics, one can better understand the criticism of Henri Poincaré targeting the *purely phenomenological interpretation* of the Lorentz transformation which led to the construction of the theory of special relativity as we have it today [(Poincaré, 1905); see also (Mandelshtam, 1933)]. The purely phenomenological aspect of this interpretation – to wit: for instance, the *shrinking of lengths* in the direction of motion – seemed untenable at the time, as it does, in fact, nowadays. And yet, it is still with us today, applied all of a sudden to some conventional kinematical and geometrical concepts as Einstein once proved (Einstein, 1905a). However, Poincaré raised quite a few legitimate natural-philosophical issues, unanswerable at the time – and, in fact, still unanswered ever since – from among which, one we have found as quite remarkable, and we are compelled to consider it essential to physics by its implications, even today. Quoting:

… What would happen if *one could communicate by signals which are no more luminous*, and whose velocity of propagation *would differ from that of light*? If, after having settled the watches by the optical procedure, we wished to verify the decision with the aid of these new signals, we should notice *discrepancies which would render obvious the common translation of the two*

> *positions*. And, are such signals inconceivable, if one admits, with Laplace, that the universal gravitation is transmitted a million times faster than light? [(Poincaré, 1905), p. 208; *our translation and emphasis*; *see also* (Poincaré, 1907), p. 100]

In other words: what would happen if «we know nothing certain…» – to use the general idea of Einstein's own expression – concerning the propagation of light?! In view of what was just said above, we aim to build an understanding of this specific objection with the help of Louis de Broglie notion of a physical ray.

A general alert on what this construction may mean is perceivable right away from the excerpt above, even intuitively we should say: according to our experience, a light signal does *not* give us *a length*, but obviously *a distance*. This distance can be associated with *a length*, is true, for this association was (and still is, in fact!) a standard procedure in the physics of all times, and was instituted in today's relativity starting with Einstein himself. The fact still remains, however, that what a light signal provides us is a distance. If, however, a length is materially realized as a physical object – a *meter stick* or a *moving train*, as they typically assume in any 'operational-type' of such arguments – then a signal can be propagated through it, in the manner of waves in matter: some sound waves for instance. In general, these waves have speeds of propagation through matter substantially different from those of propagation of light through ether. Yet, we are bound by our experience to conclude that they reveal indeed a length of the rod, not the distance between its ends. True, this is done in the very same manner the light reveals a distance within the ether, but the result is not a distance *per se*, like between two isolated material bodies apart from each other: it is revealed by a signal completely different, by its physical nature, from light, at least we can agree on that. Going a little ahead of us, such a length is rather associated with *the phase of a wave*, just like the wavelength of light in the physical optics. This length, though, can only be *assumed* equal to the distance we associate to it by light signals – a genuine gauging procedure, we should say – and according to Poincaré's argument, such a condition can be *contingently* broken: there is no guarantee of an absolute identity between the two quantities.

Thus, our contention is that the Poincaré's 'discrepancies' mentioned in the above excerpt are to be referred to the fact that the length, as associated to a rod by *internal signals* in matter, equals indeed the distance between its ends associated to the rod by light, but only in special conditions, for instance, *only at rest*. With a mnemonic phrase, this last case could be described as follows: in the direction of motion – and probably not only in that direction – *the length associated by internal signals to a meter stick cannot equal the distance associated to it by light*. Therefore, unlike the Marquis de Laplace, who, according to Poincaré, used an argument involving not the hypothesis of God, is true – as he declared to the Emperor Napoleon, according to an often-cited humorous incident destined to reveal the ignorance of social rulers in general – but the *hypothesis of propagation speed of gravitation*, we can use a fact of solid experience in order to construct the physical theory: the speed of sound through ponderous matter is perceptibly different from the light speed in ether. However, according to Katz's natural philosophy involving charges, this does not say more than that a piece of ether is different from a piece of matter. In a word, a fundamental problem still remains unsolved: we cannot decide this way if the ether is indeed matter or not. And the idea sustained in physics thus far, that the ether is a kind of matter, seems to contradict the everyday observation that in vacuum bodies do not encounter any resistence to their motion. This view, though, serves best to straightening our reasoning: *the length* and *the distance* can only serve as estimators of each other in measurements of the same type, but they are different in concept. The estimates are ‘exact’, if we may be allowed to use such expression, only in special cases.

Concluding therefore, we are compelled to advocate the notion that the Poincaré’s critique targets, in fact, the one-sided attitude – classical in essence, we have to admit – of promoting unreservedly into natural philosophy a thinking exclusively in terms of *length as unconditionally identical to a distance*, and *vice versa*, of course. Based on this thesis regarding the identity of the two concepts, taken as so obvious a truth of our experience, that it is not even viewed as a *thesis*, to say nothing of a view as *hypothesis*, the theoretical physics has evolved in quite a specific way, eliminating any other alternative possibilities. To wit, it uses, for instance, indiscriminately *the same* Lorentz transformation at different scales of space and time: at finite scale for *transformation of coordinates and time moments* associated to events in reference frames, as well as at infrafinite scale, for *transformation of their differentials*, in order to describe the motion.

Our immediate task is to expose some variants of Lorentz transformations, within the very concept of identity between distance and length, for they are most clearly indicating where to direct our theoretical efforts. There are some transformations that *appear to be* Lorentz transformations – at least no one contested this declared connotation of them thus far, so we adopt it as well – proposed along the history of physics for different reasons, and we judge them from such a perspective. The transformations we have in mind, were dubbed indeed ‘Lorentz transformations’, by their promoters, but they were constructed from the most general mathematical and physical points of view, *i.e.* as realizable by certain types of matrices, satisfying what appeared to be physical necessities. However, they are not Lorentz’s transformations *per se*, in the sense that they do not satisfy the groupal connectivity condition with respect to their physical parameters. Importantly enough, though, they respect this condition ‘infinitesimally’, as it were: the matrices representing them are elements of an *algebra of the classical Lorentz group*. Moreover, their study reveals one important issue connected with the concept of dipole, fundamental for the idea of quantization along the Planck’s line of thought: the matrices are just as essential to physics as the coordinates are to geometry, and need to be approached accordingly. Let us see what this is all about.

### 2.3 Alternative Approaches to Lorentz Transformation

A summary of the status of such theories would, of course, serve best the clarity of our argument. However, in order not to extend inadequately the work in matters that are, indeed, on the side of its purpose, we just indicate two works that can help anyone in making up their mind as to the status of the problem at the times we are discussing here [(Brehme, 1988) and (Ungar, 1988); see the works cited there]. It is just fair, though, to express what *we see* as the right summary of the works just cited by their essential line of argument. In our opinion, the esential problem was, all along the coming into existence of special relativity in fact, that of *associating a velocity to light along the Maxwell’s line of electrodynamical concept*: the *dilation of time intervals* and the *shrinking of lengths* were systematically brought into argument, either based on it or downright arguing for it, and, of course, this argument was never settled in a way or another. Let us, therefore, proceed directly on showing a neglected view, and a possible physical interpretation proper, in order to be used in settling the argument.

With the previous touch of the ideas of special relativity, we shall now turn back to the Lorentz transformations, more to the point, to the way they are being considered from a mathematical point of view. Grant Fowles once advanced the idea that a Lorentz transformation in the finite range of space and time should be represented mathematically by an *involutive matrix*. The physical reason invoked by Fowles for this conclusion is, in fact, still mathematical: we need to describe the spacetime background by a condition of isotropy defined as a *property of the reference frames in space* (Fowles, 1977). This requirement breaks down the group property of the matrices that realize the Lorentz transformation. Let us present, in broad strokes, the Fowles' line of thought.

From a natural philosophical point of view, the important observation here is that the idea of isotropy and homogeneity of spacetime must also be accounted for mathematically. And the proposal is that it does indeed, but by adding the concept of *orientability of the reference frames* used to describe that continuum in space. This means that the orientability becomes manifestly necessary in deciding the algebraical structure of the matrix representing a Lorentz transformation. To wit, Fowles looks for a Lorentz transformation between reference frames having the 'same handedness', *i.e*. the same orientation, if it is to speak in modern geometrical terms. So, considering *a pair of reference frames*, he comes to the conclusion that "from the isotropy of space and the nonexistence of a preferred frame" a Lorentz transformation requires an *involutive matrix* in order to be achieved: a matrix manifestly different from the identity matrix, which is its own inverse with respect to multiplication. Symbolically, this property of the transformation mimics the property usually contemplated for the reflected light signals used in operational construction of the relativity: in order to establish a distance, a signal sent by an observer should come back right away *when reflected*. Then, the Lorentz transformation too, applied twice should reproduce the original event to which it is applied, if the spacetime is 'orientable':

$$\boldsymbol{A} = \boldsymbol{A}^{-1} \quad \therefore \quad \boldsymbol{A}^2 = \boldsymbol{1} \tag{2.3.1}$$

Here $\boldsymbol{1}$ is the 2×2 identity matrix, as usual. Therefore, mathematically speaking, the ensemble of Fowles' matrices representing the Lorentz transformations is what mathematicians call a *semigroup* containing the inverses of its elements (the associativity is still in effect in this set of Lorentz matrices, it is even imposed from a physical point of view!)

The structure of such a matrix is 'two-dimensional', if we may say so, of the type revealed by us before for the proper Lorentz transformation [see equations (2.1.3) and (2.1.7)]: it can be algebraically established up to two arbitrary parameters, involved in a linear combination of two basic matrices. This fact can be ascertained just by noticing that the last of the relations (2.3.1) can be taken as a Hamilton-Cayley equation for the matrix $\boldsymbol{A}$. In general, the Hamilton-Cayley algebraical matricial equation, allows us to give a 'reducing recurrence' so to speak, for a certain matrix: the power of the matrix equal to its order can be calculated *linearly* in terms of the successive descending powers of that matrix, including zeroth power, which is to be taken as identity matrix no matter of its order. Therefore, in terms of matrices, as independent elementary objects of mathematics, the recurrence can be, indeed, characterized as linear. However, from the larger perspective of the matrix entries, that recurrence relation appears actually as nonlinear, at most algebraical, to be more specific. In the case of the 2×2 matrices the recurrence relation allows writing any function of a matrix as a Lorentz matrix of the kind exhibited by us in the previous sections, in some special conditions, of course.

The perspective just presented can be illustrated right away for the case above. Indeed, in the case of a 2×2 matrix, the Hamilton-Cayley equation can be written in the form:

$$\boldsymbol{A}^2 - (A+D)\cdot\boldsymbol{A} + \det(\boldsymbol{A})\cdot\boldsymbol{1} = \boldsymbol{0} \quad where \quad \boldsymbol{A} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

It is thus seen that the square of the matrix $\boldsymbol{A}$ is a linear combination between the identity matrix, which, as we just mentioned, must be considered as $\boldsymbol{A}^0$ and the matrix $\boldsymbol{A}$ itself, which is naturally $\boldsymbol{A}^1$. However, speaking in terms of the very entries of the matrix $\boldsymbol{A}$, this connection appears as nonlinear, of course: that much can be said from the very fact that the entries of $\boldsymbol{A}^2$ are homogeneous quadratic forms in the entries of $\boldsymbol{A}$. Even though this is just an obvious mathematical fact, we need to draw attention on it, for it is important on many accounts along our present work. Continuing on along this line, and taking the second of the equations in (2.3.1) as a Hamilton-Cayley equation, the entries of a matrix representing a Fowles' Lorentz transformation must satisfy a system of two algebraic conditions:

$$A+D=0, \quad \det(\boldsymbol{A}) \equiv A\cdot D - B\cdot C = -1$$

These two constraints leave, obviously, only two of the entries arbitrary. And Fowles makes here a *choice*:

$$D=-A, \quad B\cdot C = 1-A^2 \tag{2.3.2}$$

so that $\boldsymbol{A}$ can be written as a matrix involving just two arbitrary parameters:

$$\boldsymbol{A} \equiv \begin{pmatrix} A & b\sqrt{1-A^2} \\ \dfrac{\sqrt{1-A^2}}{b} & -A \end{pmatrix} \tag{2.3.3}$$

Here $A$ and $b$ are real numbers, with $A$ between $-1$ and $1$. Then, he eliminates even this arbitrariness through an algorithm involving three distinct logical steps in his reasoning, which, we think, need to be noticed as such, for they are important in guiding our reasoning: (1) first, Fowles also applies the Lorentz transformation in the *infrafinite range*, so that along with the usual finite range transformation realized by the matrix (2.3.3) – the analog of (2.1.1) – we also have the transformation acting on the differentials of the space and time measures of the spacetime:

$$\begin{pmatrix} dx' \\ dt' \end{pmatrix} = \begin{pmatrix} A & b\sqrt{1-A^2} \\ \dfrac{\sqrt{1-A^2}}{b} & -A \end{pmatrix} \cdot \begin{pmatrix} dx \\ dt \end{pmatrix} \tag{2.3.4}$$

In fact, let us say it again, this is a customary procedure of the theoretical physics: to *apply the same transformation no matter of scale*. It needs to be mentioned, though, in the present context, because it reveals what appears to be a kind of unsecured *a priori* transition in both the case of space and the time scales. This extension of the linear action allows the conclusion that "a stationary point in either frame will have the velocity $v$ in the other". In terms of equation (2.3.4), for instance, this means $dx' = 0$, so that

$$b = -v\frac{A}{\sqrt{1-A^2}} \quad \therefore \quad \boldsymbol{A} \equiv A\begin{pmatrix} 1 & -v \\ \dfrac{1-A^{-2}}{v} & -1 \end{pmatrix} \tag{2.3.5}$$

where $v$ is considered as the ratio of the two differentials: $(dx)/(dt)$, *i.e.* an 'instantaneous velocity', so to speak. (2) Secondly, within this instantaneous velocities' framework, the Lorentz transformation *acts as a homography*, *i.e.* as a linear rational relation: if $V$ and $V'$ are calculated by the ratios of differentials of the coordinates and times, corresponding to each other by the Lorentz transformation realized by Fowles' matrix (2.3.5), then we have, according to (2.3.4), the correspondence:

$$V' = \frac{V - v}{\dfrac{1 - A^{-2}}{v} V - 1} \tag{2.3.6}$$

Thus, to an event having the instantaneous velocity $V$ in a reference frame, we have to associate the instantaneous velocity $V'$ in a reference frame moving uniformly with the velocity $v$ with respect to it. (3) Thirdly, Fowles considers the *speed of light* as an *ordinary instantaneous velocity*. Recall that in the framework of Maxwell's electrodynamics – which, again, is the Einsteinian starting reason of special relativity – the value $c$, usually *assimilated* as the speed of light, is actually the ratio between the electrostatic and electrodynamic units of charges. However, there is *a priori* no reason to assume that this physical constant may have the meaning which its units suggest. Cases are known in physics, whereby essentially different characteristics of matter have the same units. Notorious among these is the case that led to Bohr quantization: the action and the kinetic moment have the same physical units. However, this last assumption allows Fowles to assign to light signal the distinctive property that, through the Lorentz transformation its velocity, only changes the sign between two reference frames of the same handedness, remaining the same only in value:

$$-c = \frac{c - v}{\dfrac{1 - A^{-2}}{v} c - 1} \tag{2.3.7}$$

In other words, the orientability of space is thereby connected only with the sign of the speed of light: it is its magnitude that remains unchanged, just as Maxwell once introduced it (*i.e. via* $c^2$). This leads to

$$A = \gamma \quad \therefore \quad \boldsymbol{A} \equiv \gamma \begin{pmatrix} 1 & -v \\ \dfrac{\beta}{c} & -1 \end{pmatrix} \tag{2.3.8}$$

so that, in our notation conventions, the resulting Lorentz transformation given by Fowles is

$$\begin{aligned} x' &= \gamma(x - \beta t), \\ t' &= \gamma(\beta x - t) \end{aligned} \quad \therefore \quad \begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ \beta & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ t \end{pmatrix}$$

Therefore, these matrices can hardly make up a group, for the identity matrix is conspicuously missing from among them. Such a Lorentz transformation involves a 2×2 matrix which, again for our later convenience, we write in the form of a *linear combination* of two fixed matrices satisfying the property expressed in equation (2.3.1) up to the sign of their determinant:

$$\begin{pmatrix} 1 & -\beta \\ \beta & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \beta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.3.9}$$

Notice that this transformation has the same *quadratic invariant* at a finite scale as the regular Lorentz transformation [see §2.1, equation (2.1.6)]. However, this time the invariant is also secured at the infrafinite scales of space and time:

$$(x')^2 - (t')^2 = x^2 - t^2, \quad (dx')^2 - (dt')^2 = (dx)^2 - (dt)^2$$

in view of the fact that this Lorentz transformation is realized by the same matrix at finite and infrafinite scale of space and time. Notice, further, that in these conditions the second of the steps of Fowles procedure listed above, is virtually unnecessary for uniform relative motions of the reference frames and material particles, inasmuch as the equation (2.3.6) is valid also for the ratio ($x/t$) of the finite quantities, according to linear action of the Lorentz transformation at a finite scale.

However, we need to mention that, for general motions, which incidentally can be assumed as uniform only on infinitesimal ranges of time and space or, in fact, not uniform at all – being, for instance, fractal by their very nature – a connection is needed between the instantaneous velocity ($dx/dt$) and the velocity ($x/t$) upon finite ranges of space and time. In general, this last velocity is usually calculated as a mean velocity over finite intervals of time and finite space distances. For the case of uniform motions such a connection is given by the identity matrix, whereby the two velocities are simply identified:

$$dx / dt = x / t \quad \therefore \quad \begin{matrix} dx = ax \\ dt = at \end{matrix}$$

where $a$ is a factor of infinitesimal order, for accordance between space scales and time scales. From this perspective we need to notice that the Lorentz transformation can be realized by the same kind of involutive matrix even in the case where

$$dx / dt = A \cdot x / t$$

In this case, though, the motion would be hardly uniform, for we have by direct integration:

$$dx / dt = A \cdot x / t \quad \therefore \quad x = B \cdot t^A$$

where A and B are two constants. This is a characteristic of the turbulent behavior in fluids (Harvey, 1966), or the case of creep or relaxation in the solids [(Jeffreys, 1972); see also the *Wikipedia* review article on *Creep* (*Deformation*)]. According to these observations, it is quite significant that the first of these cases is taken to serve for a Madelung-type interpretation of the wave mechanics (Harvey, 1966). This, in our opinion, is a clear indication that such an interpretation is, apparently, closely connected with the transition between space and time scales. We need to insist on this property as we go along with our work.

Thus, from the perspective of Fowles' theory regarding Lorentz's transformation representation – which, as we can see from the above staging, can hardly fit into physical idea of the concept – one can say that, physically speaking, the Lorentz matrix may not be necessarily connected continuously to the identity matrix: the ensemble of Fowles' Lorentz matrices is not a continuous group but, as they say, a semigroup. Rather, one can suspect that the involvement of the identity matrix is an 'external' feature, connected with that scale transition property involved in getting this kind of Lorentz transformation. In view of the observations developed in this section, Fowles' line of thought deserves a little more attention and, naturally, an appropriate appraisal. We take it as highly significant in connection to the Planck's concept of resonator, inasmuch as it involves consideration of *a pair of reference frames* as a physical unit.

The attention we believe Fowles' idea deserves from a physical point of view is virtually nonexistent in the specialty literature. Fortunately, however, there is just one observation in the literature, and a valid point of appraisal at that, which needs to be expressly mentioned and considered, for reasons that will become clear as we go along with our case in this work. Namely, R. G. Cook has noticed right away that Fowles' argument is *a priori* destined to select actually *two* types of Lorentz transformations, not just one. Fact is, nevertheless, that one of these, the one left aside by Fowles' choice of the constant *A*, seems physically unfeasible, for it violates the causality condition (Cook, 1979). In our terms here, though, Cook's observation carries a much heavier meaning, if we may say so, which surfaces when Cook's procedure is followed with all mathematical details in constructing a Lorentz transformation.

To wit, the Cook's argument unfolds starting from the observation that the restriction imposed on the constant *A* is not, mathematically speaking, normal. To wit, the second one of the conditions from equation (2.3.2) should actually be taken as:

$$\pm B\cdot C = 1 - A^2 \qquad (2.3.10)$$

Indeed, *a priori*, *i.e.* in the very spirit of the Fowles' mathematical construction, the real parameter *A* should *not* be constrained by anything. In other words, the real parameters *B* and *C* in equation (2.3.2) can also have different signs for arbitrary real *A*, and Fowles' result is just a particular one for *A* limited to the real interval (*–1*, *1*), as mentioned, when the two parameters *B* and *C* must have the same sign. In order to account for the condition (2.3.10), Cook picked the entry *C* of the Lorentz matrix to carry the ambiguity of sign, so that, instead of the choice (2.3.2), we have the following conditions describing a 'general Fowles' construction of a matrix, as it were, to be imposed on the entries of a matrix representing the Lorentz transformation:

$$D = -A, \quad \left|B\cdot C\right| = N^2, \quad B = N\cdot c, \quad C = \pm\frac{N}{c}$$

Here *N* is an arbitrary real number, and *c* is the speed of light.

This choice is always *a priori* meaningful, *i.e.* it makes sense from a mathematical point of view. For once, the speed of light needs *not* be considered a velocity anymore, as Fowles did, so that the construction of Lorentz transformation can be decided only by its *linear action* in a finite range. This liberates us from taking in consideration the *homographic action* of the Lorentz matrix when constructing it. Secondly, this liberation has a significant consequence: it means that *it is not at all mandatory to consider the Lorentz transformation at the infrafinite scale of space and time*. And, because this may very well be a separate working hypothesis of the physics at large, Cook's choice carries the burden of a historically significant meaning, already mentioned above: *c* should be only a physical parameter, just as it was at the very time when Clerk Maxwell introduced it to our natural philosophical awareness [(Maxwell, 1892), Volume II, Chapter XIX]. Only afterwards was assumed that it can have the meaning of the physical velocity of a signal, as its units actually show, even if it is representing just the constant ratio between the electrostatic and electromagnetic units of electricity. However, the last conclusion emphasized above is meant to have a much deeper significance that appears clearly at the extension of special relativity in order to include the gravitation field. To wit: the Einstein's spacetime approach may not involve at all the scales of space and time, inasmuch as the special relativity is a cosmological theory from start.

We will return for a deeper consideration of this issue in the next chapter of the work, insisting on the general relativity.

Coming back to our present algebraical considerations, Cook's matrix representing an alleged Lorentz transformation should be of the form:

$$\boldsymbol{A} = \begin{pmatrix} A & N \cdot c \\ \pm\dfrac{N}{c} & -A \end{pmatrix}, \quad A^2 \pm N^2 = 1 \qquad (2.3.11)$$

According to a Lorentz transformation realized by this matrix, the origin of the primed reference frame satisfies the linear equation in unprimed parameters:

$$Ax + (N \cdot c)t = 0$$

so that we can assign the value $v \equiv (x/t)$ to the relative velocity of the reference frames. In this case we have, in the conventional notation of physics:

$$\frac{N}{A} = -\beta$$

so that the Cook's Lorentz matrix becomes

$$\boldsymbol{A} = A\begin{pmatrix} 1 & -\beta \cdot c \\ \mp\dfrac{\beta}{c} & -1 \end{pmatrix} \qquad (2.3.12)$$

with the factor $A$ decided, up to sign, by the second of the equations (2.3.11):

$$A^{-2} = 1 \pm \beta^2 \qquad (2.3.13)$$

This condition is, indeed equivalent to $\det(\boldsymbol{A}) = -1$, as it should for an involutive matrix. Obviously, *the second* choice of sign in the equations (2.3.12) and (2.3.13) leads *directly* to the Fowles' Lorentz matrix from equation (2.3.8) as expected, with all its consequences except one, which needs to be acknowledged at any rate, for it answers to an important question, connected to the idea that the light speed should be a regular speed. Namely: does this transformation satisfy the condition (2.3.7), that led us to the (2.3.8) and, according to Fowles, describes the 'handedness' of the reference frames?! In order to answer this question, notice that first we need to solve the equation that shows a constant instantaneous velocity, equivalent to (2.3.7):

$$\frac{c(V - \beta c)}{\beta V - c} = -V \qquad (2.3.14)$$

and this has two possible solutions

$$V = \pm c \qquad (2.3.15)$$

The second sign in this equation gives, obviously, the Fowles' case. For once, this result would mean that the sign of the constant $c$ is irrelevant for defining the handedness if this constant is considered a velocity. Indeed, not only the value of $c$ is a physical magnitude, but also its sign: what we have, physically speaking, from the Maxwell's electrodynamical approach to matters electrical, is only the square of $c$, as a function of electrical properties of the medium supporting the light phenomenon. This was, in fact, the general option of theoretical physics thus far, anyway: as we already stated above, the physical magnitude of $c$ was only introduced to physics, by the James Clerk Maxwell's work on electrodynamics only *via its square*, representing the ratio between the kinematic units of electricity and the static units. And, since this ratio is a quantity having the dimensions of the square of a speed, the Maxwell's association appears as just natural.

Much more important, however, at least for what we have to say and debate here anyway, is the first choice of the signs in the equations (2.3.12) and (2.3.13). For that choice, the Fowles' condition (2.3.14) leads to the imaginary values $V = \pm i \cdot c$ instead of the real ones from equation (2.3.15), and the trouble starts brewing as to the possibility of representing the handedness the way Fowles did. In this case, the Cook's Lorentz matrix is

$$\boldsymbol{A} = \frac{1}{\sqrt{1+\beta^2}} \begin{pmatrix} 1 & -\beta \cdot c \\ -\dfrac{\beta}{c} & -1 \end{pmatrix} \tag{2.3.16}$$

so that the transformation given by this matrix becomes

$$x'\sqrt{1+\beta^2} = x - \beta t, \quad t'\sqrt{1+\beta^2} = -(\beta x + t) \tag{2.3.17}$$

Incidentally, by analogy with (2.1.3) and (2.3.9), we need to notice that the generic matrix of this transformation can be expressed as a linear pencil generated by two involutive matrices. To wit, we have:

$$\begin{pmatrix} 1 & -\beta \\ -\beta & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.3.18}$$

This is almost the form of (2.1.3) of the regular groupal Lorentz transformation, except that the identity matrix is replaced by a traceless one, so that we still have to deal with a semigroup here, on which we shall conclude later on. From (2.3.17) we have, by a direct calculations

$$(x')^2 + (t')^2 = x^2 + t^2 \tag{2.3.19}$$

This means that the causality contained in the usual Lorentz transformation may be occasionally lost in this case. Can we say that this invariance is also present at the infrafinite scale? This is a legitimate question, insofar as we did not follow here the Fowles' procedure, and therefore we are not *a priori* entitled to extend the Lorentz transformation, even axiomatically, in the infrafinite range. We hold this question for a subsequent clarification, to be provided as we go along with our development of the present physical theory.

For the moment being, what we can say for sure is the following: in this last case of choice allowed by the Cook's options, we are *a priori* entitled to a reparametrization of the transformation (2.3.17). The parameter $\beta$ can assume any real value, so that a (re)parameterization makes sense in the form:

$$\beta = \sinh\phi, \quad x' = \frac{x}{\cosh\phi} - t\tanh\phi, \quad t' = -x\tanh\phi - \frac{t}{\cosh\phi} \tag{2.3.20}$$

This reparametrization is just as meaningful here as it is the one from equation (2.1.4.) for the orthodox special relativity, because, as we said, the parameter $\beta$ is not necessarily limited to a finite interval of real numbers, and therefore the hyperbolic sine is *a priori* allowed to represent its range. In spite of this, the new parameter $\phi$ can have a precise geometrical meaning: it can be taken as *the natural arclength along some geodesics of a Lobachevsky plane*, in the Beltrami-Poincaré representation. Indeed, the conform-Euclidean metric, describing the Riemannian twofold which represents the Lobachevsky plane in the upper complex plane, is

$$(ds)^2 = \frac{(du)^2 + (dv)^2}{v^2} \tag{2.3.21}$$

According to our conclusions from §1.2, this can be taken as the metric of a Maxwell fish-eye, which is an optical medium admitting dipoles as fundamental physical structures. Here we can add a little more, just based on special-relativistic concepts: this metric admits geodesics described by the parametric equations

$$u = \xi \tanh s \quad v = \frac{\xi}{\cosh s} \tag{2.3.22}$$

each one of them issuing from the point (*0*, *ξ*), corresponding to the 'initial position' from which we start reckoning their arclength. All these geodesics are circles of radius *ξ* and center (*0*,*0*). Therefore, according to equations (2.3.14), the Cook's Lorentz transformation can be geometrically interpreted in the following way: choose any two geodesics of the Lobachevsky plane in the Poincaré representation, for two values of the parameter *ξ*: *x* and *t* say. With these values, construct the two points on the two geodesics, having the parameters

$$u_x(\phi) = x\tanh\phi, \quad v_x(\phi) = \frac{x}{\cosh\phi}, \quad u_t(\phi) = t\tanh\phi, \quad v_t(\phi) = \frac{t}{\cosh\phi} \tag{2.3.23}$$

which correspond to *the same value* of the arclength: $s = \phi$. Then the Cook's Lorentz transformation (2.3.14) can be written as

$$x' = -u_t(\phi) + v_x(\phi), \quad t' = -u_x(\phi) - v_t(\phi) \tag{2.3.24}$$

In other words: any transformed coordinates are linear combinations (even though quite special linear combinations here, in the sense that they have a special form and correspond to the same value of parameter *ϕ*) of the coordinates along the two geodesics of the metric (2.3.15) indexed by the values of coordinate and time in the initial reference frame.

For once, the geodesics in question can be taken, for instance, as Katz's circles in the space of charge [(Katz, 1966); see also (Mazilu, 2020), §3.1], and therefore the theory can make even a physical sense after all, within a Katz-type natural philosophy regarding the charges. Provided, of course, that we can establish a correspondence *between charges and lengths*, in order to mark the geodesics appropriately. We only have to notice that the metric (2.3.21) can be incidentally taken as describing a world of charge indeed: it is the coordinate space containing the center of force of the dynamic problem describing the classical Kepler problem in the case of planetary model [(Mazilu, 2020); see especially §2.3]. This space can be organized as a Lobachevsky plane by the methods of Cayleyan geometry (see §1.3 above), and has a metric that can be reduced to the form from equation (2.3.21) (see §1.2). This reminds us of the fact that the Lorentz transformation can be seen as a rotation of the field intensities of the forces generated by the Hertz particles, at the points of action of these forces. This is a second physical consequence of the charges of particles [see equation (2.2.4) above], a fact upon which we shall return later with more details.

As to the important issue of the 'correspondence between charge and length', we are led to the conclusion that it is a matter of quantization in matter, whose expression is with us today in the form of the Newtonian forces [(Mazilu, 2022); see §4.2]. The essential observation here is that, according to an important theorem of Morton Lutzky, there should be a relation between the systems that satisfy the Ermakov-Pinney equation – the charges, according to the natural philosophy of Katz, or the light according to the natural philosophy of Fresnel – and the motions satisfying the Kepler's second law, *i.e.* the law of areas (Lutzky, 1978) [see also (Wagner, 1991)]. This is a relation which generalizes the very classical Planck's constant, conceived as the ratio between energy and frequency (Lewis, 1967, 1968). We are led to see in this connection, the ontological reason of the existence of fundamental physical structure of the world known theoretically as the Kepler motion, just as it already appeared once to Newton. Speaking of charges and of classical Kepler problem, we need to notice that a proper geometrization of the metric (2.3.21) is the Maxwell fish-eye physical medium (see §1.2), characterized by this metric and such a geometry supports the physical basis of the quantization in the case of matter (Mazilu, 2023a). Just for later convenience, let us recall that the Maxwell fish-eye medium is, in a certain way – specifically, from a geodesic point of view – equivalent to the mechanical system described by the Kepler motion [(Buchdahl, 1978); see also (Chen, 1978)]. We shall come back to this important issue of the natural philosophy in due time. For now, the previous observations take us in a different direction.

The remarks right above on the place of non-Euclidean geometry raise an important question regarding the special relativity itself: is either the orthodox Einsteinian special relativity, or the relativity constructed on the base of Fowles' Lorentz transformation, showing the same property as the non-causal Cook's Lorentz transformation from the previous section? Because, if it does, indeed, show such a property, then the weight of theoretical argument shifts upon the idea of non-Euclidean metrics, just as the general relativity requires. Except that, this time, those non-Euclidean metrics are, obviously, *a priori* available, even with a ready physical meaning at that (see §1.3 above). Thus, in constructing the general relativity we have them 'ready to be used', as it were; and, in using them as such, we surely do not have to resort on Einstein's equations *per se*, but on some equations equivalent to them, identical only within special conditions. The answer to this question is affirmative, and has already been implicitly given almost a century ago, in a work that actually inspired us in appreciating the special relativity the way we did in this very chapter of the present work.

Jean-Marie Le Roux found out that at least part of the Poincaré's objections raised to relativity as a case of classical interpretation – the way Einstein presented it originally, in that remarkable year 1905 – can be overcome with another mathematically meaningful parametrization of the regular Lorentz group of matrices (Le Roux, 1933). And that parametrization is still actual, vividly discussed in fact, even today, by the people involved in special relativity developments. Notice, indeed, that a periodic function of real argument, like the *sine* or *cosine*, can accomplish the task of a meaningful reparametrization of the orthodox Lorentz matrix quite naturally. Indeed, the values of such trigonometric functions are limited to the interval $(-1, 1)$ by default, as it were. Thus, Le Roux noticed that, if instead of (2.1.4.) one takes the *a priori* perfectly admissible transformation of parameter:

$$\beta \equiv \sin\varphi \quad \therefore \quad \boldsymbol{A} = \begin{pmatrix} \dfrac{1}{\cos\varphi} & -\tan\varphi \\ -\tan\varphi & \dfrac{1}{\cos\varphi} \end{pmatrix} \tag{2.3.25}$$

one can handle the relativity in terms of more 'palatable' concepts, so to speak. What is really the case intended to be made by Le Roux can be found by consulting the original work (Le Roux, 1933), because here we are interested in what we think is a more important aspect of the parameterization (2.3.25). Notice, for what is worth here, that the previous Cook's case from the end of the last section, involves taking $i\beta$ instead of $\beta$ in the Fowles' result. Likewise, if instead of $\varphi$ we take $(i\cdot\phi)$ in (2.3.25), we get the Lorentz transformation parameterized as in equation (2.3.12), in view of the fact that $\sin(i\cdot\phi) \equiv i\cdot\sinh\phi$ and, most importantly, that we have to use an imaginary light velocity $i\cdot c$ instead of the real $c$.

However, our main interest in connection with this approach concerns an observation relating the very structure of a physical clock, like the harmonic oscillator, for instance, or a resonator, or even the free particle, for that matter. Namely, we have noticed that, according to a Katz-type natural philosophy, the charge itself, like the light once, for Augustin Fresnel, is bound to behave like harmonic oscillators [(Mazilu, 2020); §4.4]. That is, the trigonometric functions in the description of matter are just as natural as they are in the description of light starting from diffraction phenomena. Let us stop though, for a moment's historical notice, that turns out, for us at least, to be highly significant.

On this occasion we believe worth our while an observation destined to straighten some historical records. Notice that the Le Roux's matrix $\boldsymbol{A}$ from equation (2.3.25), put forward for the common awareness in the year 1933, is the one discovered by Enrique Loedel Palumbo fifteen years afterwards (Loedel, 1948), which was rediscovered by Henri Amar even later (Amar, 1955). The corresponding geometric picture of this transformation is usually baptized Loedel-Amar diagram in the specialty literature. We cannot but join the old grievance once uttered by both Amar and Loedel, on the occasion of straightening the priority of their discoveries [see *American Journal of Physics*, Volume **25**(5) (1957), pp. 326 – 327], about the lack of communication of scientific works due to the natural language barriers. As, fortunately, we got the good chance to live in the present environment of electronic communications, we were lucky enough to have discovered the works of Jean-Marie Le Roux. For once, this is the reason that we think it will do a good justice to all the parties involved, the (re)name of Le Roux-Loedel-Amar for the matrix from equation (2.3.25). We just notice the different reasons of introducing this parameterization, as invoked in the three works referring to it, worth themselves of study, and in depth comparison at that. And while we are on this streak of straightening the things historical, notice that even the Le Roux's theory is of a previous inspiration, from the works of Édouard Guillaume on the foundations of special relativity and physics, in general, for which Einstein himself apparently did not subscribe [see (Guillaume, 1920) and the works cited there].

As far as we are concerned here, the work of Le Roux seems to us most attractive, for it presents *a pairs of Lorentz observers* as a continuous group with two parameters, the *common time* and the *common phase*, along the following idea. In the parameterization (2.3.25), the Lorentz's transformation (2.1.1) leads to two relations

$$t - t' = (x + x')\tan\frac{\varphi}{2}, \quad x - x' = (t + t')\tan\frac{\varphi}{2} \tag{2.3.26}$$

In the spirit of Jean Le Roux, we may further notice that these relations can be taken as saying that there are linear expressions in the two reference frames having the same values, $u$ and $v$ say:

$$t' + x'\tan\frac{\varphi}{2} = t - x\tan\frac{\varphi}{2} \equiv u, \quad x' + t'\tan\frac{\varphi}{2} = x - t\tan\frac{\varphi}{2} \equiv v \tag{2.3.27}$$

The parameters $u$ and $v$ describe not a single reference frame, but *pairs of reference frames*, and the usual coordinates in each one of the two reference frames making a pair, can be written in terms of these parameters and the phase $\varphi$. For instance, we may have:

$$t\cos\frac{\varphi}{2} - x\sin\frac{\varphi}{2} = u\cos\frac{\varphi}{2}, \quad -t\sin\frac{\varphi}{2} + x\cos\frac{\varphi}{2} = v\cos\frac{\varphi}{2} \tag{2.3.28}$$

It makes sense to think of the fact that the two parameters ($u$,$v$), can be appropriated as parameters on a certain surface carrying the initial coordinates: the idea of surface is, again, mandatory even for the special relativity. This reminds us of the fact that the roots of relativity are deep into the Lorentz's theory of the electric matter. Suffice it, for now, to say that the physical connection between light and matter is tied up with the concept of resonator, involving a pair of charges. More about the issue will be said in due time.

Priorities aside, notice, however, in this respect, that ($\tan\varphi$) can be taken as *the mean* of a family of statistical ensembles described by a quadratic variance distribution function, whose *standard deviation* is $1/(\cos\varphi)$, where $\varphi$ is the parameter of the family. As we repeatedly noticed, and occasionally even documented (Mazilu, 2010), this type of statistical distributions is the one staying at the foundation of the Planck quantization of the light phenomenon (Mazilu, 2022). Then, the fact that Le Roux's parametrization places this kind of distribution at the very foundation of relativity becomes significant by itself: the Lorentz transformation connected to the matrix (2.3.25) has the form of a 'statistical specification', if we may be allowed to say so, of the values of coordinate and time of an event. As usual in the statistical practice, a value of a given quantity is expressed as the 'mean, plus or minus the standard deviation', obtained based on an ensemble of measurements. And so are the two Lorentz-transformed coordinates obtained by transformation with the matrix (2.3.25):

$$x' = -t\tan\varphi + \frac{x}{\cos\varphi}, \quad t' = -x\tan\varphi + \frac{t}{\cos\varphi} \tag{2.3.29}$$

Regardless of the signs, in view of our previous developments, each one of these two formulas has the aspect of a statistical specification: average ± standard deviation. This would mean that the quantization itself, to which the Planck's statistics is referring, is also in the very nature of things material, and following this route we may be able to find the right procedure of quantization to be applied to matter (Mazilu, 2023a).

Thus, given an event ($x$, $t$) in a reference frame, it can be located in any frame moving uniformly with respect to it by a recipe like that from equation (2.3.23). Only, in the present case, we are not to use the Beltrami-Poincaré metric (2.3.21): such a frame is described by the conform-Lorentzian metric

$$(ds)^2 = \frac{(du)^2 - (dv)^2}{v^2} \tag{2.3.30}$$

whose particular geodesics of the same center are given by equations

$$u = \xi\tan s, \quad v = \frac{\xi}{\cos s} \tag{2.3.31}$$

with $\xi$ constant for each geodesic [(Mazilu, 2020); see §5.4, equations (5.4.21) *ff*]. Then, in specifying the Lorentz transformation given by (2.3.25) we can use the following recipe: take two geodesics of the same center, with parameter $\xi$ having the values $t$ and $x$, respectively, and choose the points corresponding to the same value of the arclength of geodesics, $s = \varphi$ on both of them, according to the formulas

$$u_1 = x\tan\varphi, \quad v_1 = \frac{x}{\cos\varphi}, \quad u_2 = t\tan\varphi, \quad v_2 = \frac{t}{\cos\varphi} \tag{2.3.32}$$

The pair of linear combinations $(-u_1+v_2, -u_2+v_1)$ then represents an event belonging to the invariant $(x_2-t_2)$, of Le Roux, for the same value $\varphi$ of the arclength along the two geodesics.

Now, as noticed before (see §1.2), the noncausal Lorentz transformation of Cook can be connected to the geometry of the Maxwell fish-eye realm, which is a conform-Euclidean geometry. Inasmuch as the causal theory considered by Le Roux is, in fact, the theory of special relativity, referring therefore to a distance associated to a length by the properties of propagation of the light or a similar phenomenon involving propagation, we can infer that the noncausal theory should be referring to a *length associated to a distance by internal properties of matter* in the sense of Poincaré. Therefore, it is referring in fact to the very physical description of matter, in its most intimate manifestation, like, for instance, the nuclear matter of the planetary model of atom. It will be, therefore, our task here to ascertain its true place within a physical theory, and this will be done *via* the de Broglie's waves. But, before anything, we need to establish, in a more specific way, the connection between the Fowler-Cook approach of special relativity, and the continuous groups orthodox approach. As a trained eye can guess right away from the manner the things are presented here, this connection involves the relation between some involutory matrices and the matrices of the proper Lorentz transformations. In more general mathematical terms, this concerns the connection between a continuous group and its generator algebra. A first portion of the solution to this task just follows, wherein what we are taking as a *fundamental analogy* of knowledge – the analogy between the two *grand relativities* of the human kind, namely the *Galilean* and *Einsteinian* relativities – is made explicit with the assistance of the classical differential theory of surfaces. And, in order to fulfil this task, we need some preliminary concluding observations on what we already got this far.

### 2.4 A Conspicuous Connection in Special Relativity

The quintessential algebra of the transformations related to this mathematics, generically called *Lorentz transformations*, is staged by matrices like (2.1.3), having a common *linear* structure, as we said:

$$\boldsymbol{L}=\lambda\boldsymbol{1}+\mu\boldsymbol{J},\quad \boldsymbol{J}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \boldsymbol{1}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.4.1}$$

In connection with the historical observation from the previous section, we can have here the case of Loedel, where the parameters $\lambda$ and $\mu$ are given as trigonometric functions of the *aberration angle*. In the case of equation (2.1.3) or, better yet (2.1.5), which is more suggestive in context, they are given by hyperbolic correspondents of those functions.

As an analysis by Loedel shows (Loedel Palumbo, 1955), the hard part of the problem of physics here is to get over the idea of free fall – that inspired the construction of the general relativity – with its central directional aspect. For once, this kind of motion reveals a contradiction between the classical idea of rigidity of a physical structure and the special-relativistic physics. On the other hand, the central directional aspect of the motion is difficult to comprehend in the case of a multitude of centers of force, asking for a kind of democratic 'equality of rights', as it were.

The analysis of Loedel just mentioned also suggests that the solution of this problem is to be found in the *concept of a surface*, the way this is conceived for the de Broglie's idea of interpretation, *i.e.* by portions of surface intersecting fluxes of rays. As a matter of fact, Lorentz's characterization of the electric matter amounts to the very same: conceiving a surface which describes the condition of electric neutrality of matter, in order to be able to define the concept of charge (see §1.5). Within this idea, we offer here a solution starting from the observation that the Lorentz matrix $\boldsymbol{L}$ from equation (2.4.1) can be written as a square, based on a firm mathematical theorem: every such matrix is the square of linear combinations of the decomposition matrices of $\boldsymbol{J}$. Let us explain the reasons, and some of the terminology involved in this statement.

The connection between Lorentz matrix from equation (2.4.1) the matrix $\boldsymbol{J}$ entering its linear structure is univocal, and determined, up to a point, by the two possible actions of the matrix, which, as we shall see later on, allow us to characterize it even quantitatively: *linear action* in the two-dimensional case, and *homographic action* in the one-dimensional case. With respect to these two actions the matrix $\boldsymbol{L}$ has precise numerical characteristics, represented by the *eigenvalues* and *fixed points*, respectively, that remain unchanged during actions. In the case of matrix $\boldsymbol{L}$, these are the solutions of the quadratic equations

$$\det(\boldsymbol{L} - x\boldsymbol{1}) = 0 \quad \textit{respectively} \quad \boldsymbol{L}(x) = x \tag{2.4.2}$$

that is, the values:

$$\lambda \pm \mu \quad \textit{respectively} \quad \pm 1$$

respectively. Here $\boldsymbol{L}(x)$ denotes the homographic action of the matrix $\boldsymbol{L}$. Notice that the eigenvalues are properties of the *linear action* of the matrix, exclusively, while the fixed points are those of the *homographic action* of the matrix $\boldsymbol{J}$ entering the linear form along with the identity matrix. This is the general property of the Lorentz matrix written as a family (2.4.1): it has a unique correspondent involutive matrix $\boldsymbol{J}$ in its linear expression, with the same fixed points, but its eigenvalues are independent of this matrix. The question remains, though: what is so special about this matrix $\boldsymbol{J}$, to make it noticeable in the case of Lorentz transformation? Going a little ahead of us, we have no problem in answering right away: it is the only possibility of rationally introducing the concept of surface in describing the Lorentz transformation. The rest of this chapter is dedicated to describing such a possibility. However, in order to make the idea more comprehensible, we should proceed gradually, easing, as it were, our way into the subject.

First, notice that $\boldsymbol{J}$ represents an *involution*, as in the cases of Fowles' and Cook's Lorentz matrices. This means that, applied twice on an object, it reproduces the object on which it is applied, no matter of the action represented by the matrix:

$$\boldsymbol{J} \cdot \boldsymbol{J} = \boldsymbol{1} \quad \therefore \quad \begin{array}{l} \operatorname{tr}(\boldsymbol{J}) = 0 \\ \det(\boldsymbol{J}) = -1 \end{array} \tag{2.4.3}$$

The right hand side of this equation reproduces the conditions on the matrix $\boldsymbol{J}$ resulting from the left hand side, considered as a Hamilton-Cayley identity for the matrix: it must have a zero trace and unit determinant, up to a sign, just like in the equation (2.3.2) for the Fowles' case, or the equation (2.3.11) for the Cook's case of a Lorentz matrix. But the most important consequence of these algebraical considerations is the existence of two other involutive matrices, $\boldsymbol{I}$ and $\boldsymbol{K}$ say, having, basically, the same algebraical properties as in equation (2.4.3), – *i.e.* null trace and unit determinant up to a sign – which are simply factors of a unique decomposition by multiplication of the matrix $\boldsymbol{J}$:

$$\boldsymbol{J} = \boldsymbol{K} \cdot \boldsymbol{I} = -\boldsymbol{I} \cdot \boldsymbol{K}, \quad \boldsymbol{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{K} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.4.4}$$

Then, one can verify right away that the product of two linear combinations of ***I*** and ***K***:

$$\boldsymbol{M}(\boldsymbol{x}) = x_1\boldsymbol{I} + x_2\boldsymbol{K}, \quad \boldsymbol{M}(\boldsymbol{y}) = y_1\boldsymbol{I} + y_2\boldsymbol{K} \tag{2.4.5}$$

is a matrix that can be identified with ***L*** from equation (2.4.1) provided

$$\lambda \equiv -x_1 y_1 + x_2 y_2, \quad \mu \equiv x_2 y_1 - x_1 y_2 \tag{2.4.6}$$

In calculating the eigenvalues of ***L***, we need the trace and the determinant. These are:

$$\mathrm{tr}(\boldsymbol{L}) = 2\lambda, \quad \det(\boldsymbol{L}) = \lambda^2 - \mu^2 \tag{2.4.7}$$

Therefore the eigenvalues of ***L*** are $(\lambda \pm \mu)$, and we have their product as:

$$\lambda^2 - \mu^2 = (x_1^2 - x_2^2)(y_1^2 - y_2^2) \tag{2.4.8}$$

proving that the pair (***I***, ***K***) is closely connected to the structure of the set of matrices ***L***: the Lorentz quadratic form representing the determinant of ***L*** from equation (2.4.1) is the product of the Lorentz quadratic forms representing the determinants of the two ***M***'s from equation (2.4.5). This property of the quadratic forms – that is, of being the product of two *similar* quadratic forms – was used extensively by Adolph Hurwitz in a significant work, and is what we call the *Hurwitz*'*s property* of the quadratic forms (Hurwitz, 1898).

The three involutions ***I***, ***J*** and ***K*** exhibit the closure property of the algebra of 2×2 matrices of null trace, making a vector space out of it. First, they are linearly independent vectors:

$$m_1\boldsymbol{I} + m_2\boldsymbol{J} + m_3\boldsymbol{K} = \boldsymbol{0} \quad \Leftrightarrow \quad m_1 = m_2 = m_3 = 0 \tag{2.4.9}$$

Thus, any involutory 2×2 matrix, ***V*** say, can be considered *a vector* having the components $v_1$, $v_2$, $v_3$, in the 'reference frame' given by the matrices ***I***, ***J***, ***K*** taken as the base vectors of a reference frame to be used in 'expressing linearly' the elements of algebra of involutory matrices. Indeed, the linear combination of the three base vectors can be written in the form:

$$\boldsymbol{V} = v_1\boldsymbol{I} + v_2\boldsymbol{J} + v_3\boldsymbol{K} \tag{2.4.10}$$

and this 'linear decomposition' is unique: two vectors having the same components are necessarily identical, and *vice versa*, according to equation (2.4.9).

The whole point of these observations is that the triad of matrices (***I***, ***J***, ***K***) can be considered as a *fixed reference frame* in the linear space of the involutory 2×2 matrices [for mathematical philosophy of such reference frames see (Dubois-Violette, Kerner, & Madore, 1990); an interesting physical point of view is particularly described in (Shchepetilov, 2003)]. In view of their matrix multiplication table

*The First Factor*

| × | ***I*** | ***J*** | ***K*** |
|---|---|---|---|
| ***I*** | $-1$ | $-\boldsymbol{K}$ | ***J*** |
| ***J*** | ***K*** | $1$ | ***I*** |
| ***K*** | $-\boldsymbol{J}$ | $-\boldsymbol{I}$ | $1$ |

(2.4.11)

we have the following 'orthogonality' conditions, generalizing equation (2.4.4):

$$\boldsymbol{IJ} + \boldsymbol{JI} = \boldsymbol{JK} + \boldsymbol{KJ} = \boldsymbol{KI} + \boldsymbol{IK} = \boldsymbol{0} \tag{2.4.12}$$

based upon which we can infer that they form a set of linearly independent vectors, as above.

There are a few interesting properties of this approach of the Lorentz transformation, which make it particularly attractive for physics, from a point of view connected with what we have elaborated this far in the present and previous works. First, the geometrical norm of the vector (2.4.10) – that is, its square – is simply the determinant of the corresponding matrix, up to sign:

$$\det \boldsymbol{V} \equiv v_1^2 - v_2^2 - v_3^2 = -\boldsymbol{V} \cdot \boldsymbol{V} \tag{2.4.13}$$

This quadratic form can be used in the construction of an absolute geometry (see §1.3) describing the three fundamental physical quantities correlated with the Newtonian forces, that is: the gravitational mass and the two charges [(Mazilu, 2020); see §3.1]. In that case, the components of vector $\boldsymbol{V}$ are the magnitudes of the three Newtonian forces naturally connected to gravitation and the two charges, electric and magnetic. The vector itself can be taken as the magnitude of such a force: in case of equilibrium this magnitude is 'zero', which means that the quadratic form from equation (2.4.13) should be zero. This is the *absolute* of such a *Cayley-Klein geometry*. Physically, it represents an *equilibrium ensemble of material particles serving for interpretation*.

In contemplating such a generalization, it is tempting to similarly consider the case where the Lorentz matrix from equation (2.4.1) is constructed with the involution $\boldsymbol{I}$ instead of $\boldsymbol{J}$ (the case with $\boldsymbol{K}$ instead of $\boldsymbol{J}$ is practically identical with the one already discussed above). Then, using the matrix $\boldsymbol{I}$ in order to construct such linear combinations, we shall have another linear family of matrices, and these two families are mutually exclusive, in the sense that any two matrices, belonging to different family, have a commutator proportional to $\boldsymbol{K}$. Nevertheless, coming back to our discussion of the family based on $\boldsymbol{I}$, this can be obtained as a product of two matrices $\boldsymbol{M}(\boldsymbol{x})$ and $\boldsymbol{M}(\boldsymbol{y})$, linear combinations of $\boldsymbol{J}$ and $\boldsymbol{K}$, as follows:

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{x}) &= x_1\boldsymbol{J} + x_2\boldsymbol{K} \\ \boldsymbol{M}(\boldsymbol{y}) &= y_1\boldsymbol{J} + y_2\boldsymbol{K} \end{aligned} \quad \therefore \quad \boldsymbol{L} \stackrel{def}{=} \boldsymbol{M}(\boldsymbol{x}) \cdot \boldsymbol{M}(\boldsymbol{y}) = (x_1 y_1 + x_2 y_2)\boldsymbol{1} + (x_1 y_2 - x_2 y_1)\boldsymbol{I} \tag{2.4.14}$$

having complex eigenvalues ($\lambda \pm i\mu$) with obvious analog notations for $\lambda$ and $\mu$, to be calculated from the relations

$$\mathrm{tr}(\boldsymbol{L}) \equiv 2\lambda = 2(x_1 y_1 + x_2 y_2), \quad \det(\boldsymbol{L}) \equiv \lambda^2 + \mu^2 = (x_1^2 + x_2^2)(y_1^2 + y_2^2) \tag{2.4.15}$$

where the Hurwitz's property is, again, obvious. This turns out to be a purely Euclidean case.

As we just said above, the absolute quadric of this geometry is liable to represent, in fact, a static ensemble of identical Hertz material particles in equilibrium, each of them described by three physical quantities known to be connected to *Newtonian forces in equilibrium at any distance in any direction*: gravitational mass ($v_1$), the electric charge ($v_2$ or $v_3$) and the magnetic charge, complementary to electric charge ($v_3$ or $v_2$). This equilibrium ensemble is a fictitious notion, of course, – it has no reality whatsoever, according to the experience defining the finite world we inhabit – just as fictitious as the notion of Hertz particles, or classical material points for that matter, able to generate *simultaneously* the three Newtonian forces: there is no such particle in the finite world of our experience, for these forces do not act simultaneously with comparable intensities at the same space scale. However, following this very point of view, which involves *only* our imagination, the quadratic form (2.4.13) has two other fundamental virtues that allow its *connection with the reality* given by our experience. First, if the matrix $\boldsymbol{V}$ is taken as realizing a transformation, then it can represent an amplitude in the sense of Louis de Broglie (*loc. cit. ante*, §2.1), but involving exclusively fundamental quantities related to a Hertz particle. Indeed, its determinant – or the quadratic norm thereof, up to sign – is, according to a de Broglie-type of natural philosophy, the measure of a density given by this transformation and, being a quadratic form, makes an amplitude out of the very matrix ***V***. On the other hand, in a real world such as the world revealed to our intellect by experience, the quadratic form (2.4.13) is never null – the Newtonian forces exist, a fact revealed to us by motion – but can be either positive or negative *depending on the space scale* where we are considering this world as a universe.

To wit, at the finite space scale of our experience, where, as a rule, the charges prevail over gravitational mass by their Newtonian forces, the quadratic form (2.4.13) has a certain sign, say negative, just to settle the ideas. By the same token, at the infrafinite scale we can infer that it should also have a negative sign according to existing observations: there are no noticeable gravitational forces between elementary particles. On the other hand, at the transfinite scale of our world, where the gravitational mass prevails by its Newtonian force over electric forces, our imagination enters again the stage, and thus we can logically infer that the quadratic form has the opposite sign. And this inference is also plainly supported by astronomical observations. The bottom line here would then be that the sign of the quadratic form in equation (2.4.13) *is characteristic to the space scale where we are describing the universe*. This fact has an important impact on the geometry of the physical magnitudes of the bodies in the universe, which should be different at different space scales.

### 2.5 Framing the Idea of Surface into a Fundamental Analogy

We come now to one of our main points with the present work: the most important conceptual capability of this approach of the special relativity is the one that makes Lorentz's and de Broglie's ideas on the *role played by a surface in physics*, part and parcel of it. To wit: the matrices ***I***, ***J***, ***K*** can – actually, they *must* we should say, in view of the ideas just mentioned above – be *connected with the existence of a surface*. They can be, indeed, thus connected, and in the process of connection, the whole algebraic theory outlined in the previous section stands as it is. For, assume that we have a surface described, as usual [see, for instance, (Struik, 1988)] in two parameters, *u* and *v* say, and we use a certain location (*u*,*v*) on this surface in order to explain the universe around us. This is, actually, what we regularly do: in crafting physics, we use a location on Earth, specifically, the location we inhabit, *viz*. the location supporting our life. This is the basic realistic view to be taken when it comes to doing physics, and our idea is that we have to recognize it as such in the very construction of any physical theory. Special relativity, in the previous take, allows us to accomplish this need. More than this, the relativity, in Einstein's take, even obligates us to adopt the point of view.

To be specific in our argument, let us frankly recognize that we are hardly doing physics *in a point of the universe*, as usually claimed in physical theories. By the very reason of our possibility of existence, we are doing physics *in a place of the universe located on the surface of Earth*. Implicitly or, on occasion, even explicitly, this specific location makes an imprint upon that physics, determining its character and, in our opinion, we have to consider this circumstance and to assess it accordingly, when trying to extend the physics into describing the universe around us. It is, again, in our opinion, not wise at all to declare right away that, when doing physics, we are located in a point of the universe, thereby erasing *a priori* a whole chain of information that needs to be represented in modeling the physical location we inhabit.

Such a removal of an important piece of information has actually a reason that may count as an excuse after all: it allows avoiding the necessity of theoretically representing that piece of information. For, most of the times we do not have the information we thus bypass, or even if we have it, the physics might not be able to tell us how to bring it into the theories we create. Thus, it is a lot easier to bypass it by imagination, in case we know it explicitly, than to consider it a reality. Surely, sooner or later the removed piece of information strikes back, obligating us to properly complete the natural philosophy we have thus created, admittedly according to a momentary necessity. Often times, though, we do not have at our disposal means for doing this completion, so we further remain at the disposal of some mind inventions, incidentally based on our experience, however most of the times speculations framed axiomatically into a full-fledged theory. Just for a future benefit, however, we can even characterize the Earth's surface in a 'Newtonian style' as it were: it is the surface limiting the 'orb' of Earth, *i.e*. that surface admitting exclusively sideways *uniform* motions, such as those once ascribed by Newton to an 'intelligent Agent'.

The relativity, as we presented it in this chapter, offers us a way to carry the necessary information regarding location on Earth into theory, and this is what we should like to call *de Broglie*'*s way*. The reason for this appellation rests with the fact that Louis de Broglie was the one instituting in some detail the idea of surface in his physical connection with the concept of light ray (de Broglie, 1926b,c). The de Broglie's line started, as well known, from relativistic reasons allowing a certain quantization in matter. However, it is only the idea of surface, as applied in the physical description of the classical concept of light ray, that allowed him do describe the diffraction phenomena based on the concept of material particle [(Mazilu, 2020); see especially Chapter 2]. That the idea of de Broglie belongs to relativity can be shown explicitly on this occasion, by the fact that it opens an understanding of what we like to call the *fundamental analogy* between *Galilean* and *Einsteinian relativities*, which gave, and is likely to continue giving great perspectives for the natural philosophy at large.

Speaking of the surface of Earth – that is, the surface of our secular experience, the experience which decides the finite scale of the space and time of the world we inhabit – we can take notice of the fact that the possibility exists of including its geometry into the very structure of the three involutions represented by matrices $\boldsymbol{I}$, $\boldsymbol{J}$, $\boldsymbol{K}$. Indeed, we can define these matrices to account for any position of a surface, in general, given the possibility of its coordination by two parameters, $u$ and $v$ say. More to the point, we can define the three involutive matrices depending on these two parameters:

$$\boldsymbol{I} = \frac{1}{v}\begin{pmatrix} u & -u^2 - v^2 \\ 1 & -u \end{pmatrix}, \quad \boldsymbol{J} = \frac{1}{v}\begin{pmatrix} u & -u^2 + v^2 \\ 1 & -u \end{pmatrix}, \quad \boldsymbol{K} = \begin{pmatrix} -1 & 2u \\ 0 & 1 \end{pmatrix} \tag{2.5.1}$$

satisfying all the algebraic relation from the previous §2.4, with *no change or additions* whatsoever. Whence the possibility to construct a special relativity that would depend explicitly on the location $(u,v)$, chosen, in particular, even on the surface of Earth, using, instead of the matrices from equation (2.4.10) – liable to suggest doing physics in a point in space – the matrices given in equation (2.5.1), allowing an 'intermediation', if we may, of that physics by the Earth's surface, as is the case in reality. At rigor, the usual relativistic case would appear, in this construction, as only particular: the matrices from equation (2.4.10) are to be obtained from those of equation (2.5.1) for the particular values $u = 0$ and $v = 1$, so that we reserve for those matrices the notations $\boldsymbol{I}_0$, $\boldsymbol{J}_0$ and $\boldsymbol{K}_0$, in order to show that they are particulars on an incidental surface, like that of Earth. This may mean either a special choice of the position on the Earth's surface – which is highly conventional indeed, and thus may be considered as *purely subjective* in a physical context – or else, possibly, they are corresponding to a *special gauging of the surface* itself. However, there is a much more important consequence of this observation, that goes deeper into the structure of physics as we know it, actually concerning its own possibility. Obviously, it is only mandatory to insist on this consequence.

Indeed, let us proclaim, once again, this by and large theoretically unrecognized truth: it goes without saying that *we* are doing physics *not* in a point in space, as usually considered in any of the modern physical theories, but on a multitude of positions on the surface of Earth, depending on the location we inhabit on that surface. This allows us to set some order among our *ideas of invariance*. Indeed, if from the conclusions of doing physics on so many positions on Earth's surface, we were able to extract properties valid for a generic position to be assigned for the Earth in the universe, this should be the *consequence of a fundamental relativistic invariance*, of the kind made first known to our intellect by Galileo Galilei. From the point of view of our awareness of the Earth's position in the universe, Galilei's relativity can be précised by saying that the conclusions of experiments we are able to perform on Earth *are independent of the position on the surface of Earth* where they are performed. However, this kind of independence is just a part of the whole concept of invariance, namely the one with reference to the Earth only. Honestly, not even that part is covered completely though, for it ignores the kinematical aspect of displacement from a position to the other, which was, in fact, the main initial aspect of the Galilean relativity.

The physics of our times completes the one of Galilei's times with the awareness that the situation remains the same *no matter of the location of Earth in the universe*, which is a conclusion of an Einsteinian type relativity brought into analogy. For, the times of Galilei are long past, and meanwhile we got the knowledge that the Earth, in its journey, may never be twice in the same place in the universe. Perhaps accidentally this may happen, indeed, but this is just an exception of which *apparently* we may never be aware, so that we do not even know if it proves the rule or not, like any respectable exception! Now, in order to account *relativistically* for the position on the Earth's surface, we need to construct the Lorentz transformation corresponding to *that position* on the surface. Therefore, for such a construction we cannot use the matrices from equations (2.4.1) and (2.4.4), but those from equation (2.5.1), which have the theoretical capability to explicitly account for the position on the surface, with this position represented somehow by expressions depending on the parameters $u$ and $v$. The triad (2.5.1) is a kind of *generalized Shchepetilov reference frame*, assuming an extension of the meaning of the triad from equation (2.4.4), denoted from now on with $\boldsymbol{I}_0$, $\boldsymbol{J}_0$, $\boldsymbol{K}_0$, which, with due adjustments of course, can be seen as an *original* Shchepetilov reference frame [see §2 of (Shchepetilov, 2003)].

This kind of reference frame has a deeper meaning from the perspective of analogy of the two grand relativities, and this meaning concerns mainly the Galilean relativity. Indeed, consider this: perhaps the classical Galilean relativity *per se* is immune to a change – its time is long gone, as we said! – but the modern physics can certainly benefit from this analogy. However, the generalized Shchepetilov frame helps us in realizing this benefit in connection with the fundamental problem set above, namely that of representing the location of our existence into the physical theory. To wit: from a mathematical point of view, physical results valid *everywhere on Earth* should depend on the parameters $(u,v)$ *via* some *invariant expressions* involving these parameters. On the other hand, physical results valid *everywhere in space* should *not depend in any way* on the parameters $(u,v)$ of a location on Earth's surface. Not even by invariants along the Earth's surface, since one can figure out right away that the Earth's surface may evolve as the Earth moves through space, and its surface invariants may depend on its current position in space, so that may not be surface invariants any further. The first of these conditions – that of representing the experiential facts of our knowledge – belongs to Galilean part of the analogy, the second one, *i.e*. that of representing facts valid everywhere in the universe – to the Einsteinian part.

This view of the two kinds of relativities has also deep consequences on their connection by analogy. First of all, it says that this connection cannot be taken directly, as in the usual procedure in the current theoretical physics, that is, with the limit $c \to \infty$ involved in the passage from one to the other, for that is just a particular approach. One always needs a surface for constructing a Lorentz transformation, if it is to take the extension of matter into consideration, and this is the place where the physics of Louis de Broglie, involving the physical properties of the wave surfaces, enters the stage. While revealing the role of this physics in its broad strokes is a task we follow gradually in the present work, for now let us just say that the explicit presence of a surface at the very heart of the physical theory fills in naturally for the need of formulating that theory in such a way that we can mathematically account for its independence of the location on it. The basis of such formulation is delineated as follows.

Assume, indeed, a relativity based on the Lorentz transformation realized by the matrix $\boldsymbol{L}$ from equation (2.4.1), however with the Shchepetilov matrix $\boldsymbol{J}$ taken from the reference frame (2.5.1). The two invariants of this matrix, given in equation (2.4.7) – that is the trace and the determinant, and therefore the eigenvalues $\lambda \pm \mu$ – *do not depend on the parameters* $(u,v)$. Therefore the eigenvalues of the Lorentz matrix *are completely independent of the location on surface*, and can fill in for those fundamental quantities 'valid everywhere in space', as required by the analogy between the two grand relativities. In other words, the Lorentz matrix, as such, has *two distinct eigenvalues*, and these are *independent of the position on the surface*. In this respect it has a distinguished physical parentage worth recalling right now, for our future benefit on the guidance of our reasoning.

First, there is the *Fresnel's theory of light*, whereby the force generated by light within its supporting medium (the ether) is independent of the direction of propagation and is parallel to the wave plane [see, for a clear explanation of this issue, (Poincaré, 1889), §151]. There, the role of surface is, obviously, played by the wave surface, as needed in the de Broglie's theory. Then, a second example to be cited here along the same lines – an exampe that we consider the most important among the modern physical examples of such a theory – is that referring to the *half-spin of particles*. This appears to be a purely quantum problem, involving the eigenvalues in a fundamental way: two eigenvalues depending in no way on the direction in space [for a thorough documentation see (Schwartz, 1977)]. In this case the surface in question is simply the regular geometrical unit sphere of the three-dimensional space that we inhabit. In modern physics, we found that the property is certainly required for the prototype gauge fields – *the Yang-Mills fields* – which thus generalize naturally the fundamental property of their ancestors the Fresnel and Maxwell light fields. This way, we have a firm reason to consider these fields as the only rightful candidates for the universal gauge fields of our universe.

The question of the origin of a parameterization for the Lorentz's matrix can be just naturally answered, exclusively in connection with the surface, in the manner that follows here. One has to recall that the Galilei's relativity was originally referring to *the motion* on the *quiet Earth's surface*: even if we are in motion on the surface of Earth, the results of experiments we perform are not affected, *provided that motion is uniform*. Just *geometrically speaking*, this means, first and foremost, a variation of the parameters $u$ and $v$ around a certain location on Earth's surface, *regardless of the way it is done*: uniformly, uniformly accelerated, or any other imaginable way, whatsoever. Consequently, the matrix $\boldsymbol{J}$ from the corresponding Lorentz's matrix $\boldsymbol{L}$ is changing with the location on surface, and we know that this change is described by a coframe of the **sl**(2,R) algebra [(Mazilu, 2020); equation ( 4.2.27)]. In the case of ***J*** alone, this coframe is given by the differential forms:

$$\omega^1 = \frac{du}{v^2}, \quad \omega^2 = -2\frac{udu - vdv}{v^2}, \quad \omega^3 = \frac{(u^2+v^2)du - 2uvdv}{v^2} \tag{2.5.2}$$

Their discriminant, which is also an absolute metric (*loc. cit. ante*, §4.3), is given by

$$\omega^1\omega^3 - (\omega^2/2)^2 = \frac{(du)^2 - (dv)^2}{v^2} \tag{2.5.3}$$

Denote $(d\phi)^2$ this metric, thus suggesting that it can represent a phase $\phi$, for it is non-dimensional by its very algebraical construction. The important point to be noticed about this approach, is that the motion can be *uniform* on our surface, indeed, but this characteristic of the motion can be theoretically satisfied only *along the geodesics of the metric* (2.5.3), in the following sense. The parametric equations of geodesics can adopt, with a proper choice of the origin of phase $\phi$, the convenient form:

$$u(\phi) = u_0 + v_0 \tan\phi, \quad v(\phi) = \frac{v_0}{\cos\phi} \tag{2.5.4}$$

This property, which is a rational one in context, can only be contemplated by the presence of surface in the theoretical scenario. In the genuine case of Earth, and within spherical model, it says that the condition is satisfied only for the great circles of the sphere, but we shall need to continue on this statement with some further specific explanations. In any case, for the geodesics of a metric like that from (2.5.3), the rates of differential forms with respect to the variation of phase are constants, given by the relations:

$$\frac{\omega^1}{d\phi} = \frac{1}{v_0}, \quad \frac{\omega^2}{d\phi} = -2\frac{u_0}{v_0}, \quad \frac{\omega^3}{d\phi} = \frac{u_0^2 + v_0^2}{v_0} \tag{2.5.5}$$

In words: if $\phi$ is taken as 'time' along the geodesics of the metric (2.5.3), the three rates of variation of the curvature parameters, represented by the coframe components are constants. As one can convince oneself right away by eliminating the parameter, geodesics (2.5.4) are hyperbolas on the surface:

$$v^2 - (u - u_0)^2 = v_0^2 \tag{2.5.6}$$

The physical interpretation of these geodesics will be discussed later in connection with the classical idea of interpretation. For now, let us just take notice two important points of this result: first of all, comes the observation that along the geodesics of the metric generated by the variation of the matrix ***J***, a corresponding matrix ***I*** is constant. In other words, the rates of 'uniform motion' with respect to the 'time' $\phi$, along these geodesic are given by the entries of the matrix ***I*** for specified values of the parameters $u$ and $v$. Secondly, notice that a certain geodesic of the metric (2.5.3) is sufficiently parameterized by the two values $u$ and $v$ or, at rigor, by algebraic expression involving just these two values. From the perspective of the physics determined by the space scale transition this is a fundamental property: we venture to say that this is the mathematical property that made the relativity possible in the Einstein's take. We shall come back to this issue in concluding the present work.

These geodesics may or may not be paths along the original surface – the Galilei's surface of quiet sea upon which the ships are circumnavigating or, in general, any surface appearing to our practical wits as coordinated by the parameters $u$ and $v$, whatever these may be – but, the gist of the special theory of relativity, as it appears from the equation (2.5.4), is that the parameters $u_0$ and $v_0$ should be somehow correlated with a Lorentz transformation.

As we already noticed, they certainly do have a mathematical meaning by the choice of the 'initial conditions' on those geodesics. However, according to equations (2.5.5) these initial conditions appear as correlated with the entries of a matrix of type ***I*** from the involutions (2.5.1). Then, these conditions alone may also have a physical significance in connection to phenomena of the kind of those involved in electrodynamics. After all, the special relativity is a consequence of classical electrodynamics, let us not drop this fact out of our sight when doing physics! Accordingly, many other conclusions may come out from their hiding, just naturally we should say, if we also consider a type of Lorentz matrix constructed based on the involution ***I*** from among those from equation (2.5.1). In other words, we must consider, as a first instance, the case of Lorentz's ***L*** as given in equation (2.4.14), along the same line of reasoning: the existence of a surface, and even with similar results for that matter.

Now, the matrix (2.4.14), being based on ***I***, has two *complex* eigenvalues ($\lambda \pm i\mu$) – here, obviously, we put $\boldsymbol{L} = \lambda \cdot \boldsymbol{1} + \mu \cdot \boldsymbol{I}$, as in the previous case of the matrix ***J*** – and these are, again, independent of the values of the parameters $u$ and $v$: they are properties belonging to the space that contains the surface. A Galilean relativity on the surface coordinated by these parameters would involve not the variation of the matrix ***J***, but the variation of the matrix ***I*** from equation (2.5.1), for which the coframe corresponding to (2.5.2), and representing the motion on surface is given by the differential forms:

$$\omega^1 = -\frac{du}{v^2}, \quad \omega^2 = 2\frac{udu + vdv}{v^2}, \quad \omega^3 = -\frac{(u^2 - v^2)du + 2uvdv}{v^2} \tag{2.5.7}$$

The absolute metric of this coframe is the quadratic form:

$$\omega^1\omega^3 - (\omega^2/2)^2 = \frac{(du)^2 + (dv)^2}{v^2} \tag{2.5.8}$$

We recognize, in this quadratic differential the Beltrami-Poincaré metric of the hyperbolic, or Lobachevsky plane. As we already mentioned it here, one can show, based on the mandatory conditions of the existence of closed Kepler orbits in the classical dynamics [(Mazilu, 2020); §§2.3 & 4.1], that this surface can be taken as located in the *interior of a coordinate space containing the center of force* in the classical planetary model. Let us repeat the analysis related to the metric from equation (2.5.3) for this case, which is more palatable, as it were, being a well-known classical case.

Just like in the previous case, let $(d\varphi)^2$ be the metric (2.5.8), this time suggesting another phase $\varphi$, obviously different from the previous one, that plays the part of a new 'time'. The geodesics of the metric (2.5.8) can be conveniently expressed by a hyperbolic parameterization, counterpart of the one used in equation (2.5.4):

$$u(\varphi) = u_1 + v_1 \tanh\varphi, \quad v(\varphi) = \frac{v_1}{\cosh\varphi} \tag{2.5.9}$$

These are proper cycles on our surface, having the equation

$$v^2 + (u - u_1)^2 = v_1^2 \tag{2.5.10}$$

which is the counterpart of (2.5.6). In this case, the counterparts of the rates from equation (2.5.5) are:

$$\frac{\omega^1}{d\varphi} = -\frac{1}{v_1}, \quad \frac{\omega^2}{d\varphi} = 2\frac{u_1}{v_1}, \quad \frac{\omega^3}{d\varphi} = -\frac{u_1^2 - v_1^2}{v_1} \tag{2.5.11}$$

Comparing this result with that from equation (2.5.5), one can notice a remarkable duality, if we may say so, whose first part was presented right above. First, along the geodesics of the metric (2.5.3) an involution $\boldsymbol{I_0}$ from the reference frame (2.5.1) is preserved, while along the geodesics of the metric (2.5.8), an involution $\boldsymbol{J}_1$ is preserved, where the indices represent the indices of coordinates $(u,v)$:

$$\boldsymbol{I}_0 = \frac{1}{v_0}\begin{pmatrix} u_0 & -u_0^2 - v_0^2 \\ 1 & -u_0 \end{pmatrix}, \quad \boldsymbol{J}_1 = \frac{1}{v_1}\begin{pmatrix} u_1 & -u_1^2 + v_1^2 \\ 1 & -u_1 \end{pmatrix} \tag{2.5.12}$$

Secondly, in view of this observation, we are certainly entitled to choose for the parameters $u_1$ and $v_1$ of the cycles (2.5.10) values along the geodesics (2.5.4), and the equation (2.5.10) of those cycles becomes:

$$(u - u_0 - v_0 \tan\phi)^2 + v^2 = \frac{v_0^2}{\cos^2\phi} \tag{2.5.13}$$

This is a family of circles having two common points: $(u_0, v_0)$ and $(u_0, -v_0)$, and with the angle between them given by $\phi$. Assuming that these two points are the locations of two different charges of identical magnitude and opposite sign, *we have here a dipole structure*. Then the circles from equation (2.5.13) are the lines of the force field characteristic to this dipole, be it electric or magnetic, and this circumstance is liable to give us the possibility of an interpretation from the point of view of the physics of optical rays, which is the spiritual feudality given to Louis de Broglie in order to develop upon it the theory of quantization on matter. Notice that, judging by the equation (2.5.13), the surface in question is part and parcel of a Maxwell fish-eye optical medium (see §1.2). And, continuing to judge by the implications of this fact, we can proclaim that the grand analogy asks just naturally for the Planck's quantization!

In view of the concept of fluid structure of the electric medium, used so much in matters of interpretation involving the classical theory of electricity, we need to reveal another feature of the equation (2.5.13): it may be taken as the surface appearance of a material string in the electric medium representing the nucleus of the planetary atom. The surface dipole phenomenon was noticed for the first time in water, and described in theoretical details by Professor Robert Mitchell Kiehn – *may he rest in peace*! – who also baptised it: currently the phenomenon is known as a *Falaco soliton* (Kiehn, 2001). The physical appearance is a pair of vortices on the surface of the fluid, connected by a coherent tube structure in the mass of the fluid, lasting for a long time if the fluid is in a quiet state, ideally forever. This must be the case in the nuclear matter too, if it is to persist in judging by analogy: the currently observed solar mass ejections, for instance, or the structure of the active galactic nuclei, may admit the same explanation. For details on the theoretical physics of these phenomena one can consult the 2004 update of the work just cited (Kiehn, 2004).

## Chapter 3 The Scale Faculty of General Relativity

A hallmark of the grand analogy is, of course, the occurrence, and even the existence we might say, of the general relativity. However, the grand analogy entertains here an idea manifestly in contradiction with our experience at large, which is, however, seldom acknowledged as such. Namely, if the Earth, as a ship, is the analogous of the sailing ship on the quiet sea, or a submarine ship in quiet underwater, it cannot have a uniform motion through the universe. Indeed, this is a fact of solid experience: as we already mentioned, the mankind became aware of the fact that the true motion of Earth is a compound of many, practically an infinity of motions, an amalgam of the structure of which we cannot be aware. And yet, the very same experience shows that it is only the uniform motion of such a ship cannot be perceived by experiments done on it, whence the commonly accepted conclusion that the Earth should move uniformly through ether. Fact is that that, quantitatively speaking, this impossibility of perception was assigned to the relative motions of the Earth, as described by specific velocities.

This idea, in our opinion, put a real halt to theoretical physics: even today the theoretical physics is under its spell, and cannot make a significant step in building and understanding the experimental or astronomical facts *as they are*, but with reference to a *uniform motion of the Earth* through the universe. As we said, this fact, however, blatantly contradicts the experience, since the Earth cannot possibly move uniformly through the universe: it is impossible, according to experience, for Earth to have uniform motions with respect to all possible reference frames. From this point of view, the general relativity brought a substantial contribution to the logical structure of the analogy, and thereby a manner of completion of the classical natural philosophy: it brought the geodesic motion to the attention of our spirit and, with a proper generalization of the concept of time, the uniformity of motion can be, indeed, justified. For once, as we have seen at the close of previous chapter, with the benefit of the concept of surface, the idea of invariance of measured things – embodied by eigenvalues – can be maintained in the picture for arbitrary motions if these are represented by infinitesimals, no matter how these infinitesimals are quantitatively accomplished. If they are accomplished through a motion along some geodesics, then they are proportional with a time, indeed. This chapter aims at showing just how deep this connection can run, both in the history of mankind as well as in our spirit.

It appears that, from a natural-philosophical point of view in general, the Einsteinian theory of relativity is founded upon a unique general principle. Indeed, the physics underlying Lorentz transformation is ultimately asking to respect the everyday observation that bodies move freely as a whole, so that *the interpretation* of their physical structure must respect this fact of our experience. On the other hand, when it comes to the actuality of forces acting upon bodies, the gravitation seems to be indicating that the freedom is broken up by the variation of the velocity: in a free fall the bodies are all moving with *the same acceleration* on limited portions of their path, not with the constant velocity. The analysis of Enrique Loedel Palumbo, mentioned by us before, is precisely referring to this situation (Loedel, 1948, 1955). Just summarizing it for now, the essential idea is that the Einsteinian relativity, as a whole, contains the 'mathematical principles' of a 'natural philosophy' aimed at understanding the physical structure of the matter represented by the bodies *in motion*. Let us take this idea along the relativistic approach as just presented here.

In order to introduce the matters of this chapter, we recall the conclusive observations of the §2.4, amounting to the fact that an equilibrium ensemble of particles serving for interpretation is only a figment of our imagination: *it has no reality whatsoevver in our experience*. So, classically speaking, we have no possibility of including it in the theory as a concept: again, it is only the general relativity that gives us such a possibility (Israel & Wilson, 1972). The present chapter of our work includes, again, a short story of the general relativity showing how this conceptualization became possible. Along this line of thinking, it aims at making a fact, by and large unappreciated in the modern physics, recognizable: a scale transition was mathematically *defined by Albert Einstein in general relativity*, – on the occasion of undertaking the cosmological problem – which *represents a manifest continuity of our knowledge*. That definition of its creator, places the general relativity in a new light, if we may say so: it is not a 'quantum leap of our knowledge', as usually claimed, but rather an expression of continuity of that knowledge: the theoretical physics needs to take due notice of it. To wit: the scale transition defined by Einstein, continues the idea of scale transition that asked for the special relativity, which, in turn, continues a metric property contained implicitly in the classical mechanics. This time, however, the scale transition involves the electric properties of the matter exactly as the special relativity does it, but within the universe at large. The metric property in question, is in turn the one that suggests the very idea of wave, as this is necessary in completing the concept of interpretation in the Einsteinian stand. And the completion under consideration at this point is due, no doubt, to Louis de Broglie! So, let us see what is all this story about.

In mathematically rendering his chief thesis of the general relativity, namely that the *matter controls the metric of spacetime*, Albert Einstein, and after him anybody else for that matter, actually described the gravity in an entirely classical way. To be more precise, according to this description, the metric tensor of the spacetime continuum must be a solution of some partial differential equations involving the curvature of spacetime: the Einstein's field equations. As Einstein presented them, these equations naturally replace the classical Poisson equation, describing the classical Newtonian gravity from a continuum point of view, which, in context, appears as just a particular case of them. Such an approach, just like its classical counterpart, inherently *asks for boundary conditions in spacetime*, like any problem involving differential equations. However, this time the boundary conditions, involving the metric tensor of the spacetime, became an essential issue for Einstein, and he always relates to them in a way or another, in almost all of his discussions on the problem of gravitation.

The Newtonian approach of the concept of matter *per se*, in its most striking aspect necessary for sustaining the Einsteinian thesis, namely that the matter *does not equally fill the space* at its disposal, is circumvented by Einstein, in quite a natural way too, we should say, but still, only mathematically. Quoting:

> ρ is the *mean density of matter*, calculated for a region which is *large* as compared with the distance between neighbouring fixed stars, but *small* in comparison with the dimension of *the whole stellar system*. [(Einstein, 1917a), *footnote on the second page of the article*; *our Italics here*]

Therefore, even though the matter, as we perceive it, is not continuous in space, *we have to take it as continuous*, with a density *estimated* from what we are able to perceive as matter at a given time. Now, from the very same mathematical point of view, in calculating a mean density this way one has to ask, first and foremost, for the knowledge of the metric tensor, and then for some measure of 'the whole stellar system', describing as precisely as possible the spacetime extension conditions of that system. This seems to be an awkward job: acording to Richard Feynman, for instance, to cite a name of first-class repute, one can flatly talk about the *impossibility* of calculating the density this way (Feynman, 1995). Against all odds, however, the general relativity uses this classical concept of density, and even with theoretically sizable results at that. Thus, while the reason for such an unsecured, but quite successful use of the concept can be justified by the uncontested logic of general relativity as a physical theory, we find for it an 'objective' reason in fact: it turns out to have a positive return after all, again, only if it is considered from a mathematical perspective. This return consists of the addition of yet another differentia to the concept of density, above and beyond Newtonian concept: *the cardinality*. From this point of view, the Newtonian concept of density was, indeed, incomplete, and the Einstein definition can be considered as objective as it gets. But the general relativity has to pay a price for this, a price of continuity, as it were, which for its founder was unacceptable!

Here physics proceeds according to the apparently sound idea that *the matter is always a physical structure*, no question about that: one has *to count* the matter formations in order to calculate its density by the above prescription, so these should be physically perceptible, otherwise one cannot count them. It is in this respect that the concept of density becomes manifestly uncertain: by counting, it starts depending on the space scale where the counting is done, through the specific physical structure chiefly made available to our experience at that space scale. To wit: Einstein has always considered *the stars* as fundamental constituents of the matter in universe. Today, however, the stars are out of such a picture, and the galaxies are considered as fundamental matter formations at a cosmological scale, involving the universe at large. Indeed, we simply cannot travel to different places of our world in order to estimate the local density of matter independently of the position of Earth. Thus, the estimation of density asks for homogeneity and isotropy of the universe with respect to the matter density, and, in the spirit of the objectivity, the universe can only be considered isotropic with respect to galactic matter formations. Logically then, one would expect that the theoretical description of a scale transition would involve, first and foremost, *a transition from stars to galaxies*: the physical structure of a galaxy, considered as an ensemble of stars, needs to be invariant when referred to the transition of scale. This requirement mathematically assumes, among others, an idea of general continuum for the structure of the galaxy itself. However, from a proper mathematical perspective, one can safely say that the general relativity of Einstein is still limited nowadays only to the idea *countable* sets of matter formations. This is a rational idea, indeed, however it cannot be but just a starting point of the general *concept of cardinality*, judging from the perspective of a mathematical philosophy: the countability seems to be insufficient for the constructions of our intellect, and yet the physics appears to have gotten stuck with it!

The case, however, cannot remain at this stage, indeed: from the point of view of other qualities of matter involving the notion of continuity, one has to apply the general observations once made by Riemann, in order to be possible to rationally construct a geometry (Riemann, 1867). Our idea in this respect, is that physics has to accept the Schrödinger's mathematical philosophy referring to the geometry of colors, when it comes to the description of matter by its apparently continuous qualities [(Schrödinger, 1920); see also (Mazilu, 2020), especially the Chapter 5, and the literature cited there]. This approach would involve generalizing the countability of the ensembles of physical interest to the broader mathematical concept of cardinality involving ensembles of *the powers of continua*. According to Nicholas Georgescu-Roegen this is the essential mathematical condition necessary in any scale transition whatsoever (Georgescu-Roegen, 1971), and we also take it as essential in physics. From the perspective of general relativity, nevertheless, a certain liberation from the grips of the notion of density is needed, in order to realize such a scale transition. Strange enough, the very Einstein's original ideas allow for such an emancipation.

Related to this issue, it should seem necessary, in the spirit of Einsteinian natural philosophy, an *a priori* geometry that could render the *variation of the metric tensor* in such a way as to eliminate the matter represented by the density of a physical structure from the scenario. That geometry though, does not appear to be the usual Euclidean geometry. This fact has already been shown in great detail even from the beginnings of the Einstein's general relativity (Flamm, 1916), in connection with one of the first general relativistic models of the matter generating a gravitational field: *a sphere of ideal incompressible fluid* (Schwarzschild, 1916). It is this last work that we shall consider now, in connection with the concept of interpretation, for it allows us to land a certain order within the ideas about the *concept of matter in physics*, and in an apparently quite natural way at that. To wit: the work of Karl Schwarzschild is referring to the concept of incompressible fluid, as seen from the perspective of Einstein's natural philosophy, and with this notion we see the implicit content of the modern concept of *interpretation of a continuum representing the matter that fills a space*.

What is missing here, in order to make this concept useful for wave-mechanical purposes is, obviously again, the concept of wave. However, a fluid continuum is as close as it gets to assuming this concept. After all, it was used by Erwin Madelung in his exquisite interpretation of the wave mechanics itself (Madelung, 1927), but even the classical approach of the theory of the perturbations in fluids implicitly contains the mathematical possibility of introducing the waves with interpretative purposes [(Mazilu, 2020); see §2.3]. As a matter of fact, the Louis de Broglie's construction of the ray interpretation of optics is all about a physical theory of the rays distinctly based, in their physical details, upon the fluid theory. So, the Schwarzschild sphere of ideal incompressible fluid, will take us along a path where the Einsteinian theory itself proves to be all about the *interpretation of the very spacetime continuum*, but without waves and, therefore, without matter in it. This may sound strange after all, in view of the fact that Einstein always placed the stakes on the presence of matter in a physical theory, but the story is intriguing, to say the least, so we present it in some specific details.

We think that, in order to make our story more graspable, it is better to start with a note of the illustrious Felix Klein, from a letter addressed to Einstein himself, that makes our epithet 'intriguing', given above, quite understandable. This note takes Karl Schwarzschild's work just cited above, in an atypical, rather strange context, considering the regular way of physics even after that very moment of its existence. Quoting, therefore, without further ado:

> In order to give a physical turn to my letter after all, *I note that de Sitter's $ds^2$* appears implicitly already in Schwarzschild's paper of 24 February 1916. One just has to set $\chi_a = (\pi/2)$, c = 2, R = $\sqrt{(\kappa\rho_0/3)}$ in formula (35) there in order to have de Sitter's $ds^2$. Formula (35) relates, of course, to

> the *interior* (*original Italics here*, *n/a*) of the *sphere at rest* considered by Schwarzschild *of gravitating liquid of constant density*. Formula (30) is thus applicable, which yields p = $-\rho_0$, hence *a steady pull*. [*The Collected Papers of Albert Einstein*, Volume 8, Princeton University Press, Document 566; *our Italics, except as specified*, *n/a*]

In the first place: why do we find the context of this observation 'intriguing'!? It is by now a familiar fact in theoretical physics, that Willem de Sitter's $ds^2$ mentioned by Klein here is referring to a special Einsteinian construction of the metric of spacetime, whereby *the matter may not even exist* from the Newtonian point of view. This is, obviously, contradicting the very essence of Einsteinian natural philosophy. Now, Felix Klein aims, as he expresses it himself in the above excerpt, at 'giving a physical turn' to that *empty spacetime*, which is why he brings in the Schwarzschild solution here, in spite of the fact that this solution is referring *explicitly* to some *matter filling a spacetime*, not at all to an *empty spacetime*.

Indeed, that solution provided by Karl Schwarzschild to Einstein's equations, and invoked by Klein in this letter to Einstein, is originally referring to a special kind of matter, as we said, namely to *a sphere of ideal fluid of constant density*, and this, in our opinion, makes all the difference. For, if Felix Klein uses such a metric in order to give physical reason to an *empty spacetime*, this fact cannot be taken but only as *an interpretation of such a continuum*, the way this last concept was defined by Charles Galton Darwin, excluding, of course, the idea of wave (Darwin, 1927). This moment of knowledge therefore reveals the characteristic of *general relativity as an interpretational theory*. In fact, that 'steady pull' signaled by Klein in the excerpt above, highlights, in our opinion, this very circumstance, for there is no negative pressure of a classical liquid, at least not in a natural state of the world around us: this kind of pressure can only be connected with a variation of density, for special constitutive properties of the liquid.

In order to properly assess the issue thus raised by our intellect, let us explain in detail what we see in *this moment of human knowledge*, and then build up our argument accordingly. Because this is, indeed, one of the rarest instances in the history of human knowledge when our spirit faced fundamental issues raised by the intellect at a crucial moment of our existence, and was compelled to assume an attitude that literally changed the future thinking process of humanity. It has to be recognized as such in our natural philosophy, for it is one of those moments that changed the future of our life, and without war and violence at that: it involved just our spirit, not the multitudes, and, as such, proceeded without spills of blood and deaths, in a time when the multitudes on Earth lived under the sign of blood spilled and in the shadow of death.

### 3.1 Einstein's Problem and Solution

The trail of concerns here, was opened by Albert Einstein's *Cosmological Considerations* (Einstein, 1917a), which was dedicated to the… cosmological problem, obviously. As we already have mentioned, Einstein associated the cosmological issue, as he usually would do for gravitation – actually for any physics' problem, in general – with the *boundary conditions* for the metric tensor of spacetime. On this occasion, though, a situation has occurred: according to the classical view of the problem of solution of partial differential equations, the boundary conditions had to be set at the *edge of the universe*, due to the very cosmological character of the problem. But the edge of the universe is hazy, to say the least, and the boundary conditions had to be invented. The only secure way to create reliable boundary conditions was to judge by a condition of invariance, like in the case of the radiation, whereby the Wien's displacement law provides such a criterion of invariance to space scale transition (see §1.1 of the present work). In the words of Einstein himself, such boundary conditions cannot be appropriated the way this choice would be usually done, for they *are not invariant with respect to the extension of the space occupied by matter*, whose dimensions were (and actually still are!) incessantly changing. Quoting:

> In *my treatment of the planetary problem* I chose these limiting conditions in the form of the following assumption: it is possible to select *a system of reference* so that *at spatial infinity* all the gravitational potentials $g_{\mu\nu}$ *become constant*. But it is by no means evident *a priori* that we may lay down *the same* limiting conditions *when we wish to take larger portions of the physical universe into consideration*. In the following pages the reflexions will be given which, up to the present, I have made on this fundamentally important question. [(Einstein, 1917a); *our Italics*, *a*/*n*]

A subjective touch is transparent here: Einstein may have felt himself compelled to accommodate, among others, the rapid evolution of astronomical discoveries, enlarging almost 'daily', as it were, the human knowledge, and with it, obviously, the size of the perceived universe. Otherwise, why would he think of 'the same limiting conditions', for these are almost certainly nonexistent within his formulation of the problem?! Going a little ahead of us, we can say that the *Cosmological Considerations* actually represent a forthright admission of the fact that one cannot construct limit conditions for the universe, appropriate enough in order to fulfill the requests of Einsteinian philosophy regarding the metric tensor. To shorten the tale, the overall conclusion of that work is that in order to consider it as a viable cosmology, the general relativity should be referring to a *universe conceived as a finite space filled with matter*, downright in the genuine Newtonian meaning of this statement [see (Mazilu, 2020); especially Chapter 4]. The time, however, remains utterly undecided here! Therefore one cannot talk about a spacetime *per se*, and this is a big issue for the theory of relativity. And, fortunately we should say, this issue landed an alert among mathematicians, physicists, and astrophysicists alike, with great consequences on the future of theoretical physics, and the knowledge at large, in fact. Willem de Sitter was the recognized echoing critical voice of this assembly of people.

Indeed, against all odds Einstein did not appear disposed to give up his mathematical way of approaching the problem of gravitation, *i.e.* in the classical style, as shown above, by a system of partial differential equations which, naturally, necessitate boundary conditions. Accordingly, he found a method *to avoid* the problem of boundary conditions for the metric tensor, *by making them virtually unnecessary*. Specifically, he has split up the spacetime continuum back, into space and time – the '3+1 formalism' as we know it today, and use currently in doing theoretical physics – and considered that *the space resulting from this division* must be finite, because in fact even from the point of view of our experience, which again, decides the definition of the finite world we inhabit, there is no other possibility. What, then, can be the relation between the spacetime and space proper? Mathematically speaking that relation boils down to an *embedding of space in an incidental spacetime* of prescribed geometry. However, as it turns out during its application, the procedure leaves the corresponding *problem of time* in suspension, which, obviously, is a notable disagreement with the tenets of the special relativity.

According to these last precepts, the spacetime (*sic*!) should be the world arena, not the space, and Einstein's procedure reduces the time to its dynamical meaning of a parameter of continuity, of an incidental motion at best. On the other hand, though, and more importantly for us, Einstein makes a choice of this embedding procedure that is entirely in the spirit of our subject matter here, *viz*. the idea of *invariance to scale transition*.

Referring the interested reader to some genuine works for detailed mathematical considerations [for the geometric justification of the method see (Weyl, 1923), §§39 *ff*; see also (Cartan, 2001), Chapter 18], it suffices, for now, to say that Einstein sees the accomplishment of such an embedding of space as *a restriction of a four-dimensional Euclidean manifold* of quadratic type from algebraic point of view, to a hypersphere of constant 'radius' $R$ in a four-dimensional space. Even from this starting point of the Einstein's work, one can begin asking oneself about the particular choice of a quadratic form in four dimensions: does it have any reason at all? We have an affirmative answer for this: it can be taken as the metric of the background continuum, which, starting with Riemann, was a quadratic form. Einstein closely followed this idea, which seems to have an imanent reason after all: in fairly general conditions, no matter of the algebraic form of the manifold in question the metric, is always quadratic. Anyway, let us follow the original idea, in order to show what is the fact of the matter: hopefully, the underlying reasons will become clearer as we go along with our expounding of the subject. The equation of the original hypersphere is taken, by analogy with the sphere from the regular Euclidean case, in a canonical form, expressed analytically as a sum of squares:

$$\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = R^2 \tag{3.1.1}$$

In this expression, the canonical coordinates of 'events' are denoted $\xi_\mu$. In broad strokes, the embedding is accomplished as in the known classical case of embedding a surface within a regular space: the metric of hypersurface is constructed by restricting the four-dimensional metric to that of the hyperplane $\xi_4 = const$. Then, *if the four-dimensional metric is also Euclidean*:

$$(ds)^2 = (d\xi_1)^2 + (d\xi_2)^2 + (d\xi_3)^2 + (d\xi_4)^2 \tag{3.1.2}$$

using the equation (3.1.1), we can get the metric tensor of the three-dimensional space as

$$(ds)^2 = \gamma^{ij} dx_i dx_j, \quad \gamma^{ij} = \delta^{ij} + \frac{x_i x_j}{R^2 - \sum_k x_k^2} \tag{3.1.3}$$

where $\xi_k \equiv x_k$ are the coordinates of a position in the chosen hyperplane of the hyperspace. In this equation, our 'contravariant' writing of the metric tensor is not merely sanctioned by the dummy indices rule of summation, but has what we consider as *a physical reason*, in the very classical physics of the planetary model, indeed, however not that physical reason invoked by Einstein himself. This issue shall be straightaway explained in this very chapter.

For the moment, though, let us take notice of the fact that the quantities accessible to measurement are here the *contravariant coordinates* with respect to the metric (3.1.3). These can be constructed by raising the indices, the way this operation is usually done in the very formalism of the general relativity, using the metric $\boldsymbol{\gamma}$ of the ambient space obtained *via* the imbedding operation, with the result

$$x^i \equiv \gamma^{ij} x_j = R^2 \frac{x_i}{R^2 - \sum_k x_k^2} \tag{3.1.4.}$$

These coordinates are essential for a *stochastic physics* of matter, for instance within the natural-philosophical framework once designed by Carlton Frederick for a kind of general relativity (Frederick, 1976). Worth mentioning, in this connection, is one of the major of Frederick's theses, declaring that one among the components of the metric tensor plays the part of the *wave function*. This may be too much when taken *a priori*, but the gist of the thesis, as we shall see presently, is that the stochasticity applies to coordinate spaces, and that these reproduce *a geometry of the metric tensor*.

However, for now, saying 'accessible to measurement' for the case of coordinates may seem somewhat arbitrary indeed, if only in view of the fact that we have not even defined yet what the 'measurement' itself may mean, according to the well-established custom of axiomatics in physics. Concerning the coordinates (3.1.4.), though, we have a prescription of evaluation, which specifies them as 'gauged coordinates' in a precise physical sense, everywhere in the universe, in the world of the small, as well as in the world of the large. In other words, *the prescription is universal*, even though in the framework of the classical physics, but sufficiently compelling in order to determine us to take it into consideration for doing the job. Besides, this prescription is referring to the very same "treatment of the planetary system" from the perspective of which Einstein chose the reference frame in deciding the 'limiting conditions'. However, it has the advantage of being naturally correlated to the problem that guided the spirit of the great Newton into the ultimate invention of all times: that of the forces. The problem itself, in question, is the dynamical Kepler problem of planetary motion.

Our observation is that the Einstein's choice *is implicit* in the very mathematical treatment of the Kepler problem by classical dynamics. From this perspective, the procedure followed by Einstein shows that he just respected, up to a point, is true, a natural course of knowledge, provided we consider this course as an *objective* dynamics of ideas. This very fact can make his procedure a right one, in the first place. However, there is more to it, and even in a mathematical way for that matter. Indeed, we can allow for a gauging procedure, and define the corresponding coordinates, by writing

$$\xi_k = \frac{x_k}{R}, \quad \xi^k \equiv \gamma^{kl}\xi_l = \frac{x^k}{R}, \quad \gamma^{ij} = \delta^{ij} + \frac{\xi^i \xi^j}{1-\xi_k \xi^k} \tag{3.1.5}$$

Now, in the section $x_3 = 0$ of the metric space thus obtained by Einstein, the coordinates are defined up to a constant scale factor, and take a known form. If we choose those coordinates in the following manner:

$$x_1 \equiv w_1, \quad x_2 \equiv w_2, \quad R^2 \equiv (\kappa / \dot{a})^2$$

it is pretty obvious that the metric tensor (3.1.3), describing this section of the space is, up to a factor, the inverse of the matrix of quadratic form representing a Keplerian orbit as a function of the *initial data* of the corresponding dynamical problem [(Mazilu, Agop, & Mercheș, 2019), equation (4.3) *ff*]. The initial data of this particular problem are to be recognized as those 'just transverse projections' invoked by Newton in his letter to Bishop Bentley (see §2.2), occurring at the moments when the matter in its fall towards a center of force reaches the 'right orb'. So, from a natural philosophical point of view, with this cosmogonic Newtonian moment we have a manifest continuity between the two theories of the universe, Newtonian and Einsteinian. The eigenvalues of the inverse matrix in question are, therefore, *the magnitudes of the semiaxes of the orbit*, so that they effectively represent *length gauges* in the very meaning of the word. This observation has far-reaching consequences, from both mathematical, as well as physical points of view, some of which will be touched as we go along with the present work.

For once, however, we have to notice the manifest attitude of contemporary physics at large, which does not give up the idea of motion in discussing the general concept of matter. More to the point, the physics considers today, as it always did in fact, *the matter as a physical structure*, even within the very concept of interpretation. Einstein himself, in the work now under our scrutiny, does not make an exception, and we know that he had to face harsh consequences for that. For once, he had to refer the general state of the universe to a *static condition*, and this static condition cannot be realized but only by ensembles of *material points at rest* with respect to each other. And since the physical rest cannot be defined but with respect to an already existing motion, the points at rest must carry the mark of the motion they had before coming to rest. Thus, one can see the natural-philosophical reason of Einstein's prescription for the metric tensor of space, to which, according to our views, a classical explanation can be given as above: *the metric tensor of space contains the characteristics of those motions from which the material points derived their rest*, namely the Kepler motions.

Einstein followed such an idea closely. In fact, this is plainly just an interpretation – again, without waves in the picture – which he needed in order to base the whole physics on it. And the foundation upon which he builds is the following, quoting his own words:

> The most important fact *we draw from experience* as to the distribution of matter is that the *relative velocities of the stars are very small as compared with the velocity of light*. So I think that for the present we may base our reasoning upon the following approximative assumption. *There is a system of reference relatively to which matter may be looked upon as being permanently at rest*. With respect to this system, therefore, the contravariant energy-tensor $T_{\mu\nu}$ of matter is, by reason of $ds^2 = g_{\mu\nu}dx_\mu dx_\nu$, of the simple form
>
> $$\left.\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho \end{matrix}\right\} \qquad (3.1.6)$$
>
> The scalar $\rho$ of the (mean) density of distribution may be *a priori* a function of the space coordinates. But if we assume *the universe to be spatially finite*, we are prompted to the hypothesis that $\rho$ is to be *independent of locality*. On this hypothesis we base the following considerations. [(Einstein, 1917a); *emphasis added*, *a*/*n*]

The resulting mathematical construction of the physics based on this philosophy is usually taken today as a *static universe* in the specialty literature. As one can see from this excerpt, Einstein himself takes it as a *fact of experience*, therefore undisputable as it were, and always judges the different cosmological alternatives with reference to this 'static world'. Our opinion is that such a position asks for too much from a model universe since, as a fact of experience, what Einstein presents here is *only an interpretation*, and quite a particular one at that: the matter of this universe is conceived as an ensemble of material points at rest with respect to one another. Classically, this condition is taken as *meaning no forces between the material points*. However, it can also be taken as meaning an ensemble of material points in *static equilibrium* in terms of forces acting between them (Israel & Wilson, 1972), which is exactly the *Lorentz condition in defining matter* [(Lorentz, 1892); see also Chapter 3, especially §3.2 of (Mazilu, 2020)], a condition that generated the special relativity, as we have shown previously in Chapter 2, §2.2 here. In this instance a drawback becomes pretty obvious: *if the forces are not chosen appropriately*, the Einstein method of embedding the space in the manifold of events may not work properly.

In the case of the Kepler problem, however, which we take as serving to provide a mathematically natural example for Einstein's embedding procedure, the coordinates are determined by the *initial conditions* of the motion describing the classical orbit. Incidentally, we should keep in mind that these initial conditions are expressed in *velocities*. They determine quantitatively, *via* some integrals of the dynamical problem of motion – specifically, the so called *Laplace-Runge-Lenz vector* – the maximal space extension of the *source of forces* sustaining the motion from a classical dynamical point of view. Such a dynamics characterizes *exclusively* the field of forces having a magnitude inversely proportional with the square of distance in the Euclidean space. So, if we think, with Einstein, of this region *as of a closed world* – even in the physical sense, as a Wien-Lummer cavity, or an Einstein elevator – it should *not* be, by any means, a universe free of forces, in any of moments of its existence. From this point of view, we need to conceive a static world as an ensemble of material particles with Newtonian central forces between them, for it is only for this kind of forces that one can think of some kind of invariance with respect to a scale transition. To wit: we have here the Berry-Klein type of scale transition (Berry & Klein, 1984), from the perspective of which, only the Newtonian central forces are transitive [(Mazilu, 2020); see Chapters 4 and 6 there]. After all, this is the whole point of the Loedel's critique of the equivalence principle in the formulations used by Einstein himself (Loedel, 1955). And, from this point of view, Loedel's proposal is liable to carry with it the germs of an idea of quantization in matter, by a procedure that parallels the Planck's procedure of quantization in light (Mazilu, 2022). As we shall show later, this is, indeed, the case, at least if we follow the history of natural philosophical ideas a little closer.

Leaving, however, for the time being, the problem of universality of this kind of gauging open, we can still conclude that it should be valid for the planetary atom just as much as it is valid for the planetary system proper at the cosmic scale, and perhaps for some other related physical systems, like the spiral nebulae, for instance. In any case, this is the physical 'prescription' for the covariant coordinates of Einstein, mentioned before, and implicitly the prescription of the metric tensor of space, given in equation (3.1.3). The gist of the method should be that it is referring to a *finite space*, a 'coordinate space' if it is to use Darwin's phrase: a space extended around the center of force of the Kepler problem, with the extension measured, at least in the classical case, by the eccentricity of the orbit. So, if we are to think of a model universe, we cannot avoid such a physical image, but there is a catch: the very same human experience guiding Einstein's thought, tells us that such a world may be *homogeneously charged* from electrical point of view, exactly as the Lorentz model of the electrical matter demands. It appears to us that the evolution of general relativistic point of view is an illustration of the fact that, objectively speaking, our knowledge followed a path leading to the conclusion that *the charge cannot miss from the cosmological picture* [(Mazilu, 2020); see Chapter 3].

At this point we find appropriate – and even necessary, we should say – a digression concerning the Fresnel's physical theory of light, which brings us to the crux of physical connotation of the above attitude towards Kepler problem. That theory needed gauging from the very start, and this gauging was based, as known, on the concept of *ellipsoid of elasticities* of the medium supporting the light. The physical motivation rests upon the fact that a classical dynamics in the theory of light – which, at the time of Fresnel, would make a true physical theory out of it – is only fortuitous. Recall, indeed, that Fresnel's essential accomplishment was that he incorporated *the local phenomenon of diffraction* into the phenomenology of light, in order to 'update', as it were, the classical phenomenology based on just *reflection* and *refraction* phenomena [(Mazilu, 2020), *passim*]. The newly added diffraction phenomenon revealed periodicities which, in turn, brought into physical theory the *idea of phase* and thus the trigonometric functions with it. With the trigonometric functions, the second order differential equation is naturally part and parcel of the *mathematical rendition* of the theory, and with such an equation the second principle of a dynamics, involving *elastic forces*, comes forward just as naturally from a *physical point of view*. Provided, of course, that either we are not interested in the *inertia forces*, or these forces involve a different mechanism of their existence: that is, other than merely the action at a distance. In hindsight, based on the Berry-Klein gauging theory (Berry & Klein, 1984), we can say that we have to be a little more cautious when it comes to this quite unsecured interpretation of the light phenomenon. As a matter of fact, the historical development of physics shows, in our opinion, that this should be the case, indeed.

Useless to say, Einstein did not follow, in any of his works on the relativity, the problem of motion the way we just sketched it here: that is, with reference to a particular case of embedding, which naturally – that is: by the very mathematical course of the solution to the Kepler dynamical problem – ensues from the classical dynamics. Instead, Einstein just noticed that the tensor (3.1.3) does not satisfy the field equations for the prescription (3.1.6) of the energy tensor, which he has taken as undisputable, and consequently he has followed the classical way of solving the Seeliger paradox for the Poisson equation: it is this way that he chose in introducing the gravitation in the first place, so we have to recognize that he was consistent with his very own natural philosophy. Which, in the case in point here meant the adding a 'cosmological term' to his fundamental tensor $G_{\mu\nu}$, thus changing it into:

$$G_{\mu\nu} - \lambda g_{\mu\nu} \tag{3.1.7}$$

The *cosmological constant* $\lambda$ was, indeed, introduced here simply in the spirit of Einstein's initial procedure of constructing the general relativity [(Einstein, 1916b), §16]. To wit: he was extending that procedure in order to replicate the mathematics of the 'classical' Seeliger amendment for the Poisson equation, and thus to save his fundamental natural-philosophical prescription, which basically states that 'matter prevails over geometry'. In other words, Einstein followed the problem of correlation between field and matter, consistently, just as he did it earlier, on the occasion of building the general relativity, but with (3.1.7) as *the fundamental tensor* in the field equations, instead of just $G_{\mu\nu}$.

Classically, as we said, this correlation is given by the equation of Poisson, which makes the potential a fundamental characteristic of the field, provided the density of matter is not a problem. However, from this point of view, a Newtonian universe is doomed to nonexistence. For, such a universe is spatially finite, 'although it may have an infinite mass'. In this universe, the radiation of stars may well travel radially outwards *with no return*, and so may, in fact, the very fundamental matter structures do, for instance by a sort of statistical process, of the kind we know today as a *Penrose process* of extraction the energy, but from the black holes (Penrose, 2002). Quoting Einstein again:

> We might try to avoid this peculiar difficulty by assuming *a very high value for the limiting potential at infinity*. That would be a possible way, if the value of the gravitational potential *were not itself necessarily conditioned by the heavenly bodies*. The truth is that we are compelled to regard the occurrence of any great differences of potential of the gravitational field as contradicting the facts. These differences must really be of so low an order of magnitude that *the stellar velocities generated by them do not exceed the velocities actually observed*. [(Einstein, 1917a); *our Italics here*, *a*/*n*]

This excerpt suggests that Einstein may have felt that *the potential in matter asks for a separate quantitative definition*. Maybe by something like an *Emden-Fowler equation* for instance, as in the later Thomas-Fermi method of nuclear physics, if it is to maintain the guise of a physics based on partial differential equations in the picture. While the incentive of such a way of reasoning comes, again, from a nuclear theory, thus having kinship with the classical Kepler problem, in the times we are talking about, such a situation was nevertheless unbearable for Einstein, as the last sentence of this excerpt shows.

### 3.2 Willem de Sitter's Solution

This is the point where we must give the floor to the significant critical voice of illustrious astrophysicist Willem de Sitter, which seems most suitable in unveiling the true nature of the whole involvement of some great minds of this moment of human knowledge (de Sitter, 1917). For details one can also follow (de Sitter, 1916), especially the *third paper* of that series of articles. It is the time, indeed, to reiterate the fact that, from the point of view of a scale transition, the Einstein's procedure of embedding the *manifold of positions – viz*. the space – into the *manifold of events – viz*. the spacetime – is akin to Fresnel's procedure of construction the wave surface from infinitesimal pieces [see (Hamilton, 1841) for a geometrically unitary presentation of the construction, in the spirit we appropriate it here]. That is, it carries the special significance of the *mathematical transition from infrafinite to finite scales*, and *vice versa*, if it is to use the suggestion of Nicholas Georgescu-Roegen [(Mazilu, 2020); see Chapter 6 there]. Namely, the Einstein's choice of the procedure of embedding, reveals what is physically of interest in the embedding: the geometrical structure of the space of events at the *infrafinite level* [equation (3.1.2)] should be, from a metric point of view, *the same* as the geometrical structure at *finite* and *transfinite levels* [equation (3.1.1)]. This choice is consistent with the choice made initially by the special relativity and represented in equation (2.1.8). True, the identity is just a particular case of invariance with respect to the transition of scales, but it is an invariance nevertheless. Also true, the space positions are just particular type of events – to wit: they are *simultaneous* events, according to what seems to be an obvious natural point of view – but events, nevertheless.

However, while reasonable from the point of view of a customary natural philosophy suggested by the metric geometry, such a procedure breaks a particular kind of symmetry apparently imposed by the theory of special relativity. This symmetry asks for considering *the four coordinates* as equivalent, so that the time coordinate and the location coordinates play a similar part in the theory, and this is reflected in the covariance of field equations. The *time sequence* defining the common time of the positions in space, nevertheless, *becomes completely* *arbitrary in Einstein's procedure*, and this might give our intellect an unwarranted freedom which, when taken, may lead to some paradoxes. As, indeed, has happened later, on the occasion of the well known case of the Gödel's universe, when the identity between the *compass of inertia* and the *compass of gravitation* came into question from the very cosmological point of view (Gödel, 1949, 1952).

This is why Willem de Sitter has applied Einstein's procedure *to an ancillary five-dimensional finite quadratic manifold*, in order to preserve the 'relativistic' symmetry manifested by the equivalence of the *space and time* coordinates, in the exact form in which that symmetry appears for the regular case of the three-dimensional space of positions from the original Einstein's case. Thus, formally speaking, de Sitter uses an *a priori five-dimensional* quadratic manifold instead a *four-dimensional* one, in order to get a four-dimensional manifold as the result of embedding, according to the very rule put forward by Einstein's relativity:

$$\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2 = R^2 \tag{3.2.1}$$

In case the resultant coordinates $\boldsymbol{x}$ *are locating events* indeed, one of these coordinates must be imaginary, in order to respect the very prescriptions of special relativity, in describing the spacetime geometrical continuum. Assuming, therefore, the same type of invariance in the transition from the finite to infrafinite scales, *i.e.* an invariance taken exactly in the form assumed by Einstein himself, as in equation (3.1.2), the metric of the *spacetime* can be written in a form analogous to (3.1.3):

$$(ds)^2 = g^{\mu\nu} dx_\mu dx_\nu, \quad -g^{\mu\nu} = \delta^{\mu\nu} + \frac{x_\mu x_\nu}{R^2 - \sum_\alpha x_\alpha^2} \tag{3.2.2}$$

The indices run this time through four values, which is the reason we chose the Greek letters for them, for the Latin indices are always reserved by us for the three-dimensional case. The observation of Felix Klein, from the excerpt that started our presentation of this issue, is referring, formally as we said, to this kind of embedded metric. For, based on this entirely legitimate, according to Einstein's idea of scale transition, approach – and thus defending the very Einsteinian relativistic manner of procedure when it comes to the formal equivalence of the space and time coordinates – de Sitter found an apparent shortcoming of the Einstein's fundamental thesis regarding 'matter prevailing over geometry in the universe'. And this started that short wonderful period of time during which an exchange of ideas took place, that historically came to be known as *the Einstein-de Sitter debate*. Let us consider the case in a little more detail here [for a proper support on the general argument see (Weyl, 1923), especially §39 of this German edition of the renowned work of Hermann Weyl, nonexistent in any of the previous editions and their future translations].

In order to do the job of applying the general relativity to cosmology, Willem de Sitter followed closely the Einstein's very own path, which started with the observation of *impossibility of proper boundary conditions* for the metric tensor of the spacetime. His first move in the choice of some cosmological boundary conditions would be, of course, an *a priori* metric tensor of the spacetime at infinity. Apparently, though, he nurtured the idea that the matter is missing there, for the Mach's principle in the Einstein's expression was contradictory, to say the least. First of all, the very roots of relativity, in its first instalment as special relativity, would suggest a quadratic metric of the form given in equation (3.1.2), as the first move into positing the *empty spacetime metric*: that is, empty of what we consider as matter according to the tells of our senses. This was, indeed, the result of special relativity, which constitutes, in the Einstein's original approach (Einstein, 1916b), the very starting point of the general relativity, anyway. To wit: Einstein's task was, according to his natural philosophy starting from considerations of electrodynamics, the one of constructing the general-relativistic metric in order to include the gravitation in the picture. Such a metric needed to be conceived as a metric of spacetime, and Einstein's starting point was the observation that the special relativity can be considered a metric theory within spacetime, for the metric tensor:

$$\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \qquad (3.2.3)$$

Regarding this choice, both Einstein and de Sitter agree on a conclusion worth noticing: it is *not* a proper choice. The reason: it requires *a special reference frame*, namely the frame of the kind mentioned by Einstein in the excerpt above. Obviously, from cosmological point of view, the conditions at the boundary of the universe have to be *the same no matter of the reference frame*. In hindsight, we can even say more, just based on general natural-philosophical arguments: the tensor (3.2.3) *does not represent a spacetime empty of matter*, inasmuch as the light itself *can be considered as matter*, at least according to some points of view in physics, if not with all of them. However, as de Sitter concluded, the weight of a sound argument is not based on the materiality of light, but on its connection with the Mach's principle.

Indeed, we are here to discuss a specific historical reason, not ideas of a general philosophical nuance, so that, giving finally the floor to Willem de Sitter, we quote those historical reasons of the time moment when the debate took place:

> … the desire has arisen to have *constants of integration*, or rather boundary-values at infinity (*for specifying the values of the metric tensor*, *a/n*), which *shall be the same in all systems of reference*. The values (3.2.3) do not satisfy this condition. The most desirable and the simplest value for the $g_{\mu\nu}$ at infinity is evidently *zero* (*original Italics here*, *a/n*). EINSTEIN has not succeeded in finding such a set of boundary values, and therefore makes the hypothesis that *the universe is not infinite*, but spherical: *then no boundary conditions are needed*, and the difficulty disappears. From the point of view of the theory of relativity it appears at first sight to be incorrect to say: *the world is spherical*, for it can *by a transformation analogous to a stereographic projection* be represented in a *euclidean space*. This is a perfectly legitimate transformation, which leaves the different invariants *ds*, *G* etc. unaltered. But even this invariability shows that also in the euclidean system of coordinates *the world, in natural measure, remains finite and spherical*. If this transformation is applied to the $g_{\mu\nu}$ which EINSTEIN finds for his spherical world, they are transformed to a set of values which at infinity degenerate to
>
> $$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \qquad (3.2.4)$$
>
> It appears, however, that the $g_{\mu\nu}$ of EINSTEIN's spherical world (and therefore also their transformed values in the euclidean system of reference) do not satisfy the differential equations originally adopted by EINSTEIN, viz:

$$G_{\mu\nu} = -\kappa\left(T_{\mu\nu} - (1/2)g_{\mu\nu}T\right) \tag{3.2.5}$$

EINSTEIN thus finds it necessary to add another term to his equations, which become

$$G_{\mu\nu} - \lambda g_{\mu\nu} = -\kappa\left(T_{\mu\nu} - (1/2)g_{\mu\nu}T\right) \tag{3.2.6}$$

Moreover it is found necessary to suppose *the whole three-dimensional space to be filled with matter*, of which the total mass is so enormously great, that compared with it all matter known to us is utterly negligible. This hypothetical matter *I will call the* "*world matter*".

EINSTEIN *only assumes* three-dimensional space to be finite. It is in consequence of this assumption that in (3.2.4) $g_{44}$ remains 1, instead of becoming zero with the others $g_{\mu\nu}$. This has suggested the idea *to extend EINSTEIN's hypothesis to the four-dimensional time-space*. We then find a set of $g_{\mu\nu}$ which at infinity degenerate to the values

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \tag{3.2.7}$$

Moreover we find the remarkable result, that now *no* "*world matter*" *is required* [(de Sitter, 1917); *our Italics, except as indicated*].

Let us stress it again: starting with a concept of 'smeared' density, Einstein arrives to the idea that the static universe he constructed has to be *filled* with matter. As we see it, this last term implies a concept of continuity of matter above and beyond that of Newton, whose natural-philosophical thesis extracted from experience was that 'the matter does not fill the space', to put it in the most general terms. This was, for Newton, the very reason for defining the density, in the first place, as a 'attribute of the matter in filling the space at its disposal', if we may say so: the matter is continuous, but has different 'degrees of continuity', reflected by its density. However, in Einstein's case, the kind of continuity enticed by 'filling of space' would deny the very possibility of defining the density by counting, which obviously was, and still is actually, a fact of experience: the universe is, certainly, not filled with matter, at least not at all space scales. Something must be wrong in Einstein's conclusion, and de Sitter got the culprit in Einsten's digression from the very ideas of relativity, as conceived by himself. For, by the very relativistic reasons, de Sitter found that, using the Einstein's modified equations (3.2.6) for the metric (3.2.2), transcribed in an appropriate relativistic spirit, of course, there is a nontrivial solution satisfying (3.2.7) for null energy tensor, therefore, in the absence of matter, but with a non-zero cosmological constant. Parametrically, the solution of de Sitter is characterized by the values:

$$\rho = 0, \quad \lambda = 12/R^2 \tag{3.2.8}$$

where $\rho$ is the density of 'world matter', and $\lambda$ is the cosmological constant. Consequently, there should be no matter *filling* the spacetime, when proper boundary conditions (3.2.7) are used. Let us provide some details of the mathematical procedure toward this conclusion, for they are certainly helping us in making up our own mind about this important moment of our knowledge.

First, there is the essential issue of the world-matter, in hindsight quite important, but only in view of the *concept of inertia*. De Sitter's explanation of this concept is staggering by its clarity. Quoting:

If *all matter were destroyed*, with the exception of one material particle, then would this particle have inertia or not? The school of Mach requires the answer *No*. If, however, by "all matter" is meant *all matter known to us*, stars, nebulae, clusters, etc., then the observations very decidedly give the answer *Yes*. The followers of Mach are therefore *compelled to assume the existence of still more matter*. This matter, however, *fulfils no other purpose* than to enable us *to suppose it not to exist*, and to assert that in that case *there would be no inertia*. This point of view, which denies the logical possibility of the existence of a world without matter, I call the *material postulate of relativity of inertia* (*emphasis in the original here*, *n*/*a*). The *hypothetical matter* introduced in accordance with it, I call *world-matter* (*emphasis in the original here*, *n*/*a*). Einstein originally supposed that the desired effect could be brought about by *very large masses at very large distances*. He has, however, now convinced himself that this is not possible. In the solution which he now proposes, *the world-matter is not accumulated at the boundary of the universe*, but *distributed over the whole world*, which is finite, though unlimited. Its *density* (in natural measure) *is constant*, *when sufficiently large units of space are used to measure it*. Locally its distribution may be *very unhomogeneous*. In fact, *there is no essential difference between the nature of ordinary gravitating matter and the world-matter*. Ordinary matter, the sun, stars, etc., are only condensed world matter, and it is possible, though not necessary, to assume all world-matter to be so condensed. In this theory "inertia" *is produced by the whole of the world-matter*, and "gravitation" *by its local deviations from homogeneity*. [(de Sitter, 1916), *the third paper*; *emphasis added*, *except as mentioned*, *n*/*a*]

Obviously, we have to deal here with fictitious processes, pure figments of our imagination: one cannot imagine a real process in which the whole matter of the universe 'is destroyed', except for one single particle, in order to check that inertia exists or not. But the whole enormity of assumption is most apparent when one invents a thing just to suppose that it does not exists, and takes this for reality! If this invention is indeed a fact of the Newtonian natural philosophy, then one can appreciate indeed Einstein's completion of this philosophy, and his definition of density of matter connected to it. This is the density from equation (3.2.8). However, the Mach's principle is redundant, to say the least. Regarding the path to that result: again, when it comes to mathematics, de Sitter somehow 'felt' that the Einstein's procedure of embedding *is correct as a general philosophy*, but it is *not applied correctly*. To wit, for a relativist it is important that the spacetime, *not the space*, should be properly described by the equivalence of coordinates. Even in spite of the fact that the five-dimensional quadratic manifold (3.2.1) *remains in suspension for now*, regarding its origin and physical meaning!

Let us see how de Sitter got his result. Hereafter we adopt the spherical symmetry, as de Sitter did, with a proper system of coordinates mapping the space, as used by Einstein himself, where, by means of the equation (3.1.3) the metric of his spacetime becomes:

$$(ds)^2 = c^2(dt)^2 - R^2[(d\chi)^2 + \sin^2\chi \cdot (d\Omega)^2], \quad (d\Omega)^2 \overset{def}{=} (d\theta)^2 + \sin^2\theta(d\varphi)^2 \qquad (3.2.9)$$

Notice, in this expression, the character of the space, as reflected in its metric, expressed by the second term in the equation of the spacetime metric: optically speaking the space can be represented as a Maxwell fish-eye medium, as the quadratic differential form in the square brackets [compare §1.2, equation (1.2.15)]. Thus, the Einstein's static universe cannot be arbitrary, for the events are correlated by light. In this transcription we used the definitions of the Cartesian coordinates with respect to the spherical angles of colatitude and longitude ($\theta,\varphi$), within following notations:

$$x_1 = r\sin\theta\cos\varphi, \quad x_2 = r\sin\theta\sin\varphi, \quad x_3 = r\cos\theta, \quad r = R\sin\chi \tag{3.2.10}$$

When applying the same treatment as the one leading to the metrics (3.1.3) or (3.2.2), we first need to replace the last relation in (3.2.10) by three corresponding relations, as follows:

$$R^2 - r^2 - x_4^2 \equiv R^2\cos^2\omega, \quad x_4 = R\sin\omega\cos\chi, \quad r = R\sin\omega\sin\chi \tag{3.2.11}$$

After some calculations, with $r$ and $x_4$ from this equation, and with $x_k$ from (3.2.10), the equation (3.2.2) gives the metric of this spacetime in the form

$$(ds)^2 = \cos^2\omega\cdot c^2(dt)^2 - R^2\{(d\omega)^2 + \sin^2\omega\cdot[(d\chi)^2 + \sin^2\chi(d\Omega)^2]\} \tag{3.2.12}$$

where $(d\Omega)^2$ is the line element measure of the usual unit radius sphere, as in equation (3.2.9). The result (3.2.8) of Willem de Sitter goes with the Einstein's equations of this metric. It motivated the de Sitter's conclusion of a certain *lack of physical meaning* of the metric tensor. Quoting:

> We can also *abandon the postulate of Mach*, and replace it by the postulate that at infinity the $g_{\mu\nu}$ or only the $g_{ij}$ of three-dimensional space, shall be zero, or at least invariant for all transformations. This postulate can also be enounced by saying that it must be possible for the whole universe *to perform arbitrary motions*, *which can never be detected by any observation*. The three-dimensional world must, in order to be able to perform "motions", i.e. in order that *its position can be a variable function of the time*, be thought *movable in an* "*absolute*" *space of three or more dimensions* (*not* the time-space x, y, z, ct; *Italics here are from original, n/a*). The four-dimensional world requires for its "motion" *a four-(or more-) dimensional absolute space*, and moreover an *extra-mundane* "*time*" which serves *as independent variable for this motion*. All this shows that the postulate of the invariance of the $g_{\mu\nu}$ at infinity *has no real physical meaning*. *It is purely mathematical*! [(de Sitter, 1917); *our Italics except as indicated*, *n/a*]

A harsh conclusion for the Einsteinian natural philosophy, if we recall that this one places the stakes precisely on the fact that the physical meaning of the metric tensor of spacetime is a key point in the very physics of matter existing in that spacetime. In an Einsteinian context, the Mach's postulate guarantees the fact that inertia is caused by the matter spatially located beyond the local experimental accessibilities, and de Sitter found that, according to a 'correct' Einsteinian doctrine that matter must be nonexistent.

The de Sitter's conclusion recorded above, can therefore be taken to show that this cannot be the case, if it is to judge – still within the framework of the Einsteinian natural philosophy! – from the point of view of a metric of the universe, which seems relativistically more appropriate. And when we say 'more appropriate', we have in mind a relativistic mentality, basically manifested in conceiving the manifold of events in its general understanding, *i.e*. without including the idea of simultaneous events in order to describe a state of space *per se*. Add to this the necessity of the time 'as independent extra-mundane variable', and one can understand that the Einsteinian natural philosophy is certainly doomed, indeed, by the physical theory of general relativity, and that in its very own terms, actually, if de Sitter's conclusion is respected.

The first reaction of Einstein, after the analysis of the de Sitter's work, was a natural one: there are so many uncontrolled new parameters in a multidimensional space theories, that the culprit may be in their very definition. And he went on to notice that those conclusions of de Sitter may, in fact, be unsubstantiated, insofar as the metric tensor of the de Sitter's spacetime does not satisfy *in the whole universe* some natural requirements of the theory (Einstein, 1918). Indeed, by the transformation:

$$\sin\omega\sin\chi = \sin\zeta, \quad \tan\omega\cos\chi = \tan(i\eta), \quad x_4 = R\zeta, \quad r = R\eta \tag{3.2.13}$$

the metric (3.2.12) can be recast into the form (de Sitter, 1918):

$$-(ds)^2 = \frac{1}{R^2}(d\zeta)^2 - R^2 \cdot [c^2\cos^2\zeta(d\eta)^2 - \sin^2\zeta(d\Omega)^2] \tag{3.2.14}$$

For $\zeta$ and $\eta$ real, this de Sitter transformation is legitimate, for his $x_4$ coordinate, equivalent, 'by permutation' as it were, to the space coordinates, is imaginary: $x_4 \equiv ict$. Therefore, in the metric (3.2.14) of such a universe that contains the matter, all of the coordinates are real. The calculation of the contravariant metric tensor, necessary for the construction of the Einstein's field equations, requires that the determinant of this tensor should be everywhere and any time nonzero. From (3.2.14), this determinant comes down to

$$g = -\sin^4\zeta\cos^2\zeta\sin^2\theta \tag{3.2.15}$$

with obvious singularities for $\zeta = 0$, $\theta = 0$ and $\zeta = \pi/2$. Therefore, in the points of this spacetime with de Sitter's coordinates having such values, the metric tensor is not an invertible matrix, so that the gravitational Einstein's equations are not valid. Einstein notices that the first two singularity points are removable by a proper choice of the space coordinates; however, the third one persists, and one cannot see any possibility to remove it within de Sitter's embedding procedure, so that he concludes:

> If the De Sitter solution were valid everywhere, it would show that the introduction of the "λ-term" *does not fulfill the purpose I intended*. Because, in my opinion the general theory of relativity is a satisfying system only if it shows that the physical qualities of space are *completely* (*original Italics here, n/a*) *determined by matter alone*. Therefore, no $g_{\mu\nu}$-field must exist (that is, no space-time continuum is possible) without matter that generates it.
>
> In reality, the De Sitter system (2) [*the metric* (3.2.14) *here, n/a*] solves equations (1) (*these are the vacuum Einstein equations having a cosmological term:* $G_{\mu\nu} - \lambda g_{\mu\nu} = 0$, *a/n*) everywhere, except on the surface r = (π/2)R. There – as in the immediate neighborhood of gravitating mass points – the component $g_{44}$ of the gravitational potential turns to zero. The de Sitter system does not look at all like a world free of matter, but rather like a world whose matter is concentrated entirely on the surface r = (π/2)R. This could possibly be demonstrated by means of a limiting process from a 3-dimensional to a surfacelike distribution of matter. [(Einstein, 1918); *our Italics, except as indicated*, *n/a*]

As one can take notice right away, Einstein has gotten a new problem here, namely the one of dealing with the cosmological constant, called to save the day, as it were, for the general relativity in matters cosmological. As it turns out, according to the findings of Willem de Sitter, this new constant tends *by itself* to ruin the philosophy it is called to save – the very Einstein's natural philosophy – 'from the inside', rather than saving it. Anyway, Einstein's overall conclusion is that, *until contrary proven* with unquestionable certainty – which, according to him, would mean to prove that the singular (hyper)surface is an illusion due to the particular choice of the 'mapping procedure' – represented by the coordinates used in representing the hypersurface representing the world – the de Sitter universe *is a universe containing matter*, and thus the main thesis of the Einsteinian natural philosophy is actually in no real danger.

That 'proof' came, indeed, but not quite as 'contrary' as one might think. First, Willem de Sitter himself has shown that the singularity signaled by Einstein is *physically unreachable* [(de Sitter, 1916), the third paper of that series; see also (de Sitter, 1917)]: the light would take an infinite time interval to reach it, therefore *a fortiori* it is not physically accessible to matter formations. This might not be an unrealistic conclusion, were we to consider the classical acceptance of the Mach's principle: after all, the inertia for instance, is due, according to this principle, to that matter *which is out of any possible physical reach*, just like the matter of de Sitter's singularity signaled by Einstein. However, insofar as we are under the spell of the action at a distance, we might have to face the dilemma of *a force of inertia propagating with infinite velocity*, which was very uncomfortable at the times we are talking about. After all, the de Sitter's discussion was, indeed, based on a clear distinction between inertia and gravitation, but in what concerns the involvement of light in this physical reasoning we may have to recall that the idea was rejected from the very beginning – it has been agreed that the boundary values (3.2.3) are not to be taken into consideration! – so that a comparison between light and matter might not be in order here. Anyway, as we see it, this was just the course of explanation adopted by Felix Klein. To wit: by keeping the light in a physical scenario, just as it was initially introduced by the special relativity's precepts. This meant, implicitly, that *light was to be considered as a form of matter*.

Fact is though, that concerned with this issue, Einstein inquired for ideas here and there, approaching some of the German-speaking mathematicians of the time (see, in this connection, *The Collected Papers of Albert Einstein*, Volume 8, Princeton University Press; especially the English rendering of the volume, having the correspondence connected to the historical moment of Einstein-de Sitter debate, commented on pp. 351–357, and the whole correspondence translated, as indicated there). Among the mathematicians, Felix Klein himself – long and reputably concerned with the issues of non-Euclidean geometry [see (Klein, 1891, 1897) for documentation], especially with its cosmological connection – has given an answer related to the *a priori* choice from equation (3.2.1), used by de Sitter in applying the Einstein's procedure: it is valid *a priori* for the whole five-dimensional space, *i.e.*, with no limitations whatsoever due to the embedding method. That is, there is a five-dimensional quadratic form, containing *one negative term* though, and representing a four-dimensional manifold in a five-dimensional background space, having a pseudo-Euclidean metric of exactly the same signature as the quadratic form (*loc. cit. ante*, *Document 566*). From a scale transition perspective, Klein's answer covers two or three points of physical interest, deserving to be acknowledged at any rate [for the completeness of certification see §1.3 above; for details one can consult (Klein, 1918, 1919)].

First of all, Klein upholds the idea that the equation (3.2.1) of the manifold chosen by Einstein to represent the finite scale must always be *a problem of non-Euclidean geometry* (as in our Chapter 1, §§1.3 and 1.4 here). Apparently, he finds mathematical backing for this idea in the fact that Einstein's procedure of construction of the metric may be taken as an expression of the existence of an *absolute geometry*, and insists mainly upon the group-theoretical aspect of the problem (Klein, 1918). Such a geometry can be, indeed, constructed *a priori*, as a

Cayleyan geometry based on an absolute having any geometrical form, in any dimensions. It can be a homogeneous quadratic form of any signature, so that there can be an Einsteinian cosmological metric in three dimension, as conceived by Einstein himself, just as well as a de Sitter metric based on purely relativistic idea regarding the physical equivalence between time and space coordinates. In hindsight, we can say even more: there can be a Cayleyan geometry based on an absolute algebraically represented by *any homogeneous function* of coordinates (Barbilian, 1937). For instance, it may be found that the problem of isotropy of the universe, as described in terms of the variation of the so-called Hubble parameter (Misner, 1968) can be considered as a problem of metric geometry in a Cayleyan framework referred to a natural cubic form as absolute, that mimics the volume of a Cartesian reference frame (Mazilu, Agop, & Mercheș, 2019).

Secondly, inasmuch as the mathematical construction may be taken as a purely fictional thing – as, in fact, all things mathematical can! – Klein hastens to indicate a 'physical proof', as it were, in that the metric thus obtained is a *particular Schwarzschild solution for a sphere of ideal fluid* (Schwarzschild, 1916). This, in our opinion, is the message of the excerpt from the letter addressed by Klein to Einstein, and reproduced by us in the introduction of the present chapter, which started our discussion. In order to sustain this opinion, we need to notice that there is, in Klein's 'demonstration' from that excerpt, a potential danger arising with the idea that this may be thought of as a kind of a 'backfiring' argument. Indeed, the Schwarzschild solution in Klein's argument is acquired *via* Einstein's equations *with no cosmological constant* and *in the presence of matter* in the form of an incompressible fluid. In other words, by citing the Schwarzschild solution in this instance, the advocate adopts, in fact, a plain Einsteinian argument! That is, taking it as an argument in supporting the de Sitter's philosophy, may be considered as proving *a priori* the very Einstein's philosophy. However, *the same argument works equally in reverse*, *i.e.* in showing that a solution of Einstein's equation for the presence of matter – the Schwarzschild solution for a sphere of incompressible fluid – can be reproduced as a solution of a universe without matter, but with a $\lambda$-term. In this case, the Schwarzschild solution can be taken as offering only an *interpretation* of the de Sitter universe, in the sense of Darwin's definition for the necessities of wave mechanics – short of the concept of wave, of course – and this is just the point of view we want to promote. One might say that Felix Klein actually professed the wave-mechanical interpretation *avant la lettre*, as it were!

Everything hinges, in the Klein's argument, on the physical legitimacy of non-Euclidean geometry, and this is the main point at issue. As we said, it is known today that such a geometry can be built *a priori*, therefore *independently of any physics*, in order to represent, mathematically, some physical *conditions of confinement* of a general natural-philosophical character [(Mazilu, 2020); see Chapter 4 there]. It can be built as a Cayleyan geometry (see, for illustration, but not only for that, §3.4 below) and, as we have also shown in this very chapter (see §3.1), apart from mathematical details to be explained as we go along with the work here, the classical Kepler problem 'endorses' Einstein's idea in a specific physical way, involving the matter indeed, in one of its most essential features. To wit: the components of the metric tensor adduced by us in demonstrating the feasibility of Einstein's embedding procedure can be taken, in fact, as 'integrals of motion', coming naturally from the mathematical treating of the classical Kepler motion as a classical dynamical problem. Fact is that the de Sitter's metric, in any one of its instances, can be obtained as a Cayleyan metric in the sense indicated by Felix Klein [see (Castelnuovo, 1931); see also (Du Val, 1924) for geometrical details on the de Sitter's world], and the absolute of this geometry has a remarkable physical reality, to be discussed later in this work.

The third of Felix Klein's points to be mentioned here, is that, as a consequence of the non-Euclidean geometry, he was able to indicate the possibility of eliminating that arbitrariness of the 'extra-mundane' time, by a functional representation of the *time sequences* necessary in rendering the de Sitter's spacetime as a space of events. In so doing, he has the merit of using *a continuous function of the ratio of coordinates* on submanifolds of the manifold of events (3.2.1), representing the finite scale of the world. This 'functional approach' to the concept of time sequences can be taken as a warning sign that such a concept, in general, needs the idea of wave at its very foundation, a realization of which came a few years later with Louis de Broglie's work. Let us give some details in order to get a better grip on the subject. These details follow, in fact, Felix Klein's own argument.

Notice that the metrics (3.1.3) and (3.2.2) can be obtained from (3.2.9) and respectively (3.2.12) by an usual 'mapping' as de Sitter himself shows it: projection of a 'sphere' to a 'plane' (de Sitter, 1917). Consequently the metric in this dilemma, namely (3.2.14), is indeed intimately related to such a 'mapping' procedure, which may be liable to put it into question. From this perspective, Klein notices that if one takes the *real coordinates* as given by the parametric equations (Klein, 1919):

$$\begin{gathered}\xi_1 = R\sin\zeta\sin\theta\cos\varphi, \quad \xi_2 = R\sin\zeta\sin\theta\sin\varphi, \quad \xi_3 = R\sin\zeta\cos\theta, \\ \xi_4 = R\cos\zeta\cosh(c\cdot\eta), \quad \xi_5 = R\cos\zeta\sinh(c\cdot\eta)\end{gathered} \tag{3.2.16}$$

*naturally* satisfying the *quadratic constraint* assumed implicitly by de Sitter

$$\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 - \xi_5^2 = R^2 \tag{3.2.17}$$

the de Sitter metric from equation (3.2.14) becomes the metric form with constant coefficients, satisfying the Einstein's conditions of *formal invariance* when passing at the infrafinite scale:

$$-(ds)^2 = (d\xi_1)^2 + (d\xi_2)^2 + (d\xi_3)^2 + (d\xi_4)^2 - (d\xi_5)^2 \tag{3.2.18}$$

In other words, this geometry is formally identical in the finite and the infrafinite ranges, and therefore satisfies the Einstein's choice for a general 'physical embedding', which thus turns out to be a 'law' – the law of 'metric invariance' at the transition from the finite scale to infrafinite scale, and vice versa – once it is not postulated as Einstein did.

The possible *time sequence* in this universe will be given by a time coordinate to be calculated by formula:

$$ct = R\ln\frac{\xi_4 + \xi_5}{\xi_4 - \xi_5}, \quad \eta \equiv \frac{t}{2R} \tag{3.2.19}$$

which gives the ratio of the two coordinates involved as a 'solitonic solution', if it is to use terms closer to our times [(Mazilu, 2020); see §3.1 there, equation (3.1.26) *ff*]. This induces Felix Klein into further noticing the encounter of two observers of this universe, which he renders in quite a pictorial way, to the effect that they will always argue on the common time [see also (Klein, 1918)]:

> It is amusing to picture how two observers *living on the quasi-sphere* and *equipped with differing de Sitter clocks* would squabble with each other. Each of them would assign finite time ordinates to some of the events that for the other would be lying within infinity or that would even show imaginary time values. [*The Collected Papers of Albert Einstein*, Volume 8, Princeton University Press, Document 566; *our Italics*, *a/n*]

A matter of concern still remains, nevertheless, for Klein, expressed by the fact that he based his construction *directly* upon a non-Euclidean geometry. Having no reference to an Euclidean geometry which, according to Einsteinian philosophy, would suggest the absence of matter, the solution presented by Klein could be easily construed as just a mathematical tweaking having no physical basis. Then again, taken as such, it could be very easily placed, as we said, into the category from which it was supposed to pull the de Sitter's singularity. This is, we think, the reason why Klein was not wrong at all in suggesting a *physical argument* based on Schwarzschild solution, as signaled above. As it turns out, though, he is right in choosing this solution, indeed, but only *as an interpretation* as we said, and leaving in reserve the fact that the *a priori* dimensionality five can be justified from a physical point of view. However, in order to better understand this issue, we need first to see the connection of the problem with the Cayleyan geometry, and then elaborate a little upon the very Schwarzschild solution in question. Let us, for now, just say a few words on this very solution.

### 3.3 The Schwarzschild Matter Sphere: Framing de Sitter's Idea

The Schwarzschild solution is basically the one nowadays well-known for the foundation of the modern concept of black hole. In this instance, however, it represents the "gravitational field of a homogeneous sphere of finite radius, consisting of an incompressible fluid", as Karl Schwarzschild himself states. An excerpt from his original work could, again, be useful in clarifying not only the necessity of this solution within the framework of natural philosophy, but even the general attitude adopted by physics ever since, and *made possible only by the advent of the general relativity*, with all its problems that specifically motivated the human mind. Quoting:

> As another example *concerning Einstein's theory of gravitation*, I calculated the gravitational field of *a homogeneous sphere of a finite radius*, *consisting of incompressible fluid*. The specification "consisting of incompressible fluid", is necessary to be added, due to the fact that in the framework of the relativistic theory, *gravitation depends on not only the quantity of the matter*, *but also on its energy* and, for instance, *a solid body having a specific state of internal stress would produce a gravitation different from that of a liquid*. [(Schwarzschild, 1916); *our Italics*]

Once again, notice that the work of Schwarzschild is "concerning Einstein's theory of gravitation", and this speaks clearly of the fact that the reference of Felix Klein to it, in support of de Sitter solution cannot be but an interpretation. The other emphasis in this excerpt contains the clearest essential difference between the new natural philosophy, brought about by the Einsteinian general relativity, and the old Newtonian natural philosophy: *it is the internal state of the matter*, *with all its details*, *that counts in controlling the external gravitational field*, not just the ponderous mass. In hindsight, the ponderous mass only controls the reference states of static equilibrium serving for interpretation. One may say that this is the real feature brought by cosmology to bear upon the general relativity: it can explicitly describe the moment of interpretation in physical theory. And this plainly justifies the Klein's procedure of using the Schwarzschild solution in a place where, apparently, it has no reason to show up. This conclusion is, in fact, quite a positive addition coming out of the new train of ideas brought by the relativity into the natural philosophy in general. It is this addition that made possible the idea of a gravitational collapse, for instance, usually accompanying the different presentations of the concept of black hole, to mention just one of the most significant spiritual achievements of the physics of last century. Not only this, but a deeper thinking shows us that the concept of vacuum needs to be revised in its essential differentiae.

With the idea of interpretation, there is a sound reason for taking the Schwarzschild solution – which is explicitly referring to matter as a homogeneous incompressible fluid – as physically justifying the de Sitter system, which is obviously referring to a universe *without* matter, according to Einstein's theory involving the cosmological term, once its density turns out to be zero. In fact *the homogeneous fluid should be considered here as giving an interpretation to matter*, *entirely analogous to the classical interpretation of its archetype*, *the ether*. Which is what Poincaré, for instance, felt necessary in matters of electromagnetic ether (Poincaré, 1900), in order to complete, in an appropriate manner, the theory of Lorentz referring the electrical matter. What, in our opinion, should be concluded from this story as a physics' task, is what Poincaré already did in his 1906 *Theory of Electron* from *Rendiconti di Palermo*: *construct an internal state of stress based on the idea of interpretation*. Whence the necessity of an explicit theoretical consideration of that 'internal state of the matter' mentioned by Schwarzschild, for which a proposal will be advanced in the present work in the form: *the Schwarzschild fluid must be*, *in fact*, *a Lorentz electric fluid* [(Lorentz, 1892); §§57, 66 and 67; see also §1.3 of the present work].

On the other hand, a fluid is the principal ingredient, if we may, in the interpretation of ether in the spirit of Samuel Earnshaw (see §1.3 above). Therefore the Planck's vacuum must be interpreted this way. However, this interpretation speaks of a special kind of vacuum, in need to be understood correctly on occasions. This vacuum is defined not by missing matter, but, on the contrary, by the presence of a matter which only lacks some of the possible properties. It seems worth our while elaborating a little on this statement, since it is very important for understanding the ideality of the concept of physical vacuum.

Let us think, just for the moment being, of how the general relativity was possible: classically speaking, the forces of electromagnetic nature between the physical structures of matter are overwhelmed by gravitation at a cosmic level (Weyl, 1952), so that they are simply unnoticeable. Obviously, so they were too for Newton, otherwise the concept of universal gravitation could not be with us today. So, the general relativity came to be created as a theory of universal gravitation in vacuum: logically, this vacuum was *the absence of matter which does not have a gravitational mass*. However this does not mean that this kind of matter could not have the *neglected electromagnetic properties*. These properties started the Einsteinian special relativity in the first place, and they continued to haunt the general relativity, in the form of the *electromagnetic vacuum*, from its very beginning, in a Maxwell-Einstein theory, as they call it today. Likewise, in the microscopic world the electromagnetic forces appear to be dominant: a vacuum is then to be conceived as *matter missing the gravitational mass*. It is this kind of matter that cannot be logically conceived in physics nowadays, except in the form of light: the concept has been started by Augustin Fresnel.

Coming back to our subject here, one can, in short, say that the Schwarzschild's result concerning the physics involved in the general-relativistic problem of an ideal fluid sphere, is the following. First, the equation of state of the sphere of incompressible *fluid of density* $\rho_0$ *and pressure* $p$ – the equation (30) mentioned by Klein in his first excerpt reproduced by us in the introduction to this chapter – is calculated by Schwarzschild as a benefit of introduction of an angular coordinate $\chi$ – which can count as a *phase* in the economy of general relativity, for it is hardly an ordinary angle – by an 'equation of state' as it were, of the form:

$$\rho_0 + p = \rho_0 \frac{2\cos\chi_a}{3\cos\chi_a - \cos\chi} \tag{3.3.1}$$

This equation is referring to a gravitational field *in the interior of the sphere* described by a metric which later came to be termed as the 'interior Schwarzschild metric':

$$(ds)^2 = \frac{(3\cos\chi_a - \cos\chi)^2}{4}(dt)^2 - \frac{3}{\kappa\rho_0}\{(d\chi)^2 + \sin^2\chi\cdot(d\Omega)^2\} \tag{3.3.2}$$

where $(d\Omega)^2$ is the unit sphere metric. This is the equation (35) mentioned by Klein in the excerpt that started the present story. For the gravitational field in the exterior of the sphere, Schwarzschild gets the 'exterior solution':

$$(ds)^2 = (1-\frac{\alpha}{R})(dt)^2 - \frac{(dR)^2}{1-\alpha/R} - R^2\cdot(d\Omega)^2, \quad R^3 = r^3 + \rho \tag{3.3.3}$$

Here $r$ – the radial coordinate proper – and $\alpha$ are given by the expressions:

$$r^3 = \left(\frac{\kappa\cdot\rho_0}{3}\right)^{-3/2}\left\{\frac{9}{4}\cos\chi_a\cdot(\chi - \frac{1}{2}\sin 2\chi) - \frac{1}{2}\sin^3\chi\right\}; \quad \alpha = \left(\frac{\kappa\cdot\rho_0}{3}\sin^2\chi_a\right)^{-1/2} \tag{3.3.4}$$

with the constant $\chi_a$ calculated for the value of the radius $r_a$ of the fluid sphere.

In the interest of a proper understanding of the position of solution (3.3.3), mention should be made that this is the one usually taken today as the prototype metric in the discussion of the modern concept of black hole. It is formally the same as the one obtained by Schwarzschild himself in a previous work on the gravitational field described in the general relativistic way by Einstein equations for the *classical material point*. The only difference with respect to that case is in the expression of $\rho$ entering the equation (3.3.3): in the case of fluid sphere it is defined by equation

$$\rho = \left(\frac{\kappa\cdot\rho_0}{3}\right)^{-3/2}\left\{\frac{3}{2}\sin^3\chi_a - \frac{9}{4}\cos\chi_a\cdot(\chi_a - \frac{1}{2}\sin 2\chi_a)\right\} \tag{3.3.5}$$

while for the classical material point $\rho = \alpha^3$, the value corresponding to $\chi_a = \pi/2$.

Karl Schwarzschild closes his paper on the fluid sphere of which we are talking here, with a series of observations, that, again, we find quite remarkable. However, what startles most among these observations is Schwarzschild's conclusion, which warrants entirely, from the point of view of natural philosophy, the embedding procedure of Felix Klein, as described above. Namely, if *in the metric* from the equation (3.3.2), which describes the material sphere of ideal fluid, *we take* $dt = 0$, we get the metric of *the space proper*, *within which the matter thus described from the physical point of view – i.e.* interpreted *via* a homogeneous incompressible fluid whose internal state is thermodynamically characterized by a pressure $p$ etc. – *resides*. It is on this point that Schwarzschild concludes with ones of the most penetrating observations about the coordinate space of the close realm of existence of the Earth itself, imagined as a sphere:

> This is the line-element of the so-called non-Euclidean geometry of a spherical space. The spherical space geometry *holds also in the internal region of our sphere*. The curvature radius of such a spherical space is $\sqrt{(3/\kappa\rho_0)}$. *Our sphere has formed not all of the spherical space, but only a region in it*; *this is because* $\chi$ *cannot grow up to* $\pi/2$, *but grows up only to the boundary limit* $\chi_a$.

> Concerning the Sun the curvature radius of the spherical space, which determine the geometry of the interior of the Sun, would be equal to about 500 radii of the Sun...
>
> It is an interesting result of Einstein's theory that it calls for the *reality within gravitating spheres* of the geometry of *spherical space*, which *hitherto had to be regarded as a mere possibility*. [(Schwarzschild, 1916); *our Italics*]

In the case of Klein's transition to the de Sitter metric, the coordinate $\chi$ covers the entire *a priori* range at its disposal: $\chi_a = \pi/2$. As the density is physically decided by the equation of state (3.3.1), we have $\rho_0 = -p$, *i.e.* in the verbage of Klein, 'a steady pull'. This, in our opinion, bestows a significance of universality upon Schwarzschild's fluid, necessary in accomplishing any interpretation *via* this kind of fluid: *the density of matter has to be taken as equivalent here to a stress state*, it is not only characteristic to the ponderous matter. According to this principle, the de Sitter continuum can indeed be interpreted as a Schwarzschild fluid in 'steady pull', as Klein states it in his *letter 566* to Einstein, from the Princeton Volume 8 cited above. Taken as such, the Schwarzschild's solution can rightfully offer, as we said, an interpretation even to *the ether*, therefore to *a vacuum*, and so much the more to an electromagnetic vacuum for that matter, therefore to a de Sitter world, as Felix Klein once suggested.

The physics called upon by Klein's interpretation of de Sitter's world has still many other connotations related to the very Schwarzschild solution allowing that interpretation. For once, we have here the first modern reference to the idea of *space with matter* in a certain state, which can be appreciated as a theoretical-physical structure, once it has a space extension 'measured' by the planets' orbits. Indeed, that '500 radii of the Sun' represents a spherical space around the Sun, enclosing the internal planets of the solar system up to just about the mid-distance between Mars and Jupiter. This is manifestly an essential example of a physical structure, insofar as it represents matter penetrated by space. Now, if the non-Euclidean geometry is the one that dominates the space in which the solar matter exists, this geometry should be of a special type: it should be *a hyperbolic geometry of the second kind*, in a modern nomenclature apparently due to Klein himself, *i.e.* a Lorentz geometry in the jargon of the modern theoretical physics, but for a *one-sheet hyperboloid* [see (de Sitter, 1916), the *Third Paper*, footnote on page 10; for the modern connotation of the geometrical theory see also (Duval & Guieu, 2000)]. From the formal point of view of the absolute geometry, however, this is not to say that it should be any different from the spherical geometry mentioned by Karl Schwarzschild and used by Felix Klein in his calculations, for they are formally identical [see (Pierpont, 1928) for the analytic geometry of the ruled quadrics].

We cannot close the presentation of this issue of interpretation without marking the Einstein's own closing position in the debate around de Sitter's findings. This position is usually qualified as ambiguous, to say the least: it is claimed that Einstein never publicly acknowledged the fact that he was not right, nor he published his corrected position in the critique of de Sitter's work. We do not find his attitude any different from that he had on the occasion of *Cosmological Considerations*. A quotation from the letter of Einstein, written in response to Klein's observations from the *Document 566* cited above, seems to show a manifest consistency of attitude:

> De Sitter's world is, *in and of itself*, free of singularities and its space-time points are all equivalent. A singularity comes about only *through the substitution providing the transition to the static form* of the line element. This substitution changes the *analysis-situs* (*emphasis in the original here*, *a/n*) relations. Two hypersurfaces

$$t = t_1 \text{ and } t = t_2$$

> intersect each other in the original representation, *whereas they do not intersect in the static one*. This is related to the fact that, *for the physical interpretation, masses are necessary in the static conception*, *but not in the former one*. My critical remark about de Sitter's solution needs correction; *a singularity-free solution for the gravitation equations without matter does in fact exist*. However, *under no condition could this world come into consideration as a physical possibility*. For in this world, *time t cannot be defined in such a way that the three-dimensional slices t = const. do not intersect* one another and so that these slices are equal to one another (metrically). [*The Collected Papers of Albert Einstein*, Volume 8, Princeton University Press, Document 567; *our Italics, except as mentioned*]

*In cauda venenum*! … considering these conclusions of Einstein as the 'tail' of the whole story: one has to pay due attention to *the essential passage to a static world*, for in there is the culprit lurking. According to this brief account we appreciate that Klein has shown, in fact, that an interpretation should be necessarily connected to the de Sitter's conception: 'the former one', in the excerpt above. We also must appreciate that the general relativity has the virtue of offering such an interpretation. Finally, it says that the 'static world' may not even need masses for interpretation, for it is actually referring to the very masses as simple quantities of matter.

Rarely, if ever, is it mentioned the fact that classical dynamics works essentially based on the same principle: one cannot define the Newtonian forces but only based on Kepler laws. In this case, though, the definition of force is pending on a fictitious material point generating the elastic force located in the center of Keplerian ellipse, where the matter is manifestly absent [(Mazilu, Agop, & Mercheș, 2021); Chapter 5]. The Fresnel's physical theory of light works based on the same principle: there is no matter along the light ray in order to generate the elastic force necessary to generate dynamically the vibrations of light. Fact is that Einstein never abandoned his position in this argument, which he may even have considered of no consequence for the Einsteinian natural philosophy. Whatever was essential has already had been published by now, and in hindsight we seem to have a corollary that may be formulated like this: *we have to deal here with an interpretation*, and, as such, the issue must be turned to an entirely another forum of discussion.

Fact is that the *cosmological definition of matter needs a static stance*, and Einstein's contention is that this stance cannot mean a physical system unless it has the geometrical form of a quadratic manifold at any scale of the world: infrafinite as well as finite or transfinite. However, more importantly for an interpretation based on the Einstein ideas, the Schwarzschild solutions seem to be most appropriate. One of them, that is the solution from equation (3.3.3), has already proven its usefulness in the definition of the possible coordinate systems, even independently of the reference frames [see (Fronsdal, 1959, 1991); also (Bertotti, 1959b)]. It helped in developing an important physics of the black holes, with all its cortege of significant concepts that make part of the modern theoretical physics. As to the metric from equation (3.3.2), its presence was significantly less conspicuous, due to a large extent to the remarkable properties of its particular case represented by the metric (3.3.3). In our undertaking, the Schwarzschild theory, by and large, indicates two important things: first that *a Cayleyan geometry* seems to be working here with an absolute having, say, the radius of about '500 radii of the Sun', to use the words of Schwarzschild himself. Secondly, the matter in its physical manifestations is internal to this absolute in a precise way: an inaccessible region, comprising the Sun, and an accessible region containing some *internal* *orbits* of the kind once mentioned by Madelung on the occasion of his interpretation of the wave mechanics (Madelung, 1927). These are to orbits of the four interior planets of the solar system, where the density of matter is maximal. Further geometrical considerations here will give us the physical meaning of the Bohr-Sommerfeld quantization, as we go along with our development.

### 3.4 The Cayley-Klein Geometry of de Sitter's Universe

One can only guess that what we call here Einstein's embedding procedure [equations (3.1.1) and (3.1.2)] is hardly an Einstein original procedure, ranking equal to, say, the equations of gravitation in originality. Just a perusal of Harry Bateman's *Electrical Waves* (Bateman, 1915), with its rich bibliography, mostly to a specific literature that occurred after Riemann, can convince us of the fact that the rapid evolution of the non-Euclidean geometry could not leave this procedure untouched. Einstein just adopted an existing geometrical procedure, and if there is any originality in this adoption, it must to be sought for somewhere else. We see it in the choice of the quadratic form having the same signature for both the finite and infrafinite scales of the four-dimensional world, with all its consequences. The most important of these consequences is the abstract definition of the quadratic manifolds, which may be taken as abstract with respect to geometrical customs – the symbols involved may not be necessarily coordinates – but quite concrete in physical terms: they can represent charges in the electromagnetic matter. But… there is a but in adopting such representations!

Notice that the relations (1.2.19) and (1.2.24), can be associated in an Einstein-type of embedding procedure, which would mean that the Einstein's embedding equations (3.1.1) and (3.1.2) are actually referring to an optical universe of the Maxwell fish-eye type, having the conform-Euclidean metric (1.2.12). This relationship can be extended to generality with reference to the charges in a Katz-type of natural philosophy, whose association with one another in a monopole and, further on, in a dipole, assumes a four-dimensional Euclidean-type algebraic manifold [(Mazilu, 2020), §4.4]. Then, if we can *dispose of the idea of projection* connected to (1.2.19) and (1.2.24) − which seems quite particular in the context of a cosmological theory − and thus get Einstein's embedding results *independently of projection*, we have a manner to pass from the static charges to the currents represented by their differentials using the concept of surface [see §1.3, equations (1.3.12) *ff*]. In this case, the embedding procedure, which, mathematically would involve a transition between finite and infrafinite scales of the world, would turn out to be equivalent to the transition between the static charges and the currents, which is the case that broke through with the Maxwellian electrodynamics. We search, therefore, for such a liberating procedure from the grip of projection, and this is offered by the idea of Cayleyan geometry.

For once, it is the de Sitter's idea of an empty universe which can help us: if in the equations (3.1.1) and (3.1.2) the coordinates are real numbers, they can be taken as representing charges, just as well as any other physical quantities. Assume, now, that the Einsteinian hypersphere is projected from its center (0, 0, 0, 0) on one of the two tangent hyperplanes $x^4 = \pm R$. The two projections, on the north and south poles, are identical. We shall work this example in detail, however by suppressing one of the space coordinates, $\xi_4$ say, in order to get a well known case serving only for guiding purposes. Then, based on this example we shall try to extend the results to four dimensions, in order to be able to appraise the de Sitter's idea in a right way. Thus, we have the sphere centered in origin and radius $R$:

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = R^2 \tag{3.4.1}$$

and project it onto the upper tangent plane $\xi_3 = R$ (the so-called north pole; $\xi_3 = -R$ being, obviously, the south pole of the sphere) from its center. Now, let us say that $(x, y, R)$ are the coordinates of the point in the tangent plane on which the point of coordinates $(\xi_1, \xi_2, \xi_3)$ of the sphere is projected from the origin of sphere. Then the projection procedure is described by the system of equations

$$\frac{\xi_1}{x} = \frac{\xi_2}{y} = \frac{\xi_3}{R} =: \lambda \tag{3.4.2}$$

where $\lambda$ is a parameter; the colon in the last equality shows that the last ratio *defines* the parameter $\lambda$. Now, if the Euclidean metric of this continuum reproduces the signature of the quadratic form (3.4.1), then in terms of the coordinates of the plane we can write this metric as:

$$\sum_k (d\xi_k)^2 = (x^2 + y^2 + R^2)(d\lambda)^2 + \lambda^2 \left\{(dx)^2 + (dy)^2\right\} + 2\lambda d\lambda (xdx + ydy) \tag{3.4.3}$$

Using (3.4.1) and (3.4.2) for calculating $\lambda$, we get:

$$\lambda^2 = \frac{R^2}{x^2 + y^2 + R^2} \quad \therefore \quad 2\lambda d\lambda = -2R^2 \frac{xdx + ydy}{(x^2 + y^2 + R^2)^2}$$

so that the metric (3.4.3) becomes:

$$\sum_k (d\xi_k)^2 = R^2 \left\{ \frac{(dx)^2 + (dy)^2}{x^2 + y^2 + R^2} - \left( \frac{xdx + ydy}{x^2 + y^2 + R^2} \right)^2 \right\} \tag{3.4.4}$$

This is an *infinitesimal distance*, and thus can play the part of a metric. The expression from the right hand side shows that it can be calculated by *Laguerre*'*s formula* for distance. Let us show the calculations, in detail.

If we denote by $X$ a point in this three-dimensional space, then a coordinate representation is given by a triplet of numbers representing the point in the sense of Cartan. We can define them as follows: memorize them somehow, and then carry them everywhere and realize the position in any place *via* an adequate reference frame. A slight change in notation is in order here, serving to the fact that we intend to use from now on the lower indices for the points, rather than to coordinates:

$$X \overset{def}{=} (x, y, z) \tag{3.4.5}$$

Note that in this association, $X$ is not necessarily taken as a vector: it is just a triple of numbers. We shall build the geometry based on the properties of the quadratic form from the left hand side of equation (3.4.1):

$$(X, X) \equiv x^2 + y^2 + z^2 \tag{3.4.6}$$

The condition of having $X$ as a real point is $(X, X) > 0$, even if this quantity is unspecified by the idea of a sphere in space, or something like that. The condition $(X, X) < 0$ defines, from a geometrical point of view some purely imaginary points. However, from a physical perspective, such points can even be only partially imaginary, so to speak. The physical interpretation depends only on the condition $(X, X) = 0$, and this can always make sense in physics, pending a *condition of quantization*. For instance, in our case here it can represent the condition of equilibrium of Newtonian forces within the static ensembles of particles serving for interpretation. Then stochastic processes can be defined, in order to assimilate the fundamental physical quantities generating the three forces, with lengths, serving to realize the Cartan's program of constructing a geometry. These stochastic processes are to be defined, for instance, in terms of some specific Lewis-Lutzky invariants [see (Mazilu, 2020), §4.3], and thus they should constitute the mathematical expression of the necessary memory in the global problem, analogous to the classical inertia. In such a case only the mass can be imaginary, but from the point of view of the equilibrium Newtonian forces – the static forces serving for interpretation – it can always be taken as imaginary.

In order to construct this *absolute geometry*, which we often have called here *Cayleyan* after its inventor's name (Cayley, 1859), and now will settle for the name *Cayley-Klein geometries*, in order to respect the historical order of things mathematical [see for details and discussion (Klein, 1897)], we take the quadratic form (3.4.6) as a norm for the points in our space of points. It induces an internal multiplication, or a *dot product* of points by the *polarization* process:

$$(X_1, X_2) \overset{def}{=} x_1 x_2 + y_1 y_2 + z_1 z_2$$

with an obvious correspondence between the indices of points and the indices of coordinates. This dot product helps us in characterizing a straight line in space – the essential concept necessary in constructing a metric. The straight line joining the points $X_1$ and $X_2$ is the locus of the points

$$X = \lambda X_1 + \mu X_2 \tag{3.4.7}$$

with $\lambda$ and $\mu$ variable numbers representing some *homogeneous parameters*, coordinating the position of points along the straight line. This straight line intersects the absolute quadric $(X, X) = 0$ in two points, having the homogeneous parameters partially determined by the quadratic equation:

$$(X, X) \equiv \lambda^2 (X_1, X_1) + 2\lambda\mu (X_1, X_2) + \mu^2 (X_2, X_2) = 0$$

'Partially determined' means here 'up to an arbitrary factor'. For, indeed, we cannot extract from this equation, but only the ratios of these homogeneous coordinates – usually taken in analytical geometry as *non-homogeneous point coordinates* along the line – in the form of the roots of equation, *i.e.*

$$t \equiv \frac{\lambda}{\mu} = -\frac{(X_1, X_2)}{(X_1, X_1)} \pm \frac{\sqrt{(X_1, X_2)^2 - (X_1, X_1)\cdot(X_2, X_2)}}{(X_1, X_1)} \tag{3.4.8}$$

As it turns out, though, these two ratios are just enough for our purpose to build a metric of the space.

Let us explain the general philosophy of this construction: a metric usually represents in geometry the *distance between two infinitesimally close* points, so that what we need first is to define a *distance* between points in general. Now, we do not have in our experience but the Euclidean distance: any other such quantity goes by analogy. The quantity that reduces to distance between two points in the Euclidean case turns out to be the *cross ratio* of four points on a straight line: *two of these points* are fixed and used as *a reference frame on the line*, while any other two points are taken as the current pair of points between which we calculate the distance. The two points from equation (3.4.8) can be taken as the reference frame along the straight line defined by equation (3.4.7), and with them we can construct the cross ratios involving all pairs of points of the line. Then the distance is given, up to a numerical factor, by the *logarithm of such a cross ratio* (the so-called *Laguerre's formula*, as announced before). Let us see how this philosophy really works.

Given the two points $X_{1,2}$ the straight line joining them contains all the points of the form $X = tX_1 + X_2$, using for location a non-homogeneous parameter. In order to define the distance between the two points, we can choose *arbitrarily* two other points on the line, $X_{3,4}$ say, to play the part of a reference frame on the line, to which we need to refer any point belonging to the line. Then the cross ratio of points on line is simply defined as the cross ratio of the corresponding non-homogeneous parameters $t$. So, we settle for the definition:

$$(X_1,X_2;X_3,X_4)\overset{def}{=}\frac{t_1-t_3}{t_1-t_4}:\frac{t_2-t_3}{t_2-t_4} \tag{3.4.9}$$

of the cross ratio of our four points on the line. The Laguerre distance between the first pair of points is simply proportional to the logarithm of this quantity. It depends, of course, on the second pair of points of the cross ratio, but this ambiguity can be substantially reduced if we refer the construction to the absolute of space. First, notice that according to equation (3.4.7), the parameter $t$ has the values $t_2 = 0$ for the point $X_2$, and, correspondingly, the value $t_1 = \infty$ for the point $X_1$. In view of this, the cross ratio (3.4.9) takes the simple form

$$(X_1,X_2;X_3,X_4)=\frac{t_4}{t_3} \tag{3.4.10}$$

The two points $X_3$ and $X_4$ must offer us the advantage of allowing a *standardization*, if we may say so, of this construction. To this end, they must be chosen, as we said, on the absolute, since every pair of points in space has a corresponding pair of points on the absolute, these being the points where the corresponding straight line containing our points intersects the absolute. With this choice, the corresponding parameters $t_{3,4}$ are then given by the two ratios from the equation (3.4.8), so that equation (3.4.10) becomes

$$(X_1,X_2;X_3,X_4)=\frac{(X_1,X_2)+\sqrt{(X_1,X_2)^2-(X_1,X_1)\cdot(X_2,X_2)}}{(X_1,X_2)-\sqrt{(X_1,X_2)^2-(X_1,X_1)\cdot(X_2,X_2)}} \tag{3.4.11}$$

which is, indeed, sufficient for defining a metric by the Laguerre formula. This ratio, however, is complex of unit modulus so it cannot serve the intended purpose, which requires reality of the distance. The conclusion can be ascertained from the fact that the quantity under the sign of square root is always negative for real vectors in the Euclidean space. Nevertheless, according to Felix Klein, even with this cross-ratio, we can still construct a differential version of the distance by Cayley's method, *viz*. a metric of space (Klein, 1897). Indeed, the distance according to Laguerre's formula is only *proportional* to the logarithm of the cross ratio, and therefore it involves an arbitrary constant. The logarithm of the cross ratio from equation (3.4.11) is a purely imaginary complex number, so that, if we choose the proportionality constant as a purely imaginary complex number the things are getting in order. Thus, the Laguerre distance given *via* the logarithm of the cross ratio (3.4.10) can be represented, indeed, by the distance given *via* the logarithm of cross ratio (3.4.11): first, the ratio of the two expressions involved in this last equation is a purely imaginary complex number and, secondly, we are at liberty to choose an imaginary number as the constant that multiplies the logarithm defining the Laguerre distance.

With these observations we can construct a differential version of the distance – a metric of space. Thus, assuming that the two points $X_1$ and $X_2$ are infinitesimally close $X_1 = X$, $X_2 = X+dX$, we can calculate the necessary quantities in equation (3.4.11) as

$$(X_1,X_2)^2-(X_1,X_1)\cdot(X_2,X_2)=(X,dX)^2-(X,X)\cdot(dX,dX)$$

Now, in the *real domain*, we can accept that the quantity $(X,dX)/(X,X)$ is an infinitesimal quantity of the first order, while $(dX,dX)/(X,X)$ is an infinitesimal quantity of the second order. Thus the cross ratio (3.4.11) can be expanded and, to first infinitesimal order of infinitesimals, it is

$$(X_1,X_2;X_3,X_4)=1+2i\sqrt{\frac{(dX,dX)}{(X,X)}-\left(\frac{(X,dX)}{(X,X)}\right)^2}$$

The logarithm of this quantity is given, to the same first infinitesimal order, by the second term from the right hand side, which is, of course, a purely imaginary number, as we just said. Then, we can set things in order by Klein's recipe: multiply the logarithm with a purely imaginary constant quantity, $i{\cdot}R$ say, in view of the fact that the metric *per se* is defined up to an arbitrary scale factor. Thus the *Cayley-Klein* – or *absolute* – *metric* of this geometry can be finally written in the form of a quadratic differential:

$$\left(\frac{ds}{R}\right)^2 = \frac{(dX,dX)}{(X,X)} - \left(\frac{(X,dX)}{(X,X)}\right)^2 \tag{3.4.12}$$

with $R$ an *arbitrary* real quantity. This equation is a regularly considered form of the Cayley-Klein metric, with reference to any absolute of space. It turns out that this expression is also valid in *larger conditions of space definition*: *complex points, general definition of the absolute as a quadric in this space, etc*. Dan Barbilian, to mention a notable case, used it for the cases where $(X, X)$ is a homogeneous polynomial of *arbitrary* degree – a *quantic*, in algebraic phraseology – thus generalizing the metric (3.4.12) even further (Barbilian, 1937).

However, as long as the absolute is a quadric – *i.e.* a general surface specified by an equation quadratic in the coordinates – using the properties of the dot and cross products of the real vectors in space, the metric (3.4.12) can be written in the form:

$$\left(\frac{ds}{R}\right)^2 = \frac{(X\wedge dX, X\wedge dX)}{(X,X)^2}, \quad X\wedge dX \overset{def}{=} (ydz-zdy, zdx-xdz, xdy-ydx) \tag{3.4.13}$$

The previous results obtained by projection are to be found here by assuming that one of the coordinates – specifically $z$ – is constant, specifically $R$, so that, in such a case, we have

$$X\wedge dX \overset{def}{=} (-Rdy, Rdx, xdy-ydx) \tag{3.4.14}$$

If we apply this to the metric (3.4.13), we get the result:

$$\left(\frac{ds}{R}\right)^2 = R^2\frac{(dx)^2+(dy)^2}{(x^2+y^2+R^2)^2} + \left(\frac{xdy-ydx}{x^2+y^2+R^2}\right)^2 \tag{3.4.15}$$

which, no question, coincides with (3.4.4) up to a factor, but reveals an interesting position of the metric of the *Maxwell fish-eye* (1.2.18) for the two-dimensional case. Before commenting on this result let us go over to one more dimension, as promised.

So, let us now replicate the previous case for the three-dimensional space as carrier of the Cayley-Klein metric instead of the two-dimensional plane. We just have to get started from the Einstein's quadratic with four terms instead of that from equation (3.4.1):

$$\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2 = R^2 \tag{3.4.1$'$}$$

Formally then, nothing changes from what has already been done so far, except the final formula (3.4.13): there is no possibility of a vectorial connection between the dot- and cross-products in the four-dimensional case. Indeed, the dimension of the cross-product space here is six, and therefore the cross-product – which is obviously a skew-symmetric tensor – cannot be assimilated to a vector in four dimensions. However, an identity of the same kind with the one from the three-dimensional case – specifically involving quadratic quantities – is also maintained in this case, under the name of *Lagrange identity*, whose concise form in the four-dimensional case is (Pierpont, 1928) [see (Barbilian, 1974), for a nice demonstration in the most general case]: the sum of squares of the *six* 2×2 *different determinants* $S_{\alpha\beta}$, made out with the columns number $\alpha$ and column number $\beta$ of the 2×4 matrix table, constructed from the coordinates of two points:

$$\begin{pmatrix} x_1 & y_1 & z_1 & t_1 \\ x_2 & y_2 & z_2 & t_2 \end{pmatrix} \tag{3.4.2'}$$

can still be expressed by the difference between the product of the norms of the two lines of the table, as represented by the left hand side in equation (3.4.1′), and the square of the dot-product of these lines, as induced by the norm thus defined. To wit, we have:

$$\sum S_{\alpha\beta}^2 = (X_1,X_1)\cdot(X_2,X_2)-(X_1,X_2)^2,\quad S_{\alpha\beta} = \begin{vmatrix} X_{1\alpha} & X_{1\beta} \\ X_{2\alpha} & X_{2\beta} \end{vmatrix} \tag{3.4.16}$$

Here, as in equation (3.4.5), we understand that the point $X$ is specified by the four coordinates

$$X \overset{def}{=} (x,y,z,t) \tag{3.4.17}$$

Within this convention, the metric can be written as in equation (3.4.13), with

$$X \wedge dX \overset{def}{=} (ydz-zdy, zdx-xdz, xdy-ydx, xdt-tdx, ydt-tdy, zdt-tdz)$$

Making $t = \pm R$, as in equation (3.4.14), we get the Cayley-Klein metric in the form

$$\left(\frac{ds}{R}\right)^2 = R^2\frac{(dx)^2+(dy)^2+(dz)^2}{(x^2+y^2+z^2+R^2)^2} + \sum\left(\frac{ydz-zdy}{x^2+y^2+z^2+R^2}\right)^2 \tag{3.4.18}$$

with summation over positive permutations of the three coordinates $x$, $y$, $z$, whose first term is:

$$\left(\frac{ds}{R}\right)^2 = R^2\frac{\langle dx|dx\rangle}{(R^2+\langle x|x\rangle)^2} \tag{3.4.19}$$

and second term is the square of the differential area in space, up to a factor, of course:

$$\left(\frac{ds}{R}\right)^2 = \frac{R^2}{2}\frac{\mathrm{tr}(|x\rangle\langle dx|-|dx\rangle\langle x|)^2}{(R^2+\langle x|x\rangle)^2} \tag{3.4.20}$$

No doubt, the metric (3.4.19) is a Maxwell fish-eye metric, just like that from the guiding two-dimensional example: the equation (3.4.18) matches (3.4.15), and so the theory provides now now the metric in the form once used by Constantin Carathéodory in his optical researches (see §1.2 of the present work).

Now, the promised comment may be in order, regarding the great advantage of the emancipation from the classical chains of the projection mapping. Notice first, that Willem de Sitter might not have been entitled to reject the idea of a spherical world on the grounds of invariance, for the projection – and, therefore, an incriminating coordinate transformation – is eliminated from among the procedures [see de Sitter's commentary in the excerpt before the equation (3.2.4)]. Secondly, this emancipation from the idea of mapping, gives a very much needed freedom to mathematical formalism, in order to be used for physical purposes. To offer a significant example, for once we have a *principle of confinement*, like the one used in the Berry-Klein theory of gauging of the classical forces (Berry & Klein, 1984): particles contained in a spherical enclosure, interacting with this enclosure *via* forces acting in the infrafinite range of time, *i.e. collision forces*, but 'at infinity', as it were, in view of the properties of the absolute. Such an absolute can even be generalized to an ellipsoidal surface, with virtually the same results (Klein, 1984). Physically one can think of applying these principles to a Wien-Lummer cavity containing light and matter, or to an Einstein elevator in order to describe the gravitation field. We are thinking of using them in describing the physics of brain (Mazilu, 2023b).

Now, while the results above are still fresh in our mind, we think the time is ripe for a pertinent observation that can regulate, to a great extent, what follows from the present work. Bartolomé Coll, in the later times, drew attention upon the fact that the initial Riemann's program of construction of a *n*-dimensional manifold having as metric a quadratic form in the differential of coordinates (Riemann, 1867) requires $n(n-1)/2$ functions that can be determined by the physical nature of the manifold itself (Coll, 1999, 2007). Coll's observation is that the gravitation can be described as a *universal deformation*, starting from a flat metric. This replicates the historical case of Einstein's construction of general relativity (Einstein, 1916b), and finds encouraging support in the fact that for the three-dimensional case ($n = 3$) it can be 'legiferated', as it were, by a geometrical theorem (Coll, Llosa, & Soler, 2002).

Now, the case of the Cayley-Klein metric (3.4.18) is particular for this theorem, *viz*. the *Theorem 1* (or *Theorem 2*) of (Coll, Llosa, & Soler, 2002), for a metric tensor of the general form:

$$g_{ij} = \Phi\cdot(\delta_{ij} - \xi_i\xi_j), \quad \Phi \equiv (1+\Sigma_k\xi_k^2)^{-2}, \quad \xi_i \equiv R^{-1}\cdot x_i \tag{3.4.21}$$

with an obvious renotation of the coordinates. The metric tensor of this form is typical in the physical description of the mediums like the Maxwell fish-eye, and can be used in introducing the gravitation indeed, by a method involving the name of great Maxwell again, but in connection with a realization of electrodynamics. The method will be described in concluding this work. For once, a few other observations are in order.

Stating our results here in terms of Coll's universal deformation, we must say that any cosmological metric of Einsteinian type, must be the universal deformation of a Maxwell fish-eye metric. This statement has a tremenduous impact into description of the physics of universe. In order to catch a glimpse of this importance, we just need to notice that the universal deformation provided by the Cayley-Klein geometry always brings in a metric of the type needed for the cosmological boundary conditions of Einsteinian general relativity [see §3.1, the equation (3.1.5) *ff*]. On the other hand, going a little ahead of us, we can say that, if the light is the carrier of information in this universe, then its physical structure *is decided* (*sic*!) this way: Einstein did not ask too much from the light; he just did not ask enough!

### 3.5 A (Re)Assessment *via* the Grand Analogy

We can say that the Maxwell fish-eye metric is the *background* part of the Cayley-Klein metric (3.4.18). And, when we articulate 'background' here, we have in mind even a solid physical meaning: this metric describes in a purely Einsteinian natural philosophical way, the physics of the universal supporting medium of our existence. Based on this observation, we shall try to reassess the key point of the Einstein-de Sitter debate, in order to be able to extract from this historical moment what we think as the right moral. With absolute geometry, the moot point of a five-dimensional world is out of picture. Using the constant *R* for the normalization of the coordinates, in order to bring the Maxwell fish-eye metric to the dimensionless form used in §1.2, we get for the metric (3.4.19) the known expression (1.2.15), which is the space metric used in equations (3.2.9), (3.2.12) and in the

Schwarzschild solution (3.3.2), serving for interpretation in general relativity. As for the rest, given in (3.4.20), of the Cayley-Klein metric (3.4.18) – the *Coll's universal deformation* part of the metric, as we would like to call it, constructed according to Cayley-Klein geometrical recipes (see §3.4 above) – a simple but a little tedious calculation gives the result

$$\sum\left(\frac{ydz-zdy}{x^2+y^2+z^2+R^2}\right)^2=\left(\frac{r^2}{1+r^2}\right)^2(d\Omega)^2 \tag{3.5.1}$$

where $r = \Sigma(x^2)$ and $(d\Omega)^2$ is the metric of the unit sphere, as usual. Adding this term to the Maxwell fish-eye metric (3.4.19) and using the transformation (1.2.14) to replace $r$, gives (3.4.18) as

$$\left(\frac{ds}{R}\right)^2=\left(d\frac{\phi}{2}\right)^2+\sin^2\frac{\phi}{2}\cdot(d\Omega)^2 \tag{3.5.2}$$

Thus, under a universal deformation according to Cayley-Klein geometry, *the space metric does not change its algebraical form*, only the characteristic *phase* defined by equation (1.2.14) is reduced by half. The character of the metric does not change by this Coll universal deformation. If, according to Coll's main thesis, the gravitation is responsible for the deformation, then this property of the Maxwell fish-eye metric is of essence. Let us show in what sense we understand this statement.

With a little stretching of the mathematical notions, this metric description of a universe – known as the *static universe* of Einstein ever since the times of historical debate – is actually a particular case of a *hologram*. First of all, just by being a projection, such a description surely qualifies as a 'photography'. With this observation, Willem de Sitter's objection can be better understood. Originally, this objection was referring to the fact that the prescriptions of relativity, which require a four-dimensional spacetime as an arena of reality, were simply forgotten by Einstein in this case (de Sitter, 1917). But from the point of view of the concept of a hologram, one can see immediately that this objection can be actually split in two subordinate parts. First, the one essentially stressed by de Sitter himself, is that the manifest relativistic equivalence between space coordinates and time coordinate was lost in Einstein's construction: a three-dimensional universe cannot possibly describe but only a space filled with matter, and the time can only be a continuity parameter of a classical nature. As a matter of fact, with the metric (3.1.3) one forgets completely about time, for we only have a static universe, as mentioned above. So, de Sitter applied the Einstein's procedure to a five-dimensional abstract space, and projected this abstract space onto a four-dimensional spacetime, thus preserving the Einstein's method, but considering, on this occasion, the spacetime physical, according to the precepts of relativity, not the space alone as Einstein did. Here the second subordinate objection surfaces, as to the possible physical meaning of the assumed five-dimensional manifold. And, while we are on it, notice that neither (3.1.1) nor (3.1.2) may have a physical meaning, if not conceived as the manifolds of events, as in special relativity. The basic agreement of both Einstein and de Sitter is that this case cannot be accepted. However, there is a little subtlety in the relativistic equivalence of the time and space coordinates, that cannot be better illustrated than by the concept of a hologram, as we shall show right away.

Indeed, in order to do this job in utmost generality, let us use the previous idea – suggested, as a matter of fact, by Einstein's projection method – of the universe as a hologram. Certainly, as we just said, by such a projection the universe qualifies as a photography. In this case, the point of Willem de Sitter's objection to this method can be associated with the fact that, a universe being three-dimensional, the projection must be a *bona* *fide* hologram, rather than just a simple photography. At rigor, such a hologram can be considered as a set of photographies of the same object, taken simultaneously at different 'phases', as it were, which is, in fact, the way a hologram is technically realized. To further elucidate this issue, while making our case along, notice that Einstein chose the universe $\xi_4$ = constant, when he would be completely free to choose also any of the other three possibilities $\xi_k = constant$, with *k* taking the remaining values in turn. The correctness of the choice he made is, nevertheless, *a priori* guaranteed by symmetry: a *sphere filled continuously with matter*, is the same any way you look at it! However, as de Sitter has noticed in the excerpt above, a simple projection leads to Euclidean case, and the theory gets stuck with it. If the symmetry is somehow broken, for instance by an interpretation of matter with particles having no homogeneous numerical density inside the sphere, all of the four Einstein's projections are needed *simultaneously* in order to describe the resulting universe as a *bona fide* hologram.

This was the case signaled by Willem de Sitter and, relativistically speaking, it arises no matter of the conditions of symmetry and continuity of matter, *i.e*. even in the ideal conditions of classical physics or relativity, where the interpretation involves just classical material points. However, in the case of relativity, the coordinates and the time must be equivalent from the point of view of any perspective of projection, so to speak. In his argument, de Sitter considered the Minkowski metric of the spacetime continuum. Then an Einstein projection at *constant time* would give the universe proper of classical mechanics, as a continuous set of *simultaneous* points, if an incidental interpretation is in place. All the other three remaining projections break the symmetry: they are, each one of them, *Lorentz three-dimensional worlds* different only by the pair of space coordinates used in projection [see (Mazilu, 2020), §3.1 for a formal illustration of such a world, in the case of charges]. So, if it is to apply any considerations of a symmetry here, only these three last projections can be, in the last resort, considered as equivalent. The universe must then be, according to Einstein's recipe, necessarily described at least as a 'binary hologram', if we may say so: one of the 'photographies' is the regular Einsteinian static world at a given relativistic time, the other is a Lorentzian three-dimensional world, which is a plane world, but 'in motion', as it were. Mention should be made, in connection to the historical case in point, that all of the parties ever involved in the Einstein-de Sitter debate did not consider but a *single* 'photography' instead of a hologram, and thus many conceptual details were left for the future to be decided. In other words, a burden falls on us: it is our duty to properly assess what that moment represents indeed for physics!

These natural-philosophical considerations, are mostly issues of theoretical physics proper, to be deferred to further appropriate works [see (Mazilu, 2023a) for initial details]. They certainly become necessary, for instance, in problems of physiology of brain and heart, where the concept of hologram serves for the definition of the memory (Mazilu, 2023b). What we need to extract from this tale is the description of a universe in general, as a concept: it is, indeed, *a hologram*. And it is legal, we have to admit, to realize it as a single 'photography', as the protagonists of the historical debate did in fact. However, such a hologram must contain *any number* of 'photographic phases' depending on the structure of matter contained in the universe to be described. For, the problem is, in fact, if we can *dispense with the idea of projection* and describe a hologram as a whole, like its name implies, in order to be able to discern 'the phases' in it. These phases need to be used in describing the universe as a physical structure, which, in turn, must be used for prediction purposes. The solution to this problem was virtually provided by Felix Klein even before Albert Einstein became… Einstein, but it was particularly obvious on the occasion of Einstein-de Sitter debate, for which Klein was the distinguished 'moderator', as it were. It is offered by the concept of *absolute* or *Cayleyan geometry*, presented in the previous section. In order to properly understand this issue, we first need to explain the coming to being of the *concept of hologram*.

We take as beyond any reasonable doubt the fact that the wave concept of wave mechanics, at least as it was initially conceived by Louis de Broglie, originates in the physics of light. Time and again, along the evolution of wave mechanics towards its condition of a free-standing physical method, people have returned upon the concept of wave, refining it over and over. However, we should take notice that the refinements in question went, by choice, only about mathematical subtleties. And, in spite of the fact that they were not particularly intended for that purpose, those mathematical subtleties had mostly the final effect of harmoniously incorporating into the existing idea of wave some *old concepts* usually deemed as being only of historical interest for the physics of light. For, they were directly descending to us from the times when the physics of light itself would come to being from the maze of mechanics and geometry. Those old concepts regarding waves had an actuality for the physics of light at the historical moment when they were revealed to physics, but their prominence faded away in time, outshined, as it were, by the rising tide of calculation techniques which, naturally, attracted the physicists and lured the natural philosophers and historians of science with a promise of 'democracy', if we may be allowed to use this word here. To be more precise, that 'democracy' meant what it actually means even today: accessibility to physics, regardless the reason, for all those who can handle a little mathematics. Incidentally, this may count as a progress, indeed, and is usually taken as such in almost all modern historical assessments. However, mention should be made that this progress scarcely touches the natural philosophy: very few are the individuals who, capable of handling mathematics, are at the same time capable to assume the position of an authentic natural philosopher.

The explanation of this phenomenon of our knowledge – that is, regarding the refinements of concepts, and all – is as simple as it gets: like any other *concept of physics*, the *wave itself is a product of analogy*. This, in fact, seems to be the best occasion to elaborate a little on *our understanding* of the concept of analogy itself, since it might be out of the usual line in some respects. Generally speaking, the *analogy* is defined by a *standard* to which the *analogue*, *viz*. the *object concept*, in case we are talking of concepts, is referred, in a process of comparison and assessment of differentiae. Apparently, though, there is an objective law of evolution of our knowledge, namely that *if* in the process of analogy the standard concept is not present in the object concept *with all its differentiae*, the analogy does not work properly, that is, it does not properly serve its intended purpose. This lack of appropriateness is manifested through the fact that the analog concept itself objectively evolves to its defining finality by the tendency of including in its own idea *all* the differentiae possessed by its standard of reference, no matter if this one actually *had* them, or *only had to have* them at the inception moment of the process of analogy. Now, most of the times – if not all of them! – the concepts of human knowledge are actually incomplete. To wit: they cannot be present *as standards* in any analogy whatsoever, with *all* their differentiae, simply either because the man is not aware of those differentiae, or they are themselves incomplete as ideas. This basically means that the human concepts 'grow differentiae', so to speak, only when they are used in analogies: a certain differentia is to be recognized as such in the *analog concept* – even though it is not present in the standard one – and therefore needs to be transferred back to the standard in order to complete it as a concept.

It is, perhaps, worth presenting this conception, at variance with a distinguished historical case in the logic: Gotlob Frege's celebrated request of 'meeting halfway'. This historical case is, in our opinion, an example showing that the natural philosophy – science in general – goes beyond the everyday experience, to which the natural language is immanently confined. Quoting:

> … By a kind of *necessity of language*, my expressions, taken literally, *sometimes miss my thought*; I *mention an object*, when what I *intend is a concept*. I fully realize that in such cases I was *relying upon a reader* who would be *ready to meet me halfway* – who does not begrudge a pinch of salt.
>
> Somebody may think that this is an artificially created difficulty; that there is no need at all to take account of such *unmanageable thing* as what I call *a concept*… [(Frege, 1960); *our emphasis*, *a/n*]

The Frege's 'object' needs to be taken as equivalent to our 'standard concept' above, for an object is the word designating a concept presented to our wits by the senses: that specification of Frege to the effect that he was 'intending a concept' leaves no doubt about the fact that a concept needs more than senses in order to have the differentiae embodied in an object. We need to recall, however, that the natural philosophy – science, in general – goes beyond words, for our experience implies the necessity of explaining the objects themselves.

Present case in point: in expounding the Louis de Broglie's concept of wave [(Mazilu, 2020), *passim*], the standard concept – that is the light, which, no doubt can be taken as an object of our experience – is almost always used *with no reference* to the old details concerning the initial description of the light, in the times when its concept was itself an object concept for analogy. However, we need to mention that the idea of wave in the case of matter occurred to de Broglie only in connection with the possibility of attaching a frequency to a material point (de Broglie, 1923) *suggesting* a periodic motion that can be assigned to that material point. Louis de Broglie saw a wave in this periodic motion, liable to make out of the classical material point a 'wave phenomenon called material point' according to his very own expression (de Broglie, 1926c). The first article from that 1923 series (de Broglie, 1923a) opened the 'Pandora's box', as it were, of the wave-particle duality. While the train of issues was triggered by the idea that the matter's fundamental formations (specifically the electrons) may have a wave associated to them, what strikes us most is the fact that all those three works of de Broglie from that fateful year 1923 are mainly concerned with *issues connected to light*, from which the concept of wave is borrowed: diffraction phenomenon, and the interference explaining it. This means that the very concept of wave, taken as standard in the de Broglie's wave-particle analogy, would miss many fundamental differentiae in the original Fresnel's physical theory of light that instituted the diffraction as a phenomenon in its general phenomenology.

Fact is that Louis de Broglie remained always concerned with the properties of light, as inferred from the properties of matter, rather than with the properties of matter *per se*. To wit: after setting the things as straight as possible with all issues he could perceive in connection with the frequency association to a material point *via* energy (de Broglie, 1925), he had to face the essential one among these, associated with the proportionality of intensity of light with the square of its amplitude. This occurred in the times when the issue became critical through the works of Erwin Schrödinger, who presented quantization in matter as a problem of eigenvalues (Schrödinger, 1928, 1933). Thus, de Broglie realized the fact that the solution of the problem should be associated with the modern idea of propagation, represented by d'Alembert equation (de Broglie, 1926b,c); however, in order to finalize the solution, he had to put in order the details related to the old concept of a light ray.

Indeed, it is only this concept that would make us capable of comprehending an idea of corpuscular intensity needed for the probabilistic interpretation of the wave function, through its differentia of *transport*. And it was on this occasion, that some missing points of the Fresnel's physical theory of light had to be recognized into a new description of the concept based on an approach of the diffraction phenomenon. This approach had, first, to arrange physical properties for the wave surface, up to the point where it became a potential of velocities. Then the idea of flux of particles through this surface became essential, so that the density of particles had to be taken as proportional to the square of the amplitude of the wave. The moral of this story is that Louis de Broglie's theory had to 'update', as it were, the description of diffraction phenomenon according to Fresnel's physical theory of light, and this update went as far as *to include into theory the old concept of ray*, as it appeared to Newton and Hooke way earlier than the time of Fresnel (Mazilu, 2020).

Now, as we see it, the fact that the initial idea of de Broglie sprung from special relativity, points out towards the consistency of another analogy discussed by us here in some detail, namely the one between Galilean relativity and Einsteinian relativity – the 'grand analogy' of modern physics, as we would like to call it. From this perspective, the de Broglie's idea of duality reveals the important role that the Earth's surface – as a 'cradle of civilization', if we may be allowed to use a poetic description – has to play in our knowledge, a role just as important as its secular one of supporting the life on Earth, and hints to a connection, by *continuity* as it were, between the two kinds of relativity, Galilean and Einsteinian. First of all, this 'grand analogy' clarifies the concept of general relativity as presented right above in this very chapter.

More to the point: the Galilean relativity needed the idea of a smooth surface, upon which the reference frames would be able to move uniformly. According to our experience this can be realized on a quiet sea, indeed, but *only using our imagination*, since a quiet sea *per se* does not exist in the reality revealed to our intellect by the facts of daily experience. No doubt, this idea of surface was suggested by an already existing notion of reference frame, namely *the sailing ship*. This is essentially a *limited space* – a *coordinate space*, one can say, having in mind what physics made in time out of it – whose counterpart in the case of ether was realized only later in physics, again, in connection with the light phenomenon. To wit, it was associated with those physical properties of the light that led to the idea of quantization. It is the *Wien-Lummer cavity*, used in studying the thermodynamical properties of light, *i.e.* those properties that led to the Planck's archetypal quantization procedure [see (Wien, 1900); see also (Lummer, 1900); the work that launched the idea of cavity in the experimental physics of light is a joint work of these two authors (Wien & Lummer, 1895)].

Then questions aroused by the analogy between the two relativities, can be elucidated. A first one, comes right away: do we have at our disposal the counterpart of a ship, in the Earth's journey through the universe? The answer is affirmative, on account of the fact that the *equivalent of a classical ship is simply a Wien-Lummer cavity*. According to this view, the idea of ship – and, with it, the concept of reference frame – needs to be completed with some further differentiae, for such a reference frame is more like a *modern submarine ship*, so to speak, rather than a regular sailing one. In fact, this is one of the best documented physical facts in the recent history, theoretically as well as experimentally: the space around Earth contains radiation having the Planck's spectral distribution (Fixsen, Cheng, Gales, Mather, Shafer, & Wright, 1996). Let us explain this statement.

It is known, indeed, that all of the spectral energetic distributions in the case of light, Planck's included, of course, must obey the *Wien's displacement law* (see §1.1 for further details). This law, in turn, is an expression of independence of the spectral distribution of the dimensions of the enclosure containing the radiation that happens to represent the light in that enclosure (Mazilu, 2010). This kind of invariance is manifestly a *space scale transition invariance*: the one that allows us to posit that the laws of radiation found in a terrestrial laboratory, are also laws of radiation of the world at large, acting, therefore, in any coordinate space whatsoever. And the NASA work results published in the year 1996 (*loc. cit. ante*) prove, at the highest possible technological level ever reached by man, that the background radiation around Earth has a Planck's spectrum, just like any kind of light studied on Earth inside Wien-Lummer experimental cavities. This, we believe, shows that the idea of a 'ship' is handy for the general relativity, in the form of a coordinate space described by us based on Earth surface. Therefore, the analogy between Galilean relativity and Einsteinian relativity can also be extended for general relativity on this account, provided, of course, the second basic term of this analogy exists, in the form of a… sea: *the analog of the quiet sea*! Here, the Willem de Sitter's idea gets into play, with its *world without matter*.

The de Sitter moment of our knowledge acquires, therefore, a fundamental meaning: it introduces a *charge sea*, of the kind usually illustrated in physics by *Dirac* or *Fermi* '*seas*' [see (Dirac, 1935), for an early encounter of the theory of fields and particles with the de Sitter's world]. One can come up with this idea, just from Einstein's considerations: the homogeneous quadratic four-dimensional manifold used by him for the embedding procedure is simply what the geometers call a *Veronese variety*, that can be taken as representing Katz's natural philosophical ideas regarding the charge [see §3.4 above; see also (Mazilu, 2020), §§ 3.1 and 4.4]. This *sea of charge is*, according to de Sitter, *not matter just yet*: the Einstein's precepts of natural philosophy do not apply here. Only starting from this point on, the matter is coming into being, and the grand analogy tells us just how this can happen: in this sea of charge – obviously, a continuum of the kind constructed by Einstein, and revealed by the analysis of de Sitter – the *motion of reference frames*, with elementary particles playing the part of 'submarines', *cuts waves and leaves wakes*, as it were, just like the ships in the still water of a Galilean quiet sea.

Indeed, as our experience plainly shows, there is no motion of ships without wakes on the surface, or within a quiet sea. These wakes are lasting for periods dictated by the physical properties of the water. It is, of course, our duty, then, to show how these wakes represent, for instance, the *inertia of matter* or, more generally, the *memory of the world*, imprinted into de Sitter background, as it were, just like the wakes representing the memory of a motion of the sailing ships in the case of a quiet sea. For once, this brings to our attention a valid differentia of the of physical *concept of memory*: it would mean that *the memory is intimately connected to the concept of motion*. It is, perhaps, *a third differentia* of this concept, besides the equation of motion and the trajectory: it imprints memories into electrical background. And, the grand analogy shows that within a process of interpretation, this idea can be accounted for by the *concept of turbulence*. Until further elaboration, though, we only have to take due notice of the fact that the analogy between Galilei's relativity and Einstein's relativity works just perfectly even on this account.

If anything, the Einstein-de Sitter debate, described by us above, has a positive connotation, even though we have to express it… negatively: there cannot be a position *per se* of the Earth in the universe, just as there *cannot be any cosmological boundary conditions for the metric tensor*, in the sense of Einstein. Thus, in assigning the metric tensor, we have to limit ourselves to the use of *classical integrals of motion*, and work with them in the way prescribed by Einstein in building his cosmology. It is for this case that the de Sitter's metric tensor (3.2.7) can get a physical meaning in connection with the eccentricities of the cosmic motion of the Earth's system and this is the most we can do in order to accomplish an Einsteinian-type natural philosophy. As we have shown above, from this perspective the Einstein's recipe simply means a prescription of the cosmologic metric tensor according to a manifest continuity of our knowledge: it only sanctions the fact that it does not make any sense to talk of closed Newtonian orbits otherwise.

On a positive note, this conclusion also means that, just like Newtonian dynamics, the general relativity should work only with the prescription of position of a central point of attraction inside the core of the Earth. Such a position, however cannot be given but only with a certain probability in a region of space filled just with the matter *per se*. That region contains matter that needs to be described by the same statistics as the one used by Planck for the realization of the quantization of light: *the Earth itself is a Wien-Lummer cavity*, existing at the cosmic scale in another Wien-Lummer cavity, existing… and this hierarchy goes on and on, both ways, in infrafinite as well as transfinite space ranges, in view of the scale-transcendent Wien's displacement law. This should be the message, as it can be read in the above-mentioned NASA results: we take it as such anyway. It is, as we said, one of the most significant discoveries of the humanity in the last times: *results of our experience*, *therefore valid in the finite space range*, *and proved to be valid in the infrafinite range by the Planck*'*s statistics leading to quantization*, *turn out to be valid at a transfinite scale of space*. In other words, the quantization seems to be *the only* universal law of physics!

On the other hand, thus understood, Einstein's prescription carries a heavy mathematical meaning, that needs to be noticed right away. Namely, it sets the metric tensor as a matrix function of integrals of motion describing the classical motion – the components of the Runge-Lenz vector – in the associated Kepler problem of the dynamical explanation of the Moon's motion, for instance, if it is to consider the case of Earth. As Willem de Sitter would put it, this turns out to be a right cosmological prescription of the metric tensor, after all. But it is by no means a prescription at infinity *per se* or, if it is considered as such, this infinity has to be dealt with by the mathematical procedures of the absolute geometry. Indeed, the Einstein's prescription of the *space* metric should be, somehow, equivalent to a Cayleyan prescription of the metric (see §1.3), which appears as a particular case of Coll prescription. This was, indeed, noticed right away by Felix Klein on the occasion of reading Einstein's *Cosmological Considerations* [see (Klein, 1918), for his almost obsessive insistence on the point that physicists should take notice of the fundamental concepts of geometry, in their construction of a spacetime theory] as already remarked before. In order to better grasp the close connection between the Lorentz transformation and the Cayleyan geometry, one can also consult the extensive work of Jean-Marie Le Roux, elaborated within just about the same time period [see (Le Roux, 1922), especially §§26–31], and the already cited work of Guido Castelnuovo written a decade later (Castelnuovo, 1931).

On a note that reminds us of Gottlob Frege, we need to mention that the previous manner of (re)assessment incites, and is, in fact, incited itself, by some pertinent observations on the meaning of certain words uttered, on the occasion of the old Einstein-de Sitter debate that we presented here, especially by Willem de Sitter. On these, we think, it is appropriate to insist by the way of closing the present chapter, in order to recall ourselves that the word remains yet the main tool of understanding in the world of humans. It is possible that originally these words would have been associated with a completely different meaning. However, they helped us in establishing a kind program of the present work, exactly as they were written and reached us, so that we feel, in a way, even obligated to explain them according to their original mathematical association.

In order to associate them with what, in our opinion, is a proper understand, one must start from the idea that the Einstein's natural philosophy can be summarized by the following two points, unfortunately unrecognized as such in the modern physics: 1) *a length is perfectly equivalent to a distance*, *as long as the electrodynamics is involved in our judgments*, and 2) *if the gravitation enters the play the two magnitudes – length and distance – are no more equivalent*. These aspects of the Einsteinian natural philosophy where particularly stressed during the old debate, especially by the commonly recognized circumstance that the whole Einstein-de Sitter argument is centered around the fact that the metric tensor from equation (3.2.3) *is not a proper choice* for the cosmological boundary condition of the metric tensor. The deep reason, at least as we see it, is that it cannot represent a metric tensor, because the relativistic quadratic form itself is not a metric *per se* but an *estimation*, that can be even taken as a *statistical sampling estimation*. To wit: that quadratic form is referring either to the *estimation of length*, *sampling an equivalent distance*, or to the *estimation of distance*, *sampling an equivalent length*. The metric *per se* cannot be constructed but *only cosmologically*, in the manner indicated by Einstein, and when it comes to boundary conditions we have to take in consideration de Sitter's philosophy. Going a little ahead of us here, we can say that the metric associated with the tensor (3.2.3) represents only a statistical sampling *variance* of either the distance when the length is measured, or of the length when the distance is measured. However, geometrically speaking, a variance cannot be always a metric. It may be of interest to take heed – as we shall actually do here – of the fact that a variance is taken as such a metric in the original 'Riemannian' geometrical spirit (Riemann, 1867), for the case of *the theory of colors* [(Silberstein, 1938, 1943); see also (Silberstein & MacAdam, 1945)].

Secondly, in our opinion, the definitions (3.2.10) and (3.2.11) of coordinates, mathematically support the thesis that 'the world is *not at all* spherical', which is contrary to Einstein's cosmological thesis. To wit: by its very definition, $r$ depends on a further variable – again, *a phase* – that can itself depend on the direction in space, but it is always connected to the space extension of the relevant Einstein 'instanton', previously denoted by $R$. In this take, the equation (3.2.10), for instance, presents the radial coordinate as a *statistical estimate* of $R$, while (3.2.11) is, in the same sense, even more precise. The whole point of these definitions appears to be that the *radial coordinate can be different in different directions*: from the perspective of a general theory of surfaces, the universe is, indeed, *not at all spherical*, except in its mathematical description.

Now, all this having thus been stated, de Sitter's words to the effect that the 'whole universe *performs arbitrary motions*, *which can never be detected by any observation*', should direct our thinking not toward an idea of 'hidden', as they are usually taken, but toward that of 'impossible to be observed'. For, this last way of thinking leaves us the possibility of description by the concept of scale transition. There is a fundamental example here, provided by our experience: the existence of *instantaneous* arbitrary motions, that is *motions performed at the infrafinite scale of time*, can be assessed only by the observation of their offshoots, such as the existence of atomic transitions, for instance. They just cannot be described according to regular rules of describing the motion, that's all. More to the point, they cannot be described by the classical dynamics, within Newtonian natural philosophy but, as we try to show here, they can be described within Einsteinian natural philosophy, *as geodesic motions*. A sample of such a description has already been offered in the previous §2.5.

When taking the scale transition standpoint, we can even sustain an argument to the effect that the '*extra-mundane* "*time*" which serves *as independent variable for this motion*' makes physical sense after all. This kind of time is mentioned by Willem de Sitter in connection to Einstein's static world. And, far from being a nuisance, it should, on the contrary, be taken as a 'primary commodity', as it were, for the theoretical physics at large. The general idea is, again, that if the universe should be holographic at any rate, in the sense mentioned by us before – namely that, as a physical object it is placed in a region that cannot be but a *de Broglie limit of a wave surface within a ray* – the extra-mundane time cannot be but *the phase of a certain wave*. It is in this capacity that the Einstein's arbitrary time is actually a commodity rather than a nuisance, as we just said, for it allows us to define the time in a direct connection with space scale transition. All these observations will be developed as we go on with our discourse. For the moment, let us concentrate on a *concept of instanton* enticed by such an 'extra-mundane time'.

# Chapter 4 Holography: Coexistence of Light and Matter

The previous presentation of the Einstein-de Sitter debate shows that it places a particular emphasis on the *concept of interpretation*, which, in our opinion, was, and still is the main issue of the theoretical physics. Not only this, but it seems to us that the case even shows that the general relativity *needs* to include in its curriculum a special chapter dedicated to interpretation. More to the point, the general relativity, at least as it stands today, proves to be all about the interpretation. And it did, indeed, include an explicit reference to interpretation, as the historical facts brought about by us above show, at least partially if we may say so: the issue, obviously, was not solved in a conceptual way, as it were, but rather historically, by evolution of the corresponding mathematical discipline created through the general relativity. This chapter, therefore, is intended to be all about the conceptual solutions aroused along this historical evolution.

## 4.1 The Ernst Physics

There was a case along the evolution of the theory of general relativity when it revealed the capability of solving the problem of interpretation not in terms of coordinates, not in terms of symmetries, but in terms of coordinates related to symmetries. This case presented a solution to the problem of interpretation in the exact classical way: the existence of *static ensembles* of particles, maintained in equilibrium by the natural forces connected to their classical properties, that is, gravitational mass and charges (Israel & Wilson, 1972). Obviously such ensembles are fictitious along with their elements, insofar as the three natural forces can never be in equilibrium at any space scale. This means that, as the Einsteinian natural philosophy reveals, the limiting static universes are just logical instruments to work with, never to be realized in the world of our senses.

One might say, that the general relativity has grown the capability of describing such fictions by the very natural philosophy it entices. However, the work of Israel and Wilson just cited above, seems to have an apparently fresh hint of circumventing the indeterminacy related to the metric tensor of general relativity with profitable outcome regarding the general ideas of measurement and interconnection between the theory of nuclear matter and gravitation. This really means a way to describe the real world we inhabit. So let us show the essentials of the Israel's and Wilson's work.

In short, the starting point of the work of Israel and Wilson is, indeed, just the fundamental assumption we have chosen for the basis of our natural philosophical approach of the physics here, as well as anywhere else for that matter. In a way, therefore, we are even obligated to follow this work, for it has the fundamental problem of *static forces* in sight almost explicitly, as a natural possibility of existence of the static ensembles of Hertz material particles necessary for interpretation. Thus, it reveals, in the framework of the general relativity, that necessary static solution required by Einsteinian natural philosophy, and transposes it into general relativistic terms, without any requirement regarding the symmetry of the space coordinates used. The only such requirement regards the time coordinate: *the metric of spacetime must be stationary*, *i.e.* it possesses just one Killing vector related to the 'symmetry of time', as it were, which from a mathematical point of view actually counts as an incentive for considering Einstein's '*3+1 split*' of the spacetime. Let us quote, therefore, the original words:

> Coulomb's law and Newton's law of gravity *are formally identical* apart from a sign. Hence, classically, any *unstressed distribution of matter* can, *if suitably charged*, be maintained in *neutral equilibrium* under a *balance between the gravitational attraction and electrical repulsion* of its parts. Indications that this *obvious Newtonian fact* has a relativistic analog, first emerged when Weyl obtained a particular class of *static electromagnetic vacuum fields* (Weyl, 2012), later generalized by Majumdar (1947) and Papapetrou (1947) to remove Weyl's original restriction to axial symmetry, and further studied by Bonnor (1953, 1954) and Synge (1960). The Papapetrou-Majumdar fields are to all appearances *the external fields* of static sources whose *charge* and *mass are numerically equal* (in relativistic units: $G = c = 1$). That they are indeed interpretable as external fields of *static distributions of charged dust* having *equal charge* and *mass densities* has been shown by Das (1962), who has examined corresponding interior fields. [(Israel & Wilson, 1972); *emphasis added*, *and bibliography suitably updated when necessary*, *n/a*]

It is, indeed, 'an obvious Newtonian fact' but the classical physics could never present the problem this way; nor does it even the Hertz's update of mechanics, in spite of a rational definition of the concept of material particle [(Hertz, 1899), p. 45]: the Newtonian natural philosophy does not know of static ensemble of equilibrium under Newtonian forces. Having in mind the results arising from the Einstein-de Sitter debate, as we presented them in the previous chapter, and the work of Israel and Wilson from which we excerpted the text above, we can say that the general relativity is *the only* branch of theoretical physics which is *forced* to settle for the problem of static ensembles by an idea of equilibrium of forces. The way it solves this problem is not complete, indeed, but the approach shows why, and this is one of the most intriguing achievements of the physics of our times: *it sees motion where there is none*!

The Israel and Wilson's work starts with the general observation that a stationary spacetime metric – a metric with a single Killing vector, where all of the components of the metric tensor of this spacetime do not depend on the time coordinate – can be conveniently written in the form

$$(ds)^2 = f\cdot(dx^4 + \omega_l dx^l)^2 - f^{-1}\cdot(\gamma_{mn}dx^m dx^n) \tag{4.1.1}$$

Here the summation convention over repeated alternating indices is adopted. Based on this, in an appropriately chosen system of physical units (Weinberg, 1972), the *Einstein's field equations* for the electromagnetic field as the only form of matter – the electrovacuum, as they call it – can be written as:

$$G_{\alpha\beta} = -8\pi T_{\alpha\beta} \tag{4.1.2}$$

As Israel and Wilson show in detail [see also (Perjes, 1971) and (Mazur 1983)], it turns out that these equations take the form of a system of nonlinear partial differential system:

$$f\nabla^2\mathcal{E} = \nabla\mathcal{E}\cdot(\nabla\mathcal{E} + 2\Psi^*\nabla\Psi),\quad f\nabla^2\Psi = \nabla\Psi\cdot(\nabla\mathcal{E} + 2\Psi^*\nabla\Psi) \tag{4.1.3}$$

Here the symbols are used according to our overall convention: the Greek indices run from *1* to *4*, while Latin indices run from *1* to *3*, representing *space indices*, the index *4* being reserved for the *time coordinate*. The spacetime metric tensor is here defined by

$$g_{44} = f, \quad \gamma_{mn} = g_{4m}g_{4n} - g_{44}g_{mn}, \quad f\omega_k = g_{4k}$$

and the three-dimensional corresponding metric ($\gamma_{mn}$) is used to raise and lower the indices in space coordinates operations. As we said, the property of stationarity means that all these components do not depend explicitly on the time coordinate. Then a potential 4–vector ($\boldsymbol{A}, A_4$) ≡ ($A_\gamma$) describes the electromagnetic field whose intensities are given by its *covariant curl*:

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$$

This electromagnetic field contributes to the only energy tensor $\boldsymbol{T}$ of the problem, having the components

$$-4\pi T_{\mu\nu} \equiv g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} (F^{\alpha\beta} F_{\alpha\beta})$$

and the tensor $\boldsymbol{G}$ correlated with it in equation (4.1.2) is the Einstein tensor of the metric field defined so as to be used in Einstein's field equations like (3.2.5) above:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Here $R_{\alpha\beta}$ is the Ricci tensor of the metric, and $R$ is the corresponding scalar invariant of the curvature. In terms of these symbols we then have in equation (4.1.3):

$$\Psi \equiv A_4 + i \cdot \Phi, \quad \mathcal{E} \equiv f - \Psi^* \Psi + i \cdot \phi \tag{4.1.4.}$$

where $\Phi$ is a *magnetic potential* and $\phi$ is an arbitrary function. Once we know the three functions $\mathcal{E}$, $\Phi$ and $\phi$ we are able to construct the three-dimensional Ricci tensor corresponding to the metric ($\gamma_{mn}$) by formula

$$-f^2 R_{mn}(\boldsymbol{\gamma}) = (1/2)\mathcal{E}_{(m}\nabla_{n)}\mathcal{E}^* + \Psi\nabla_{(m}\mathcal{E} \cdot \nabla_{n)}\Psi^* + \Psi^*\nabla_{(m}\mathcal{E}^* \cdot \nabla_{n)}\Psi - (\mathcal{E} + \mathcal{E}^*)\nabla_{(m}\Psi \cdot \nabla_{n)}\Psi^* \tag{4.1.5}$$

where the round parentheses mean symmetrization with respect to indices specified by them, of the monomial in which those indices participate. For more details one can also consult (Mazur, 1983).

It has been noticed quite a few times along history ever since, that the problem of gravitational field in the general relativistic formulation could be solved if the logic would be taken a little out of the usual line, as it were. Such would be, for instance, the case if the space metric $\boldsymbol{\gamma}$ would be allowed to be arbitrary. This circumstance would give us the possibility to choose it conveniently. In fact, Israel and Wilson notice that equations (4.1.5) may be taken as *compatibility conditions* between a *preselected* space metric and the complex fields $\mathcal{E}$ and $\Psi$, rather than field equations *per se*. They found that, in the particular case of an *a priori flat space* metric, the compatibility conditions amount to a single linear relation

$$\Psi = a + b \cdot \mathcal{E}, \quad ab^* + ba^* = -1/2 \tag{4.1.6}$$

and the whole construction comes down to solving the Laplace equation:

$$\nabla^2 \epsilon = 0, \quad \epsilon \equiv (1 + \mathcal{E})^{-1} \tag{4.1.7}$$

equivalent to the system (4.1.3). By equation (4.1.6) the gravitational field determines an electromagnetic field – or *vice versa*, of course. This electromagnetic field is however not a *transition field* as we usually know it from the quantum mechanics of classical atom, but rather reflects the omnipresence and permanence of the very gravitational field, *as in the case of de Sitter continuum*. Charles Misner and John Archibald Wheeler admirably captured these attributes of the gravitational field – actually of the space itself taken as an acting physical entity in the sense of Einstein's general relativity – and studied in depth their meaning for physics in a work that made epoch ever since it was published (Misner & Wheeler, 1957). Along the present work we are only interested in making clear that the equation (4.1.6) is actually a *mark of a measurement process*, by showing its relevance for the case of a de Sitter universe in general, and especially for the planetary model.

The system of partial differential equations (4.1.3) was first introduced to physics by Frederick Ernst, for the significant case of the stationary axial symmetric – involving, therefore, two Killing vectors: one symmetry for time, and one for space – vacuum and electrovacuum fields (Ernst, 1968). This is why the complex potential $\mathcal{E}$ from equation (4.1.4), defined, according to equation (4.1.7), up to a linear rational transformation of a complex harmonic function, usually carries his name ever since: *Ernst potential*. Even on that pioneering occasion, Ernst has noticed the fact that a functional relation between the pure gravitational and pure electromagnetic complex potentials solves the problem of gravitational field in a convenient background. But Ernst went a little further: just about the same time he proved that the above Einsteinian theory yielding the equations written here as (4.1.3) and (4.1.5), is still reducible to a variational principle, even when applied for a more general case of time dependent and nonaxially symmetric fields (Ernst, 1971). Thus, in the general case we have a variational principle, whereby the gravitation and electromagnetism are *separated*, so to speak, in the variational principle producing the field equations:

$$\delta \iiint \left\{ R(\boldsymbol{\gamma}) + \frac{\gamma^{mn}(\nabla_m \mathcal{E})(\nabla_n \mathcal{E}^*)}{(\mathcal{E}+\mathcal{E}^*)^2} \right\} \sqrt{\det(\boldsymbol{\gamma})} \cdot d^3\boldsymbol{x} = 0$$

Here $R(\boldsymbol{\gamma})$ is the scalar curvature of the metric $\boldsymbol{\gamma}$. We can see now that *only in a flat space* or, with some minor provisos, in *a space of constant Riemannian curvature*, this principle involves exclusively the Ernst's complex potential for, in that case, the curvature is zero and we have:

$$\delta \iiint \left\{ \frac{\gamma^{mn}(\nabla_m \mathcal{E})(\nabla_n \mathcal{E}^*)}{(\mathcal{E}+\mathcal{E}^*)^2} \right\} \sqrt{\det(\boldsymbol{\gamma})} \cdot d^3\boldsymbol{x} = 0 \qquad (4.1.8)$$

In other words, only in cases where the gravitational field defines an electromagnetic field by a linear relation like (4.1.6), or *vice versa*, the *gravitational field is described exclusively by a complex potential*. The Ernst equations (4.1.3) thus reflect the physical interdependence between fields, which is reduced by these equations to the classical mathematical problem of a *harmonic mapping*, as represented in equation (4.1.8).

Indeed, in this case, the equation (4.1.8) shows that one can define the Einsteinian gravitational field by harmonic mappings (Eells & Sampson, 1964). Richard Matzner and Charles Misner were the first to notice that the variational principle (4.1.8) is actually an answer to the problem of *harmonic maps* (Matzner & Misner, 1967), a fact explicitly recognized and amplified later on by Misner himself in a general account for the harmonic mappings models in physics (Misner, 1978). From this point of view, the equation (4.1.8) simply describes a harmonic mapping from the Euclidean space to the Riemannian $\mathfrak{sl}(2, \mathrm{R})$ manifold. This fact is much better palpable if, instead of Ernst potential $\mathcal{E}$ we use the field variable defined by $h \equiv i \cdot \mathcal{E}$, where $i$ is the imaginary unit of the complex numbers, so that the equation (4.1.8) becomes

$$\delta \iiint \left\{ \frac{\gamma^{mn}(\nabla_m h)(\nabla_n h^*)}{(h-h^*)^2} \right\} \sqrt{\det(\boldsymbol{\gamma})} \cdot d^3\boldsymbol{x} = 0, \quad h \overset{def}{=} i \cdot \mathcal{E} \tag{4.1.9}$$

Obviously this variational equation describes a harmonic map between the ordinary flat space of metric $\boldsymbol{\gamma}$, and the complex half plane possessing the Beltrami-Poincaré metric, exactly as in the case of Kepler motion (see §1.2 of the present work). But here the physical interpretation is a little bit different.

The complex potential $h$ seems more appealing to us, inasmuch as it is much closer to the way this $\mathfrak{sl}(2,\mathrm{R})$ geometry is built, from many points of view, including even a physical one. Indeed, equation (4.1.4.) for the case of null electromagnetic field (pure gravitational field, as it were) we have, according to previous algebra:

$$h = -\phi + i \cdot f$$

so that the real part of the potential is an arbitrary 'phase', as it were, while the imaginary part is an essential component of the metric tensor itself, of the spacetime continuum:

$$v \equiv f = g_{44}$$

This has always the nice quality of being positive and also has a physical 'fixed point' unity (light speed), features required by both the geometry of the upper half-plane in the Poincaré representation of hyperbolic plane, and by the relativity itself. By this very fact we might say that the Beltrami-Poincaré metric should be physically legitimated. And it has an attractive theoretical advantage: this metric can be given *a priori*, by as a Cayley-Klein metric, based on the unit plane disk as an absolute (see §3.4 above). This means that we can use it, for the necessities of mathematical descriptions of any fictions serving for both the necessities of interpretation, as well as for the field description.

The evolution of theoretical physics along the last part of the previous century plainly confirms this conclusion. A significant example will be provided in this very chapter, with the case of the archetype gauge fields: the Yang-Mills fields, a natural successor of the Maxwellian electromagnetic fields. Another attractive theoretical point of the complex potential $h$, is that the differential equations to be obtained as the Euler-Lagrange equations that give the solution to variational principle (4.1.9) – the 'Ernst equations' of the problem, as one calls it in this specific case – take the relatively simple form

$$(h-h^*)(\nabla^2 h) - 2(\nabla h)^2 = 0, \quad (h-h^*)(\nabla^2 h^*) + 2(\nabla h^*)^2 = 0 \tag{4.1.10}$$

where the gradient and the Laplace operator are both taken in the metric $\boldsymbol{\gamma}$ of the ambient space. These equations are the whole point of our elaboration thus far. They allow remarkable solutions of the metric tensor for empty space, having direct connection with de Sitter background discussed by us in the previous chapter.

In order to exhibit one of these solutions – the one that inspired us most – we have to notice first that the Ernst equations (4.1.10) are obtained from the variational principle (4.1.9), and this principle involves a Lagrangian constructed simply by replacing the differentials with the components of the gradient operator in the Beltrami-Poincaré metric form of the hyperbolic or Lobachevsky plane:

$$(ds)^2 = \frac{dh \cdot dh^*}{(h-h^*)^2} \quad \rightarrow \quad L = \frac{1}{2}\frac{\nabla h \cdot \nabla h^*}{(h-h^*)^2} \tag{4.1.11}$$

Vincenzo Benza and Piero Caldirola took notice that the geodesics of this metric – the semi-circles in the upper complex plane, having the center on the real axis of the plane – can also be obtained from a Lagrangian constructed based on the same idea as the *harmonic mappings*: replace the differentials with the derivatives with respect to a variable, $\varphi$ say, playing the part of affine parameter of the geodesics. That is, the equation of the geodesics is formally a… 'Ernst equation' in a 'space of one dimension', as it were:

$$L = \frac{1}{2}\frac{h'(\varphi)\cdot h'^{*}(\varphi)}{(h-h^{*})^{2}} \quad \to \quad (h-h^{*})\cdot h''(\varphi) - 2[h'(\varphi)]^{2} = 0 \tag{4.1.12}$$

where a prime means differentiation with respect to the designated independent variable. Based on this observation they devised a method of solution that, in our opinion, is the key to a concept of interpretation (Benza & Caldirola, 1981) [see also (Benza, Morisetti, & Reina, 1979) for the description of the method in the case of a metric having even two Killing vectors: stationarity in time and axial symmetry in space]. To wit: assume those solutions of (4.1.10) that depend on coordinates of the background through $\varphi$. Then, we have after some calculations:

$$(h-h^{*})\cdot h''(\varphi) - 2[h'(\varphi)]^{2} = -\frac{\nabla^{2}\varphi}{(\nabla\varphi)^{2}}\cdot(h-h^{*})\cdot h'(\varphi) \tag{4.1.13}$$

Now, if the parameter $\varphi$ is a harmonic function of space coordinates, the Ernst equation is compatible with the equation of geodesics, and then its solution is given by the parametric equations of the geodesics. If $\varphi$ is not a harmonic function, this construction is not possible. However, if *we are at liberty to choose a parameter* which is harmonic in coordinates, denoting it $\phi$ say, the equations of the geodesics are, indeed, Ernst equations. To wit, the equation (4.1.13) gives:

$$(h-h^{*})\cdot h''(\phi) - 2[h'(\phi)]^{2} = 0, \quad \nabla^{2}\phi = 0 \tag{4.1.14}$$

and $h(\phi)$ is a solution of Ernst equation. This situation asks, nonetheless, for a connection between the two parameters, imposed by the fact that $\phi$ needs to be harmonic. In order to find this connection, assume a change of parameter from $\varphi$ to $\phi$, which means:

$$\nabla^{2}\phi = \phi''(\varphi)\cdot(\nabla\varphi)^{2} + \phi'(\varphi)\cdot\nabla^{2}\varphi$$

so that $\phi$ is a harmonic function of space, if it is connected to $\varphi$ by the differential equation

$$\frac{\phi''(\varphi)}{\phi'(\varphi)} = -\frac{\nabla^{2}\varphi}{(\nabla\varphi)^{2}} \tag{4.1.15}$$

Thus, if $\phi''(\varphi) = 0$, $\varphi$ itself can be taken as a harmonic function. A few details on the practical use of this method are worth our while, inasmuch as they help us in better understanding the concept of instanton.

The metric (4.1.11) is the one from equations (2.3.21) or (2.5.8), in a 'complex disguise', so to speak. Indeed, they differ just by a constant factor, for we have:

$$\frac{(du)^{2}+(dv)^{2}}{v^{2}} \equiv -4\frac{dh\cdot dh^{*}}{(h-h^{*})^{2}}, \quad h \stackrel{def}{=} u+iv$$

The relevance of this metric for the theory of gravitation rests upon the fact that the three differential forms (2.5.7), generating it as a $\mathfrak{sl}(2, \mathrm{R})$ metric, represent *three conservation laws*. There is a mathematical reason for this [see (Schutz, 1982), for instance]: the metric possesses three Killing vectors, known to represent conserved quantities. These conserved quantities are given by the projection of *the momentum forms*, obtained from the metric taken as a Lagrangian, along the Killing vectors of the metric: they turn out to be the differential forms from equation

(2.5.7), indeed. We shall need here such constructions along this work, and will return to this issue in due time. For now, however, we just indicate a 'brute-force' method, as it were, in order to prove that the differential forms (2.5.4) represent indeed conserved quantities along geodesics of the Beltrami-Poincaré metric. Namely, in real terms, the equation (4.1.14) of the geodesics, splits into two differential equations:

$$vu'' - 2u'v' = 0 \quad and \quad vv'' + (u')^2 - (v')^2 = 0 \tag{4.1.16}$$

where the accent represent derivative with respect to the parameter $\phi$ along the geodesics. Now, one can verify by calculations that when transforming the differentials (2.5.7) into rates along geodesics – by replacing the differentials $du$ and $dv$ of the variables with the derivatives $u'$ and $v'$ respectively – these rates are constant under the constraints given in equation (4.1.16). For the relevance of this construction in the problem of gravitation one can consult [(Kinnersley, 1974), §2.4]. Now we can be more explicit on the meaning of the method of Vincenzo Benza and his collaborators.

Indeed, if the constants $c^1$, $c^2$, and $c^3$ represent the three conserved quantities along a geodesic, this one is an object that can be considered as a 'point' of coordinates ($c^1$, $c^2$, $c^3$), located in the $\mathfrak{sl}(2,R)$ Riemannian space. Let us see what is such an object, geometrically speaking. Along the geodesic we have:

$$\frac{du}{v^2} = c^1 d\phi, \quad u\frac{du}{v^2} + v\frac{du}{v^2} = c^2 d\phi, \quad (u^2 - v^2)\frac{du}{v^2} + 2uv\frac{du}{v^2} = c^3 d\phi \tag{4.1.17}$$

which is a compatible system in the differentials $du$ and $dv$, provided the determinant from expression

$$\frac{du \cdot dv \cdot d\phi}{v^2}\begin{vmatrix} 1 & 0 & c^1 \\ u & v & c^2 \\ u^2 - v^2 & 2uv & c^3 \end{vmatrix} \tag{4.1.18}$$

vanishes. This gives a linear relation among the three 'coordinates' ($c^1$, $c^2$, $c^3$):

$$c^1(u^2 + v^2) - 2c^2 u + c^3 = 0 \tag{4.1.19}$$

which, in the complex $h$-plane represents a circle centered along the real axis $v = 0$:

$$(u - c^2 / c^1)^2 + v^2 = \Delta / (c^1)^2, \quad \Delta \equiv (c^2)^2 - c^1 c^3 \tag{4.1.20}$$

The upper half of this circle is the geodesic of parameters ($c^1$, $c^2$, $c^3$). Obviously, in order that this figure is real, we need to have $\Delta > 0$. The bottom line is that a geodesic is a point of coordinates ($c^1$, $c^2$, $c^3$) located in a three-dimensional space on a plane of equation (4.1.19) containing the origin of the space.

Now, a parametrization of the geodesic (4.1.20) that satisfies the conditions (4.1.17) is:

$$u(\phi) = \frac{c^2}{c^1} + \frac{\sqrt{\Delta}}{c^1}\tanh\phi, \quad v(\phi) = \frac{\sqrt{\Delta}}{c^1}\frac{1}{\cosh\phi} \tag{4.1.21}$$

and therefore $\phi$ is defined in terms of $u$ and $v$ by the following linear combination:

$$e^{\phi} = \frac{u}{v} + \frac{\sqrt{\Delta} - c^2}{c^1}\frac{1}{v} \tag{4.1.22}$$

This condition of compatibility of the system (4.1.17) needs further interpretation, but for now we concentrate on a solution suggesting some connections we already used in this work.

However, what we assume to be an 'Ernst physics', *viz*. a physics based on the existence of a complex potential, has an entirely different incentive. This incentive is connected to an idea of continuity between the

Newtonian and Einsteinian natural philosophies, as revealed by the analysis of the classical Kepler problem. We shall dedicate later more attention to this problem but, for now, just notice that a solution of the problem (4.1.14) can be written directly as

$$h = i\frac{\cosh\phi + e^{-i\alpha}\sinh\phi}{\cosh\phi - e^{-i\alpha}\sinh\phi}, \quad \nabla^2\phi = 0 \tag{4.1.23}$$

with $\alpha$ real, but otherwise arbitrary. While this solution can be verified, again, 'brute-force', as it were, by a little longer, but nonetheless direct calculation, our source of inspiration in getting it is classical, if we may say so: its existence is the guarantee of existence of the Kepler motion, which is the cornerstone of the modern physics. It is worth reproducing here the reasoning, for it is connected with our assessment on the Einstein-de Sitter debate, made in §3.4 above.

We shall come back to this issue in due time, to wit: on the occasion of description of Yang-Mills fields, which turn out to be a remarkable expression of the Ernst physics. However, for now, we base our story on previously published results [(Mazilu & Agop, 2012), Chapter 4; (Mazilu, Agop, & Mercheş, 2019), Chapter 4; [(Mazilu, Agop, & Mercheş, 2021), Chapter 6]. The fact of the matter is that the solution (4.1.23) of the Ernst equations, for a harmonic function $\phi$, gives a harmonic application from $\mathfrak{sl}(2,R)$ to the Euclidean space. The 'Ernst physics' of this application starts with observation that the quantity $\tanh\phi$ represents the eccentricity of a Kepler orbit at the location where $\phi$ is calculated, having an arbitrary orientation described by the parameter $\alpha$. Imagine a toroidal canal surface, representing an electron extended spatially in motion around nucleus. One of the constitutive material particles of this electron, is located in a point inside a sphere playing the part of absolute for the Cayley-Klein geometry of our electron. That particle is prone to motion on one of the orbits of an ensemble having the eccentricity $\tanh\phi$, and stochastic, equally probable, orientations given by $\alpha$. This is exactly the situation envisioned by Newton on the occasion of invention of the forces [(Newton, 1974), Book I, Section II, Corollary III of Proposition VII], therefore a classical Newtonian situation. As it turned out, the Yang-Mills fields are an expression of the Ernst physics too (Forgács, Horváth, & Palla, 1981): plainly an Einsteinian situation. We add nothing more on this occasion, that is, other than the Ernst physics turns out to be a key for understanding our very knowledge. The reasons for this statement will follow as we go on with our work.

### 4.2 An Account of Light in Terms of 2×2 Matrices

The previous section reveals, among others, the remarkable part played by the homographic transformation in the economy of general relativity: details apart, the metric (4.1.11) from which the Ernst Lagrangian is constructed, turns out to be the invariant metric of the homographic action of 2×2 matrices with real entries on the complex dual variable ($h$, $h^*$). According to Benza-Caldirola theory, the parameter of geodesics of the Beltrami-Poincare metric (4.1.11) must be a phase describing the algebraic expression of the complex potential $h$ as in equation (4.1.23). When comparing this result with equation (1.2.16), a twofold conclusion of physical interest emerges. First of all, the metric (4.1.11), and therefore the Einstein's vacuum field equations of the case, are referring to a Maxwell fish-eye medium, describing the relativistic background. In view of the fact that this metric can be obtained as an absolute metric, of the kind shown in §3.4, it must count *a priori* as a cosmological metric like the one put forward by Einstein himself (see §3.1). This would mean the existence of a four-dimensional manifold of Euclidean type in which the space as a three-dimensional Riemannian manifold of constant curvature is embedded. And this four-dimensional manifold is, according to de Sitter results, the continuum of charge. Here comes the second conclusion by comparison with equation (1.2.16), bringing the classical argument of existence of closed orbits in the Kepler problem: such a continuum of charge must be the continuum represented by the space extension of the center of force in the Kepler problem, that is the nucleus of the planetary atomic model.

There are quite a few logical issues marring the conclusions we just noticed, and these issues, in our opinion, determined the natural philosophy to take the path it followed, in order to reach us as it did. Two of these are worth mentioning, for they are of essence for our argument. They both derive from the problem of nucleus: a continuous distribution of charge. First, comes the thesis of Dewey Bernard Larson – *may he rest in peace*! – making a case *against* the nuclear atom (Larson, 1963), which must be realistic indeed, except that it is misdirected. In fact, none of the theses of Larson can be understood within the idea of a *physical* structure of the atom: we need to assume the concept of scale transitions. A continuum can only be related to motion, assuming that this motion takes place instantaneously. In other words it is effected at a given space scale – finite, infrafinite or transfinite – however, in a different scale of time. The problem is to decide in what consists that difference between the scale of time and the scale of space and this is simply decided by the *inequivalence between length and distance*! We shall deal with these issues in due time. Just summarizing, for now: at the cosmological scale, where the point locations and time moments are decided, the differentials of lengths are *distributions* of charges along the lines, generated by instantaneous motions. This is why we can deal today with one-dimensional relativity (see §§2.3 and 2.4).

The second issue that determined the path of natural philosophy, is best illustrated by reference to Louis de Broglie: in his argument that concerns us here, he is referring to the spacetime continuity by mixing, as it were, two essentially different points of view: the 'global' continuity of events, as understood in the special relativity, and the continuity representing the motion, which must be, at least in its classical take, a 'differential' continuity. For instance, $(x,t)$ represents, from a relativistic point of view, an event in a two-dimensional manifold supporting the idea of 'globality'. The global continuity is here represented by *transformation between events*: Lorentz transformations, in the case of Louis de Broglie's physics. When talking of a motion, $x$ must be a continuous function of $t$, and in order to associate a velocity to the motion, this continuity should even be differentiable. In general, the differential spacetime continuity is represented by a connection between the *differentials* of the coordinate and of time. In a global representation, this connection can be a kind of Lorentz transformation only in the cases where it is linear and homogeneous – in order to be compatible with the differential continuity of the motion – so that it can be expressed by a matrix.

By reasons of the kind of those just signaled here, one should make, therefore, a clear distinction between *the Lorentz transformation* proper, and the *transformation between differentials*. This means that just as the Lorentz transformation is usually considered a transformation of time moments and positions, where the time is contemplated as a sequence of moments, the transformation between differentials must be regarded as somehow independent, not always obtainable simply by a differentiation. In other words, it is only the transformation between differentials that takes *the time and the coordinates in a* '*measured*' *continuity*, so to speak, necessary for the description of motion by a speed. In order to grasp the right connection between the two kinds of continuity, and, moreover, even to justify it from all the proper points of view, we shall reproduce here, firstly, the mathematics presented by Victor Lalan a long time ago, and based exclusively on the theory of continuous parametric groups (Lalan, 1937). Then, secondly, once we are able to discern what the group theory asks for in the problem of implementing the idea of continuity, we shall use this point of view for the transformations of differentials themselves. In order to become more familiar with the general theory of the families of parametric transformations in the form used by Victor Lalan himself, one can also consult some early works on the application of the theory of continuous parametric groups in the problems of relativity and geometry [see, for instance, (Mandelshtam, 1933) and (Vrânceanu, 1962b)].

Victor Lalan took notice of the fact that in the case of a one-dimensional physical theory, whereby the events are located by a space coordinate whose values are assigned using an *external reference frame* and a time moment whose values are assigned with an *external clock*, the *invariance with respect to the origin of events* imposes right away the fact that the possible transformations must be of the form:

$$x' = A(x - vt), \quad t' = Bx + Ct \tag{4.2.1}$$

Here the coefficients $A$, $B$, $C$ depend on the relative velocity $v$ of the reference frame in which the coordinates are $(x',t')$ with respect to the reference frame where the coordinates of events are $(x,t)$. This means that the transformation assigns coordinates of the events in any reference frame moving uniformly with respect to the original one, once we have these coordinates in this last reference frame.

Operationally, though, the procedure is not that simple. For, suppose we want to physically *check* the coordinates $(x',t')$; in this case the trouble starts brewing. First, we need to have for the primed coordinates the very same physical means we have in the initial coordinates. In this one-dimensional case we are essentially talking of the same kind of physical clock used into establishing the time sequences. However, the time sequences indicated by two different physical clocks – supposed to be attached to the two reference frames – are *a priori* completely different, and we need to make sure that the moments from the two time sequences corresponding to each other *via* transformation (4.2.1) are the right ones. Or else, the transformation (4.2.1) should be taken as *assigning* the events $(x',t')$ to known $(x,t)$, and there is no problem: what we measure then, are just the coordinates *provided by this transformation*, and the case is closed. Otherwise we need to synchronize the two clocks, and the physical synchronization was always a place where the arbitrariness moved in, disguised in the form of assumptions regarding the correspondence between two time sequences.

However, coming back to the pure mathematical problem here, there are some constraints under which this one-parameter family of transformations should be a continuous one-parameter group, usually known as *Lie*'*s theorems*. Understandably, the group parameter is simply the relative velocity $v$ of the reference frames, which is assumed to vary continuously: once chosen a frame for reference, the relative velocity with respect to this frame is in a one-to-one correspondence with the ensemble of reference frames moving uniformly with respect to it. Then, in order to be a group, the family of transformations should contain the identity transformation, obviously for $v = 0$. The simplest way to satisfy this condition comes down to assuming that $A$, $B$, $C$ depend on the relative velocity in such a way that, within *the first order* in the value of relative velocity, we have

$$A(v) = 1 + \alpha v, \quad B(v) = \beta v, \quad C(v) = 1 + \gamma v \tag{4.2.2}$$

with $\alpha$, $\beta$, $\gamma$ – constants. $\beta$ and $\gamma$ here are *not* to be confused with the parameters thus designated in the special relativity, as in the case of previous Chapter 2: they are simply taken as parameters with no other particular assignment. That is, no other assignment than the one that might result from the necessities of the present chapter. Then, the differential equations of the continuous group can be obtained in the first order in parameter *v*: they will express the continuous evolution of the very group transformation with respect to this continuous parameter. However, this very circumstance raises an issue again, for the transformation may not be linear in parameter, and may even involve more than one parameter.

In order to deal with this issue, notice that from the perspective of scale transition, the differential group equations mean the transition between *finite* and *infrafinite* scales of both space and time. In fact, we may not be much out of the truth by saying that this scale transition is the one that dominates the mathematics of actual physics, at least up to this point in time. The groupal condition imposes a kind of factorization (Mandelshtam, 1933): the derivatives of the group variables with respect to the group parameter are *linear* functions of coordinates. By *a change of the parameter of continuity*, the differential equations of the group can be reduced to such a linear form:

$$\frac{dx'}{d\varphi} = \alpha x' - t', \quad \frac{dt'}{d\varphi} = \beta x' + \gamma t' \tag{4.2.3}$$

Without this change, a linear form of the group equations is only *approximate*, as we just said. Here $\varphi$ is what they usually call a ‘canonical parameter’ of the transformation: it is *not the relative velocity itself*! The requirement of continuity entails that for $\varphi = 0$ we should have the coordinates $x$ and $t$ in the ‘initially’ chosen reference frame. The clear advantage of this choice of parameter, is that the system (4.2.3) can be integrated directly, by the following method (Vrânceanu, 1962b): write it down in the form

$$\frac{dx'}{\alpha x' - t'} = \frac{dt'}{\beta x' + \gamma t'} = d\varphi \tag{4.2.4}$$

and then construct, based on this writing, exact differentials of the form:

$$\frac{m dx' + n dt'}{(m\alpha + n\beta)x' - (m - n\gamma)t'} = d\varphi \tag{4.2.5}$$

that may represent the differential of the canonical parameter. Then the requirement becomes obvious, that the left hand side of the equation (4.2.5) must be an exact differential, and this condition takes a nice algebraic form. To wit: the equation can be, indeed, integrated directly, if the coefficients of the variables in the denominator are proportional to those of the differentials in the numerator, *viz*.:

$$m\alpha + n\beta = \lambda m, \quad m - n\gamma = -\lambda n \tag{4.2.6}$$

for in that case the left hand side of (4.2.5) is manifestly an exact differential. Thus, in such a case we have, within the chosen initial condition for coordinates,

$$mx' + nt' = (mx + nt)e^{\lambda\varphi} \tag{4.2.7}$$

and in order to adequately write down the transformation (4.2.1) everything comes down to finding the proper constants $m$, $n$ and $\lambda$. Obviously, for real parameters $\alpha$, $\beta$ and $\gamma$ there are only two possible values of $\lambda$ for which (4.2.6) has nontrivial solution, and these are given by the roots of the quadratic equation

$$\lambda^2 - (\alpha + \gamma)\lambda + \alpha\gamma + \beta = 0 \tag{4.2.8}$$

representing the condition of compatibility of that system. Using one of these roots in the equation (4.2.7), the transformation (4.2.1) can be written in the form:

$$x' - ct' = (x - ct)e^{\lambda\varphi}, \quad c(\alpha - \lambda) = \beta \tag{4.2.9}$$

Then, the equation (4.2.8) means also two possibilities of the very same algebraical nature for the new parameter *c*, which obviously should have the physical meaning of a *velocity*, and these possibilities are given by the quadratic equation equivalent to (4.2.8):

$$c^2 - (\alpha - \gamma)c + \beta = 0 \tag{4.2.10}$$

Assume now that the roots of this equation *are different* and, for once, *real*. The finite transformation that concerns us can be drawn from the two specimens of equation (4.2.9), corresponding to the two values of *c* provided by equation (4.2.10):

$$x' - c_1 t' = (x - c_1 t)e^{\lambda_1\varphi}, \quad x' - c_2 t' = (x - c_2 t)e^{\lambda_2\varphi} \tag{4.2.11}$$

We can better handle this transformation if it is given in the form of a matrix equation:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & -c_1 \\ 1 & -c_2 \end{pmatrix}^{-1} \begin{pmatrix} e^{\lambda_1\varphi} & 0 \\ 0 & e^{\lambda_2\varphi} \end{pmatrix} \begin{pmatrix} 1 & -c_1 \\ 1 & -c_2 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \tag{4.2.12}$$

so that the representative matrix is the 2×2 table

$$\frac{1}{c_1 - c_2} \begin{pmatrix} c_1 e^{\lambda_2\varphi} - c_2 e^{\lambda_1\varphi} & c_1 c_2 (e^{\lambda_1\varphi} - e^{\lambda_2\varphi}) \\ e^{\lambda_2\varphi} - e^{\lambda_1\varphi} & c_1 e^{\lambda_1\varphi} - c_2 e^{\lambda_2\varphi} \end{pmatrix} \tag{4.2.13}$$

From equation (4.2.11) we have, obviously:

$$(x' - c_1 t')^{\lambda_2} (x' - c_2 t')^{-\lambda_1} = (x - c_1 t)^{\lambda_2} (x - c_2 t) e^{-\lambda_1} \tag{4.2.14}$$

which means that this algebraic expression is an *invariant of the family of transformations*. In order to grasp something of a possible meaning of this invariant expression, notice that if $\lambda_1 + \lambda_2 = 0$ it becomes the *usual quadratic Lorentz invariant*. Indeed, in this particular case, one can disregard the power in equation (4.2.14), and take the invariant as a Lorentz quadratic form. However this quadratic form is not referring to the original (*x*, *t*) variables, but to some special linear transforms of these two variables, determined exclusively by the two invariant speeds $c_1$ and $c_2$. In short, the quadratic invariant is:

$$\left( x - \frac{c_1 + c_2}{2} t \right)^2 - \left( \frac{c_1 - c_2}{2} \right)^2 t^2 \tag{4.2.15}$$

This becomes the known Lorentz invariant only under the supplementary condition $c_1 + c_2 = 0$, which is satisfied by particular transformations with $\alpha = 0$ in equation (4.2.3).

But let us stop here, for a little while, and take due notice of the fact that the form of the groupal invariant (4.2.14) was the one chosen by Laurent Nottale in order to introduce through it his considerations regarding *the scale relativity* [(Nottale, 1992); *the 2003 updated version*, §§3–6]. That this is a right choice in order to do such a job is an obvious thing in the theory developed up to this point: the groupal equations (4.2.3) from which this invariant originates are, like any continuous group equations, *manifestly* scale transition equations. However, nothing proves here that expression (4.2.14) is a *scale transition* invariant – *i.e.* it can be written as such in the infrafinite or transfinite range – and this condition remains to be further analyzed as a distinct possibility.

Before going any further, though – and while the case is fresh in our mind – let us assume now the other class of *different* roots of equation (4.2.10), namely the roots that *are complex*, to wit:

$$c_1 = c_2^* \equiv c_R + ic_I \quad \lambda_1 = \lambda_2^* \equiv \lambda_R + i\lambda_I \tag{4.2.16}$$

Here a star means complex conjugation, as usually in physics. The whole theory just presented above also goes unchanged here. However, it is worth marking the fact that the invariant (4.2.14) has to be written, this time, as a genuine *phase factor*, *viz.*:

$$e^{2i(\lambda_R\phi_R+\lambda_I\phi_I)}, \quad \phi_R \equiv \tan^{-1}\frac{c_I t}{x-c_R t}, \quad \phi_I \equiv \frac{1}{2}\ln\left\{(x-c_R t)^2 + c_I^2 t^2\right\} \tag{4.2.17}$$

where the parameters are connected by equations

$$c_R = \beta\frac{\alpha-\lambda_R}{(\alpha-\lambda_R)^2+\lambda_I^2}, \quad c_I = \beta\frac{\lambda_I}{(\alpha-\lambda_R)^2+\lambda_I^2} \tag{4.2.18}$$

The 'Lorentzian condition' which, as we have seen before, comes down to $\lambda_1 + \lambda_2 = 0$, has to be written this time in the form $\lambda_R = 0$, in which case these speeds are

$$c_R = \beta\frac{\alpha}{\alpha^2+\lambda_I^2}, \quad c_I = \beta\frac{\lambda_I}{\alpha^2+\lambda_I^2} \tag{4.2.19}$$

and the *invariant phase factor* constructed with these values is:

$$e^{2i\lambda_I\phi_I}, \quad \phi_R \equiv \tan^{-1}\frac{c_I t}{x-c_R t}, \quad \phi_I \equiv \frac{1}{2}\ln\left\{(x-c_R t)^2 + c_I^2 t^2\right\} \tag{4.2.20}$$

Therefore, the invariance is literally understood here 'up to an arbitrary phase factor', involving the phase $\phi_R$. No question then, this is the second one of Cook's cases (see §§2.3 and 2.4), involving an 'unorthodox relativity', as it were, and the groupal theory of relativity, as presented here, makes it somewhat 'legal'. At some point of our discussion we shall have to insist on this aspect of the physical theory. For now, let us just notice that from a groupal point of view, the relativity needs *two* signal velocities, in case we are bent on an operational definition of terms. In this case, the medium supporting the signal is constitutively anisotropic from the point of view of the elastic properties, and this is why such a relativity should be termed as *anisotropic*. However, the elastic properties of the medium are not the only incentives for considering the anisotropy, as we shall show right away. But, let us come back to our main argument of this section of our work.

A quick comparison between the matrix of the transformation (4.2.1) and the matrix (4.2.13), gives the connection between the *velocity parameter v* of the family of transformations, and the *canonical parameter φ*, in the form:

$$v = \frac{c_1c_2(e^{\lambda_1\varphi}-e^{\lambda_2\varphi})}{c_2e^{\lambda_1\varphi}-c_1e^{\lambda_2\varphi}} \quad \therefore \quad e^{(\lambda_1-\lambda_2)\varphi} = \frac{c_1}{c_2}\frac{v-c_2}{v-c_1} \tag{4.2.21}$$

Using the last equality from this equation, we can transcribe the additivity property of the canonical parameter of this continuous group, $\varphi$: $\varphi_3 = \varphi_1 + \varphi_2$, as a composition property of the corresponding velocities, *i.e.*:

$$\frac{v_3-c_2}{v_3-c_1} = \frac{c_1}{c_2}\frac{v_1-c_2}{v_1-c_1}\frac{v_2-c_2}{v_2-c_1}$$

After due calculations, we have the *Lalan's relation of composition of velocities* [(Lalan, 1937), equation (10′)]:

$$v_3 = \frac{v_1 + v_2 - (c_1 + c_2)\dfrac{v_1}{c_1}\dfrac{v_2}{c_2}}{1 - \dfrac{v_1}{c_1}\dfrac{v_2}{c_2}} \tag{4.2.22}$$

which reduces to the usual relativistic rule for $c_1 + c_2 = 0$ (Einstein, 1905a). Consequently, either this equation or the equivalent one resulting from it – by the transformation given in equation (4.2.8) – for the eigenvalues $\lambda_{1,2}$ of the transformation matrix, must have a special physical importance, which can be revealed *via* the theory of groups. In order to conclude this mathematical line, we can say once again that there are, indeed, *by law* as it were, *two limit speeds* for the description of the kinematics of the classical material points, as long as this kinematics has to respect those relativistic precepts associated with the property of scale transition of the special relativity. For, as we have already mentioned quite a few times before, the differential equations of a group are, indeed, *an expression of the scale transition in physics*. In cases where these speeds are equal in magnitude but of different algebraical signs, we have the special relativity with its regular Lorentz transformations and, consequently, with everything that follows logically from this observation.

In order to realize where, from a physical point of view, the importance of the groupal transformation resides, we have to assume an operational point of view. So, let us first notice the form of the transformation matrix (4.2.13) in the special case of Einstein's relativity. It is:

$$\begin{pmatrix} \cosh(\lambda\varphi) & c\sinh(\lambda\varphi) \\ (1/c)\sinh(\lambda\varphi) & \cosh(\lambda\varphi) \end{pmatrix} \quad with \quad \begin{matrix} c_1 = -c_2 \equiv c \\ \lambda_1 = -\lambda_2 \equiv \lambda \end{matrix} \tag{4.2.23}$$

As we have noticed before, this is clearly a regular Lorentz transformation which, however, admits a physical interpretation, as we shall show here in due time. For now, though, notice that in the case of complex roots, the equation (4.2.21) becomes

$$v = \frac{c \cdot c^* \sin(\lambda_I\varphi)}{c_R \sin(\lambda_I\varphi) - c_I \cos(\lambda_I\varphi)} \quad \therefore \quad e^{2i\lambda_I\varphi} = \frac{c^*}{c} \cdot \frac{v - c}{v - c^*} \tag{4.2.24}$$

where $c \equiv c_R + ic_I$. As we have seen above, the 'Lorentzian case' is given this time by $c_R = 0$, so that, in this specific case, the equation defining the velocity becomes:

$$v = -c_I \tan(\lambda_I\varphi) \quad \therefore \quad e^{2i\lambda_I\varphi} = -\frac{v - ic_I}{v + ic_I} \tag{4.2.25}$$

This is a genuine case of *phase to be associated with a moving particle*. In view of the observation that the existing Einsteinian relativity is, in fact, an interpretation, but without the necessary wave in the picture, the phase factors above are to be associated with material particles: they need to be considered, for instance, as some *de Broglie phase factors*.

Taking this case for granted, a conjecture presents itself quite naturally, capable to put things in order for a logical theory of physics: the case of complex roots in special relativity *defines the phase factor that should be considered as a result of collapsing of wave function after measurement*. For, according to Charles Galton Darwin, even though the theoretical results cannot be presented but in a 'wave language', as it were, the experiment always involves particles, and we need an interpretation, in order to 'translate' the wave-theoretical results into 'particle facts'. In these conditions, the phase factor (4.2.25) should be the *de Broglie wave factor* associating a frequency to the 'wave phenomenon called material point', in order to 'report the case', as it were. Thus, even though for the time being we do not know anything about the wave function, outside the optical analogy, of course, that much we can figure out: the phase factor to which the wave function reduces after measurement has to be a de Broglie phase factor, just *in order to account for measurement*. This realization allows us to take the conclusion from an 1995 article of Boris Kayser and Leo Stodolsky to a new natural-philosophical level that does not deny the existence and the role played by the wave function, but only aims at making more precise the role and status of the phase factor. Quoting the conclusions of that work:

> … the superiority of the amplitude approach seems clear. This is not only because of the greater simplicity of calculation, deriving from the fact that the *amplitude is invariant* while *the wavefunction is not*. By choosing to focus on the *probability amplitude*, something attached to a *process* (*original emphasis*, *a*/*n*), we avoid the conceptual difficulties associated with a reified, "existing" wave function. The latter is now relegated to the role of a secondary quantity, one describing the changes in the amplitude. Since there is no longer any wavefunction to "collapse" in the first place, the psychological and philosophical discomforts associated with the said "collapse" disappear. Also, on a pedagogical level *there is less danger of the frequent confusion of probability amplitude waves with the waves of some physical entity like the electric field*. Concerns with bizarre constructs like enigmatic parallel universes, worlds haunted by restless prowlings of possibly dead cats, disappear, cats and all. The best answer, finally, to the "question of the collapse of the wave function" *is that there is no wave function* [(Kayser & Stodolsky, 1995); *our emphasis, except as noticed*, *a*/*n*].

Obviously, when our authors speak about amplitude, they understand here the classical amplitude of probability, on which alone the theoretical physics of the last century imposes conditions of invariance. It is not without interest, though, finding the conditions of transition in space and time connected with the transiting matter, for those conditions of invariance are referring to matter only. Keeping the phase factors in our view, directs our thinking towards one of the most important theoretical issues of contemporaneity, along the line that the wavefunction needs not be eliminated at all, but only its phase needs to be reconsidered. And the holographic phenomenon from the case of light proves to be liable to help us in understanding this issue, and even offer a solution to it: *in short, the holography is a universal phenomenon of the world we inhabit*.

Among the many modern spiritual phenomena connected to the present-day theoretical physics' concepts, there is a theory explaining the collapse phenomenon of the wave function by the so-called *protective measurements* [see (Aharonov, Anandan, & Vaidman, 1993, 1996), and the most recent collection of original works we are aware of: (Gao, 2014)]. In broad strokes, a protective measurement is a measurement where a field intervenes in order to protect the physical wave function representing the space extension, from the necessary collapsing in a sequence of measurements. Occasionally, the gravitational field was suggested by these authors as playing such a fundamental role here, in saving the wave function from collapse. What we argue here, is that the phase factor is there from the very beginning, and the collapse phenomenon is a reality, indeed, representing, in fact, the necessary scale transition in the measurement phenomenon: *just like in the classical case of light*, *which is the quintessential phenomenon transiting the scales of our world*. The problem, in our case, is to understand and express theoretically the involvement the gravitation *directly* in the scheme of relativity, as we already suggested in Chapter 2 (see §2.2). In the present context – that is, in the context of the continuous group theory as presented right above – this errand of theoretical physics has already been carried out a long time ago in the moder era, by Vladimir Grigorevich Boltyanskii – *may he rest in peace*! The idea is thereby suggested that the gravitation regulates even the electromagnetic phenomena which, as we have seen, stay at the very foundations of special relativity (Chapter 2 above), and even involved a specific analysis by Einstein himself (Einstein, 1919). The way we see it, one cannot possibly avoid this way of thinking in the theoretical physics at large, so let us, therefore, present the special ideas of Boltyanskii in the problem of connection between the gravitation and relativity.

### 4.3 Vladimir Boltyanskii: a Way of Describing the Gravitation

It is now time to draw some 'methodological' conclusions, if we may say so, from the display above, in order to apply them in the case of differentials. In the groupal presentation of Victor Lalan, the Lorentz geometry, as expressed by a quadratic metric form, is obtained only as a particular case where the two speeds that describe the light – therefore, from the most general perspective of the de Broglie association wave-particle, a special propagation! – are equal in magnitude and opposite in sign, as in Cook's type relativity [see §2.3, equation (2.3.15)]. The problem arises, however, if in the cases where the two propagation speeds are different, the geometry can still be Lorentzian, and under what conditions this geometry preserves the quadratic character of the metric. As long a the metric is quadratic, the case would not involving some 'unsecured' scale transition, as it were, from the finite scales to infrafinite ones, both in space and time. After all, this is the spirit of Einstein's cosmological considerations (see §3.1). Vladimir Grigorevich Boltyanskii gives a positive answer to this question, using a linear transformations, indeed, which acts upon the *differentials of coordinates and time* though – therefore at the infrafinite scale of space and time – not on the coordinates themselves (Болтянский, 1974). The *certified* importance of the works of Boltyanskii on this subject rests also upon the fact that he makes out of the problem of gravitation a *problem of control* (Boltyanskii, 1981), whereby the control is exerted through the intermediary of a… wind, as it were, specifically a kind of *Zermelo wind*, if it is to take into consideration the physical phrase used in such instances for a long while [see (Gibbons & Warnick, 2011) for the details and history of the concept]. And here one has what we see as the most important feature of Boltyanskii's approach: the 'motion' to which this control is referring is generated by a 'gravitational level' *formally calculated by the sum of the two limit speeds*, as in the Lalan's case. The main physical fact making such an assumption an essential point of this theory is that *in the special relativistic case proper*, *there is no gravity*! This is, after all, just the Einstein's original thesis, no question about that, but coming here with a mathematical formalism ready to be used physically. And we shall indicate a possible way of using it. Let us, therefore, briefly render the development of essentials of the idea of Vladimir Boltyanskii.

To start with, we write the transformation of the differentials in the form of a linear transformation:

$$dx' = Adx + Bdt, \quad dt' = Cdx + Ddt \tag{4.3.1}$$

and this can be taken as a *homographic*, or *Möbius transformation* between what may appear to be some instantaneous velocities, i.e. the ratios of the differentials of space coordinate and those of time coordinate:

$$\frac{dx'}{dt'} = \frac{A(dx/dt) + B}{C(dx/dt) + D} \tag{4.3.2}$$

In this instance of the transformation, we have to deal with a *homography between the magnitudes of instantaneous velocities*, which generalizes the de Broglie's homography between phase and group velocities, for instance. Boltyanskii uses the equation (4.3.2) in order to get the most general transformation (4.3.1) algebraically. The basics of the manner in which this can be done were shown in the Chapter 2, §2.3 of the present work, but let us repeat the procedure here, for there is an important point of difference.

First of all, Boltyanskii takes notice of the fact that such a homography is well defined by the condition of existence of two speeds invariant by homography, plus the condition that *a classical material point maintains its identity at rest*, *i.e. at zero velocity, as well as at any arbitrary speed*, $v$ say. While we are on it, let us notice that this last condition is the only one that can insure the physical fact that a classical material point, or a Hertz material particle for that matter, can be taken as a reference frame. In the infrafinite space range, more to the point in the microscopic world of our daily experience, where the instability of matter is regular, this condition is essential from a theoretical point of view: one cannot use as reference frames but stable particles. For now, let us notice that in the conditions just stated above, using (4.3.2) we can write, indeed, the system of linear algebraical equations:

$$c_1 = \frac{Ac_1 + B}{Cc_1 + D}, \quad c_2 = \frac{Ac_2 + B}{Cc_2 + D}, \quad v = \frac{B}{D} \tag{4.3.3}$$

where $c_1$ and $c_2$ are the two invariant speeds, and $v$ is the velocity of motion of the material point, which, by the transformation, is corresponding to its rest velocity *0*. In other words, the transformation itself essentially represents a moving material point starting from rest. For the two invariant speeds, Boltyanskii actually uses the symbols $\rho$ and $\sigma$; however we continue to use the Lalan's notations, with the hope that the invariant speeds, like those of light itself, would be properties transcending the space and time scales, as in the original case of Maxwell's electromagnetics: they are the same in the transfinite, finite and infrafinite ranges. Also, we hope to exhibit some conditions in which this invariance is valid, and these conditions would involve some *fields*.

The solution of system (4.3.3) above, completely determines the homographic action (4.3.2) of the matrix in question. Indeed, the linear-fractional action (4.3.2) is well defined just by three of the entries ($A$, $B$, $C$, $D$) of the corresponding matrix, one of these entries being superfluous in the homographic action of the matrix. On the other hand, though, the linear action (4.3.1) of the matrix, is not completely defined this way, *i.e.* by only three parameters. More precisely, it is defined only up to an *arbitrary factor*, so that Boltyanskii writes the linear transformation in the form:

$$dx' = D\left\{ vdt + \left[ 1 - v\left( \frac{1}{c_1} + \frac{1}{c_2} \right) \right] dx \right\}, \quad dt' = D\left( dt - \frac{v}{c_1 c_2} dx \right) \tag{4.3.4}$$

where, this time, $D$ is the entry of the matrix that in the solution given by the system from equation (4.3.3) turns out to become an arbitrary factor. By a direct calculation involving the transformation (4.3.4), we get the invariance of a *quadratic metric* – not of the *general form* metric as in equation (4.2.14)! – under the additional condition that the transformation has a unit determinant, which settles the constant $D$. Thus, the metric is quadratic from the very beginning, no need to assume any further relations, and can be written as:

$$(dx)^2 - (c_1 + c_2)(dx)(dt) + c_1 c_2 (dt)^2 \equiv (dx - f \cdot dt)^2 - c^2 (dt)^2 \qquad (4.3.5)$$

Here, we have used the very Boltyanskii's original notations $f$ and $c$ for the quantities to be calculated according to the following formulas:

$$f \equiv \frac{1}{2}(c_1 + c_2), \quad c \equiv \frac{1}{2}(c_1 - c_2) \qquad (4.3.6)$$

A comparison between equation (4.3.5) and the equation (4.2.15) from the finite Lalan's case, shows that the transition from finite to this infinitesimal case must be effected under the condition $\lambda_1 + \lambda_2 = 0$. Then, the great merit of Boltyanskii may rest upon the fact that he finds a physical interpretation of the condition $c_1 + c_2 = 0$: *the absence of gravitational field*. Let us document this idea a little closer. Among other things, Boltyanskii obtains the general rule of composition of velocities, as given by Victor Lalan, and reproduced by us in equation (4.2.22), but with the important observation that any *two* reference frames have relative velocities with respect to each other, satisfying the condition

$$\frac{1}{v_1} + \frac{1}{v_2} = \frac{1}{c_1} + \frac{1}{c_2} \qquad (4.3.7)$$

Therefore, if $c_1$ and $c_2$ are absolute constants, as their Maxwellian ancestor is usually considered, the *harmonic mean* of the relative velocities must be a constant – *for motion* as well as *for propagation*, we should add – well defined by these absolute constants: *the relative velocities are different*, so that, in general, there is no reciprocity between reference frames! But, if there is reciprocity for propagation, *i.e.* $c_1 + c_2 = 0$, there is also reciprocity for motion, *i.e.* $v_1 + v_2 = 0$, and *vice versa*. Seems just normal: if the invariant velocities obtained based on the transformation of the differentials are those from Lalan's theory – which is referring to the coordinates themselves, not to their differentials – then these velocities must, nevertheless, satisfy to some further restrictions. One can even say that, for an analogy between de Broglie waves and the light waves one has to pay a price, represented by the condition from the equation (4.3.7). The important fact here is that this price can, indeed, be payed – may be not quite in full, though, but it can be payed anyway! – only by an addition to the condition of scale transition. Noticing that $v_1 + v_2 = 0$ is the usual case of the material points in vacuum, the equation (4.3.7) can be viewed as saying that even the invariant velocities are referring to such points. In this case it tells us much more about the very *cause* which might determine the breaking of this classical vectorial symmetry.

Indeed, noticing that for $f = 0$ the metric (4.3.5) goes over into the usual Lorentz metric, Boltyanskii advanced the idea already mentioned above (see also §§2.3 and 2.4) which, again, seems quite natural in this context: the condition $c_1 + c_2 \neq 0$ *should be due to the gravitational field*. Therefore, this should be *the correct way* of introducing the gravitational field directly from the special theory of relativity, *i.e.* with *no intervention whatsoever of the Mach's principle*. After all, according to Willem de Sitter, the Mach's principle has to be avoided at any rate! Starting from this idea, Vladimir Boltyanskii translates the regular problem of motion of a classical material point into a problem of control, as we said, which makes sense even in secular terms. Indeed, one can say in no ambiguous popular terms that *the gravitational field controls the motion on a certain direction*. This property is manifestly the classical condition of *existence of any field* for that matter. Quoting, however, the original words, the conclusion sounds:

> The velocity of any motion is restricted in all reference frames $X_\alpha$ by inequalities like $\rho \leq v_\alpha \leq \sigma$ (the limit values correspond to light motions, the intermediary ones – to «gravitational motions»), i.e. $-c \leq v_\alpha - f \leq c$. If we introduce the parameter $u = v_\alpha - f = \dot{x}_\alpha - f$, we get the relations
>
> $$\dot{x}_\alpha = f + u, \quad |u| \leq c \tag{5}$$
>
> Thus, the consideration of all possible motions in a frame $X_\alpha$ leads to the *controllable object* (5). The light motions are optimal (by the rapidity of action) trajectories of this object. Notice that passing from a frame $X_\alpha$ to another frame $X_\beta$ the number $f = (\sigma + \rho)/2$ does not change, i.e. the equation (5) is *invariant* with respect to transition from a frame to another, so that this invariance appears as a consequence of the relativistic postulate ($\rho$ and $\sigma$ are the same in all reference frames) [(Болтянский, 1979); *our translation, original Italics and captions*]

The theory can be extended to the case of three-dimensional velocity vectors (Boltyanskii, 1995), which allows for a generalization of the metric (4.3.5) to a stationary metric of the spacetime, in the so-called "3+1 form" invoked by Einstein in connection with the cosmological problem (see Chapter 3, §3.1 above), that proves to be necessary to both the theory of the Ernst's complex potential (§4.2 above) and, for instance, to the membrane model of the black holes [(Price & Thorne, 1988); for details see (Mazilu & Porumbreanu, 2018)].

This fact gives us the occasion for an essential observation: in our experience the gravitation is manifested as an acceleration of any *free radial motion* – free fall as they regularly call it – on Earth. Now, the relativistic point of view here, is usually expressed in the manner of Wolfgang Rindler, whereby the motions are represented by lines on a one-sheeted hyperboloid, and, obviously, so are represented the propagations (Rindler, 1960). In this case, in the interpretation process *the matter can be represented by matrices offering transformations between* '*light motions*', and this fact is made possible only by the Boltyanskii's kind of considerations. The great merit of the theory of Boltyanskii rests, therefore, with the accomplishment of this possibility, which, from our perspective, offers the explanation that follows, for the connection between light and matter.

In order to make our point, notice that in a Boltyanskii-type approach there is an apparent inconsistency: we have at our disposal only two of the invariants connected with the idea of a homography, so that it may seem that the Boltyanskii's form of the matrix from equation (4.3.4) is not quite 'unequivocally' associated to a particle. For once, it does not say us what is happening with the particle in the interval of velocities between $0$ and $v$. Indeed, in this approach we are associating invariantly only the two invariant speeds, playing the part of propagation velocities. Boltyanskii's equation (4.3.3) introduces these speeds, $(c_1, c_2)$ in our notation, through the roots of the quadratic equation

$$Cv^2 + (D - A)v - B = 0, \quad v \equiv dx/dt \tag{4.3.8}$$

derived from equation (4.3.2) under condition $v = v'$: for any *pair of reference frames* there are two such invariant velocities. However, if the action is not involutive, as in §2.3, there is a third invariant parameter of a homographic action, which in Boltyanskii's procedure sketched by us above, is replaced by an *ad hoc* assumption. It is the third relation from equation (4.3.3), namely the velocity associated to zero velocity serving to identify *kinematically* the reference frame: unlike the two speeds, this *is not an invariant* of the transformation. Its necessity, even within Boltyanskii's approach as it is, would suggest, though, that the construction must be 'broken' in order to be completed with the intervention of a third particle in need to be referred to this bi-frame, as it were. This can be done, indeed, but within a proper mathematical framework. And a proper mathematics would ask here for a consistent treatment: this means a structure of the entries of the matrix realizing the transformation, *given exclusively in terms of the invariants of the homographic action*, which thus can be taken as some 'coordinates' identifying our particle. Out of these three invariant 'coordinates', the original Boltyanskii's procedure uses just two: the invariant velocities. A proper third invariant of the homographic action of a given matrix, is the characteristic cross-ratio of the homographic relation, *which is a constant for a given homography*. In cases where the transformation is involutive, this constant is $-1$. But in general, by its very definition, in this specific case we have for that cross-ratio an expression of the form:

$$k \equiv (v, v_0; c_1, c_2) = \frac{v - c_1}{v - c_2} : \frac{v_0 - c_1}{v_0 - c_2} \tag{4.3.9}$$

where $v_0$ is any possible velocity of the particle. According to this formula, the right invariant association of Boltyanskii, would be, instead of the velocity $v$ from the third relation (4.3.3), the cross-ratio $(v, 0; c_1, c_2)$, that is, *the exponential factor* from equation (4.2.21). Now, if we arrange the equation (4.3.9) conveniently, in the form of an action like that from equation (4.3.2), and identify the coefficients accordingly, we get the following system of equations for the entries of the matrix realizing the homographic action in terms of the 'invariant coordinates':

$$\frac{A}{c_2 k - c_1} = \frac{B}{(k-1)c_1 c_2} = \frac{C}{1-k} = \frac{D}{c_1 k - c_2} \tag{4.3.10}$$

This system also defines the linear action (4.3.1), even if only up to an arbitrary factor, by the characteristic invariants of its homographic action. Mention should be made, again: this family of transformations with three parameters is a characteristic to *pairs of reference frames*, understood in the usual manner of Lorentz transformations. However, this time, such a doublet is invariantly defined, and the reference to it does not break its 'symmetry', as it were. Unfortunately, the 'reference' in question is hard to define within the operational procedures of the special relativity. Fortunately, though, on the other hand, there is another way to define it, perhaps not quite so 'operational', however endorsed by pure mathematics.

Notice, indeed, that the definition of our doublet is only connected to the transformation between the two events: in a given physical background – ether, matter, field, etc. – any two events associated by the transformation are, in fact, referred to such a doublet. In this respect, the definition (4.3.10) reveals an interesting form of the Boltyanskii matrix, able to show what the reference to the doublet it describes may physically mean. Namely, if we disregard the arbitrary multiplicative constant from the definition of its entries according to this system, the matrix can be written in the form of a linear combination of two singular matrices:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \sim \begin{pmatrix} c_1 & -c_1 c_2 \\ 1 & -c_2 \end{pmatrix} - k \begin{pmatrix} c_2 & -c_1 c_2 \\ 1 & -c_1 \end{pmatrix} \tag{4.3.11}$$

This is a linear pencil of matrices generated by the two singular matrices, representing an ensemble of pairs of reference frames. We shall show now that between $c_1$ and $c_2$ there should always be a homographic relation: after all, this is a well-established mathematical theorem.

To start with, let us notice that in *coordinates* ($A$, $B$, $C$, $D$) a point represents a matrix. The quadratic form representing a quadric, more specifically, a hyperboloid:

$$AD - BC = 0 \tag{4.3.12}$$

is simply the locus of points of coordinates ($A$, $B$, $C$, $D$) representing the ensemble of singular matrices. Geometrically, we have here a one-sheeted hyperboloid, which is a *doubly ruled* surface: it has *two systems* of generators described by the parameters $c_1$ and $c_2$. This can be ascertained as follows: take, for instance, the first singular matrix from the linear pencil (4.3.11). The representation (4.3.10) gives the point of coordinates

$$\frac{A}{c_1} = \frac{B}{-c_1 c_2} = \frac{C}{1} = \frac{D}{-c_2} \tag{4.3.13}$$

One can take notice right away that there are *two* systems of planes whose intersections can count as *two* straight lines – the generators or the rulings of our quadric – at the intersection of which the point (4.3.13) is located on the quadric in question, namely:

$$\begin{matrix} Ac_2 + B = 0 \\ Cc_2 + D = 0 \end{matrix} \quad and \quad \begin{matrix} A - c_1 C = 0 \\ B - c_1 D = 0 \end{matrix} \tag{4.3.14}$$

So these equations describe the two systems of generators of the hyperboloid, whose parameters are the invariant speeds $c_1$ and $c_2$, *i.e.*, taking the heed of existing special relativity, some propagation speeds of this world. Now, assume a generic point of coordinates ($\alpha$, $\beta$, $\gamma$, $\delta$) say, in this space, located inside or outside the quadric (4.3.12), not on the quadric itself, to wit: $\alpha\delta - \beta\gamma \neq 0$. Thus, such a point represents a *nonsingular* Boltyanskii matrix. Its *polar plane* with respect to the quadric (4.3.12), meets this very quadric along a hyperbola having the equation

$$-\alpha c_2 + \delta c_1 - \beta + \gamma c_1 c_2 = 0 \quad \therefore \quad c_1 = \frac{\alpha c_2 + \beta}{\gamma c_2 + \delta} \tag{4.3.15}$$

In other words: an arbitrary *nonsingular* matrix represents a transformation which gives *a correspondence between the two propagation speeds*. The last equality in this equation can be taken – and we, for ones, effectively take it – as a *generalized Boltyanskii relation* between the two propagation speeds defining a doublet. Let us justify this statement of ours. The correlation (4.3.15) can be written as a linear connection between the product, the sum and the difference of the two invariant velocities of a doublet:

$$2\gamma c_1 c_2 + (\delta - \alpha)(c_1 + c_2) + (\delta + \alpha)(c_1 - c_2) - 2\beta = 0 \tag{4.3.16}$$

Any relation between the two invariant velocities is included by this general equation. For instance the Boltyanskii's gravity level $f = 0$, means basically a matrix $\boldsymbol{K_0}$ – $\gamma = \beta = \alpha + \delta = 0$ – from the Lorentz matrices of the Cook's type (see §2.4). Thus, the equation (4.3.15) actually counts as a generalized relation of Boltyanskii type. A more profound meaning of this generalization should be discussed later.

Thus, in a theory of this kind, the nonsingular matrices give correlations between the two invariant velocities of a doublet, acting just like a piece of matter in the process of refraction. Once again, the 'kind' of theory we understand here, is the 'Boltyanskii kind' involving, in general, the idea of field, just a little amended, as it were: the Lorentz transformation at the infrafinite scale should be determined by all *three invariants* of its homographic action. Again, in equation (4.3.3), Boltyanskii used just two of these invariants in his original theory: the invariant speeds, playing the part of propagation velocities. In order to count as an invariant, the third one should be the characteristic cross-ratio of the homography. Using a canonical parameter of the group, this should be an *exponential factor* like that from equation (4.2.21) or that from equation (4.2.24): *a phase factor*. This may be taken as the key point to the concept of protective measurements, as we have noticed before. The hard part of the problem is the existence of the two invariant velocities, but these may appear naturally within the $\mathfrak{sl}(2,R)$ algebra approach of relativity, of the Fowles' or Cook's type (see §2.3).

While this last problem asks for more involved considerations, for the moment we have, even though not quite so directly and, certainly, not intuitively, a confirmation of the above image of the world, in the situation of Earth itself, tantamount to Lorentz approach to electrodynamics of the moving bodies (see §2.2 of the present work). As a matter of fact, it is this situation that allows us to consider the Earth, as well as its fictitious counterpart, the black hole, as prototypes of the concept of *physical particles*. Indeed, start by noticing that, *if* in the last of the equalities (4.3.15) $c_1$ is a constant – like, for instance, the speed of light of our experience – the variation of $c_2$ is determined by the variation of the four parameters ($\alpha$, $\beta$, $\gamma$, $\delta$) according to differential equation:

$$dc_1 = 0 \quad \therefore \quad dc_2 = \omega^1 c_2^2 + \omega^2 c_2 + \omega^3 \tag{4.3.17}$$

where $\omega^k$ are the known differential forms making the components of the $\mathfrak{sl}(2,R)$ coframe [see (Mazilu, 2020); equation (4.2.27)]:

$$\omega^1 = \frac{\alpha d\gamma - \gamma d\alpha}{\alpha\delta - \beta\gamma}, \quad \omega^2 = \frac{\alpha d\delta - \delta d\alpha + \beta d\gamma - \gamma d\beta}{\alpha\delta - \beta\gamma}, \quad \omega^3 = \frac{\beta d\delta - \delta d\beta}{\alpha\delta - \beta\gamma} \tag{4.3.18}$$

Such mathematics is, indeed, prone to describe the situation of Earth, where $c_1$ is the Maxwellian speed representing the *ratio between the electrodynamic and electrostatic units of force*: it does not have to be considered more than that, and the Maxwell's association of it with a speed acquires a new meaning. It means that for the universe around Earth, *this ratio should be a constant*, and all the speeds of propagation, in the matter as well as in the vacuum – are to be referred to this constant. Just as they were, indeed, historically speaking: this is, after all, an unquestionable fact of our experience. To wit: this is how the relativities have come to being in the Einsteinian physics.

Then, a fundamental problem of theoretical physics would certainly be how to attach the equations (4.3.17) and (4.3.18) to the de Broglie's region from the case of Earth. Provided, of course, we know something about the existence of such a region which, again, is only a figment of our imagination, just like the Poincaré fluid in the case of interpretation of the electromagnetic ether (Poincaré, 1900). We have such an example of region in the case of black holes: it is a region where the electrodynamics' equations are valid, helping us in explaining physically the so-called *membrane paradigm* (Price & Thorne, 1986, 1988). This suggests a host of 'subordinate' analogies connected with the grand one, between Galilean and Einsteinian relativities. In general, if between these speeds there is a correspondence given by the homography from equation (4.3.15), the equation (4.3.17) represents the whole spectrum of velocities of the signals serving to establish the lengths, for those kinds of matter existing in the vacuum characterized by the light of velocity $c_1$. In other words, the $\mathfrak{sl}(2,R)$ coframe from equation (4.3.18) simply represents *the matter existing in a certain vacuum*, characterized by the ratio $(c_1)^2$ between the electrostatic units of electricity and the electrodynamic units.

### 4.4 A Modern Account of the Mutuality of Fields

The Ernst's approach of the field equations of general relativity reveals a 'mutuality', as it were, between the fields, for instance in the form given in equation (4.1.6) according to Israel and Wilson. In view of the conclusions of the Einstein-de Sitter debate, the whole point of this theory can be expressed by a law amounting to the fact that between the *boundary conditions* for the metric tensor, and the actual *metric tensor per se*, a harmonic principle of application should be involved. In other words, a harmonic principle gives the field equations *starting directly from the boundary conditions*, which, classically, are simply 'initial conditions of the classical Kepler problem, in disguise', so to speak, as given, for instance, in equation (4.1.23). The only proviso, according to Ernst's own mathematics (Ernst, 1971), is that the background space must be an Euclidean manifold.

As we see it, everything in the man's possibility of explanation of the world we inhabit depends on the proper consideration of the Planck's concept of resonator, as a fundamental structure of the modern theoretical physics. Now, for most people this statement can be taken as peremptory, since that subjective phrase: 'as we see it' hardly really matters in a comparative logical argument where others may see something else. However, the intuitive basis of our statement is encouragingly simple, therefore quite appealing: if the fundamental structure of the world must be experimentally verified at any rate, the verification depends on the existence of dipoles, and this is our reason for this statement. For then, the only experimental device serving this task should always be *a Wien-Lummer enclosure*: it is the only device *known to provide experimental evidence* transiting the scales of the world, in both time and space, due to the nature of the object of study, that is of the light. For, as we repeated quite a few times by now, our essential tenet in this respect is that the light is the only phenomenon that transits the scales of the world.

Then the issue of describing the gravitation must be solved within the same kind of experimental physics, and for it another experimental device must be considered along the same lines of transition between the scales of the world: *the Einstein elevator*. It was also realized experimentally – incidentally, we take here the modern cosmic space experience of humanity as 'experimental physics' – but, apparently, not at any scale of the world. However, the account of theoretical physics of the last half of the previous century, which follows here, suggests an equivalence of the two fundamental devices of our experience. In its turn, this equivalence is able to show what is the kind of interdependence of the fields that satisfy it. And thus, the story that follows becomes our reason for the statement above regarding the fundamental structure of the world, so that it may be taken not quite as peremptory as it may seem at the first sight, after all.

At the infrafinite scale of the world, the three matrices $\boldsymbol{I}_0$, $\boldsymbol{J}_0$, and $\boldsymbol{K}_0$ from the §2.4 above, provide a basis for the *isotopic spin*, represented as an involutive matrix like the one we have in equation (2.4.10). It seems, therefore, worth our while exploring here the idea that Einstein's line leading from electrodynamics to special relativity, is also the line that led to the modern idea of Yang-Mills fields – that is, generalizing the Maxwell fields on the basis of a quantum philosophy – which are the quintessential gauge fields of the modern theoretical physics. There is, nevertheless, a change in emphasis in doing this, that shows where and how the Einstein's infrafinite scale intervenes, and how the finite and transfinite scales are to be described based on this idea. For a proper documentation, let us first quote the *whole abstract* of the original 1954 epoch-making work of Chen-Ning Yang and Robert Mills, that means so much for the today's theoretical physics:

> It is pointed out that the usual principle of invariance under isotopic spin rotation *is not consistent with the concept of localized fields*. The possibility is explored of having *invariance under local isotopic spin rotations*. This leads to formulating *a principle of isotopic gauge invariance* and the existence of a ***b*** field which *has the same relation to the isotopic spin* that *the electromagnetic field has to the electric charge*. The ***b*** field satisfies nonlinear differential equations. The quanta of the ***b*** field are *particles with spin unity*, and *electric charge* ±*e* or *zero* [(Yang & Mills, 1954); *our italics*, *n*/*a*].

It is best, in view of the statements contained in this excerpt, to explain *our incentives* in considering here the Yang-Mills fields, in connection with this agenda of the renowned work just cited. Our contention is that the choice of the basis (2.5.1) is the one that fills in for the *nonlocality of the gauge fields*. With the initial choice (2.4.1) for the characterization of the isotopic spin [see (Yang & Mills, 1954); especially the literature cited there], only the 'rotation' is sought for. However, on the occasion of that choice, a few items were conceptually omitted, that have been partially touched by the evolution of physics in different directions ever since. These became issues that remind us, once again, that we are *not doing physics* in a point in space, but always on a surface, and this is a fact to reckon with in building our understanding of the world we inhabit.

The first one of these issues, coming to our mind right away, for it is conspicuously present into modern theoretical physics, is that the definition of the Yang-Mills fields, just like the definition of their ancestors, the Maxwellian fields, *asks for the concept of surface in a precise way*: the fields should be part and parcel of the physical definition of that surface [(Yang, 1977), and the literature cited there; see also (Gu & Yang, 1977)]. Secondly, we have the observation that the isotopic spin is not compatible with the idea of *localized fields*, but, like its Maxwellian ancestor, it is leading to the idea of *gauge invariance*, which, nevertheless, can be explored only locally: there is not an extant general form of the frame (2.5.1) that allows for the definition of localized fields from a nonlocal perspective. And thirdly, there is not, in the modern physics, *a characterization of the quanta of the Yang-Mills fields* as 'quanta', in the sense made possible by the Maxwellian ancestor of them. What we understand, in this statement by 'made possible' is mainly referring to 'possible by the same means as those of Planck's procedure of quantization, leading to the idea of quanta of light: physical, mathematical, and statistical'. Such a 'possibility' is contrary to today's actual view in theoretical physics, whereby in the process of quantization of the gauge fields one follows mainly the *second quantization* procedure. While these three missing points of the initial proposal of the epitome gauge field will be gradually touched by us, what we got thus far allows for an observation regarding the first two points from the definition of the gauge fields: the choice (2.5.1) is a particular one among the general cases connected with the concept of surface.

The reason for this situation seems to us clear, for it emerges *explicitly* in the subsequent evolution of the Yang-Mills theory into a gauge theory: unlike its Maxwellian ancestor, it is characterized by nonlinear equations of motion, which make a necessary statics of the matter rather hard to understand. In understanding this statement, a first observation is instrumental, regarding the analogy between the general relativistic philosophy and the gauge fields theory: the nonlinearity triggers self-interaction of the fields, a fact made notorious in a quite striking fashion for the physics of modern times, by the case of *solitons* (Scott, Chu, & MacLaughlin, 1973). The self-interaction, allows an idea that the field themselves generate the physical quantities which, as we know from the classical physics, generate it in turn – the charges, the mass etc. – and thus eliminates the necessity of sources of fields. Further on, along with the concept of sources, the nonlinearity thus eliminated the classical idea of a *statics*, which turned out to be virtually disposable, thus making the external sources from the case of electromagnetism obsolete. Quoting:

> … In the absence of external sources of isotopic spin, the $\boldsymbol{b}_\mu$ field *interacts with itself*, since the $\boldsymbol{b}_\mu$ field *possesses an isotopic spin and hence is self-generating*. In this latter characteristic, the $\boldsymbol{b}_\mu$ field is different from the electromagnetic field, which *is described by linear equations* in the absence of other fields. (The nonlinear equations describing the self-generating $\boldsymbol{b}_\mu$ field *are in some respects similar to the equations of general relativity*.)
>
> We seek in this paper to find a solution of the (unquantized) $\boldsymbol{b}_\mu$ field in the absence of other interacting fields. Our aim is then similar to that of Born and Infeld [*the celebrated article on nonlinear electrodynamics* (Born & Infeld, 1934), *a*/*n*), except that they started with equations which were *written down on a more or less ad hoc basis*. [(Wu & Yang, 1969); *emphasis added*, *n*/*a*]

Those 'respects' in which the equations describing the $\boldsymbol{b}_\mu$ field 'are similar to the equations of general relativity' proved to be equivalent, from an essential point of view, as we see it, to the Ernst equation leading to the Maxwell-Einstein vacuum field equations (see §4.1 above). Consequently, what follows in this section is a short story of the path followed by theoretical physics to this conclusion.

We start this story with a fundamental work of C. N. Yang, that we have found quite explicit in explaining the task of the gauge theory of Yang-Mills type (Yang, 1977). Quoting, therefore:

> There has been great interest in recent years in *sourceless gauge fields*. *A self-dual gauge field is sourceless*. We shall consider *a flat space*
>
> $$(ds)^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2 \qquad (4.4.1)$$
>
> and a *self-dual* SU(2) *gauge field*. The main purpose of this Letter is to show that *the condition for self-duality of the field* can be integrated once, resulting in *a set of Laplace-like equations for three unknown functions*. All *considerations are local in character*, and do not refer to global properties. [(Yang, 1977); *emphasis added*, *a*/*n*]

Notice, from the very beginning, a hint of analogy with the general relativity: the metric (4.4.1) has to be compared with the Einstein's choice from equation (3.1.2). Both of them represent a geometric situation *at an infrafinite scale of physics*, being expressed in differentials. This tells us that either the Yang-Mills generalization of the classical electromagnetic fields is not described by equations involving the field propagation *per se*, like its classical Maxwellian ancestor or, as in Yang & Mills original characterization, the existence of the isotopic spin fields at the infrafinite scale 'is not consistent with the concept of localized fields'.

Fact is that the story of physics of the later times spins around this last line, involving, at the finite scale of the world, the Einstein's idea of representation by quadratic form (see §§3.1 and 3.2 above). This situation deserves a logical explanation: the striking success of the general-relativistic natural philosophy, as well as that of the Yang-Mills fields theory, lead mathematically to the idea of some common roots. This idea is obviously connected with the metric (4.4.1) which is formally identical with (3.1.2), and this can be taken as symptomatic. It started with Newton's theory of 'nascent moments' or 'genitae' of the finite quantities [(Newton, 1974), Book II, Section II, Lemma II]. Leaving aside the numerous points of criticism raised along history, and judging Newton in a Voltairian perspective 'by the question he asks', the differentials in general are the nascent moments of the quantities: the set measures on the quantities just about to be born and are not to be treated as the finite quantities about to come to being from them. The point at issue of all critics of Newton, however, was, in our opinion, pertinently analyzed by the great philosopher Georg Wilhelm Friedrich Hegel in his *Science of Logic* [(Hegel, 2010), Book One: *The Doctrine of Being*, Section II: The *Magnitude* (or *Quantity*), especially Chapter 2: The *Quantum*]. For our purpose here, the results of this analysis can be summarized as follows: while for the genitae there is a whole philosophy to justify their logic, in the case of finite quantities there is no logic at all. The finite quantities are simply 'data', and this should tell everything about their definition. This is the philosophy of C.-N. Yang, who then proved that the quadratic form in finite variables can be obtained *via* an integration, representing quasiparticles.

As we can see it, this situation was intended to be stopped with Einstein's equation (3.1.1), which defines – in the Hegelian spirit, as it where – the ideal quantities representing the infinity, with respect to which *only* (*sic*!), are we allowed to define the finite quantities. This is the whole logic of Einstein theory: it should be based on cosmology, that alone allows us to define the finite quantities, the same way as these last ones allow us to define the infinitesimal quantities. Geometrically, the procedure is involved enough, as the Einstein' and de Sitter's debate show us, but it can be framed in a Cayley-Klein geometry (see §3.4 above) and thus subsumed to a Kleinian logic, that has anything to do with physics: *the Ernst physics*, as we called it here. Once again, the Yang-Mills fields, in the C.-N. Yang's take, are a brilliant illustration of this kind of physics. So, let us come back to the streak of discussion of the present section.

From an Einsteinian point of view, the problem is quite simple: the general relativity started from the idea of Newtonian force fields, but it could not construct them as reference. This basically means that, according to Einstein's natural philosophy, these force fields can only be fields at infinity. On the other hand the Yang-Mills gauge theory allowed the construction of such fields as self-interacting gauge fields since the field equations are nonlinear. So, the problem of *construction of such fields at a finite scale occurred* (Wu & Yang, 1969), *playing the part of static fields*, and its solution involved, indeed, quadratic forms (Marciano & Pagels, 1976), but also turned out to involve cubic forms (Uy, 1976) or, in general, a function satisfying a nonlinear equation of Klein-Gordon type, for the so-called '$\phi^4$ field' (Corrigan & Fairlie, 1977):

$$\Box\phi = C\cdot\phi^3 \tag{4.4.2}$$

Here '□' is the *d'Alembertian*, in a quite familiar notation. A special solution of this equation can be written as:

$$\phi(x_1,x_2,x_3,x_4) = a_0\cdot\left[(x_1)^2+(x_2)^2+(x_3)^2+(x_4)^2+\lambda^2\right]^{-1} \tag{4.4.3}$$

with $a_0$ adequately chosen and $\lambda$ arbitrary. This represents a *pseudoparticle* of 'instanton' type, indeed (Belavin, Polyakov, Schwartz, & Tyupkin, 1975). Remarkable enough, this solution represents the finite scale physics, as described based upon the idea of gauge fields. This statement is based on the idea of comparison with Einstein's condition (3.1.1), which can be taken as a constraint on the field. But the implications are by far deeper than that.

Everything, in the theoretical physics of Yang-Mills fields of the second half of the 20th century, revolves around this concept of pseudoparticle. On this state of the case, a discovery of C. N. Yang, reported in the work from which we took the excerpt above (Yang, 1977) allowed the connection of the concept of Yang-Mills gauge field with the Ernst equations (Forgács, Horváth, & Palla, 1980, 1981). This concurrence of circumstances indicated us a manner of connection – *mutuality*, as we would like to call it – between light and gravitation that will be explained here based on the grand analogy of physics. However, in order to explain it properly, let us first present the concepts from a mathematical point of view.

As we said, Chen-Ning Yang's definition for the fields generalizing the classical Maxwell ones (Yang, 1974, 1977), explicitly hints towards the concept of surface and even further, towards the concept of a phase connected with this surface. That is, the self-dual field is described by a set of four potential vectors ***b*** – not just one potential vector like in the case of the classical electrodynamics – represented as a 3×4 matrix, with the field components being the entries of this matrix:

$$\boldsymbol{B}_{\mu} \stackrel{def}{=} b_{\mu}^{k} \cdot \boldsymbol{X}_{k}, \quad \boldsymbol{X}_{k} \equiv -\frac{1}{2} i \cdot \boldsymbol{\sigma}_{k} \tag{4.4.4}$$

Here $i$ is the imaginary unit of the complex numbers, $\boldsymbol{\sigma}_k$ are the regular Pauli matrices, and the summation rule is respected. The Latin indices take three values, while the Greek ones take four values in this case. The field strengths are defined by

$$\boldsymbol{F}_{\mu\nu} = f_{\mu\nu}^{k} \cdot \boldsymbol{X}_{k}, \quad f_{\mu\nu}^{k} \stackrel{def}{=} b_{\mu,\nu}^{k} - b_{\nu,\mu}^{k} - C_{ij}^{k} b_{\mu}^{i} b_{\nu}^{j} \tag{4.4.5}$$

where the entries of matrix $\boldsymbol{C}$ are the structure constants of the Pauli matrices' algebra. This algebra, usually taken as a $\mathfrak{so}3$ algebra, needs to be replaced, in our opinion, with the $\mathfrak{sl}(2,\mathrm{R})$ algebra of the matrices (2.5.1), in order to realize the de Broglie's program.

However, as it stands now, we have the following definition for the field strengths:

$$\boldsymbol{F}_{\mu\nu} = \boldsymbol{B}_{\mu,\nu} - \boldsymbol{B}_{\nu,\mu} - \boldsymbol{B}_{\mu}\boldsymbol{B}_{\nu} + \boldsymbol{B}_{\nu}\boldsymbol{B}_{\mu} \tag{4.4.6}$$

for which the condition of self-duality can be written as:

$$2\boldsymbol{F}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}\boldsymbol{F}_{\alpha\beta} \tag{4.4.7}$$

where $\boldsymbol{\epsilon}$ is the four-dimensional Levi-Civita symbol. Now, C. N. Yang has noticed (Yang, 1977) that in the complex coordinates defined by

$$\sqrt{2} \cdot y = x_1 + ix_2, \quad \sqrt{2} \cdot \bar{y} = x_1 - ix_2, \quad \sqrt{2} \cdot z = x_3 + ix_4, \quad \sqrt{2} \cdot \bar{z} = x_3 - ix_4 \tag{4.4.8}$$

the self-duality can be expressed as

$$\boldsymbol{F}_{yz} = \boldsymbol{0}, \quad \boldsymbol{F}_{\bar{y}\bar{z}} = \boldsymbol{0}, \quad \boldsymbol{F}_{y\bar{y}} + \boldsymbol{F}_{z\bar{z}} = \boldsymbol{0} \tag{4.4.9}$$

where the indices are replaced by the representative coordinates in the complex ($y$, $z$) plane. Starting from this point, we just summarize Yang's discovery, referring the interested reader to the original works for details.

First, we need to take a little closer look at the Yang's concept of a gauge field, for this is the most attractive part of the concept in these gauge fields. He just takes the Weyl's theory of the electromagnetic field (Weyl, 1952), to a new level, so to speak, the 'integral level', as he calls it, by asserting that *a gauge field is intimately connected to a path* in space time and, most importantly for us, this connection is mediated through *a phase*

(Yang, 1974). This phase can be expressed by a *group of matrices*, whose infinitesimal transformations have to match the classical condition of propagation in spacetime. At the infrafinite scale, the matrices can be written as:

$$\boldsymbol{\Phi}_{x\to x+dx} = \boldsymbol{1} + \boldsymbol{B}_\mu(\boldsymbol{x})\cdot dx^\mu \quad \therefore \quad \boldsymbol{\Phi}_{x\to x+dx} = \boldsymbol{1} + b_\mu^k(\boldsymbol{x})\cdot \boldsymbol{X}_k \cdot dx^\mu \tag{4.4.10}$$

Here $\boldsymbol{1}$ is the identity matrix, as usual, and the matrices $\boldsymbol{X}_k$ are those generating the gauge group, and satisfy the specific commutation relations of the group:

$$[\boldsymbol{X}_i, \boldsymbol{X}_j] = C_{ij}^k \boldsymbol{X}_k \tag{4.4.11}$$

where $\boldsymbol{C}$ is the matrix of structure constants of this group algebra. Then *the field strengths are related to the closed paths in spacetime*, thus generalizing Weyl's differential definition of the classical Maxwellian electromagnetic fields:

$$\boldsymbol{\Phi}_{x\circlearrowright x} = \boldsymbol{1} + \boldsymbol{F}_{\mu\nu}(\boldsymbol{x})\cdot dx^\mu dx^\nu \quad \therefore \quad \boldsymbol{\Phi}_{x\circlearrowright x} = \boldsymbol{1} + f_{\mu\nu}^k(\boldsymbol{x})\cdot \boldsymbol{X}_k \cdot dx^\mu dx^\nu \tag{4.4.12}$$

with the matrix $\boldsymbol{f}$ defined as in equation (4.4.5). Here the round arrows mean cycles starting and ending in $\boldsymbol{x}$.

Based on this concept of gauging, Chen-Ning Yang demonstrates the existence of a *generating matrix*, with which all the self-dual fields can be constructed. In the geometry of coordinates (4.4.8), the equation (4.4.9) allow for a special choice of the generating matrix, called by Yang *the R-gauge*. Using his own notations, this matrix is of the form:

$$\boldsymbol{R} \overset{def}{=} \begin{pmatrix} \phi^{-1/2} & 0 \\ R\cdot\phi^{-1/2} & \phi^{1/2} \end{pmatrix} \quad \therefore \quad \boldsymbol{R} = \begin{pmatrix} 1 & 0 \\ R & 1 \end{pmatrix}\cdot\begin{pmatrix} \phi^{-1/2} & 0 \\ 0 & \phi^{1/2} \end{pmatrix} \tag{4.4.13}$$

where $\phi$ is a real function of coordinates, and $R$ is a complex function. The self-duality equations become

$$\begin{aligned} &\phi\cdot(\phi_{,y\bar{y}} + \phi_{,z\bar{z}}) - \phi_{,y}\cdot\phi_{,\bar{y}} - \phi_{,z}\cdot\phi_{,\bar{z}} + R_{,y}\cdot R^*_{,\bar{y}} + R_{,z}\cdot R^*_{,\bar{z}} = 0 \\ &\phi\cdot(R_{,y\bar{y}} + R_{,z\bar{z}}) - 2R_{,y}\cdot\phi_{,\bar{y}} - 2R_{,z}\cdot\phi_{,\bar{z}} = 0, \\ &\phi\cdot(R^*_{,y\bar{y}} + R^*_{,z\bar{z}}) - 2\phi_{,y}\cdot R^*_{,\bar{y}} - 2\phi_{,z}\cdot R^*_{,\bar{z}} = 0 \end{aligned} \tag{4.4.14}$$

where the upper star index means complex conjugate. Now, Yang makes one further choice for an even more particular solution of these equations, satisfying the conditions:

$$\begin{aligned} &R_{,y} = \phi_{,\bar{z}}, \quad R_{,z} = -\phi_{,\bar{y}}, \quad R^*_{,\bar{y}} = \phi_{,z} \\ &R^*_{,\bar{z}} = -\phi_{,y}, \quad \phi_{,y\bar{y}} + \phi_{,z\bar{z}} = 0 \end{aligned} \tag{4.4.15}$$

which leads to a special set of potentials just surfacing within the theoretical physics at that time. Quoting:

> These potentials are exactly of the form of the *Ansatz*
>
> $$b_\mu^k = -\bar{\eta}_{\mu\nu}^k(\ln\phi)_{,\nu} \tag{4.4.16}$$
>
> which has been discussed by Corrigan and Fairlie (1977), by Wilczek (1976), and by 't Hooft (1974, 1976) in the search for sourceless SU(2) gauge fields on Euclidean space [(Belavin, Polyakov, Schwartz, & Tyupkin, 1975); (Witten, 1977); (Jackiw, Nohl, & Rebbi, 1977)]. All published self-dual gauge fields (known to the author) can be gauge transformed to the form (4.4.16), which is equivalent to (4.4.15). They are special cases of (4.4.14), which give *all* self-dual SU(2) gauge fields. [(Yang, 1977); *references updated when necessary*, *a/n*]

The Yang's theory raises a legitimate question: in what respect are the gauge fields describable by the classical idea of propagation? The theory itself gives the answer, even though it is not quite so obvious for an untrained eye: as the decomposition from the right hand side of equation (4.4.13) shows, *Yang's R-gauge is actually a propagation condition* according to the rules of the physics of optical instruments generating light beams (Abe & Sheridan, 1994). Better yet, we can explain such a condition based on the concept of holography, which was actually the inception point of Yang's approach of Yang-Mills field problem in the first place.

It is, perhaps, best to insist a little more, right on this point on the 'holographic moment', as it were, of the Yang-Mills fields, for it is indicative on the methodology of approaching this important problem and its solution. Chen-Ning Yang realized the importance of surface loops in the definition of the concept of gauge fields probably in the late sixties or early seventies of the last century (Yang, 1974). What we call here the 'holographic property' came with a multiplicity of surfaces on which these loops are represented (Wu & Yang, 1975), on which Yang recognized a kind of invariance: each one of these surfaces carried a specific field of electromagnetic type, but all these fields have the same field strength (Yang, 1977). Then we need to consider that the *field strength* is what in the world we inhabit can be taken as *physically accessible*, for instance as matter in a quantization process of the Planck type. Then, the different phases, are surfaces upon which 'photographs' are imprinted in the form of specific field potentials. This, in our opinion, is the genuine holographic property of the Yang-Mills fields, and it is, obviously, the classical acceptance of the holography. Let us elaborate further on this issue.

### 4.5 Holography as the Universal Phenomenon of our World

Taking heed of the presentation from the previous section, we extend here the holographic property to a universe in general. One of the consequences of the theory regarding the thermodynamics of radiation, is that the frequency needs to be presented as a statistic, ranking evenly with temperature (see §1.1). This section is all about how the frequency can be seen as such a statistic: we construct an ensemble having it as a representative, and this ensemble is the *holographic instanton*. On the other hand the holographic property of the frequency connects it with the property of curvature of the surfaces, which is the essential property of gauge fields.

In the case of classical ideal gas ensemble, the energy allows a statistic, taken occasionally even as a sufficient statistic (see §1.1 for details). To wit: because this energy is purely kinetic, it provides a statistic that can be associated with the temperature *via* its variance over the molecular ensemble. If it is to continue this 'statistical tradition' in order to accomplish an interpretation 'classically', as it were, inside a Wien-Lummer cavity serving for describing the thermodynamic equilibrium between matter and light, the light must be cogitated from the same statistical point of view, and this task is, theoretically speaking, a lot harder. For once, the concept of intensity of light strongly indicates that it is *the amplitude* of the light signal that has to be taken as a statistical variable. This motivated Einstein on striving to conclude upon the nature of its statistics (Einstein, 1909). And, as that work conclusively shows, in our opinion Einstein may be considered as the *only one* among physicists who grasped the true physical nature of Planck's method of quantization. Let us expound this issue a little further.

Notice, indeed, a fact of concern here: when considering the radiation as a thermodynamical system, a contradiction, already signaled in §1.1, creeps into our reasoning. This can be relegated to the fact that the formula for entropy, used by Planck in calculating the entropy of radiation [see (Mazilu, 2022), equation (2.1.2) there], is an *equilibrium formula* from thermodynamical point of view, but is used to connect two apparently different temperatures embodied in the limiting cases of Rayleigh-Jeans and Wien for the laws of radiation. In other words, since according to principles of thermodynamics we cannot physically conceive one and the same system as having a given temperature, but being composed of two physical parts existing at different temperatures, there can be no equilibrium at all: those two parts are necessarily exchanging energy between them. In fact, according to Born, the differential equation leading to his radiation law [(Mazilu, 2022), equation (2.1.9)] was taken by Planck only as an *interpolation* between the two extreme cases – Rayleigh-Jeans' and Wien's radiation laws – and thus the emphasis changes its place from thermodynamics to a mere mathematical methodology: the differential equation of Planck would appear as a purely mathematical trick. Still, the problem is not solved, since the methodology cannot be justified but by a statistical reason of independence of two sub-systems. If the radiation system at a given temperature is described by such an equation, this means that in the interaction with the environment the whole physical system behaves partly as being at high temperature and partly as being concurrently at low temperature, according to the two laws of radiation that served to Planck's reasoning. Thus, we are entitled to conclude that the temperature plays a dual role here, and even this occurrence still needs to be further explained from a physical point of view.

Planck himself, in carrying over his method of quantization, opted, as well known, for the idea of *resonator*, whose existence is not only allowed, but is even imposed we should say, by the established Kirchhoff's laws of equilibrium radiation as, in fact, Einstein himself has noticed. However, as we have shown in §1.2 here, the concept of Planck's resonator asks for a special optical medium, a fact that Einstein could not take in consideration at that moment in time. It is from this historical perspective that we define what we like to call the universal property of the world we inhabit, namely *the holography*. In designating this property as universal, we have in mind, first of all, that it is characteristic to both light and matter: it is, indeed, a property of the world we inhabit, more accurately speaking. In describing it we follow closely one recent work of ours, describing the holography in a particular instance (Mazilu, 2023).

The idea of particle – predominantly that of photon – appears, according to discussion right above, to reside upon the *correlation* between the two fundamental 'sub-processes' of the light just mentioned, considered separately as components of a thermodynamical process representing the light. It is quite significant then, in sustaining this conclusion – and Albert Einstein himself took notice of this in the work just cited above (Einstein, 1909) – that he established the physical properties of a quantum only *based on just one* of the two parts of radiation of the light process, namely the Wien's law of radiation (Einstein, 1905b). This may be taken as a particular case of the general explanation of the quantization, and, actually, we take it as such. That one part of the light process considered by Einstein for quantization represented, nonetheless, a *single temperature*, indeed, from the two possible temperatures involved in the light process here, and thus it was 'legal', as it were, from the point of view of equilibrium thermodynamics. In general, however, *i.e.*, considering the whole process that represents the light, a particle becomes an agency of transition between the light at high temperature and the light at low temperature, so to speak. In modern terms this can be translated thus: insofar as the light is sometimes considered as the *modus essendi* of the *vacuum* – like in the case of Planck's quantization, for instance – the particle accomplishing its interpretation can just as well be seen as the *modus essendi* of the *vacuum tunneling* process. Fact is, that the multiplicity of vacuum – the existence of an infinity of vacua simultaneously – seems to be an already settled issue today, at least from a modern theoretical point of view (Jackiw & Rebbi, 1976). In this case, however the vacuum would appear as absence of the physical properties of matter (see §3.3 above), not as the absence of the matter itself.

Just about this kind of conclusions are documented in the works of Einstein addressed to the theory of thermal radiation (Einstein & Hopf, 1910): the pursuance of the Einstein's former idea as presented just above, did not lead to conclusive results on the issue at hand. Citing the original conclusion:

> Therewith is also proved *the validity* of equation (1) and *the impossibility of constructing a probability theoretical relation between the coefficients of the Fourier series that describes the thermal radiation* [(Einstein & Hopf, 1910a); *emphasis added*, *a*/*n*]

The 'equation (1)' mentioned here is *a law of probability* whereby the elementary probabilities of different amplitudes in the Fourier series representing the light are *statistically independent*. Implicitly, in their statistical dependence, we see the Einsteinian idea on the possibility of describing the matter influence on the light. Quite obviously and, of course, naturally, the light is conceived here in the manner of heat: the name of Fourier tells everything in this sense. For, ever since the times of Joseph Fourier, the light, just like the heat, was represented by a series whose terms are considered harmonic oscillators, and Einstein and Hopf were looking, in the work just cited, for the law of probability of these components in the series. A theoretical statistical idea of description was found by them possible, however only based on components of the Fourier series representing the light that are statistically independent. This fact erases *a priori* any possibility of statistical description of the correlation between the components of Fourier series representing the light, and thus brings with it the *a priori* impossibility of considering the correlations between the two extremal ends of the light spectrum. Quoting again:

> One has wanted to find the reason *why all exact statistical analyses in the field of radiation theory lead to Rayleigh's law in the application of this approach to the radiation itself*. With some justification, Planck brings up this argument against Jeans's derivation. However, *in the above derivation* (*i.e. the derivation of Einstein* & *Hopf*, *a*/*n*) there is no question whatsoever of a somehow *arbitrary transference of statistical considerations to radiation*; the energy equipartition theorem was applied *only to the translatory motion of oscillators*. But the successes of the kinetic theory of gases demonstrate *that this law can be considered as thoroughly proved for translatory motion*.
>
> The theoretical foundation we used in our derivation, *which is certain to contain an unfounded assumption*, is thus nothing else but *that underlying the theory of light dispersion in completely transparent bodies*. The *actual phenomena* differ from the results *deducible from this foundation* owing to the fact that *additional kinds of momentum fluctuations are discernible in the former* (*i.e.* in those '*actual phenomena*', *a*/*n*) which, *in the case ot short-wave radiation of low density* (the condition used by Einstein in 1905 to introduce the idea of *light quanta*, *a*/*n*), *enormously overwhelm those obtained from the theory*. [(Einstein & Hopf, 1910b); *emphasis added*, *a*/*n*]

With the perspective of one century over this issue, let us take another look at it, starting, however, with another point of view regarding the statistical principles serving to our approach. If we may be, again, permitted to express the idea, in remembrance of the divine Voltaire, let us «judge Einstein by the questions he asks, rather than by the answers he offers», and try to give an answer ourselves, where his answer seems to us unsatisfactory or incomplete. Luckily for us, we have at our disposal plenty of 'shoulders of giants' to stand upon!

As we have seen in §1.1, the Wien's displacement law is a criterion of selection for any law of radiation, and we extend it over the parameters involved in the expression of such a law. The classical theories of mechanics and electrodynamics indicate two of these parameters that can be expressed as statistics: the temperature and the frequency; the rest of the parameters involved in such a law are universal constants. The temperature is, in fact, the only parameter that can rightfully be called a statistic in this case, based on the studies of thermodynamics of the classical ideal gas: the molecules of matter are the only ones involving "translatory motion" supporting a statistics that leads to temperature. That the radiation is prone to having a temperature characterizing it, is a thermodynamical conclusion backed up by the Kirchhoff's laws of radiation: enclosed, with a gas in a Wien-Lummer cavity, the whole system reaches an equilibrium in the thermodynamical acceptance of this word. However, *there is no such statistic correlated to frequency* so that, obviously, we cannot say that an incidental statistic in this case is supported by a motion of translation, like in the case of ideal gas alone.

And yet, a case may be made for such a statistic, and we need to consider it, insofar as it has special ties with the case now under scrutiny. Besides, it is a good guiding post, as it were, on our way of construction of a statistic for frequency, and it seems to be supported by the fact that Planck's formula appears to be a statistical distribution, just like Wien's spectral density on which Einstein based his heuristic reasoning from 1905 [(Priest, 1919); see the §1.1 above]. On the other hand, the process of associating with each other of the charges from a de Sitter continuum representing the universal physical background of the world, is a stochastic process involving, in the case of a resonator, *two random phases*: one for the electric charges and one for the magnetic charges [(Katz, 1965); see also (Mazilu, 2020); Chapter 3]. In matters of Newtonian force equilibrium, therefore *a fortiori* in matters of interpretation, the charges are associated with a phase [*loc*. *cit*. *ante*, equation (4.4.1) to (4.4.3)]. Thus, the stochastic process of *charge association* is described by a second order differential equation of oscillator type:

$$q''(\phi)+q(\phi)=0 \quad \therefore \quad q(\phi)=e\cdot\cos\phi+g\cdot\sin\phi \tag{4.5.1}$$

where $e$ and $g$ are the electric and magnetic parts of the charge $q$, respectively, and the accent means differentiation with respect to the arbitrary phase $\phi$. This means that the particle possessing the charge $q(\phi)$ is characterized by a linear uniform 'motion' having the equation:

$$\xi''(\tau)=0, \quad \xi\equiv\frac{q}{\cos\phi}, \quad \tau\equiv\tan\phi \tag{4.5.2}$$

so that $e$ and $g$, for instance, *may appear as statistics correlated to this* '*uniform motion*' $\xi(\tau)$ in an incidental ensemble of such 'particles', just like in the case of the classical molecular gas. This result is known as *Arnold*'*s theorem* [(Arnold, 1988), p.44], and is by and large taken as a manner of 'transforming the harmonic oscillator into a free particle', and *vice versa* (Bernardini, Gori, & Santarsiero, 1995), thus easing out the solutions of quantal equations of motion for instance (Aldaya, Cossío, Guerrero, & López-Ruiz, 2011). One may say that there are two such 'free particle' generated by charges of electric and magnetic type, and two statistics appear here as just natural for the process of associating charges. In the spirit of our discussion right above, we may be tempted to see a connection between the statistics characteristic to the harmonic oscillator, and that characteristic to the free particle. This last one is characterized by the parameter temperature, as the molecular kinetic theory instruct us, while the first one is confuse, to say the least. Namely, while the quantum theory of radiation teaches us to tie it up to frequency as a statistic analogous to temperature, the Arnold's transformation above strongly indicates a statistic connected to the amplitude, as Einstein himself searched for, therefore to the charge. Let us concentrate on the case of charge, for it sheds a specific light upon the case of the amplitude.

We cannot but notice that in the above association of charges here, frequency does not appear yet: we have to deal only with functions, *a priori* periodical is true, and expect to define the frequency… when the time comes for it to be considered, as it were. Then, the physics' question is: how, and when could that time come?! An answer is handy right away: just the way it comes with the motion of a free particle in the Galilean environment. That is, associated with an equation of motion. And, as shown above, the equation of motion, as a concept, is 'common', in fact, for the free particle and harmonic oscillator: *they only differ by an Arnold transformation*. If we apply that recipe from equation (4.1.2) to *a genuine free particle* itself, as we have it from our experience in one dimension, we find that the Arnold transformation enjoys the properties of a special realization of the $\mathfrak{sl}(2,R)$ type structure [(Mazilu, 2020), Chapter 4, §4.2]:

$$\tau = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \xi = \frac{x}{\gamma t + \delta} \tag{4.5.3}$$

where $(x,t)$ are the coordinate and the time of the original uniform motion of the particle. This is a realization of the SL(2,R) group structure, which gives us a *possibility of interpretation* by an invariant function

$$\frac{x^2}{at^2 + 2bt + c} \tag{4.5.4}$$

suggesting, on one hand, that the Newtonian forces are the only admissible forces in the case of ensemble of particles realizing the interpretation and, on the other hand, that the group (4.5.3) is referring to a 'radial motion' as it were, of the free particles serving for interpretation, whereby $x$ plays the part of standard deviation on their ensembles (*loc*. *cit*. *ante*).

However, we still do not find any trace of the *concept of frequency* – as we know it from physics, of course, for an *a priori* periodicity exists, mathematically speaking – in this approach of the description of free particles. If we really need to touch this concept from physics' point of view, we obviously have to consider the case of a genuine oscillator, whereby the frequency comes with a differential equation involving, apparently, the elastic properties of the continuum to be interpreted. This idea was instated in physics even from the times when Fresnel added the *diffraction phenomenon* to the phenomenology of light, which classically – that is, as inherited from Newton's and Hooke's theory of the light rays – included just the phenomena of *reflection* and *refraction*. From this perspective, one can say that by 'updating' this way the classical phenomenology of light, Fresnel just *gave an interpretation* (Fresnel, 1821, 1826, 1827) to the old Hooke's ideas on the description of light phenomenon in material media [see (Hooke, 1665), pp. 53 – 69]. And when addressing the issue that way, a thing became obvious, indicating the insufficiency of the diffraction phenomenon in completing the phenomenology of light: the essential concept describing the light phenomenon is that of a *phase*. The frequency can come out of this game *only as an invariant*, specifically related to an SL(2, R) type action of the group of 2×2 matrices, as indicated in equation (4.5.3) for the time. In other words, in order to introduce the frequency we first need to properly introduce the phase, and connect the frequency to it in a way that completes even the Fresnel's diffraction theory: according to Louis de Broglie's theory of diffraction, this theory misses at Fresnel a few essential points that imposed the concept of quantization (Mazilu, 2020).

To this end the phenomenology of light needs to be 'updated' once more, over the diffraction phenomenon added by Fresnel: the phenomenon to be added this time is the *holography*, and the job has already been done, *a posteriori* as it were, based on the Louis de Broglie's ideas, therefore basedon the very quantization procedure (Gabor, 1948, 1949, 1950). This fact can be easily accomplished along the line of reasoning that follows here, strictly connected to the properties of the second-order differential equation. Namely, the ratio, say $\tau$, of the two fundamental solutions of such a kind of equation, entering the Arnold's theorem for the case of an undamped harmonic oscillator of frequency $\omega$, defines this frequency by a third order nonlinear differential equation involving its *Schwarzian derivative*, acording to a formula that can be readily verified by direct calculation:

$$2\omega^2 = \{\tau,t\}, \quad \{\tau,t\} \stackrel{def}{=} \left(\frac{\tau''}{\tau'}\right)' - \frac{1}{2}\left(\frac{\tau''}{\tau'}\right)^2 \equiv \frac{\tau'''}{\tau'} - \frac{3}{2}\left(\frac{\tau''}{\tau'}\right)^2 \tag{4.5.5}$$

Then, if $\tau$ is the time imposed by the oscillator on the corresponding free particle according to Arnold's theorem, it is by no means unique: *any homographic function of $\tau$ with constant coefficients corresponds to the same frequency*. This is a mathematical property of the Schwarzian derivative [see (Needham, 2001), Chapter 5, §XII, Ex. 19(v)]. In other words, there is an ensemble of oscillators, which has *a priori* the cardinality of a three-dimensional continuum, corresponding to the same frequency defined as in the first of the equations (4.5.5). If we call *phase* such a *homographic function of time*, this property is plainly a definition of the holographic phenomenon in the original Gabor's acceptance.

Our specific task now translates into substantiating this statement in the very spirit of Einstein's philosophy just described above, that led to the quantization of light. It is expected that the statistical as well as geometrical properties of procedure will transpire just naturally as we follow the task. Notice that a typical term of the *Fourier series* of the kind taken by Einstein and Hopf to represent the light in a Wien-Lummer cavity, should be of the form of a complex signal (Einstein & Hopf, 1910):

$$q(t) = A(t)e^{i\theta(t)} \tag{4.5.6}$$

Here, the *amplitude* $A(t)$ and the *phase* $\theta(t)$ are to be considered as random continuous variables. These two numerical characteristics may be supposed real and continuous, just for the sake of argument for now; the things may be more complicate but, in order to settle our ideas, this is the basic situation. We try to associate a frequency to such a complex signal, and this amounts to an association of $q(t)$ with a solution of the second order differential equation dynamically describing the harmonic oscillator. So, we must establish an equivalence, which, in real terms, comes down to a system of two differential equations for amplitude and phase:

$$\ddot{q} + \omega_0^2 q = 0 \quad \therefore \quad \begin{aligned} &\frac{\ddot{A}}{A} + \omega_0^2 = \dot{\theta}^2 \\ &2\frac{\dot{A}}{A} + \frac{\ddot{\theta}}{\dot{\theta}} = 0 \end{aligned} \tag{4.5.7}$$

The second one of the equations from the right hand side here, gives immediately:

$$A^2\dot{\theta} = const \tag{4.5.8}$$

which is a kind of Kepler's second law, in the form usually known as the *area law*, suggesting a periodic motion for the amplitude itself, when compared with the details of a classical Kepler problem. Of course, here this means that we shall need a kind of interpretation of this amplitude, and such an interpretation involves, as we have shown above, the classical idea of free particle. This was, indeed, realized on the occasion of a description of the collective motions in the case of nuclear matter (Goshen & Lipkin, 1959). Therefore, taking for guidance the second of the classical Kepler laws, we may need to revise even the concept of particle, since the equation (4.5.8) is referring to a *full orbit*, and such a particle can be located in any point of this orbit, 'equally likely' so to speak. In other words, not only the position in space, but even the position of the particle serving for interpretation along its trajectory is actually a random process of the kind described by Schrödinger on the occasion of offering an interpretation for his newly introduced wave function. Quoting the very words of Schrödinger, the wave function would represent the 'strength' of existence of a particle in a given position:

> … If we like paradoxes, we may say that the system exists, as it were, *simultaneously in all the positions kinematically imaginable*, but not "equally strongly" in all… [(Schrödinger, 1928), p. 120; *our emphasis*, *a*/*n*]

According to this view, the second of Kepler laws should be a first incentive for the introduction of the stochastic element in the very classical mechanics: those simultaneous positions could not be 'kinematically imagined' but by a dynamics of Newtonian kind, based on static Newtonian forces. The 'equal strength' would then be described through the wave function introduced by Schrödinger himself. It is important to keep this idea *in reserve* for later purposes. For now, though, using the equation (4.5.8), it can be shown right away that the first equation from the right hand side of (4.5.7) gives an Ermakov-Pinney equation for the amplitude:

$$\ddot{A} + \omega_0^2 A = \frac{R_0^2}{A^3} \tag{4.5.9}$$

where $R_0$ is a real constant. The connection with the periodic motion *per se* is then as follows. Let $A$ be the composite amplitude of a two-dimensional harmonic oscillator, described by a quadratic form in the partial amplitudes of component signals varying in time according to the equation of $q(t)$ from (4.5.7), *i.e.*, in particular:

$$A^2 = q_1^2 + q_2^2 \tag{4.5.10}$$

where $q_1$ and $q_2$ are two independent solutions of the equation from the left hand side of (4.5.7). This amplitude satisfies the equation (4.5.9) with $R_0$ the constant from (4.5.8). Thus, the frequency $\omega_0$ should be associated to the components of the vector $|A\rangle$ in an obvious way, inasmuch as they are oscillators.

It is in these conditions, however, that one can conclude directly from (4.5.10) that the square of amplitude of the signal from equation (4.5.6) itself – that is, practically, the *intensity of signal* it represents, if it is to speak in optical terms – satisfies a linear third-order differential equation of known type, that can be gotten right away, based on the Ermakov-Pinney equation (4.5.9):

$$\frac{d^3}{dt^3} A^2 + 4\omega_0^2 \frac{d}{dt} A^2 = 0 \tag{4.5.11}$$

When comparing this equation with the equation of a ray as in (1.2.11), a host of conclusions emerge, that impose 'by themselves', as it were.

First of all, we have the striking conclusion, namely that Louis de Broglie was right after all, and in detail at that: the equation characterizing an optical ray is referring, indeed, to the square of the amplitude of an optical signal, just as de Broglie described it for the necessities of the physical optics. Then, because the square of the amplitude of a recorded signal is, according to de Broglie, the numerical density necessary for an incidental proper interpretation, the equation (4.5.11) should also be taken as an equation for that density. For once, this would mean that the much discussed density waves, for instance in problems of astrophysics, are the support of a physical structure of continua. This conclusion of ours is apparently secured by the fact that equation (4.5.11) is one of the fundamental equations of the theory of regularization of the classical Kepler problem. However, the optical medium of our experience, may not be arbitrary after all: as long as the light can be explained as a periodic process, it needs to be described as such a process in a Maxwell fish-eye medium at some level, for the equation (1.2.11) comes with an optical medium having the refraction index given by equation (1.2.4). And this optical medium is, indeed, a Maxwell fish-eye.

It may appear that, with this conclusion, we are rushing in a little, 'where the angels fear to tread', as they say. For once, the kind of ray described by a refraction index (1.2.4), which asks for an equation like (4.5.11), may not be universal, at least not to the same degree as the equation (4.5.11) for the mathematical model of a signal. The optical medium, described by a refraction index, such as that given in equation (1.2.4) may be very particular indeed. However, it is worth recalling that this kind of 'particular' is just mathematical here: as we have shown in the previous chapter, from a physical point of view this kind of mathematically particular medium, may prove to be physically universal after all. And speaking of the light *per se*, this may be the case, indeed.

A warning sign on this issue is the existence *Hanbury Brown-Twiss effect*: there are *intensity correlations* of the rays issuing from the same distant source of light (Hanbury Brown & Twiss, 1956, 1957). Indeed, the square of the amplitude means an intensity in the optical realm. And if an equation like (4.5.11) proves to be universal according to the general mathematical structure of a signal, then we can conclude that the medium of refraction index (1.2.4) is that necessary all-pervading medium of the classical ether type, support of every phenomenon in the world we inhabit. Anyway, at least we have a guidance in our proceedings. To wit: we need to follow the idea of a meaning of the refraction index as suggested by this ray optics, and then, more importantly, to follow the track of an equation like (4.5.11). It is, indeed, particularly important to know if such an equation appears anywhere else in physics at all, and in what conditions.

A key problem thus remains to be solved here, though, since it is directly connected to the equation (4.5.10), which, in turn, is conditioning any result declared thus far: how can we define the frequency in a proper way. This way is taken to mean a way that incorporates *the all-inclusive phenomenon of holography* from the very beginning?! This optical phenomenon is, indeed, the most comprehensive phenomenon of optics: according to Louis de Broglie's theory of light ray it concludes the phenomenology of light to the point were it can be used as such even in the case of matter. A sound solution imposes by itself through an implementation of the *idea of coherence*, and can be obtained using the 'Kepler's second law' (4.5.8), which seems to be an apt universal mathematical fact, endorsed by the theory of regularization [(Mazilu, 2020); §3.4]. Taking, then, for the *amplitude as a function of phase*, the very definition provided by the Kepler's second law (4.5.8), will be consistent with the *holographic principle* defined according to the original Dennis Gabor's ideas on coherence. For once, such a definition means that *the time variation of phase must be physically recognizable in the intensity of a certain* *wave*: the intensity is, in fact, not independent on the phase of signal. Then, proceeding just mathematically, we are able to transform the Keplerian condition (4.5.8) into a second-order differential equation for the amplitude of the complex signal:

$$A = \frac{C}{\sqrt{\dot{\theta}}} \quad \therefore \quad \ddot{A} + \frac{1}{2}\{\theta,t\}A = 0 \tag{4.5.12}$$

where $C$ is a constant. Indeed, by comparison with the oscillator's equation of motion, the right hand side of this equation *defines the frequency in terms of the phase* of a general signal, like (4.5.6), by the equation (4.5.5). Therefore, we can take this last formula as defining the frequency, thus including the holographic phenomenon into the definition of a general optical medium. Then our problem is reduced to the mathematical description of this phenomenon, and the definition of frequency according to (4.5.5) provides a natural way.

Everything thus revolves, in problems of frequency, around the definition of the Schwarzian derivative [see, for the relevant details and a comprehensive presentation of this operation (Needham, 2001), Chapter 5, §§X, XI, XII]. The outstanding property in this definition, of which we shall make much use in this work, was already mentioned here: any solution of the equation (4.5.5) *is defined up to a homographic transformation*. This would mean that the manifold of solutions of equation (4.5.5) corresponding to a general phase is three-dimensional, not in the sense of the linear superposition rule, though, but in the sense that it can be *surveyed* by locating its points with three parameters. In the superposition rules' phrasing, we rather have here a *nonlinear superposition rule* with three basic solutions of the equation [see (Cariñena, Marmo, & Nasarre, 1998), §§2, 3, especially equations (3.51–53)]. More precisely, knowing three solutions of the equation (4.5.5), a fourth one can be found right away, without any integration, because it must have a constant cross ratio with those three (see §4.3 above). In order to prove this statement, we use the general relation of transformation of the Schwarzian [see (Needham, 2001), especially Chapter 5, §XII, Ex. 19(iii)]:

$$\{\theta,t\} = \{\phi,t\} + \{\theta,\phi\}\cdot\dot{\phi}^2 \tag{4.5.13}$$

where $\{\theta,\phi\}$ is the Schwarzian derivative of the phase $\theta$ with respect to the phase $\phi$. If this derivative is null, the two phases are connected by a homographic relation [*ibidem*, Ex. 19(v)], *i.e.*:

$$\{\theta,\phi\} = 0 \quad \therefore \quad \theta(\phi) = \frac{\alpha\phi+\beta}{\gamma\phi+\delta} \tag{4.5.14}$$

so that equation (4.5.13) becomes

$$\{\theta,t\} = \{\phi,t\} \tag{4.5.15}$$

Therefore, the homographic action of the matrices 2×2 can cover the whole ensemble of solutions of the equation (4.5.5). According to this theorem, the general form of a solution for equation (4.5.5) depends on three parameters: it can be obtained from any particular solution by the group formula (4.5.14). In other words, we can locally construct the whole system of phases of a signal having a definite frequency – in technical terms: a 'frequency coherent signal', *i.e.* the kind of signals used in the *technical implementation of holography* – starting from a particular one. The whole system of phases corresponding to the same frequency – this one being defined by the equation (4.5.12), with an amplitude as in equation (4.5.10) – is the orbit through a particular phase $\theta$ of the group of real homographies. This is a continuous group with three infinitesimal generators, locally described as a $\mathfrak{sl}(2,R)$ Riemannian space. Therefore, this Riemannian space *is the local expression of the holographic phenomenon*, which here has a precise meaning: the whole system of phases corresponding to the same frequency. This gives us a possibility of interpretation – and speaking of interpretation here, we mean interpretation in the wave-mechanical sense, *whereby the phase can be associated to a particle* (Darwin, 1927) – mathematically describable in the terms that follow, mimicking the actual construction of a hologram.

First, one needs to find a 'seed phase', as it were: a phase that remains the same during the holographic process. This is the hard part of the mathematical description of the holographic phenomenon according to the previous definition involving the idea of Fourier series or even integrals: finding the phase whose information is carried over into any other phase. One can thus better appreciate the definition (4.5.5) of the frequency, which allows us a firm mathematical characterization of the coherence, and thereby a clearcut characterization of the phenomenon of holography. Let us, therefore, assume for the moment that we have found that 'seed phase', and denote it by $\theta$. The whole system of phases $\phi$ carrying the very same information is described by equation (4.5.14), with $\theta = constant$. Thus, any phase $\phi$ coherent with $\theta$ in the sense of our definition of frequency, is mathematically describable by the solutions of a differential equation of Riccati type, that *we take as defining an instanton* (Mazilu, 2020):

$$d\phi = \omega^1\phi^2 + \omega^2\phi + \omega^3 \tag{4.5.16}$$

This means that, in cases where the phase $\theta$ defines a time, the equation $\theta = constant$ defines 'an instant' of that time, and this instant is spatially described by equation (4.5.16). This equation correlates the variation of phase at the same frequency and the same seed phase with the variations of the three parameters describing the holographic phenomenon. Here the differentials ($\omega^k$) are the components of the standard $\mathfrak{sl}$(2,R) coframe [see (Mazilu, 2022), equation (4.4.3)]. If we are able to transform this equation into an ordinary differential equation with respect to a certain 'time parameter', then it gives us an expression of the phase rate to be used in equation (4.5.12), in order to define the amplitude.

Now, in most cases which *we* have encountered thus far in our study, this transformation is an easy task facilitated by the metrics of the $\mathfrak{sl}$(2,R)-type: as a rule, these metrics possess three *Killing vectors*, for which the dual rates ($\omega^k/dt$) are constants along their geodesics [see (Weinberg, 1972), Chapter 13; also (Schutz, 1982), for details in a modern mathematical spirit]. It is known, indeed – and we shall repeat the procedure in due time here for a typical case of interest, – that the differential forms of the $\mathfrak{sl}$(2,R)-type coframe are projections of the momentum forms generated *via* the metric Lagrangian, along the Killing vectors. Therefore, in such cases, the equation (4.5.16) becomes an ordinary Riccati differential equation along the geodesics:

$$\dot{\phi} = a^1\phi^2 + 2a^2\phi + a^3 \tag{4.5.17}$$

where ($a^1$, $a^2$, $a^3$) are three constants characterizing the $\mathfrak{sl}$(2,R)-type geodesics in question, and a dot over means differentiation with respect to the arclength of the geodesics. This means that a geodesic becomes a point in the $\mathfrak{sl}$(2,R)-type Riemannian space. So, according to the holographic principle formulated as above, *i.e.* based upon the idea of frequency coherence, only along such geodesics the physical theory may happen to be interpretable in the wave-mechanical sense. Of course, the process asks for an inversion of the amplitude defined by the rate of phase (4.5.17), so that the inverse of the amplitude will appear, by Arnold's theorem, as describing a free particle. Indeed, using the combination of the Kepler law (4.5.8) with the equation (4.5.17) gives:

$$\frac{r^2}{a^1\phi^2+2a^2\phi+a^3}=const, \quad A^2=r^{-2} \tag{4.5.18}$$

which represents the radial motion of a free particle, whose kinematics is described in a time provided by the phase $\phi$.

We have strong reasons to believe that, physically speaking, this should be the case: for once, according to *Wagner's theorem* [see (Mazilu, 2020), Chapter 4, §4.3] this holographic space is the realm of the free particles realizing the oscillators. The most important of these clues, though, is the fact that the holographic definition of the frequency characterizes indeed the nucleus of a planetary atom. This statement seems to us sufficiently proven as a consequence of the classical dynamical problem associated to Kepler problem (Mazilu & Agop, 2012). However, in the theory of nuclear matter *per se*, this idea comes associated with an *interpretation via* the concept of *collective coordinates* (Goshen & Lipkin, 1959). So, we need to insist on the physical aspect of the problem from the perspective of these two natural philosophical concepts. For this we need first some special geometrical considerations regarding the realizations of homographies, for they are the basis of definition of the holographic phenomenon. Inasmuch as a homography is simply defined as an action of the 2×2 matrices, specifically on phases in this case, the realizations in question involve some properties of these very matrices, as determined by their actions.

## Chapter 5 An Old Einsteinian Case for Gravitation and Atomic Structure

Einstein never gave up the idea of getting rid of the cosmological term in his equations describing the gravitation, and he started reconsidering his natural-philosophical position right after *Cosmological Considerations*, making it even 'official', as it were (Einstein, 1919). If he was consistent or not in correctly following the concepts during this reassessment, this is another question, but we are not entitled to judge him anyway, for there is no point in such a judgment: when the concepts are virtually nonexistent the spirit goes just by guess. An educated guess, is true, but still, a guess. This means that, if we *feel an urge to judge* at any rate, the words of Voltaire certainly must be held in view: *judge the man by the questions he asks not by the answers he offers*. As we see it, though, with the benefit of retrospection of over a century of theoretical physics, part of whose conclusions we just selected in the previous chapters, Einstein was right after all. The modern approach of his general relativity by the model of harmonic mappings can be taken as a solid proof of this statement, as we shall document right away. This fact, however, does not exclude that others may have been right as well. The man is just a finite being: he can grasp just a part of the truth which, as such, is universal. Then, the point of his existence, mostly in cases where the concepts are poorly defined or even nonexistent, is to relate to others negatively, as Einstein – and every theoretical physicist, in fact – plentifully did. But this is the man's 'prerogative', if we may say so, due to his finiteness: it does not mean that the very differentiae of a concept of knowledge he promotes are contradicting others' such differentiae. Let us proceed with expounding this remarkable example of our process of knowledge.

With the contribution of de Sitter, the idea may have surfaced in those old times, that the physical construction of the world necessarily involves charges. On the other hand, with the contribution of Hermann Weyl, [see (Weyl, 1923, 1952)] it became quite obvious that a 'static condition' is vitally necessary to the construction of a physical theory, but, unfortunately, it could not be properly included within its structure (in making up our mind, the documents 619 and 626 among the English translations, Volume 8, *The Collected Papers of Albert Einstein*, Princeton University Press, have substantially contributed). In our opinion, this static condition is all about the moment of interpretation of the theory of relativity, serving to connect it to the classical views, without involving the concept of wave. In this case we need an interpretation in terms of particles *exclusively* and, what may count as belittling the whole Einsteinian natural-philosophical point of view in this respect, is the Einstein's own struggle to insert a *physical* interpretation of matter where there is simply no call for it.

Indeed, *any interpretation whatsoever involves only figments of our imagination* – material points, Hertz material particles, partons, quarks, and so on; the list can be completed with many other examples of such inventions serving for the sole purpose of interpretation – but not *physical particles*. Especially in the matters regarding the cosmological term introduced 'accidentally' by Einstein, this problem became acutely pressing and, at least as far as we can understand it, even in an explicit form. To wit: incapability of our intellect to 'gauge' in a more precise way the intervention of imagination in the solution of the problems of physics otherwise than mathematically, became critical at that time and, in our opinion, remained critical even to this day. This fact, however, is due to the limited posibilities of mathematics at the very time when spirit needs it, which is quite a natural circumstance of the process of knowledge, as a matter of fact. Quoting, specifically:

> Not only the problem of matter, but the cosmological problem as well, leads to doubt as to equation (1) [*equation* (3.2.5) *of the present work*, *a*/*n*]. As I have shown in the previous paper, *the general theory of relativity requires that the universe be spatially finite*. But *this view of the universe necessitated an extension of equations* (1), with the introduction of a new universal constant λ, *standing in a fixed relation to the total mass of the universe* (or, respectively, to the equilibrium density of matter). This is gravely detrimental to the formal beauty of the theory. [(Einstein, 1919); *emphasis added*, *a*/*n*]

An observation: in the context of Einstein's work containing this excerpt 'the previous paper' would appear as (Einstein, 1916a); however, Einstein is most probably referring in fact to the *Cosmological Considerations* (Einstein, 1917a), since the first one of these two references does not seem to make any sense in this instance. Thus, he appeared to have taking for granted that the problem of cosmological boundary conditions has not an orthodox solution, but can be overcome only by conditions of statical spatial symmetry of an instance of the universe: spherical symmetry in this specific case.

Note in passing that, as the history shows, Einstein seems to have been the only one among theorists who have seen a 'grave detriment' here. Also note that the work (Einstein, 1916a) contains a formal presentation of equations of the electromagnetic field in four-dimensional arrangement needed for the development of relativity, and that an equivalent of this presentation, to wit (Lorentz, 1917) has, as Einstein himself noticed, a counterpart presentation of the same problem, but with emphasis on the forces rather than the fields. Such an approach, at the time we are talking about, proves consistency from the part of Lorentz: the concern of *equilibrium of forces* describing the stability of matter was his first incentive to create the Lorentz transformation in the first place, the transformation that led to the special relativity. However, the 1917 approach of electrodynamics along the same lines proves, in fact, to be easier to generalize along the idea of static condition in a de Broglie's concept of ray, in general. To wit: if, according to holographic principle, the fields – taking after the Yang's explanation of the Yang-Mills prototype – define the different 'phases of matter', then the forces representing the intensities of action of those fields due to their 'coherence', as it were, need to be taken as invariants. Once again, in this sense we can see the Newton's definition of forces as a first theoretical instance of the procedure of quantization.

Anyway, going back to 1919 article, we need to take notice of the fact that, after carrying the necessary analysis, Einstein came up with a set of problems that – again, in our opinion – were sound, and deserved a concentrated concern in order to be solved but, unfortunately, they were only partially solved, or even remained unsolved to this very day. Quoting:

> The above reflections show the possibility of a *theoretical construction of matter out of the gravitational field and electromagnetic field alone*, without the introduction of hypothetical supplementary terms on *the lines of Mie's theory*. This possibility appears particularly promising

in that it *frees us from the necessity of introducing a special constant* $\lambda$ for the solution of the cosmological problem. On the other hand, there is a peculiar difficulty. For, if we specialize (1) [*equation* (3.2.5) *of our present work*, *a*/*n*] for *the spherically symmetrical static case*, we obtain one equation too few for defining the $g_{\mu\nu}$ and $\phi_{\mu\nu}$ (*the skew-symmetric tensor of electromagnetic field*, *a*/*n*) with the result that *any spherically symmetrical distribution* (*original emphasis here*, *a*/*n*) of electricity *appears capable of remaining in equilibrium*. Thus the *problem of the constitution of elementary quanta* cannot yet be solved on the immediate basis of the given field equations. [(Einstein, 1919); *emphasis added*, *except as indicated*, *a*/*n*]

The rest of our present work is dedicated to showing how quantization, in the form put forward by Max Planck (see Chapter 1, §1.1) can help in solving some of the problems raised by Einstein in these conclusions. We also intend to show support of our opinion that the problems just raised by Einstein were reasonably formulated, and only partially solved, or even remained unsolved, to this day, as we said. This is, indeed, a case that we can make about the discrepancy between the possibilities of the existing, at a certain time, mathematics, and the necessities of the natural-philosophical – and even purely philosophical, we should say – spirit at a moment of our knowledge. And, if this case still remains unconvincing, an example will be presented in our conclusions, with a solution of the problem within the mathematical means of this very day and time.

### 5.1 Removing Planck's Restriction on the Dipole Structure

Edward Kasner is often cited as a reputed name in matters of cosmology (Kasner, 1921b), specifically in connection with cosmological solutions of the Einstein field equations [see, for instance, (Landau & Lifshitz, 1971), Chapter 12]. These solutions are unambiguously addressed to the *vacuum field equations* as they were conceived by Einstein, both in his initial general relativistic theory (Einstein, 1916b), and in the *Cosmological Considerations* (Einstein, 1917a) where he included the celebrated cosmological constant. As we have already shown above, Einstein himself started regretting the introduction of this cosmological constant, and with good reasons at that, according to his very own natural philosophy, which justified him in trying subsequently to eliminate it from the curriculum of general relativity. In his almost immediate attempts to get rid of this constant, which appeared to him as "gravely detrimental to the formal beauty of the theory", Einstein suggested new field equations of the form [see (Einstein, 1919), §2, equation (1a)]:

$$G_{\mu\nu} - \frac{1}{4} g_{\mu\nu} G = -\kappa T_{\mu\nu} \tag{5.1.1}$$

that may be able to 'free us from the necessity of introducing a special constant $\lambda$'. Here $\boldsymbol{T}$ is the energy tensor which, if determined exclusively by electromagnetic fields in the Maxwell's take (see §4.1 above), satisfies the identity $\mathrm{tr}(\boldsymbol{g}^{-1} \cdot \boldsymbol{T}) = 0$. This identity is also satisfied by the tensor from the left hand side of equation (5.1.1), so that this form of the field equations of general relativity can have a touch of rational basis, as it were, just like the original field equations (4.1.2). Einstein noticed that the equation (5.1.1) is a consequence of equation (4.1.2) with the tensor $\boldsymbol{T}$ determined exclusively by electromagnetic fields, "but not conversely". This statement may be taken by itself as a valid proof of the fact that general relativity, as Einstein conceived it, asks for something more than mere electromagnetic fields, but he has apparently opted for the idea that only an electromagnetic structure is prone to determine the internal physical structure of the fundamental particles. This may be true, indeed, only, as we shall see, only with a little tweak in conceiving the electromagnetic fields, however, not at all outside the Maxwellian picture.

Indeed, in carrying out the analysis, Einstein reached what we think as an 'ultimate conclusion', namely that one has to follow the idea of Gustav Mie, along the lines indicated by David Hilbert in 1915, in building a unitary physics based on his natural philosophy. Parenthetically: we have followed all these works in their English translations, printed in the collection (Renn & Schemmel, 2007). The work of Gustav Mie is only reproduced in fragments there, but just in case, one can also consider the (still partial) translation of the Mie's theory of matter due to D. H. Delphenich, in order to fill in for the missing parts of the translation of Renn and Schemmel, or *vice versa*. In order to understand why Einstein wanted to avoid "the lines of Mie's theory" (see the excerpt right above), let us describe this theory in its broad strokes.

The essential one from among the points raised by Mie's theory is the existence of a space-extended electron having an *electromagnetic internal structure*. In our context here, it helps if we take notice, once again, of the fact that this electromagnetic structure is understood in the sense of Maxwell, not in the sense of the definition of Lorentz, involving the idea of an already existent interpretation [(Lorentz, 1892); see the §1.3 above]. Along the line thus chosen, Gustav Mie even reached the idea that the *inertial* and *gravitational masses* of such an electron *are different*, which, however, could not grow into a proper fruition, due to the eternal problem impairing the theoretical physics up until the wave mechanics emerged: *the interpretation deals with figments of our imagination, not with real physical structures*, and physics cannot give up asking physical structures for any interpretation. Not even today, when it works manifestly with figments of imagination, gives it up asking for such physical structures, and if they are nonexistent in reality, it invents realities that might contain them. Anyway, it is on the occasion presented by Mie's theory of matter, that Einstein found a good opportunity, so to speak, of (re)appraisal of his own theory of general relativity, this time with the explicit hope to get rid of the cosmological constant. After a careful analysis of his original field equations of gravitation (Einstein, 1916b), he concludes:

> So if we wish to contemplate *the possibility that gravitation may take part in the structure of the fields which constitute the corpuscles*, we cannot regard equation (1) [*equation* (3.2.5) *of the present work*, *originally proposed by Einstein in 1916*, *a*/*n*] *as confirmed*. [(Einstein, 1919); *emphasis added*, *a*/*n*]

Let us say it once again: '*the fields which constitute the corpuscles*', if it is to conclude on a corpuscle structure by the case of an electron, are to be understood as electromagnetic fields described, according to Gustav Mie, *by the Maxwell equations*. So, Einstein has modified the field equations as shown in equation (5.1.1), along the lines of his proposed cosmological equations (Einstein, 1917) – equation (5.1.1) looks like (3.2.6), only with a special value of cosmological constant – and carried the analysis according to his natural philosophical line, based on these field equation. The conclusion of this analysis is altogether encouraging, in that it allows implicitly the much needed – for Einstein at least! – riddance of the cosmological term. Whence the following optimistic conclusions. Quoting, therefore:

> The scalar of curvature (*that is the invariant G*, *of the Einstein tensor **G***, *a*/*n*) *plays the part of a negative pressure* which, *outside of the electric corpuscles*, has the constant value $G_0$. *In the interior of every corpuscle* there *subsists a negative pressure* (positive $G–G_0$), the fall (*that is*, *what we currently designate today as the gradient*, *a*/*n*) of which maintains the electrodynamic force in equilibrium. The *minimum of pressure*, or, respectively, the maximum of the scalar of curvature, *does not change with time in the interior of the corpuscle* [(Einstein, 1919); *emphasis added*, *a*/*n*]

Obviously, admitting the equation (5.1.1) – which, as we said already, is essentially the equation (3.2.6) of our present work, however with a special value of the cosmological constant 'naturally' included in it – Mie's natural-philosophical ideas can be accommodated *without further hypotheses*: the 'lines of Mie's theory' can, therefore, *be avoided*. However, based on our experience with the Yang-Mills descendants of the Maxwellian fields [see §4.4 above; see also (Mazilu, Agop, & Mercheș, 2021), Chapter 4, equation (4.53) *ff*] we see in this conclusion a little different message: it is just the lines of a Maxwellian theory of the internal structure of particles that must be avoided, not those of the Mie theory *per se*. And, we have to add now, not even the whole Maxwellian philosophy, associated with the internal structure of the particles should be avoided. For once, the 'internal pressure' might be of electromagnetic nature after all, but still, "the *problem of the constitution of elementary quanta* cannot yet be solved on the immediate basis of the given field equations", as Einstein himself has noticed, indeed. Why?

The 'elementary quanta' here are *quanta of electricity*: they necessitate Planck's quantization, indeed, *but within the matter*, *not in the light*, *i.e*. not outside the matter. Besides, there is no possible restrictive physical condition for the charge of a sphere in equilibrium, in order to introduce the quantization condition: as far as we can see, this condition can only be satisfied by a 'fragmented' surface of a 'fractal' type. Incidentally, notice that there cannot even be such conditions for a smoothly closed surface: the very static Newtonian force, which the relativity tries to avoid, is the essential condition of this kind of quantization. Interestingly enough, we must notice that Gustav Mie himself touched this issue and, more importantly for what we have to say here, he approached it from the very point of view of the constitutive element of matter used by Planck himself: *the dipole* (see Chapter Four in the Delphenich's translation of the Mie's theory). Even more interesting is the fact that Mie reached the conclusion that this dipole should have a substantially different electromagnetic structure from the one we are accustomed to conceive based on the classical dipole of charges – the case considered by Planck himself – and may even have the properties of a magnet, for instance. Incidentally, this conclusion is way out of the Maxwell line of approach of the electromagnetic theory and, along with the Planck's quantization procedure, it is one of our strong incentives in (re)introducing the Lorentz's interpretative ideas in the structural physics of matter. So, again in line with the physics of the epoch we are discussing now, that is, by having a physical structure in view, Mie concludes:

> It is extremely difficult to establish more exact conditions for *the possibility of existence of such elementary dipoles*, but it should be of little use in view of *our total ignorance of the nature of the world function*. For once, I will assume *that there are such elementary dipoles*, and I will draw some consequences from this assumption. [(Mie, 1913), *our translation from German*, *emphasis added*; *one can also consult the translation by Delphenich indicated before*, *a*/*n*]

Gustav Mie's 'world function' is an extension of the energy in the form of a Hamiltonian, involving therefore the geometry of a background in its calculations. This involvement, however, amounts to associating in calculations, in a specific way, of course, the metric geometry of such a background. Therefore, it is this metric geometry of the background that should interest us here. In turn, the metric geometry obviously involves… a metric, which, in the views of the general relativity, is always assumed to be *a quadratic differential form*, and here is the point where Edward Kasner has entered the stage.

The notable undertake of Kasner's is referring to the possible quadratic differential metrics for the *vacuum field* equations – the vacuum can be considered a background by default, as it were – corresponding to Einstein's initial proposal for the field equations (Einstein, 1916b), and to the equations (5.1.1) respectively, that is:

$$G_{\mu\nu} = 0 \quad and \quad G_{\mu\nu} - \frac{1}{4} g_{\mu\nu} G = 0 \tag{5.1.2}$$

It regards the metrics allowed by these equations *assuming* that they are a sum of *quadratic differential forms* in one, two or three of the four spacetime variables (Kasner, 1925). Notice, again, that the first equation here implies the second one, but not *vice versa*, as Einstein has already noticed himself. This means that the vacuum is not the same in the two cases, but only the first of these vacua can be taken as 'universal', that is: existent in the case of matter as well as in the case of light. Two Kasner cases are important for us here, insofar as they proved to be essential for the theoretical physics of the last century.

The first significant from among Kasner's results along this line (Kasner, 1921a) is about the field allowed by *the first* of the equations (5.1.2). It states that the quadratic differential equation involving the sum of four terms:

$$(ds)^2 \equiv (dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2 = 0 \tag{5.1.3}$$

cannot take place if there is a *permanent gravitational field* in the sense of Einstein: the first of equations (5.1.2) does not provide solutions for this case. Here $x_1$, $x_2$, $x_3$, $x_4$ are the coordinates in the *Einsteinian four-manifold*. Taking them in the special relativity spirit – that is, as representing lengths and distances alike – Kasner translates this result into a kind of incompatibility between light and gravitation, of the nature of the incompatibility which led to the rejection of the background metric (3.2.3) by both Einstein and de Sitter, in the cosmological case. It is significant though, in our context, to mention that the 'permanent gravitational field' is represented in Kasner's case by a conformally flat metric of the form

$$(ds)^2 \equiv F(x_1, x_2, x_3, x_4) \cdot \left\{ (dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2 \right\} \tag{5.1.4}$$

From the point of view of the scale transition, this is the condition that the gravitational field is the same for *any* value of the infrafinite distance representing the propagation of light. Interestingly enough the Einstein's equations would tell us that the 'permanent gravitation field' thus conceived is perfectly determined at a finite scale of the world, in both space and time. Indeed, Kasner has found that the solution of Einstein's field equations $G_{\alpha\beta} = 0$, is given in this case by a nonhomogeneous quadratic form as in equation (5.1.4), where the function $F$ of finite coordinates is given by:

$$F(x_1, x_2, x_3, x_4) = \left\{ a_0 \left[ (x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 \right] + a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 + a_5 \right\}^{-2} \tag{5.1.5}$$

provided the center of this quadric is located on the quadric:

$$(a_1)^2 + (a_2)^2 + (a_3)^2 + (a_4)^2 - 4 a_0 a_5 = 0 \tag{5.1.6}$$

In this case, there may be, incidentally we should say, an admissible transformation of coordinates that reduces (5.1.4) to a sum of squares, therefore to a situation where the metric tensor is a constant diagonal matrix. Therefore, the permanent gravitational field is inexistent, indeed, since the metric is flat. In any case, with this result, the metrics include cases of the Maxwell fish-eye type in four dimensions. Notice also, from a modern perspective, as it were, that the general metric (5.1.4) is of a particular *Coll type* [(Coll, 1999); see also §3.4 above], for it misses the deformation part of a full Coll metric. This may be taken as an incentive to describe the static case: it simply corresponds to a conform Euclidean metric in four dimensions. Any universal deformation of it would then involve the presence of matter. Another immediate result of Kasner, shows that an Einstein's original gravitational field cannot be *embedded in a five-dimensional Euclidean flat* (Kasner, 1921b). In other words, one cannot use Einstein's procedure of embedding for a five-dimensional flat. This last conclusion may count as a solid reason for introducing the cosmological term, after all.

For, indeed, such a conclusion of Kasner's does not mean that the de Sitter's philosophy would not work for embedding here, since the whole mathematics supporting such a philosophy *is not based* on the first of the equations (5.1.2), but on the second. Thus, on a positive note (Kasner, 1921c), the field characterized *by the second equation* from (5.1.2), is embeddable in a six-flat, staged, in a geometrical description, by *two* unit spheres. The resulting spacetime metric is then given by a sum of two quadratic forms, each one of them involving just two of the four variables. It is this result of Kasner that, in our opinion, is of essential concern for physics at large: it liberates us from the Planck's proviso consisting of the description of dipole physical components as vibrating exclusively along the axis of dipole. This observation is helping us in creating a generalized model of dipole, good for any occasion, as it were, in the matter as well as in the light. Indeed, this liberation gives us the opportunity "to assume the vibrations *as taking place in space instead of in a straight line only*", if it is to use words of Planck himself (see §1.1 of the present work). Let us elaborate a little on this issue.

The necessity of such a liberation comes just naturally, according to our developments thus far in the present work, from the idea of a general optical medium having the characteristics of a Maxwell fish-eye: if the material particles are moving along the geodesics of this medium, these particles, among which the 'elementary quanta' of Einstein can certainly be included, must have *a fortiori* three degrees of freedom. The dipole structure should then be an Ampère element limited along its 'axis' by two portions of surfaces that must have a negative curvature if the charges are involved in the physical structure (see §§1.3 and 1.4 above). This structure can then be geometrically explained by the infinitesimal deformation of surfaces, a phenomenon induced through the presence of charges in the medium [see §1.3 above, equations (1.3.12) *ff*, and §1.4; see also (Mazilu, 2023a), §3.1]: the charges at the ends of an Ampère element get an extra-displacement as a result of the infinitesimal deformation. Such a phenomenon, considered from the perspective of the concept of instanton, can offer us the possibility to explain the classical structure of a planetary atom, in the intuitive manner that follows.

Assume, indeed, the following, more realistic image – at least we see it that way! – of a planetary atom: a spatially extended spherical nucleus of positive charge – suggested by the dynamical analysis of the classical Kepler motion – and a spatially extended spherical electron of negative charge (Dirac, 1962), considered, say just by analogy. The incentive of such an analogy can be given by the description of nucleus combined with the fact that in the reality of our experience the planetary model involves planets having their own satellites. A kinematic approach can be assumed for the uniform charge of each of these two spheres, based on the concept of instanton:

*each one of the spheres is an instanton*. Kinematically, this image even acquires an intuitive meaning: the two charges, considered attached to the Hertz particles serving for interpretation, are moving *instantaneously* over the Riemannian space of instanton. To wit: they move so fast inside respective spheres, that these regions appear, *in each and every instant of time*, *reckoned*, *of course*, *at the time scale of the Kepler motion of the electron around nucleus* [see §4.5, equation (4.5.14) *ff*], as uniformly charged with positive and, respectively, negative charges. In other words, we have *a single* type of interpretative charged material particles in each one of the regions, but that charge is acquired so fast by them in each space at their disposal, that this space appears to us as uniformly charged at the time scale of the Kepler motion.

In a Schrödinger phrasing, like that in the excerpt from §4.2 above, *the charge* is 'simultaneously in all kinematically possible states, however not equally likely'. The 'equal likeliness' of the presence of the charge inside the sphere is measured by a special probability density. The physics here resides on what we would like to call the *Feynman*'*s interpretation concept*. This concept was used by Richard Feynman just as 'a side topic', so to speak, to what he considered the right concept of interpretation assisted by the idea of quantization. However, as a benefit of the idea of scale transition, we are compelled to see in it a *genuine concept of interpretation*. Quoting, indeed:

> The wave function $\psi(\boldsymbol{r})$ for an electron in an atom, does not describe a smeared-out electron with a smooth charge density. *The electron is either here, or there, or somewhere else, but wherever it is, it is a point charge*. On the other hand, think of *a situation in which there are an enormous number of particles in exactly the same state*, a very large number of them with exactly the same wave function. Then what? One of them is here and one of them is there, and the probability of finding one of them at a given place is proportional to $\psi\psi^*$. But since there are so many particles, if I look in any volume $dxdydz$ I will generally find a number close to $\psi\psi^* dxdydz$. So in a situation in which $\psi$ is the wave function for each of an enormous number of particles which are all in the same state, $\psi\psi^*$ *can* be interpreted as the density of particles. If, under these circumstances, each particle carries the same charge $q$, we can, in fact, go further and interpret $\psi^*\psi$ as the density of *electricity* (*original emphasis here*, *a*/*n*) Normally, $\psi\psi^*$ is given the dimensions of a probability density, then $\psi$ should be multiplied by $q$ to give the dimensions of a charge density. For our present purposes we can put this constant factor into $\psi$, and take $\psi\psi^*$ itself as the electric charge density. With this understanding, $\boldsymbol{J}$ (the current of density of probability I have calculated) becomes directly the electric current density.
>
> So in the situation in which we can have very many particles *in exactly the same state,* there is possible *a new physical interpretation of the wave functions*. *The charge density and the electric current can be calculated directly from the wave functions and the wave functions take on a physical meaning which extends into classical, macroscopic situations*. [(Feynman, Leighton, & Sands, 1977), Volume III, §21–4; *our emphasis*, *except as mentioned*, *n/a*]

We reproduced this lengthy paragraph from the well known course of lectures in physics, first of all in order to get the gist of a classical interpretation: a kinematics is needed in order to describe the charge – incarnated as electron in the excerpt above – as being 'here or there or somewhere else', where 'here' means 'attached to a

Hertz material particle of the interpretative ensemble'. The prototype of this kind of kinematics was, indeed, constructed by Richard Feynman himself in his renowned and widely recognized approach of the wave mechanics (Feynman, 1948, 1949). However, the excerpt above turns out to be also useful in suggesting that the Schrödinger's interpretation of the wave function (see §4.2 above) is by no means in disaccord with the general idea of interpretation involving the concept of material particle. The wave function of Schrödinger even gains a bonus of clarity here, regarding its meaning: it represents *that instantaneous density of the charge*, which, obviously, is in a manifest way different from a Newtonian density, and was described in physics of the last century as a *probability density*. With the concept of instanton, it appears that this probability density belongs to the class of densities characterizing ensembles which have quadratic variance functions in terms of their means. In other words, the probability density in question is part of the same class of probability densities discovered by Max Planck on the occasion of the prototype quantization (Mazilu, 2022).

Thus, in order to upgrade the Feynman's interpretation into *the main* concept of interpretation of physics, we can say that 'the existing interpretative particles are charged properly at any instant, everywhere inside the Kasner spheres', with a probability described by a density given through the wave function. However, this wave function is here suggested as a construction *upon waves exclusively propagated with speeds superior to a certain experimental limit*, for instance the speed of light, as in the case of de Broglie's wave groups. Putting this issue aside for a later deeper analysis, the fact remains that the *two* instantons of the classical planetary model can be described, in a '*Kasner*'*s view*' as it were, as continuous static charges, represented as one continuous manifold of dipoles. Each one of these dipoles is *instantaneously* established – again, with 'instantaneity' defined in terms of time scale of the Kepler motion describing the model – by the connection of a material particle from one region with a material particle from the other region. This connection can only be formally described by a Lorentz transformation of the Cook's type (see §2.3 above). One can say that both the electron and the nucleus of the planetary model are two Maxwell fish-eye mediums instantaneously connected by a family of Lorentz transformation. The result is an Ampère element, which can be imagined as a continuous congruence of straight lines, representing instantaneous dipoles of an arbitrary physical nature: electric, magnetic or mixed, depending on what kind of charges are connected within an instant.

This is, in our opinion, a situation *geometrically described* by Edward Kasner's theory, and according to this description, it can be tied up with the Einstein's second set of equations (5.1.2). More to the point, Kasner's results can be summarized as follows [(Kasner, 1921c); see also (Kasner, 1925)]: if the metric form is necessarily a quadratic differential in four variables, then the Einstein equations proper [the first one of the equations (5.1.2)] allow for only an Euclidean, or a conformal-Euclidean metric. On the other hand, the Einstein's equations with a cosmological term [the second one of the equations (5.1.2)] allow only for a solution which is either Euclidean or conform-Euclidean (the de Sitter's case), or *a sum of two metrics of surfaces of constant negative curvature*. This last case can be represented geometrically as an Einstein-type embedding of a four-dimensional manifold into a six-flat represented by the union of two unit spheres, as presented above. This case, – as we see it, the most important of the Kasner cases – needs a closer attention for reasons to be presented right away.

Transcribing all the above geometrical facts into equations, the last result here can, for once, be expressed in a remarkable way, justifying the fundamental occurrence of instantons in the above imagine of the planetary atom. Namely, if on this Einsteinian manifold, the second of the equations (5.1.2) is valid, then we can only have, up to some multiplicative constants, that may be considered either explicitly or somehow included in the definition of the coordinates, the following form of the metric of the four-dimensional continuum:

$$(ds)^2 = \frac{(dx_1)^2 + (dx_2)^2}{(x_2)^2} + \frac{(dx_3)^2 + (dx_4)^2}{(x_4)^2} \tag{5.1.7}$$

This metric (re)presents the universe as formed of two 'blades' – that can be viewed as surfaces of negative curvature, *via* the methods of a Cayley-Klein geometry where the Euclidean sphere serves as absolute (see §3.4, above) – each of these 'blades' physically representing an instanton, as above (see also §4.2). Indeed, the metric (5.1.7) can be considered as the cosmological metric of an Einsteinian four-manifold embedded in a six-dimensional flat described by the equations:

$$\pm Z_1^2 \pm Z_2^2 \pm Z_3^2 = 1, \quad \pm Z_4^2 \pm Z_5^2 \pm Z_6^2 = 1 \tag{5.1.8}$$

in coordinates ($Z_1$, $Z_2$, $Z_3$, $Z_4$, $Z_5$, $Z_6$), assumed to be real. Each one of the two terms from equation (5.1.7) can be considered, indeed, as the Cayley-Klein metric of the geometry having as absolute one regular quadric of any signature from among the two equations (5.1.8) (see §3.4). In this case, mathematical physics must consider undertaking the task of describing the connection of the geometries of the two 'blades', by a mathematically proper procedure, compatible with the existence of the two Beltrami-Poincaré metrics entering the formula (5.1.7). This is what we would like to call the *mathematical physics of Kasner's geometry of a de Sitter universe*.

Apparently, the term 'blade' was used for the first time in the modern theory of general relativity by Gerald Rosen, on the occasion of a first undertake of the problem of electromagnetic fields from the geometrical point of view (Rosen, 1959). This significant work was followed right away by another one, just as significant, of Bruno Bertotti on the same subject, who undertook the term as such (Bertotti, 1959a), but also came to recognize the high significance of Edward Kasner's work on the subject (Bertotti, 1959b). In particular, in this last work Bertotti has shown that "any manifold which is a product of two surfaces of constant curvature" can be obtained as a solution of the Einstein electrovacuum equations, with cosmological term:

$$G_{\mu\nu} + \lambda g_{\mu\nu} = T_{\mu\nu}, \quad G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \quad \frac{1}{2} T_{\mu\nu} \equiv g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} + \frac{1}{4} g_{\mu\nu} (F^{\alpha\beta} F_{\alpha\beta}) \tag{5.1.9}$$

a result which inspired our writing in the equation (5.1.8) above. This writing only suggests the general idea that any surface of constant curvature can be obtained as an embedding in the sense of Cayley-Klein geometry, having one of the quadrics (5.1.8) as absolute.

Concluding this section with a final note, we need to mention this overall result of Kasner: the only fields that satisfy the second set of equations (5.1.2) correspond to a metric of *constant Riemannian curvature* (Kasner, 1921c). This justifies, in the most general manner, as it were, the Ernst's conclusion mentioned by us in the §4.1 above: there should always be a relation between the gravitation and the physical fields connected with the existence of the electricity in the universe [see equations (4.1.6) thru (4.1.8)]. Our contention goes even a little further: if the planetary model is to be described by a kind of mathematical procedure involving a unitary description of the 'continuous congruence' between two negative curvature surfaces, then a metric description of such a congruence can be taken in constructing a general harmonic map based on the same principles as the Ernst-type map. This harmonic map describes the physical fields in the Euclidean space of our existence. So, with this logic of approach at our disposal, let us concentrate on the task of describing the possible meaningful correspondences between two constant negative curvature surfaces, first from a geometrical point of view. For, the two Kasner blades of a universe cannot be really separated as in the case of Maxwellian fields [see (Rosen, 1959); also (Bertotti, 1959)]: they need to be upgraded to Yang-Mills fields, and this cannot be done without a connection between the two blades.

### 5.2 The Mathematical Physics of Kasner's Metric Geometry

The closing observations of previous section allude to a special mathematical physics of the Kasner's geometry based on the metric (5.1.7). In order to get the gist of this physics we shall start with the description of the geometry: according to the precepts of Einsteinian physics, this geometry is not independent of physics and, with the completion of Einsteinian philosophy realized by de Sitter, even the space itself must have a physical component in electricity (Misner & Wheeler, 1957). Then, our contention amounts to the simple fact that the matter makes its mark in the universe we inhabit through an action that generalizes the fundamental rotation described by the classical charges as a kind of a *duality rotation*, to use the modern theoretical terms (see §2.2).

In short, Edward Kasner reproduces the whole philosophy of general relativity, aimed towards bypassing the impossibility of a *direct* solution of the Einstein's field equations: the Einstein's field equation cannot be, *practically* speaking, solved as they stand. Then, the procedure instituted even from the early stages of Einsteinian doctrine's development, is to go on finding physically meaningful, but mathematically simpler metric tensors, on which the Einstein's field equations are enforced. The whole imbedding method of Einstein (see §§3.1–3 above) is a quintessential example. With the equations (5.1.7) and (5.1.8), Edward Kasner adds a twist on this procedure, making it liable to be turned into a universal physical method of reasoning along the lines that follow.

First of all, according to previous section of this chapter, the fundamental unit of matter – the dipole – is liberated from the Planck's constraint imposed by the idea of dynamic vibration along the axis of the dipole: each one of the two charges to be associated with each other into a dipole, are free to move, but 'instantaneously', if we may say so, on two surfaces of negative curvature. In view of the fact that the motion is instantaneous – *i.e.* it can be considered as taking place at an infrafinite scale of time – the two surfaces are carrying what at the finite scale of our experience appear as *continuously distributed charges*. Then, each point of one of the two surfaces, when associated with each point of the other surface, forms a genuine Ampère current element, as once cogitated by Joseph Liouville (see §1.5 above). The whole construction can be seen, geometrically speaking, as a congruence of lines joining the two negative curvature surfaces, so that the correspondence between charges can be described as a family of *Bäcklund transformations* [(Reyes, 2003); a good account of the Bäcklund transformation serving the physics' purposes, is provided by (Sasaki, 1979), §§4 and 5; [see also the comprehensive work (Rogers & Schief, 2002) on this subject]. With this observation we have reached a key point of the theory on which we shall be concentrating for the remainder of this work: we aim at presenting the essential manner of mathematical approaching the physical procedure of interpretation.

To start with, we can present the general philosophy almost intuitively: each one of the two parts of the Kasner's cosmological metric (5.1.7), can be cast in the form (2.5.8), either in real terms or in complex terms. Then we have two surfaces of negative curvature to be set in correspondence with each other, and this *correspondence can be constructed effectively*. Indeed, write one of the metrics in (5.1.7) as:

$$(ds)^2 = \frac{(du)^2 + (dv)^2}{v^2} \quad \therefore \quad (ds)^2 = \frac{dz \cdot dz^*}{(z - z^*)^2} \tag{5.2.1}$$

Here, the notations are as follows: $z \equiv u + iv$, is a regular complex variable, having the real components $(u, v)$, and $z^*$ is its complex conjugate. In the second form of this metric a fact become obvious, which has been made known to Poincaré by the end of 19th century. Namely, the metric (5.2.1) retains its form for the homographic *real transformations* of the complex variable $z$:

$$z \to \frac{a \cdot z + b}{c \cdot z + d} \quad \therefore \quad z^* \to \frac{a \cdot z^* + b}{c \cdot z^* + d} \tag{5.2.2}$$

This can be verified right away, by simple calculation. But then, the Riemannian geometries of the two surfaces of negative curvature involved in the Kasner metric (5.1.7) should be just naturally connected due to this very condition. Let us first describe the basics of a generic geometrical structure of the metric (5.2.1).

In view of the results of §4.5, we shall discuss this geometry based on an absolute given by a one-sheeted hyperboloid. There are some incentives to do it this way, that will be explained as we go on with the closing of our presentation. The first among these, is that this universe can be discussed directly in terms of matrices, in a manner analogous with the discussion of the Yang-Mills fields (see §4.4 above) or, better yet, with the very idea of fundamental analogy of the two relativities (see §2.5 above). Secondly, such an absolute is topologically equivalent to a torus, which is the natural shape of the canal surface representing a finite electron in its classical journey around nucleus. And last, but by no means the least of these incentives, is the fact that we have a natural geometrical connection between charges this way, as explained right above. As we see it, this connection validates the ideas of Riemann and Betti (see §1.5), thus providing a natural definition for an Ampère element, along the lines of the 'Planck's dipole as a fundamental structure of a universe'. In other words, an Ampère element is *the fundamental quantum structure of the world*, equivalent, at the infrafinite scale of the world, with a *Wien-Lummer enclosure*, or with an *Einstein elevator* necessary in describing the gravitation.

Start, indeed, with the equation (4.3.10) representing a 2×2 matrix, assumed to have the homographic action characterized by the complex numbers $z$ and $z^*$ as fixed points, and by a specific cross-ratio $k$ which, in this case, must be necessarily a complex number having unit modulus, as we have already shown in §4.5 on the occasion of an amendment of the Boltyanskii's theory of anisotropic relativity. Thus, in these conditions the equation (4.3.10), assumed to determine a general matrix which realizes the homographic action from equation (4.3.15) can be transcribed as

$$\frac{\alpha}{z - z^* \cdot k} = \frac{\beta}{(k-1)z \cdot z^*} = \frac{\gamma}{1-k} = \frac{\delta}{z \cdot k - z^*} \tag{5.2.3}$$

The resulting matrix can be considered real, insofar as its entries are defined up to a factor that can be chosen conveniently, so as to make them real indeed. However, using, for now, the entries (5.2.3) of our matrix as they are, we shall try to build a differential geometry in the absolute style, as given in the §3.4 above. To this end, we will be using as absolute the geometrical representation of the singular matrices, *i.e.* a one-sheeted hyperboloid, as we did in the §4.3. Considering the homographic action of the matrix with the entries given in equation (5.2.3), the coframe differential 1-forms, written according to the recipe from equation (4.3.18) can be written in terms of the variables $z$, $z^*$ and $k$. A direct, but a little tedious calculation gives these differential forms as:

$$\omega^1 = \frac{1}{z - z^*}\frac{dk}{k} + \frac{1-k}{k}\frac{dz - kdz^*}{(z-z^*)^2}, \quad \omega^2 = -\frac{z+z^*}{z-z^*}\frac{dk}{k} + 2\frac{k-1}{k}\frac{z^*dz - kzdz^*}{(z-z^*)^2},$$
$$\omega^3 = \frac{z \cdot z^*}{z-z^*}\frac{dk}{k} + \frac{1-k}{k}\frac{z^{*2}dz - kz^2dz^*}{(z-z^*)^2} \tag{5.2.4}$$

This coframe satisfies the Maurer-Cartan equations (1.4.16) with the structure constants given by (1.4.17), which are characteristic to an **sl**(2,R) coframe. This structure is preserved with a change in the action it describes: it is, in fact, a simply transitive action, as we shall show right away.

It is now the appropriate time, we think, to sketch, in this case, the calculations we have promised in the §4.2, regarding the Killing vectors of the Beltrami-Poincaré metrics. Namely, the differential forms (5.2.4) define some basic operators *via* the momentum 1-forms corresponding to the Lagrangian given by the Killing-Cartan metric:

$$(ds)^2 \overset{def}{=} \omega^1 \cdot \omega^3 - (\omega^2/2)^2 = \left(\frac{dk}{2k}\right)^2 + \frac{(k-1)^2}{k}\frac{dz \cdot dz^*}{(z-z^*)^2} \tag{5.2.5}$$

This Lagrangian provides the following momentum 1-forms:

$$p_k = \frac{dk}{2k^2}, \quad p_z = \frac{(k-1)^2}{k}\frac{dz^*}{(z-z^*)^2}, \quad p_{z^*} = \frac{(k-1)^2}{k}\frac{dz}{(z-z^*)^2} \tag{5.2.6}$$

These are three independent 1-forms, helping us in finding *the frame* of **sl**(2,R) algebra, that is the infinitesimal generators of the group. All we have to do, is to write the coframe (5.2.4) linearly in the components of the momentum, and then replace these very components by the partial derivatives with respect to the corresponding variable serving as lower index to the component of momentum. The final result of this procedure gives the following vectors:

$$B_1 = 2\frac{1}{z-z^*}k\frac{\partial}{\partial k} + \frac{k}{k-1}\frac{\partial}{\partial z} - \frac{1}{k-1}\frac{\partial}{\partial z^*}, \quad B_2 = \frac{z+z^*}{z-z^*}k\frac{\partial}{\partial k} + \frac{k}{k-1}z\frac{\partial}{\partial z} - \frac{1}{k-1}z^*\frac{\partial}{\partial z^*},$$
$$B_3 = 2\frac{z\cdot z^*}{z-z^*}k\frac{\partial}{\partial k} + \frac{k}{k-1}z^2\frac{\partial}{\partial z} - \frac{1}{k-1}z^{*2}\frac{\partial}{\partial z^*} \tag{5.2.7}$$

These vectors satisfy what we take as the standard structure relations of the **sl**(2,R) algebra:

$$[B_1, B_2] = B_1, \quad [B_2, B_3] = B_3, \quad [B_3, B_1] = -2B_2 \tag{5.2.8}$$

corresponding to the structure constants from equation (1.4.17).

Now, let us apply this procedure for the involution ***I*** from equation (2.5.1), whereby $k = -1$, – like for any involution in fact – so that the coframe (5.2.4) is

$$\omega^1 = -2\frac{dz + dz^*}{(z-z^*)^2}, \quad \omega^2 = 2\frac{z^*dz + zdz^*}{(z-z^*)^2}, \quad \omega^3 = -2\frac{z^{*2}dz + z^2dz^*}{(z-z^*)^2} \tag{5.2.9}$$

which coincide with the coframe from equation (2.5.7), up to a sign. Correspondingly, the frame (5.2.7) is, up to an arbitrary factor:

$$B_1 = \frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}, \quad B_2 = z\frac{\partial}{\partial z} + z^*\frac{\partial}{\partial z^*}, \quad B_3 = z^2\frac{\partial}{\partial z} + z^{*2}\frac{\partial}{\partial z^*} \tag{5.2.10}$$

We can thus conclude that, each of the two negative curvature surfaces of Kasner – the blades – involved in the metric (5.1.7), is characterized by a set of three vectors like these, satisfying the commutation relations (5.2.8). Assume $y$ and $y^*$, the complex variable describing the companion negative curvature 'blade' of the one described by $z$ and $z^*$, as in §4.4. Then the equivalent frame (5.2.10) will be written as:

$$C_1 = \frac{\partial}{\partial y} + \frac{\partial}{\partial y^*}, \quad C_2 = y\frac{\partial}{\partial y} + y^*\frac{\partial}{\partial y^*}, \quad C_3 = y^2\frac{\partial}{\partial y} + y^{*2}\frac{\partial}{\partial y^*} \tag{5.2.11}$$

The two surfaces are, obviously, isomorphous. The variations on one of them, as described by the corresponding operators (5.2.10) or (5.2.11), must satisfy some connection corresponding to this isomorphism.

Such a connection is described, in finite terms, by one or more of what we call *joint invariants* of the two surfaces, according to a theorem due to Marius Stoka – *may he rest in peace*! – [see (Stoka, 1968), Chapter II; see also (Leuci & Pastore, 1994), and (Mazilu, 2006)]. With reference to the vectors (5.2.10) and (5.2.11), this theorem can be formulated as follows: the joint invariants of the actions generated by the two frames ***B*** and ***C*** are solutions of the system of partial differential *equations of Stoka*:

$$(B_k + C_k)f(y, y^*; z, z^*) = 0, \quad k = 1,2,3 \tag{5.2.12}$$

We take this result somehow out of its usual geometrical meaning, and here we have the best occasion to illustrate what we mean by this. In geometry, the solution of this system of partial differential equations would mean families of curves in the complex plane $y$, depending on two parameters $z$, or vice versa, of course. In a physical context of the kind we have here, the function $f(y, y^*; z, z^*)$ would mean the functional form of the constraints to which the correspondence between the two Kasner surfaces having the metrics involved in the background field (5.1.7), must be submitted. While the geometrical context may suggest an incidental subordination – $z$ are the parameters and $y$ are the variables, or vice versa – the physical context suggest an 'equivalence', since the group action is what prevails here, not the geometrical shape.

As a case in point, we have the solution of the system (5.2.12), with (5.2.10) and (5.2.11) for the vectors ***B*** and ***C***, which are functions of the single algebraic form:

$$\frac{(y - z^*)(y^* - z)}{(y - y^*)(z - z^*)} \tag{5.2.13}$$

This is one of the six possible values of the cross-ratio of the four complex numbers. When this cross-ratio is constant, we have a matrix with entries as in equation (5.2.3) acting on $(y, y^*)$, as in equation (5.2.2), or vice versa: a matrix depending on $(y, y^*)$, acting on the variables $(z, z^*)$. In real terms expressed in the blade variables entering the Kasner's cosmological metric equation (5.1.7): $y = x_1 + ix_2$, $z = x_3 + ix_4$, the quantity (5.2.13) is a ratio of two quadratic forms:

$$\frac{x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2(x_2x_4 - x_1x_3)}{4x_2x_4} \equiv \frac{1}{2} + \frac{x_1^2 + x_2^2 + x_3^2 + x_4^2 - 2x_1x_3}{4x_2x_4} \tag{5.2.14}$$

which may be taken as the mathematical condition of the definition of static Yang-Mills fields (Wu & Yang, 1969). Theoretical physics knows, indeed, of different functional forms of such a condition [see §4.4 above; see also (Marciano & Pagels, 1976) for a purely quadratic case, and (Uy, 1976) for a purely cubic case of geometrical varieties to be used in the theoretical physics of the Yang-Mills static fields]. Obviously, in the case of purely geometrical purposes, it is sufficient to report the last ratio of equation (5.2.14) as a joint invariant, for then the first is automatically invariant [see (Stoka, 1968), p. 54]. However, from a physical point of view, this omission may hide a possible phase difference, for instance in the cases where the two complex numbers represent charges (see §4.5 above).

All we can say regarding the relation between physics and geometry, based on Stoka theorem applied to Yang's geometry of fields, is that the expression (5.2.14) must be *homogeneous of zero degree*. If the charges are involved here, then such an expression should be taken as representing the coefficients in the Ampère generalization of Newtonian central forces [see §1.5, equation (1.5.4)]. If we consider the situation of the Yang-Mills fields, as described by the two complex variables ($y$, $z$), according to Ernst's theory in the C. N. Yang's take (see §4.4), then the logarithm of the algebraic expression (5.2.13) is just the natural candidate for a geometrical distance between the Kasner's blades representing the de Sitter background of a universe. Then the two complex coordinates are simply charges in the Katz's acceptance of the natural philosophy of charges, and the cross-ratio (5.2.13) is the natural candidate in constructing the field intensities according to C.-N. Yang's idea (see §4.4 above).

### 5.3 The Archetype Kasner's Blades of Physics

The classical Kepler dynamical problem reaches a moment when it comes so close to special relativity that we are tempted to put the mark of identity on them. This section is all about that mark of identity. Mathematically, it can be labeled by an idea of confinement which, physically speaking, must be imposed upon velocities. Only, these velocities need to be carefully considered, for they make, indeed, an essential difference between the classical Kepler problem and the modern Einsteinian relativity. In the Chapter 3 above, we presented the essentials of this relativity. We think opportune presenting again, even if only in broad strokes, the dynamical Kepler problem, from the perspective of the planetary model described, say, more phenomenologically, in §5.1 above.

To start with, the equation of motion of the Kepler dynamical problem, and its solutions in polar coordinates ($r$,$\phi$) of the plane of motion can be written as:

$$\ddot{\boldsymbol{r}}+\frac{\kappa}{r^2}\hat{\boldsymbol{r}}=\boldsymbol{0} \quad \therefore \quad \frac{\dot{a}}{r}=\frac{\kappa}{\dot{a}}+v_1\cos\phi+v_2\sin\phi \tag{5.3.1}$$

The physical constant $\kappa$ includes in its algebraical structure both the physical properties of the two interacting bodies generating the Newtonian forces, as well as the inertial properties of the moving body. From a purely physical point of view, the last of equations (5.3.1) is, in fact, an *equation for velocities*. To wit, it connects the current velocity ($\dot{a}/r$), where $\dot{a}$ is the area constant of the Kepler's second law, to the initial velocity $\boldsymbol{v}$ of the orbit. This connection is accomplished by a translation of magnitude ($\kappa/\dot{a}$), having a physical origin, and a rotation of an angle given by the orientation of the actual position vector $\boldsymbol{r}$ of the mobile material point along its orbit. This material point can hereby be identified through its *initial velocity* ($v_1$, $v_2$), chosen to describe that orbit. For, obviously, one can see from equation (5.3.1) that a complete Kepler orbit of a material point is a trajectory that can be *labeled univocally* by its initial velocity: there is no other orbit corresponding to this initial velocity, if the physical conditions reamin the same. And, going further along this way of reasoning, a complete Kepler orbit corresponds, univocally, to a given material point, even in a 'precise' sense, if we may say so: given no further specifications, the material point having the initial velocity ($v_1$, $v_2$), can be found anywhere on the ellipse represented by the second of the equations (5.3.1), *equally likely*. This means that the parameter $\phi$ can be taken as a statistical variable, which was, indeed, the case, historically speaking, revealed by the precepts of the wave mechanics.

On the other hand, the historical development of the human knowledge contradicts the idea of closed orbits on two accounts. First, it revealed that the real orbits are never a closed curves, *viz.* ellipses, as Kepler inferred for the first time from the analysis of the data of Tycho Brahe referring to planet Mars: in reality, there is always a perihelion advance of the orbit, to say the least, and this reality came to be described by the general relativity. In a word, the Kepler ellipse is one of those figments of our imagination, helping us in settling on the problem of interpretation and its solution. As well known, the first interpretation ever in this case was given by Newton in the form of classical dynamics, whose remarkable product is the very equation (5.3.1). On the other hand, the natural philosophy of the beginning of the last century, slowly slipped with that 'equally likely' towards taking it as 'equal likelihood', thus transforming the conclusion into a 'measurable' one, according to the theory of probability. It is in this form that the wave mechanics came to contradict it: Schrödinger found that the material point in revolution can be found 'simultaneously in all kinematically possible positions on the trajectory, but not equally likely'. As known, he even provided a measure of the likelihood: the square modulus of the wave function, and this is how the angle $\phi$ started being a statistical variable.

Now, from a purely mathematical point of view, Kepler's inference is only valid if the initial velocity, having the components ($v_1$, $v_2$) in the plane of motion, is limited in its Euclidean magnitude by the physical quantity ($\kappa/\dot{a}$), which compels us to take this last physical quantity as a *limit velocity*. Indeed, the equation (5.3.1) gives the trajectory of the dynamical problem describing the Kepler motion in the field of Newtonian forces – *i.e.* central forces having the magnitude *inversely proportional to the square* of the mobile position vector with respect to the center of force – as a general *conic section* described in polar coordinates with respect to the center of force. Such a conic section happens to be an ellipse only in the cases where its label – *i.e.* the initial velocity of the material point in motion – satisfies the inequality:

$$\left(\frac{\kappa}{\dot{a}}\right)^2 - v_1^2 - v_2^2 \geq 0 \tag{5.3.2}$$

and only in those cases. Classically, this is a well-known condition of *limitation* of the velocities playing the part of labels for Kepler trajectories: it thus describes a particular ensemble of particles, out of all possible Hertz material particles from the ensemble giving the interpretation to the Hertz material point we call electron. On the other hand, it describes the eccentricity of the orbit, which is actually all we can have about it in reality. In other words, the vector of initial velocity is a *fictitious vector* that we can only be inferred from the shape and size of the orbit, which are the only parameters we can *effectively measure*. In this case the condition (5.3.2) can be taken as measuring the extension of the space accessible to the center of orbit with respect to the center of force, and this is an estimation of the space occupied by the center of force.

Indeed, speaking of the planetary atom, as described in §5.1, if the electron has a closed orbit – assuming the condition of 'closed' an ideal one, in the sense that the shape of the electron is the same after running through the orbit – then, from a classical point of view, all of the constituent material particles from the interpretative ensemble of a ‘nucleus’ of that *revolving electron* should have closed orbits in the interior of a toroid in space. This toroid can be described as a solid figure delimited by a *canal surface* generated by the shape of that nucleus. The ideal condition also assumes that the Hertz material particles of the electron’s ‘nucleus’, will have their own trajectories all along the Kepler orbit of the electron. This is a genuine *condition of confinement* of the very material particles from the interpretative ensemble of the nucleus of a material point in general: they belong to a space confined by a well-defined geometrical manifold. It is worth now, for us, describing a family of Kepler orbits, of an ensemble of material particles serving for the interpretation of the structure of electron, from this very point of view.

According to equation (5.3.1) an ensemble of material particles is described in the vicinity of a center of force by an equation of the form:

$$\frac{\dot{a}}{r}=\frac{\kappa}{\dot{a}}+\frac{\dot{a}}{r_0}\cos(\phi-\phi_0) \tag{5.3.3}$$

The particle takes this orbit at the point of coordinates ($r_0,\phi_0$) with respect to the center of force, in a plane undecided with respect to its orientation: again, such orientation is ‘equally likely’, and the physics of the last century came to deny the statement on a few accounts! But let us take the things in a reasonable direction, just in orde to make our point. Fact is that regarding the orientation of the plane of orbit, we do not know too much, given just the initial conditions as above. To wit: we only know that the plane normal would have to be given by the direction of the vector product $\dot{\boldsymbol{a}}=\boldsymbol{r}\times d\boldsymbol{r}$, between the position of the revolving point, and its infrafinite variation – the fundamental displacement, as they say – and thus, unfortunately, it has to remain undecided from a metrical point of view, just as undecided as the differential $d\boldsymbol{r}$ appears to be. However, the equation of orbit, taken in the form (5.3.3), does not contain but the magnitude of this vector product – not at all its direction – as given by the second of Kepler’s laws, so that an initial position vector can be assigned to the trajectory, that can be calculated from the initial velocity:

$$\begin{aligned} v_1&=\frac{\dot{a}}{r_0}\cos\phi_0 & & \phi_0=\tan^{-1}\left(\frac{v_2}{v_1}\right)\\ & & \therefore & \\ v_2&=\frac{\dot{a}}{r_0}\sin\phi_0 & & v_1^2+v_2^2=\left(\frac{\dot{a}}{r_0}\right)^2 \end{aligned} \tag{5.3.4}$$

Now, if in the Cartesian coordinates with respect to the center of force we calculate the coordinates of the center of the orbit itself, they are also determined by the initial conditions, according to equations:

$$\xi_c=-\dot{a}\frac{v_1}{\Delta},\quad \eta_c=-\dot{a}\frac{v_2}{\Delta};\quad \Delta\equiv\frac{\kappa^2}{\dot{a}^2}-v_1^2-v_2^2 \tag{5.3.5}$$

The equation of the orbit with respect to this center, can be simply obtained by the translation

$$x=\xi-\xi_c,\quad y=\eta-\eta_c$$

which results in the Cartesian equation of the orbit:

$$\left(\frac{\kappa^2}{\dot{a}^2}-v_1^2\right)x^2-2v_1v_2xy+\left(\frac{\kappa^2}{\dot{a}^2}-v_2^2\right)y^2=\frac{\kappa^2}{\Delta} \tag{5.3.6}$$

The semiaxes of orbit are then given by the eigenvalues of the inverse of the matrix of this quadratic form:

$$a^2 = \frac{\dot{a}^4}{\kappa^2 \lambda^2}, \quad b^2 = \frac{\dot{a}^4}{\kappa^2}; \quad \lambda^2 \equiv 1 - \frac{\kappa^2}{\dot{a}^2}(v_1^2 + v_2^2) \tag{5.3.7}$$

so that the eccentricity of the conic (5.3.6), assumed an ellipse, is given by

$$e^2 \overset{def}{=} \frac{a^2 - b^2}{a^2} = 1 - \lambda^2 \equiv \frac{\kappa^2}{\dot{a}^2}(v_1^2 + v_2^2) \tag{5.3.8}$$

Therefore, in the eccentricity of the current orbit of the Kepler motion, we can recognize a past of the particle moving along that orbit, according to the laws of the classical dynamics. However, what we have in mind with this past is a more profound meaning than these mathematical displays, which can be exhibited by going back in history, to Newton himself.

Indeed, the choice of the initial velocities determining the Kepler orbit – only regarding its shape and orientation, in fact – can be related to that *fictitious event*, whereby a 'just transverse impulse is infused', in Newton's imaginary scenario involving a particle that falls towards a center of force. Newton ascribed that transverse impulse to 'an intelligent Agent', as shown in the excerpt from his letter to Bishop Bentley, given by us in the §2.2. We, on the other hand, can see in this initial condition an opportunity to give a physical identity to that 'intelligent Agent': it is given by the static charges assigned to particles (see §2.2). To be more precise, the Newtonian scenario can be 'improved', as it were, with the proviso that in the fall of particle towards a center of force, *charges are suddenly created* on the 'particle's own orb', that rotate the static fields generated by its own existing charges, thereby generating *a current around the center of force*. Thus, the 'intelligent Agent' is not to be identified with God, as sometimes proclaimed, based, in fact, on the very Newton's words, *but to a physical law*. Like anywhere else in our life on Earth, for that matter, we have to recognize that God does not control the events in the world, as everybody seems to believe in the human society of all times: He just prescribed the law… according to which this world must work! The events *happen* according to *the law*: this statement should not be taken as a paradox. Every term of it has logical and scientific consistency by law!

However, for the moment, let us make a 'Newtonian connection', as it were, with another concept of our knowledge: the special relativity. Notice that in order *to generate a closed orbit*, the 'just transverse impulse' of Newton must imprint to the falling particle, at 'its own orb', a tangent velocity of magnitude lower than $\dot{a}/\kappa$. This fact gives us the possibility to describe the space measured through the eccentricity of the orbit around the center of force in the dynamical Kepler problem, by a Cayley-Klein geometry in two dimensions, as in §3.4. Theoretical physics knows nothing of this kind; however, it knows instead of a Lobachevsky space of velocities in the special relativity (Fock, 1959). Fact is that the Lobachevsky geometry can be presented as a Cayley-Klein geometry whose absolute is given by the spherical wave surface of light. For consistency, let us present the case along the lines of the present work.

If we denote by $X$ a point in this space of velocities, then a coordinate representation is given by a quadruple of numbers as in equation (3.4.17), with the numbers $x, y, z$ purely imaginary, making the component of the vector $i{\cdot}\boldsymbol{v}$, while the component $t \equiv c$ – the speed assigned to light:

$$X \equiv (c, i \cdot \boldsymbol{v})$$

The Einsteinian norm (3.4.1′) of these points is given by the quadratic form:

$$(X, X) \equiv c^2 - \boldsymbol{v}^2 \tag{5.3.9}$$

Thus, the points which satisfy the condition $(X, X) = 0$ – the light fronts – are geometrically shaping an absolute for this geometry, and thereby the universe we live in can be represented relativistically by the metric geometry of a Maxwell fish-eye (see §3.4 above). Among other things, the common contemporary concept is that, physically speaking – but with reference to our experience nevertheless – these light fronts represent the propagation of light if the limit velocity $c$ is a constant. Specifically, the points with positive norm (5.3.9) represent *inertial motions*, as usual, while the points with negative norm represent, for instance, *de Broglie waves* in the regular case of special relativity. The idea is that some other ensembles of Hertz material particles can likewise be described in the case of the Kepler motion.

The norm (5.3.9) induces an 'internal multiplication' of points by the 'polarization' procedure corresponding to the norm from equation (3.4.16). According to this prescription, we have for the Lagrange product of a point with its infrafinite counterpart, entering that equation, the expression:

$$X \wedge dX = [-\boldsymbol{v} \times d\boldsymbol{v}, i \cdot (c d\boldsymbol{v} - \boldsymbol{v} dc)]$$

This will give us the four-dimensional equivalent of the formula (3.4.13) as a sum of two contributionswhich we write here, intentionally, as:

$$(ds)^2 = \frac{c^2 (d\boldsymbol{v})^2 - (\boldsymbol{v} \times d\boldsymbol{v})^2}{(c^2 - \boldsymbol{v}^2)^2} + \frac{\boldsymbol{v}^2 (dc)^2 - 2c(\boldsymbol{v} \cdot d\boldsymbol{v})(dc)}{(c^2 - \boldsymbol{v}^2)^2} \tag{5.3.10}$$

Our intention of speculation is simple: if the first part of the metric appears as a genuine metric of the continuum of velocities, the second part can then be interpreted as a genuine *Coll universal deformation* (see §§3.4 and 3.5) Thereby we can read that, if the Coll's thesis *that the gravitation can be described by a universal deformation* is valid, then the gravitation is manifested by a variation of the electromagnetic properties of a de Sitter background continuum. Meanwhile, notice that this metric is not exactly the metric of a Lobachevsky space of velocities, which, in fact, is given only by the first part of the expression (5.3.10) [see (Fock, 1959), §§16,17; especially the equation (17.01) of this reference].

The main point of difference is that we assume here that the speed assigned to light is variable, while in the special relativity it is a constant. If we assume $c$ as a constant too, this condition comes down to $dc = 0$, and the Cayley-Klein metric (5.3.10) reduces, indeed, to Fock's metric. However, there is more to conclude from (5.3.10) in this case, if we take the Kepler dynamical problem for guidance. For, in this problem the norm (5.3.9) is replaced by the condition (5.3.2) of existence of the closed orbits, in which case the correspondent of the maximal velocity $c$ is $(\kappa/\dot{a})$ and this quantity is manifestly variable with the incidental interpretation particle carrying that property. Just tentatively speculating, this variability makes the classical dynamical Kepler problem essential in *a holographic universe*, for we have

$$c \equiv \frac{\kappa}{\dot{a}} = \frac{(\kappa / r^2)}{\dot{\phi}}, \quad \dot{a} \equiv r^2 \dot{\phi} \tag{5.3.11}$$

which means that $c$ here is an *intensity* connected with the phase represented by the angle of revolution of the particle in the dynamical Kepler problem [see definition (4.5.12) of the amplitude by coherence; compare also with the interpretative condition (4.5.18)].

Thus, we wrote the absolute metric (5.3.10) as a sum of two terms with the clear intent of accounting for the variability of $c$ in a particular way: in the framework of special relativity the first term from equation (5.3.10) is the the Fock's metric of velocity space, and it is a direct consequence of the Einstein's law of composition of relativistic velocities. In this undertake, one can say that the whole metric (5.3.10), itself, generalizes the relativistic metric of the velocity space, which can be obtained from (5.3.10), for instance in cases where the first component of the point $X$ is a constant, as we did before: the differential of a constant $c$ is always zero, and the second term in (5.3.10) vanishes. In other words, in this case *there is no infrafinite four-vector velocity in the relativistic physics* as we inherited it from Maxwell electrodynamics and used into Einsteinian special relativity: since the speed of light is a constant its differential is zero, and there is not an infinitesimal fourth component of acceleration, to be expressed as the variation of the corresponding component of the velocity. It is known, indeed, that in special relativity the four-vector acceleration needs to be constructed in a special way, involving the force and the proper time of the motion.

However, as equation (5.3.2) suggests, a relativistic-like description can be obtained by identifying the vector $\boldsymbol{v}$ with the *initial condition of motion*, and then taking $c \equiv \kappa/\dot{\alpha}$. Every closed Kepler orbit is thus described by some initial velocity having a magnitude smaller than this particular value. The metric (5.3.10) could therefore be applied for describing the structure of *matter contained in the nucleus of revolving body of the planetary model* of physics, but there is a drawback. Indeed, in this interpretation, the matter of the revolving material point can be taken to be a 'swarm' of Hertz material particles, as once described by Joseph Larmor in a very suggestive work (Larmor, 1900). Obviously, as we noticed above, the value of $c$ is here *variable with each particle* of the Hertz material point in revolution. And, as each particle is uniquely labelled by its initial conditions, there should exist a correlation between the limit velocity $c$ and the *label of the particle*. The nature of such a correlation can be inferred directly from (5.3.10), because this formula discloses still another condition in which the metric reduces to the usual 'relativistic' one, more general than that of maintaining $c$ a constant. This condition is given by the differential equation

$$vdc - 2cdv = 0 \quad \therefore \quad v^2 / c = const \tag{5.3.12}$$

which guarantees the vanishing of the second term in (5.3.10). In other words, in this generalization of special relativity, *there are*, *actually*, *two cases in which the metric becomes purely relativistic* in the sense of Fock.

One of these is, of course, the usual case of constancy of the value of limit velocity, which precludes the infrafinite differential measure for the time component in the realm of velocities taken as four-vectors: there are only regular finite four-vectors and infrafinite three-vectors in this case. On the other hand, there is also a case of differential measure in this realm of the four-velocities, given by equation (5.3.12), and involving the connection between the time component and space components of the four-vectors. These last velocities are 'reciprocal' with respect to the 'sphere' given by the initial conditions of the Kepler problem. Such a reciprocity is akin to a 'de Broglie duality', as it were, between the phase and wave-group velocities. However, this time *it is the inertial matter which imposes it upon light*. In general, therefore, one can assume that $c$ is not a constant, regardless of this last situation, and describe a universal… special relativity starting from a 'double' relativistic metric, to be obtained from (5.3.10) in two special cases: one for considering *the light in ether*, the other considering *the matter in ether*.

For the first case we cannot decide anything yet, but the last case can always be described, indeed, by two absolute metrics corresponding to two limit velocities. In order to get the gist of the method to accomplish such a task, let us assume the case of constant $c$: the idea here is that only after analyzing this case closely, we can properly improve on it. The absolute metric (5.3.10) turns out to be:

$$(ds)^2 = \left[1-\left(\frac{\boldsymbol{v}}{c}\right)^2\right]^{-2} \cdot \left[\left(\frac{d\boldsymbol{v}}{c}\right)^2 - \left(\frac{\boldsymbol{v}}{c}\times\frac{d\boldsymbol{v}}{c}\right)^2\right]$$

In the three-dimensional velocity space of relativity, this metric can be written in the form

$$(ds)^2 = \left(\frac{d\boldsymbol{q}}{1-\boldsymbol{q}^2}\right)^2 - \left(\frac{\boldsymbol{q}\times d\boldsymbol{q}}{1-\boldsymbol{q}^2}\right)^2; \quad \boldsymbol{q} \equiv \frac{\boldsymbol{v}}{c} \tag{5.3.13}$$

while in the two-dimensional case of the Kepler motion it can be written in the form

$$(ds)^2 = \left(\frac{d\boldsymbol{e}}{1-\boldsymbol{e}^2}\right)^2 - \left(\frac{\boldsymbol{e}\times d\boldsymbol{e}}{1-\boldsymbol{e}^2}\right)^2; \quad \boldsymbol{e} \equiv \frac{\boldsymbol{v}}{c}, \quad c \equiv \frac{\kappa}{\dot{a}} \tag{5.3.14}$$

where, this time, $\boldsymbol{e}$ suggests the *eccentricity vector* of the orbit. There is no formal difference between the two formulas, (5.3.13) and (5.3.14), except the dimension of the velocity space and the fact that this last one of this relations is controlled by gravitation according to Newtonian precepts. Keeping this last condition in reserve for later considerations, the equation (5.3.14) can be obtained from (5.3.13) just by choosing to work in one of the planes of coordinates of the velocity three-dimensional space. Perhaps it is worth mentioning again that in describing the gravitational field according to Einsteinian natural philosophy, we simply cannot consider the Newtonian case as just particular: it turns out that it is manifestly involved in the construction of the fields, at least by the boundary conditions (see §3.2).

In order to settle our ideas, we choose to work on the formula (5.3.14), in the Cartesian components of the eccentricity vector, for then it is more transparent how the space extension of matter can enter our reasoning. Indeed, the eccentricity vector $\boldsymbol{e}$, represents the relative position of the center of orbit of the dynamical Kepler problem, with respect to the center of force [see equation (5.3.5) above]. Therefore it can be taken as a rough measure of the limited *space extension of the matter generating the force field* in a planetary model. One can surely assume that the geometry of the space containing the matter which generates the force field in the classical Kepler problem is not an Euclidean geometry, but a Lobachevsky geometry and, optically speaking, a Maxwell fish-eye medium. In other words, we have a geometry of the *space containing the center of forces*: it is the hyperbolic geometry. To make this statement even more understandable, notice that the Cayley-Klein metric (5.3.14) can be cast into the Beltrami-Poincaré form

$$(ds)^2 = -4\frac{dh\cdot dh^*}{(h-h^*)^2} \equiv \frac{(du)^2+(dv)^2}{v^2} \tag{5.3.15}$$

by the transformation

$$h \equiv u+iv = \frac{e_2 + i\sqrt{1-e^2}}{1-e_1}; \quad h^* \equiv u-iv \tag{5.3.16}$$

and thereby get the general relativity by a *harmonic mapping*, *via* Ernst prescription (see §4.1 above).

This will make an understandable connection between special relativity and general relativity right within the classical framework: no assumptions of finiteness of the velocity of fields, no electrodynamics in describing the foundations. To wit: in view of the similitude between the metrics (5.3.13) and (5.3.14), the first one of these can be cast in the form (5.3.15), just like the second one, and this opens the possibility of some mathematical-philosophical sound speculations. For once, one can assume that, in a planetary model, the equation (5.3.13) is referring to the revolving spatially extended electron, by describing the particles of its interpretative ensemble. Likewise, (5.3.14) is referring, as we said, to the interpretative ensemble of the nucleus proper of the planetary model, and therefore to the Einsteinian cosmological boundary conditions for the metric tensor. These structures being geometrically isomorphous, can be taken as validating the Kasner approach to Einsteinian cosmology (see §5.1), so that the theory from the §5.2 is applicable. The bottom line: we can have a conclusion on the existence of a certain connection between charges and speeds, contained in the cross-ratio (5.2.13), and transcending the usual apparent invariance of the charges of our experience. This idea will be developed *in extenso* with another occasion, since it needs a more comprehensive discussion, for which there is no room on this occasion. For now, though, let us give a comprehensive geometrical description of the Kasner's correspondence between the two blades of the metric (5.1.7), just in order to see where it leads us from the perspective of the Ernst principle of general relativity.

### 5.4 The Kasner's Blade as a Negative Curvature Surface

There is, again, an element of arbitrariness left with the above representation of the physics of planetary model, leading to the necessity of a *reverse interpretation* (Mazilu, Agop, & Mercheș, 2021). This concept would mean simply an Einsteinian representation of 'the discrete' by 'continuum' [(Einstein, 1917a); see introduction to our Chapter 3 above], just as the interpretation proper means representation of 'continuum' by 'discrete'. It is necessary, primarily as a concept we should say – we do not have a concept for the matter density, other than the Newtonian one, of course – inasmuch as in the connection between space and matter one cannot escape the definition of Einstein for the density of matter. This means that the density is effectively dependent of the space scale of our perception of matter. In hindsight, we can even add to this physical attribute of density another important one: for the same *space scale*, the density of matter depends on the *time scale* of its observation. Anyway, this reverse interpretation is, in our context, particularly simply made possible by the Beltrami-Poincaré metrization of the hyperbolic plane. Indeed, the metric (5.3.15) is conformal to an Euclidean metric, and this last one is known to be invariant with respect to rotations in plane. At the risk of repeating well-known things, we need to point out in some detail how this arbitrariness shows up in the hyperbolic geometry, just for the sake of completeness if for nothing else. Let us state here that for what follows we have found the necessary inspiration in the works of Alexander Petrovich Shirokov – *may he rest in peace*! – and his collaborators, from among which, two are of special interest: (Shirokov, 1988), (Perelomova & Shirokov, 1990).

It turns out that in the $(u,v)$ coordinates, as defined in equation (5.3.16), this geometry exhibits a special invariance connected with what is known as the *Bäcklund transformation* fore-cited in §5.2 above. These properties are connected with the Euclidean properties of the metric (5.3.15), assumed *a priori* valid in this geometry. Following a known routine, we take a point $P$ in the hyperbolic plane as being described, like in the ordinary geometry of surfaces, by the coordinates $(u,v)$, so that its infrafinite vector variation can be written in the form [(Flanders, 1989), p. 134]

$$\boldsymbol{d}P \overset{def}{=} \begin{pmatrix} du \\ dv \end{pmatrix} = \sigma^1 \hat{\boldsymbol{e}}_1 + \sigma^2 \hat{\boldsymbol{e}}_2, \quad \hat{\boldsymbol{e}}_1 \overset{def}{=} \begin{pmatrix} v \\ 0 \end{pmatrix}, \quad \hat{\boldsymbol{e}}_2 \overset{def}{=} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{5.4.1}$$

This formula makes the fact obvious, that the hyperbolic plane is conformal to the Euclidean one, indeed, but it needs a little explanation. Notice that we denoted here the base of vectors by ($\hat{\boldsymbol{e}}_1$,$\hat{\boldsymbol{e}}_2$), indicating an *orthonormal* reference frame according to our Euclidean notation conventions, *i.e*. a frame satisfying the relations:

$$\hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2 = 0, \quad \hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_1 = 1, \quad \hat{\boldsymbol{e}}_2 \cdot \hat{\boldsymbol{e}}_2 = 1 \tag{5.4.2}$$

Obviously, the dot-product here cannot be Euclidean: manifestly orthogonal, the two vectors cannot be Euclidean *unit* vectors. However, they are *hyperbolic unit vectors*, *i.e*. unit vectors in the Lobachevsky plane, equipped with the metric (5.3.15), whereby the dot-product involves the metric tensor. That is, in general, for two arbitrary vectors, ***U*** and ***V*** say, belonging to the space equipped with a *metric* generated by the tensor ***m***, for the dot-product, or the internal multiplication, as they sometimes call it, we have:

$$\boldsymbol{U} \cdot \boldsymbol{V} \overset{def}{=} \boldsymbol{m}(\boldsymbol{U},\boldsymbol{V}), \quad \boldsymbol{m}(\boldsymbol{U},\boldsymbol{V}) \equiv m_{\alpha\beta} U^\alpha U^\beta \tag{5.4.3}$$

where ***1*** is the identity matrix as usual, and $\boldsymbol{m} \equiv (1/v^2)\cdot\boldsymbol{1}$ is the metric tensor from (5.3.15). In the right hand side of the second of these equalities, the Greek indices run over the values 1 and 2, again, according to our usual convention, and the summation over dummy indices is understood. The coframe from equation (5.4.1) is given by the two differential forms

$$\sigma^1 \overset{def}{=} \frac{du}{v}, \quad \sigma^2 \overset{def}{=} \frac{dv}{v} \tag{5.4.4}$$

The Beltrami-Poincaré metric (5.3.15) is, obviously, the sum of their squares, as in the Euclidean case.

Now, we have to insist on the Frenet-Serret formulas for the previous reference frame, in order to have a differential-geometric theory as in §1.3. Notice, to this end, that by a direct calculation, we get

$$\begin{pmatrix} d\hat{\boldsymbol{e}}_1 \\ d\hat{\boldsymbol{e}}_2 \end{pmatrix} = (\frac{dv}{v}) \begin{pmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \end{pmatrix} \quad \therefore \quad |d\hat{\boldsymbol{e}}\rangle - (\frac{dv}{v})|\hat{\boldsymbol{e}}\rangle = |0\rangle \tag{5.4.5}$$

with an obvious 'ket' notation for the column matrices. As the metric tensor is not constant, the Frenet-Serret formulas are not given by a purely skew-symmetric matrix as in the genuine Euclidean case, and this complicates a little the process of obtaining those formulas, which thus becomes more involved.

Let us find the Frenet-Serret formulas of an evolution that leaves the metric unchanged, in general. Assuming the usual rules of differentiation when it comes to the variation of the quadratic forms, a variation of the quadratic metric can be formally expressed as:

$$d\langle du|\boldsymbol{m}|du\rangle = 0 \quad \therefore \quad \langle du|\boldsymbol{g}^t\boldsymbol{m} + \boldsymbol{m}\boldsymbol{g} + d\boldsymbol{m}|du\rangle = 0 \tag{5.4.6}$$

where ***m*** is the metric tensor, $\langle u| \equiv (u, v)$ in a Dirac notation, and a *gauging* ***g*** is assumed, to the effect that the differentials of coordinates on surface are solutions of the equation:

$$d|du\rangle = \boldsymbol{g}|du\rangle \tag{5.4.7}$$

Thus, the gauging condition transfers the second differentials of coordinates onto some first differentials representing the entries of a gauging matrix. What makes a gauging condition out of the second equality (5.4.7) is the fact that it must be *unconditionally valid*, *i.e.* valid independently of $(u,v)$. This means that the equation (5.4.7) *is not* an equation of evolution: it is valid for the whole portion of surface coordinated by $(u,v)$. Then, for that portion of surface, the matrix from the second equality in (5.4.6) must be a matrix with null entries, for it is *a priori* a symmetric matrix:

$$\boldsymbol{g}^t\boldsymbol{m} + \boldsymbol{m}\boldsymbol{g} + d\boldsymbol{m} = \boldsymbol{0} \tag{5.4.8}$$

In this case one can easily figure out that the non-symmetric matrix $\boldsymbol{mg}$ entering this expression, should be given by an equation like the following:

$$\boldsymbol{m}\boldsymbol{g} = -\frac{1}{2}d\boldsymbol{m} - \boldsymbol{a} \quad \therefore \quad \boldsymbol{g} = -\frac{1}{2}\boldsymbol{m}^{-1}d\boldsymbol{m} - \boldsymbol{m}^{-1}\boldsymbol{a} \tag{5.4.9}$$

where $\boldsymbol{a}$ is an *arbitrary* skew-symmetric matrix. It involves just *one arbitrary differential parameter* in the two-dimensional case. One can say that, in this case, the last equation in (5.4.9) defines the gauging matrix 'almost' completely, knowing the metric tensor.

Now, using (5.4.9) for a Beltrami-Poincaré metric, the gauging equation (5.4.7) becomes:

$$|d^2u\rangle - \left(\frac{dv}{v}\boldsymbol{1} + a\cdot v^2\boldsymbol{i}\right)\cdot|du\rangle = |0\rangle, \quad \boldsymbol{i} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.4.10}$$

where $a$ is the mentioned arbitrary 'differential parameter'. If this differential parameter is defined such that the second order differentials are homogeneous quadratic forms in the first order differentials, the problem simplifies. For instance, if we choose $a\cdot v^3 = du$, then the matrix equation (5.4.10) represents the system of equations of geodesics of the Beltrami-Poincaré metric, as given, for instance, in the equation (4.1.16) in this case, only written in differentials, instead of derivatives:

$$d^2u - 2\frac{dudv}{v} = 0, \quad d^2v + \frac{(du)^2 - (dv)^2}{v} = 0 \tag{5.4.11}$$

We can even go a little further along this line of reasoning, with the consideration of a differential factor in equation (5.4.10) defined up to an exact, but otherwise arbitrary differential, introduced *via* an arbitrary phase, $\phi$ say, in the mathematical picture, under the defining condition:

$$a\cdot v^2 = -d\phi \tag{5.4.12}$$

Here $\phi$, our arbitrary 'phase', defines the differential factor $a$, and the minus sign is chosen only because we have in mind a later convenience. Then the equation (5.4.10) can be written in the form:

$$D|du\rangle = -\left(d\phi + \frac{du}{v}\right)\cdot\boldsymbol{i}\cdot|du\rangle, \quad D|du\rangle \overset{def}{=} |d^2u\rangle - \left(\frac{dv}{v}\boldsymbol{1} + \frac{du}{v}\boldsymbol{i}\right)\cdot|du\rangle \tag{5.4.13}$$

whence it becomes obvious that the choice $d\phi = -du/v$, which means $D|du\rangle = |0\rangle$ represents the equations of geodesics (5.4.11) written in a matrix form, and therefore characterizes the parallel transport along the geodesics of the Lobachevsky plane. According to Dan Barbilian, this choice defines 'the angle of parallelism in the Lobachevsky's plane' (Barbilian, 1938).

In order to describe a *general* reference frame of the hyperbolic plane, satisfying the conditions (5.4.2), where the dot-product is defined *via* equation (5.4.3), we define a family of two *a priori* orthogonal unit vectors depending on the phase just introduced. These vectors play the part of a *répère mobile* in the sense of Cartan:

$$\hat{\boldsymbol{e}}_1 \stackrel{def}{=} \begin{pmatrix} v\cos\phi \\ v\sin\phi \end{pmatrix}; \quad \hat{\boldsymbol{e}}_2 \stackrel{def}{=} \begin{pmatrix} v\cos(\phi+\pi/2) \\ v\sin(\phi+\pi/2) \end{pmatrix} = \begin{pmatrix} -v\sin\phi \\ v\cos\phi \end{pmatrix} \tag{5.4.14}$$

Then a direct calculation shows that we have the following Frenet-Serret equations that generalize (5.4.5), by describing the evolution of this frame in terms of the differential parameter (5.4.12):

$$D\hat{\boldsymbol{e}}_1 = \left(d\phi + \frac{du}{v}\right)\hat{\boldsymbol{e}}_2, \quad D\hat{\boldsymbol{e}}_2 = -\left(d\phi + \frac{du}{v}\right)\hat{\boldsymbol{e}}_1 \tag{5.4.15}$$

Here the following notations are used:

$$D\hat{\boldsymbol{e}}_1 \stackrel{def}{=} d\hat{\boldsymbol{e}}_1 - (\frac{dv}{v})\hat{\boldsymbol{e}}_1 + (\frac{du}{v})\hat{\boldsymbol{e}}_2, \quad D\hat{\boldsymbol{e}}_2 \stackrel{def}{=} d\hat{\boldsymbol{e}}_2 - (\frac{du}{v})\hat{\boldsymbol{e}}_1 - (\frac{dv}{v})\hat{\boldsymbol{e}}_2 \tag{5.4.16}$$

for what appear to be some *absolute differentials* of the orthonormal Lobachevsky frame: the matrix from the right hand side of equalities (5.4.15) is skew-symmetric again, that is the relations (5.4.2) are preserved by this differentiation, with the definition (5.4.3) for the dot product.

Then one can verify that, along the geodesics (5.4.11) the coframe ($\Omega^1$, $\Omega^2$), obtained by a rotation of angle $\phi$ of the coframe ($\sigma^1$, $\sigma^2$) given in equation (5.4.4):

$$\Omega^1 = \cos\phi\cdot\sigma^1 + \sin\phi\cdot\sigma^2, \quad \Omega^2 = -\sin\phi\cdot\sigma^1 + \cos\phi\cdot\sigma^2 \tag{5.4.17}$$

has the following equations of evolution:

$$d\Omega^1 = \left(d\phi + \frac{du}{v}\right)\Omega^2, \quad d\Omega^2 = -\left(d\phi + \frac{du}{v}\right)\Omega^1 \tag{5.4.18}$$

The equations (5.4.17) and (5.4.18) formally represent what is usually known as the *Bäcklund transformation* of the negative curvature surface representing the Lobachevsky plane.

Now, this transformation allows the construction of a *simply transitive* action of the SL(2,R) type group in three variables with three parameters. The construction proceeds as follows: notice, again, that the Bäcklund rotation (5.4.17) leaves the Beltrami-Poincaré metric unchanged, and this is the *arbitrariness* we are talking about:

$$(\Omega^1)^2 + (\Omega^2)^2 = (\sigma^1)^2 + (\sigma^2)^2 \tag{5.4.19}$$

On the other hand, by a direct calculation one can verify the condition

$$\Omega^1 \wedge \Omega^2 = d \wedge (\frac{du}{v})$$

Based on this relation, one can construct the following structure equations specific to a $\mathfrak{so}(2,1)$ group algebra:

$$\Omega^0 \wedge \Omega^1 = -d \wedge \Omega^2, \quad \Omega^2 \wedge \Omega^0 = -d \wedge \Omega^1, \quad \Omega^1 \wedge \Omega^2 = d \wedge \Omega^0 \tag{5.4.20}$$

where $\Omega^0$ is the newly introduced differential form: $d\phi + du/v$, whose vanishing defines the parallel transport of the Lobachevsky plane. This algebra is a Riemannian space having the metric given by the quadratic differential form (Shirokov, 1988):

$$(\Omega^1)^2 + (\Omega^2)^2 - (D\hat{\boldsymbol{e}}_1)^2 \equiv \left(\frac{dv}{v}\right)^2 - (d\phi)^2 - 2d\phi\frac{du}{v} \tag{5.4.21}$$

where for the differential symbol we have used the definition from equation (5.4.16).

The new Lie algebra is therefore a Riemannian space of negative curvature (Vrânceanu, 1967). Such a Riemannian space is the central feature of our physics here, *as a representative continuum of the matter per se*.

This feature makes the space prone to describing the process of reverse interpretation, as we said before. Let us elaborate on this aspect of the problem, for it is destined to illuminate us on the kind of forces which are described by the Riemannian geometry, and their relationship with the manifold of events, *i.e.* with the spacetime. On the other hand, one can expect to get a glimpse of the kind of mathematics we need in order to rationalize the Einstein's procedure of defining the density on account of the concept of scale transition.

### 5.5 A Geometry of Schwarzschild's Statement

The essential note distinguishing the Einsteinian natural philosophy from the Newtonian natural philosophy is best rendered, in physics' terms, through the words of Karl Schwarzschild excerpted by us in the previous §3.3. We venture rewriting them here again, just for the sake of continuity of our discussion, as it were:

> The specification "consisting of incompressible fluid", is necessary to be added, due to the fact that in the framework of the relativistic theory, *gravitation depends on not only the quantity of the matter*, *but also on its energy* and, for instance, *a solid body having a specific state of internal stress would produce a gravitation different from that of a liquid*. [(Schwarzschild, 1916); *our Italics*]

Now, we really believe that the overall task of the present work is best served with a conclusion of it, along the message contained in this note. In a word: the Newtonian natural philosophy is simply concerned exclusively with interpretation, while the Einsteinian natural philosophy is also concerned with its consequences, contained in the notion of reverse interpretation. The general idea here, of an Einsteinian extraction, is that the transition from discrete to continuum involves the concept of field in the Yang-Mills form (see §4.4 above).

To start with, the algebra spanned by the differential forms (5.4.20) characterizes a *simply transitive* SL(2,R) action in the complex range, corresponding to the transitive action (5.2.2) of the group of real 2×2 matrices. In modern terms, this transitive action serves as a quintessential example in the construction usuall called the *Selberg trace formula* [see (Selberg, 1956), especially §4] and used fruitfully in the theoretical description of the chaos (Gutzwiller, 1984, 1985, 1990). It can be realized as a transitive group action in three variables with three parameters given by the replacements

$$z \to \frac{a \cdot z + b}{c \cdot z + d}, \quad z^* \to \frac{a \cdot z^* + b}{c \cdot z^* + d}, \quad k \to \frac{c \cdot z^* + d}{c \cdot z + d} \cdot k \tag{5.5.1}$$

where $k$ is a complex unimodular factor (Barbilian, 1938). Consequently, along our line of ideas here, this mathematics tells us that *each one of the two blades* of the Kasner's cosmological metric must be described by such an endomorphism, and this gives us the possibility of establishing a complex form of the matrix realizing the action [one can consult (Mazilu, Agop, & Mercheș, 2021), Chapter 6, for details on Barbilian mathematical theory]. The frame of infinitesimals of the action (5.5.1) differs very little from the Beltrami-Poincaré frame from equation (5.2.10), involving just one more term added to $B_3$:

$$B_1 = \frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}, \quad B_2 = z\frac{\partial}{\partial z} + z^*\frac{\partial}{\partial z^*}, \quad B_3 = z^2\frac{\partial}{\partial z} + z^{*2}\frac{\partial}{\partial z^*} + (z - z^*)k\frac{\partial}{\partial k} \tag{5.5.2}$$

but satisfies the very same structural equations (5.2.8). The differential forms (5.4.20) play here a special role: they are the real components of the *absolute invariant* coframe. Indeed, the differential 1-forms

$$\Omega = \Omega^1 + i\Omega^2, \quad \Omega^* = \Omega^1 - i\Omega^2, \quad \Omega^0 \tag{5.5.3}$$

turn out to be both *right* and *left invariants* of the action (5.5.1). In the complex variables of this group action, they can be written as

$$\Omega = \frac{dz}{(z - z^*)k}, \quad \Omega^* = -\frac{kdz^*}{z - z^*}, \quad \Omega^0 = -i\left(\frac{dk}{k} - \frac{dz + dz^*}{z - z^*}\right) \tag{5.5.4}$$

Obviously, the Kasner's $y$-blade from the §5.2 must be described in the very same way, with $(z, z^*, k)$ replaced by $(y, y^*, l)$, where $l$ is a complex variable of unit modulus. Then the cross-ratio from equation (5.2.13) is not the only joint invariant characterising the correspondence between the two Kasner blades of the cosmological de Sitter solution of Einstein's equations.

This can be seen by solving the system of Stoka equations (5.2.12) for the frame ***B*** from equation (5.5.2) and its correspondent ***C*** in $(y, y^*, l)$ variables. The solution says that any joint invariant must be an arbitrary function of the following three fundamental algebraical formations:

$$\frac{(z - z^*)(y - y^*)}{(z - y^*)(z^* - y)}, \quad \frac{z^* - y}{z - y}k, \quad \frac{z - y^*}{z - y}l \tag{5.5.5}$$

out of which the first one is the cross-ratio from equation (5.2.13). While the first of these algebraical formations would suffice in describing the Yang-Mills fields according to Ernst formalism, in considering the whole two-blade picture, it proves to be insufficient: one has to add the last two expressions (5.5.5) in constructing the connection between the two blades of an Ampère element. Everything depends here on the validity of introduction of a phase in order to describe the three-dimensional situation of a Bäcklund transformation.

The action (5.5.1) represents the invariance group of a *cubic equation* having real roots (Barbilian, 1938). Its geometric relevance for us here is that the group characterizes a Cayley-Klein geometry having a one-sheet hyperboloid as absolute. On the other hand, from a physical point of view, the variables $(z, z^*, k)$ or $(y, y^*, l)$, are liable to represent what Karl Schwarzschild designated as 'specific states of internal stress', participating in producing the 'specific gravitation'. This can be shown along the lines that follow. Given a complex quantity $z$ and a unit modulus complex number $k$, as above, one can always construct three *real* quantities:

$$r_1 = \frac{z + z^*k}{1 + k}, \quad r_2 = \frac{z + j \cdot z^*k}{1 + j \cdot k}, \quad r_3 = \frac{z + j^2 \cdot z^*k}{1 + j^2 \cdot k} \tag{5.5.6}$$

that may be taken as the roots of a cubic equation. Here $j$ is taken as the cubic root of unity. Such a cubic equation can then be considered as the characteristic equation of a 3×3 matrix, whose physical meaning can be associated with a field of stress or strain existing in a universe described by Kasner's cosmological two-blade solution. *Grosso modo*, the rational explanation of such a cosmology is the following: the matter of universe appears sparsely to our experience, inasmuch as this experience is dominated by senses and is thus scale dependent: spatial and temporal scales. As we need a continuum, according to Einsteinian natural philosophy, in order to describe it, we certainly need a reverse interpretation which brings us to the Barbilian scheme of cubic space. One can see that the complex 2×2 matrix (Barbilian, 1938):

$$\begin{pmatrix} z^*k & z \\ k & 1 \end{pmatrix} \tag{5.5.7}$$

which connects homographically the real numbers ($r_1$, $r_2$, $r_3$) the the three 'standard' numbers ($1, j, j^2$), represents, in a unique manner, the cubic having the roots (5.5.6), as transforms of the standard cubic roots of unity [see for details (Mazilu, Agop, & Mercheş, 2021); Chapter 6]. In this case, $z$ and $z^*$ are the roots of the Hessian of cubic in question. The four coefficients of the cubic can be considered as homogeneous coordinates of a point in a three-dimensional space, where any cubic having real roots determines a projective structure having the topology of a one-sheeted hyperboloid [(Barbilian, 1938); see (Mazilu & Agop, 2012) for more details]: this is the ensemble of cubics having their Hessians apolar with the Hessian of real-roots cubic.

These cubics, that is the ones represented by points on what we should like to call *the Barbilian hyperboloid* in order to honor the name of its promoter, must have Hessians with real roots. Indeed, one of the three determining values of the cross-ratio of two pairs of points in harmonic range – as the roots of the Hessians should be in the case of apolarity – must be $-1$; a negative value of the cross-ratio cannot ever be realized with two pairs of complex conjugate numbers: as one can see from the first expression (5.5.5) for the pairs ($z$, $z^*$) and ($y$, $y^*$), such a cross-ratio is always positive. Thus the condition of apolarity can only be reached with two real numbers, ($y_1$,$y_2$) say, instead of ($y$,$y^*$). Then the condition of apolarity comes down to the following connection between the two real numbers

$$y_2 = \frac{(z+z^*)y_1 - 2zz^*}{2y_1 - (z+z^*)} \tag{5.5.8}$$

In other words, the two real numbers must be connected by a homography realized with the matrix

$$\begin{pmatrix} \frac{z+z^*}{2} & -zz^* \\ 1 & -\frac{z+z^*}{2} \end{pmatrix} \tag{5.5.9}$$

which is a matrix of the type ***I*** from those of the frame (2.5.1).

If the whole mathematical development thus far is meant to assist the process of inverse interpretation, then this last statement has a tremenduos importance: it means that the invers interpretation is the objective foundation of the special relativity (see §§2.3 – 5). In view of this conclusion, we need to insist a little more on the classical meaning of this side of continuous group theory. This will be done in the next chapter, by the way of an example illustrating the intermingling of the gravitation with electromagnetism.

# Chapter 6 Conclusions: the Dipole in a Unified Physics of Einstein

We just cannot overextend the present work beyond a decent measure, and so – hoping that it is not already too late – it should be the time to conclude it with some observations coming out from the previous analysis of the Einsteinian approach to physics. We must confess that, in view of our opening with our 'profession of faith', a kind of social perspective is guiding this conclusion: the word out, in the society at large, is that as time goes by, the spirits usually tend to settle for a resting peace, along with the beings that carried them on the face of Earth. Then our duty, as *rational* social beings, is to sanction this peace, perhaps with our very existence at that, if nothing else. Regarding the physics, though, as the previous presentation hopefully illustrates, it so happens that there is something else to consider: some spirits could not have been put to rest along with the men that carried them on Earth. They continue stirring us up with a sense of lack of satisfaction, when it comes to understanding them through their very creations. To be more precise, the questions they once asked have not been satisfactorily answered neither by them, nor by us all along the time. After all, this may be taken as the way how our wisdom grows! Otherwise a spirit like Voltaire, for instance, could not once utter the often-cited word of the wise: «judge the man by the questions he raises, rather than by the answers he provides to those questions». The giants on whose shoulders we are standing today, to use the words of old Newton, would not answer the questions they asked as properly as these questions have been asked, and neither did we for that matter: whence our dissatisfaction, and thereby the motivation bearing the progress of knowledge.

Apparently, Newton's word of wisdom is incomplete though: standing on the shoulders of giants one has to look *where they are looking*, in order to be able to claim having seen further! For, those giants are, surely, not blind, like the old Orion once: they do not carry somebody just in order to see for them! The idea of approach of the quantum itself by Einstein is relevant for the case (Einstein, 1917b), for it shows that a first-rate spirit cannot approach the knowledge of the world but unitarily. Indeed, as we will show through a couple of definite examples in this concluding chapter of the present work, the problem of quantization is closely connected with that of the universal regulating field of the universe – the gravitation – and Einstein, by the physics he practiced, as well as by his attitude with respect to society, touched every side of the science associated with this connection. The way we see it, life of Einstein, like that of Newton before him, is an example of attempts to show us the many directions we have to look when 'standing on his shoulders': the special relativity, the general relativity, the quantization, all seem to have a common denominator in Einstein's mind. And, certainly, they do have a common denominator with the idea of forces and dynamics already existent in the old Newton's mind at the epoch when Albert Einstein became … Einstein. So, here, in this final touch of the present work, we have an attempt to lay rest upon their spirits, beyond that social customary «*may he rest in peace*!» we uttered quite a few times along our present development. We feel, indeed, that they are ones of the very few among us that deserve more than this in order to rest in peace: namely, they need to have a report showing that we have tried, and are even able to understand about their spiritual productions what they could not understand themselves: that something which gave them the dissatisfaction that continuously fed their unrest while dwelling in this world!

### 6.1 Einstein: Quantization Condition from Gravitation's Perspective

For once, we are now in position to bring in our story another part of Einstein's spirit, this time related to the very *condition of quantization*, indeed: in the classical mecanics' take – which, specifically, means what we know today as the Bohr-Sommerfeld quantization condition – it falls in contradiction with the Planck's quantization procedure (Einstein, 1917b). As we reiterated here quite a few times by now, the idea is that this contradiction should be due to the fact that the very analogue of the Planck's constant, in the case of matter quantization, *is not* just… the Planck's constant, but a geometrical invariant of the Lewis-Lutzky type [(Mazilu, 2020), §4.3]. In our opinion, the first instance of such invariant is just the Newtonian central force, having the magnitude inversely proportional with the square of distance, whose invention thus appears the result of a quantization procedure *avant la lettre*, as it were.

It would be impossible to think that Einstein did not touch the very problem of Planck's constant in a way or another. After all, he was the only one we are aware of, who appears to have understood the essence of Planck's quantization procedure (on which we already commented in §4.5 above), as well as the most interested one, among physicists, in a clarification of the connection between gravitation and fundamental physical structures (Einstein, 1919). So, it must not apear as a surprise the fact that he was preoccupied with the very *structure of the Planck's constant* for the case of application of quantization in matter (Einstein, 1917b). The modern acceptance of the task of the old work is, however, not what we just stated above: in today's physics at large, the Planck's constant is seen as an absolute constant, ranking equal to Maxwell's $c^2$, for instance. Nobody seems to think today that the Newtonian forces, to name the most shocking of the examples, may count as an expression of quantization in nature. The modern thinking seems to have taken different directions.

Martin Gutzwiller, for instance – to name one of the founders of the modern theory of chaos, *may he rest in peace*! – based on the work just cited, sees Einstein as the "Godfather of Quantum Chaos" (Gutzwiller, 1985). As a matter of fact Gutzwiller's work can be taken as the only source of details – especially mathematical – in showing the implications of that work of Einstein from 1917. Regarding the geometry of chaos, Gutzwiller emphasized the role of what we took here as Maxwell fish-eye medium [(Gutzwiller, 1985), §§4 and 5]. But what we consider the most interesting of his achievements in connection with Einstein's work, is an improvement on the *Selberg's trace formula* in the spirit of our §5.5 above, contained in a work where he even raised a considerable apology to that old paper of Einstein (Gutzwiller, 1984). Finally, Martin Gutzwiller has a detailed mathematical and physical account of the modern problems of chaos (Gutzwiller, 1990), whereby the Einstein's role in quantum theory of matter is reiterated, with important mathematical details.

What first stirred our interest in Einstein's old work, is the book that Gutzwiller designated as 'the only reference to Einstein's paper in the ensuing 40 years' since Einstein's work has appeared [(Lanczos, 1970); this is the 4$^{th}$ edition of the book of Cornelius Lanczos, with the first edition in 1949, where he used extensively the Einstein's ideas in constructing the Hamilton-Jacobi theory]. However, people only started noticing Einstein's article after Joseph Keller's paper on the structure of Planck's constant, issued a decade later than Lanczos' first edition of the work we just mentioned (Keller, 1958). Citing Gutzwiller on this issue, one can say that…

> … Such a total neglect of an incisive comment on a 'hot subject' by the world's best-known physicist is almost beyond comprehension, in particular, since many close colleagues wrote large and learned reviews of the whole topic in the early 1920s [(Gutzwiller, 1990), p. 208]

But the neglect did not stop there: in hindsight, one can say that it continues even today, for it appears that understanding Einstein is a deeper problem than ever thought, measurable only with his personality. Citing again:

> … Remarkably, even in 1979 during all the noise of Einstein's 100$^{th}$ birthday, this particular paper barely made it into the bibliography of his most knowledgeable biographer [*Abraham Pais*, *a*/*n*; *the work in question is* (Pais, 1979)], but there is no discussion of it. [(Gutzwiller, 1984), p. 216]

'With due contrition and humility' – to use the Martin Gutzwiller's own words – we may be allowed to notice that Einstein's idea continues to not be properly understood, for it is referring to the structure of the Planck's quantum in matter, and such a view is, as we said, not recognized as necessary in physics today. This is what we are going to (partially) document in this final chapter of our work.

To start with, the book of Cornelius Lanczos which first appeared in 1949, during those '40 years of neglect', contains the Chapter VIII that seems to be written exclusively under the spell of the old Einstein's work. Some of Lanczos' conclusions are particularly illuminating regarding Einstein's tasks, which the original 1917 article may have obscured due to yet unsettled concepts at that time. So, in the hope of being more clear when it comes to those concepts, we are quoting from Lanczos:

> … Einstein (in the year 1917) gave an *astonishingly imaginative new interpretation* of the Sommerfeld-Wilson quantum conditions *by abandoning the stream lines* of the $q_k$, $p_k$ planes and operating with the *S*-function itself. Notice that the "phase-integrals"
>
> $$J_k = \oint p_k \, dq_k \qquad (6.1.1)$$
>
> can be replaced by $\Delta S_k$ – i.e. *the change of* $S_k$ *in a complete revolution* – on account of the relation
>
> $$p_k = \frac{\partial S}{\partial q_k} \qquad (6.1.2)$$
>
> Hence the quantum conditions enunciate something about the multiple-valued nature of $S_k$. Einstein now took the sum of all the quantum conditions:
>
> $$\sum J_k = \sum \Delta S_k = \Delta S = nh, \qquad (6.1.3)$$
>
> where $n = \Sigma n_k$ is again an integer. This one equation cannot replace, of course, the original set of equations. But Einstein handled this equation as a *principle* (*original emphasis here*, *a*/*n*) rather than an equation, by requiring that the multiple-valuedness of *S* shall be such that for *any* (*original emphasis here*, *a*/*n*) closed curve of the configuration space the change in *S* for a complete revolution shall be a multiple of *h*. *Taking these curves to be* $q_k$ = const. *in the case of separable systems*, the quantum condition

$$J_k = n_k \cdot h \qquad (6.1.4)$$

are immediate consequences of Einstein's principle. Einstein's invariant formulation of *the quantum condition led de Broglie* (in 1924) *to his fundamental discovery of matter waves* [(Lanczos, 1970); adapted from the original's Chapter VIII, end of §4, by inserting the equations in text, instead of their original captions; *our emphasis except as indicated*, *a/n*]

A few explanations are here in order, just to settle the ideas. First of all, in order to respect the Lanczos' notation, on this unique occasion in our present work, $h$ denotes the Planck's original constant, for which we regularly used the rationalized value, usually symbolized $\hbar$. As to the main idea of Einstein, we think that it foreshadows the modern concept of Yang's construction of the Yang-Mills fields.

Einstein's observation was directed toward the fact that everything in the procedure of quantization in matter depends on the accident that the action $S$ is separable or not:

$$S(q_1,q_2,...,q_n) = \sum S_k(q_k) \equiv S_1(q_1) + S_2(q_2) + ... + S_n(q_n) \qquad (6.1.5)$$

*i.e.* if it is a sum of terms, each one depending on a single coordinate in the phase space. This may be the reason why he was so fond of the Hamilton-Jacobi equation, after all: this equation admits a solution by separation, as above, always for the free particle in Euclidean environment, anyway. In this case, by (6.1.2) the components of momentum give a tangent vector to the component curve in the respective phase plane of the phase space. If this does not happen, the definition (6.1.2) for momentum involves values of the other coordinates and the quantization condition is complicated or even inexistent.

Einstein *imagined* that the solution depends on the *existence of the cycles* transversal to the lines of current in the phase space, and this is an essential point. What specifically drew our attention on it, is identity of the problem presented by Einstein with the problem of definition of the Yang-Mills fields in C.-N. Yang's take. This identity goes to details: one just has to follow the similitude of the Figure 2 of Einstein and the Figure 1 of (Gu & Yang, 1977) which illustrates the C.-N. Yang's definition of the Yang-Mills fields. Having in mind that Einstein discussed everything *in terms of a fluid*, and that the general relativity has, in fact, a proper moment of *interpretation*, as we have shown in the present work, – the Klein interpretation, during Einstein-de Sitter debate (see the introduction to Chapter 3) – it was tempting to identify the two apparently different problems and solutions. To wit: the lines $A_1A_2$ and $B_1B_2$ from Einstein's Figure 2, as well as the lines *AE* and *EA* of Gu & Yang's Figure 1, belong to paths within the Maxwell fish-eye medium in which the fluid particles exist and move. The issue is even better illustrated in the Figure 2 of (Wu & Yang, 1975), where an idea of posssible *flux of connections* between the two portions of surface is also suggested. And this medium can be described only in the general relativistic manner of Ernst (see §4.1) as follows.

According to Hertz's natural philosophy, one can talk of a position in space only after the incident that the position was defined: tht is, indicated by a certain material particle. The two positions of a dipole's charges are indicated by the particles moving *instantaneously* along the geodesics of the Maxwell fish-eye optical medium. The two-blade Kasner geometrical description of the vacuum, allows us a mathematical portrayal *of the state of field*. Indeed, the state of the medium, as a whole, *at a certain instant* is given by analogy with the definition of an instanton. Namely, the region of space between the two charges can be described by the three variables giving the state of field as in §5.5, $z$, $z^*$ and $k$, say. The *instantaneity*, if we may, is then generically described by an 'instanton philosophy', as we do it typically; that is, by finding an ensemble of states ($z$, $z^*$, $k$) solutions of the system of differential equations obtained by differentiation from equation (5.5.1):

$$dk = \omega^1 \cdot k(z - z^*), \quad dz = \omega^1 z^2 + \omega^2 z + \omega^3, \quad dz^* = \omega^1 z^{*2} + \omega^2 z^* + \omega^3 \tag{6.1.6}$$

This system is algebraically compatible and, moreover, has unique solution for the three differentials $\omega^k$: the action (5.5.1) is a *simply transitive* one. The three 1-forms are then found simply by solving the system (6.1.6). They are:

$$\omega^1 = \frac{dk}{(z - z^*)k}, \quad \omega^2 = \frac{dz - dz^*}{z - z^*} - \frac{z + z^*}{z - z^*}\frac{dk}{k}, \quad \omega^3 = \frac{zdz^* - z^*dz}{z - z^*} + \frac{zz^*}{z - z^*}\frac{dk}{k} \tag{6.1.7}$$

This is, again, a $\mathfrak{sl}(2,R)$ coframe, whose corresponding frame is given in equation (5.5.2). To prove this statement we proceed as in §4.5 but in order to avoid tiresome calculation it is better to use real variables ($u$, $v$, $\phi$), defined by $z = u + iv$, $k = e^{i\phi}$. Then the 1-forms (6.1.7) are:

$$\omega^1 = \frac{d\phi}{2v}, \quad \omega^2 = \frac{dv}{v} - \frac{u}{v}d\phi, \quad \omega^3 = \frac{vdu - udv}{v} + \frac{u^2 + v^2}{2v}d\phi \tag{6.1.8}$$

and satisfy the structural conditions (1.4.16). The structure constants are those from equation (1.4.17), so these structural relations are characteristic to a $\mathfrak{sl}(2,R)$ type algebra. The coframe from equation (6.1.8) just describes a Riemannian threefold having the metric (5.4.21). To be more precise, using the differentials (6.1.8) we have the quadratic metric

$$(ds)^2 \equiv 4\omega^1\omega^3 - (\omega^2)^2 = (d\phi)^2 + 2d\phi\frac{du}{v} - \left(\frac{dv}{v}\right)^2 \tag{6.1.9}$$

Now, we are in position to be able to calculate the corresponding frame for the cofrane (6.1.8) by the procedure that we have already explained in the §5.2 above, to wit: the equations (5.2.5), (5.2.6) and (5.2.7). First, using the quadratic differential (6.1.9) as Lagrangian, we calculate the differential 1-forms of the momentum for this case. The result is:

$$p_\phi = d\phi + \frac{du}{v}, \quad p_u = \frac{d\phi}{v}, \quad p_v = -\frac{dv}{v^2} \tag{6.1.10}$$

and used in equation (6.1.8) results in

$$\omega^1 = \frac{1}{2}p_u, \quad \omega^2 = -up_u - vp_v, \quad \omega^3 = \frac{1}{2}\left\{(u^2 - v^2)p_u + 2uvp_v + 2vp_\phi\right\} \tag{6.1.11}$$

Then, changing the momenta into derivatives on corresponding variables, and using the metric (6.1.9) to transform the differential forms thus obtained into operators, we get the associated frame for (6.1.8) as:

$$B_1 = \frac{\partial}{\partial u}, \quad B_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad B_3 = (u^2 - v^2)\frac{\partial}{\partial u} + 2uv\frac{\partial}{\partial v} + 2v\frac{\partial}{\partial \phi} \tag{6.1.12}$$

The notation here is intentional: these are exactly the operators from equation (5.5.2), only written in real form, with $z = u + iv$.

Let us stop for a little while, in order to better assess these results in terms of the concept of instanton. Within the framework of the old Schwarzschild's observation regarding the Einsteinian natural philosophy, they simply describe the *field content of an instanton*, indeed. The instantaneity in this field can be described by a simultaneity represented as a Riemannian threefold having the metric (6.1.9). According to the results from the previous §5.4, this threefold can be seen as a family of surfaces of negative curvature, connected with each other by Bäcklund transformations. Globally, the threefold is described by three Killing vectors (5.5.2), or equivalently (6.1.12), whose conserved quantities are the rates (6.1.7), or equivalently (6.1.8), with respect to the affine parameter of the geodesics of threefold. *These geodesics are cycles on a constant negative curvature surface*! Before continuing on with this physical image of an instanton, let us show how we understand this statement, simply by calculating the equations of geodesics.

The metric (6.1.9) is quite simple and should not ask for involved calculation when exhibiting its geodesics. However such calculation are usually longer, and we do not want to get lost in symbols, but just need some guiding results. So, we take the metric as a Lagrangian, and consider the arclength as the 'time' of problem of extremum for this Lagrangian. The Lagrange equations for $\phi$ and $u$ are easily obtained, showing that the respective rates are conserved:

$$\frac{d}{dt}\left(\dot{\phi}+\frac{\dot{u}}{v}\right)=0,\quad \frac{d}{dt}\left(\frac{\dot{\phi}}{v}\right)=0 \quad\Leftrightarrow\quad \dot{\phi}+\frac{\dot{u}}{v}=a,\quad \frac{\dot{\phi}}{v}=b \tag{6.1.13}$$

Here a dot over symbol means derivative on 'time' $t$, as usual, while $a$ and $b$ are two constants in which we recognize the rates corresponding to the components $p_\phi$ and $p_u$ of the momentum, according to the equations (6.1.10): these components are constants along the geodesics.

However, the component $p_v$ is not constant, for the Lagrangian depends on $v$ explicitly, and the third Lagrange equation of the geodesics can, after calculations, be reduced to the form:

$$\ddot{v}-\dot{\phi}\dot{u}-\frac{\dot{v}^2}{v}=0 \tag{6.1.14}$$

Using the second of (6.1.13) to eliminate the rate of $\phi$, this equation of the v-component of geodesics simplifies, and can be integrated right away:

$$\frac{d^2}{dt^2}(\ln v)=b\cdot\dot{u} \quad\therefore\quad \frac{\dot{v}}{v}=b\cdot u+c \tag{6.1.15}$$

where $c$ is a new integration constant. On the other hand, using again (6.1.13), and eliminating the rate of $\phi$, we get the equation:

$$\frac{\dot{u}}{v}=a-b\cdot v \tag{6.1.16}$$

With the first of (6.1.13), (6.1.15) and (6.1.16) we can construct the projection equation of the geodesics of this threefold onto surface of negative curvature of the threefold. Indeed, along the geodesics the Lagrangian maintains a constant value, $E$ say, and we can write, for instance:

$$\left(\frac{\dot{u}}{v}\right)^2+\left(\frac{\dot{v}}{v}\right)^2-\left(\dot{\phi}+\frac{\dot{u}}{v}\right)^2=E \tag{6.1.17}$$

So, in view of the last of (6.1.15) and (6.1.16), the equation of this projection of the geodesics in question is

$$(u-u_0)^2+(v-v_0)^2=R^2 \tag{6.1.18}$$

which represents circles of center $(u_0,v_0)$ and radius $R$, where these new parameters are defined in terms of the old parameters of integration *via* formulas:

$$u_0 = -\frac{c}{b}, \quad v_0 = \frac{a}{b}, \quad R^2 = \frac{a^2 - E}{b^2} \tag{6.1.19}$$

These are the cycles on our negative curvature surface. Let us expound a little the situation.

The third coordinate of the threefold can be determined with respect to 'time' from the second equation (6.1.13), along the following line of reasoning. Use only the constant $b$ from (6.1.13), in order to eliminate the 'time' in favor of the phase $\phi$ as independent variable. We get the following system of differential equations:

$$\ddot{u} + bv\dot{v} - \frac{\dot{u}\dot{v}}{v} = 0, \quad \ddot{v} - bv\dot{u} - \frac{\dot{v}^2}{v} = 0 \tag{6.1.20}$$

Now, eliminate the 'time' using the phase $\phi$ as independent variable, according to the following scheme of differentiation:

$$\dot{f} = \dot{\phi}\cdot f' = bv\cdot f', \quad \ddot{f} = b^2 v(vf'' + v'f')$$

where $f$ is any function of $\phi(t)$, and an accent means derivative with respect to $\phi$. The system (6.1.20) becomes

$$u'' + v' = 0, \quad v'' - u' = 0 \tag{6.1.21}$$

which means that, as functions of the phase $\phi$, the two components $u$ and $v$ are just two solutions of the universal equation (4.5.11) for the intensity:

$$f''' + f' = 0 \tag{6.1.22}$$

which is characteristic to a Maxwell fish-eye optical medium [see equation (1.2.11)]. Thus, $u(\phi)$ and $v(\phi)$ are necessarily of the form

$$u = u_0 + R\cos(\phi - \phi_0), \quad v = v_0 + R\sin(\phi - \phi_0) \tag{6.1.23}$$

where $\phi_0$ is an arbitrary phase, and $u_0$, $v_0$, $R$ are the parameters from equation (6.1.18). The connection we are searching for, between the 'time' $t$ and the 'Bäcklund phase' $\phi$ is then given by the last equation from (6.1.13):

$$b\cdot dt = \frac{d\phi}{v_0 + R\cdot\sin\phi} \quad \therefore \quad \frac{1}{2}b\cdot dt = \frac{dx}{v_0 x^2 + 2Rx + v_0} \tag{6.1.24}$$

Here, however, we have omitted the arbitrary phase $\phi_0$ as irrelevant for the present argument, and denoted the new integration variable $x \equiv \tan(\phi/2)$. Now, the integration necessary for the construction of connection between time and phase can proceed routinely, but there are a few distinct cases.

Let us consider first the case of *negative discriminant* of the quadratic from the denominator of (6.1.24). This condition means

$$R^2 - v_0^2 < 0 \quad \therefore \quad 0 < E < a^2 \tag{6.1.25}$$

where we used the equation (6.1.19). This therefore happens when the 'energy constant $E$ is positive, but always finite. In this case the connection between the 'Bäcklund phase' and the time of geodesics is given by the trigonometric equation:

$$\tan\frac{\phi}{2} = \frac{\sqrt{E}}{a}\cdot\tan\frac{t\sqrt{E}}{2} - \sqrt{1 - \frac{E}{a^2}} \tag{6.1.26}$$

The opposite case of (6.1.25) is given by

$$R^2 - v_0^2 \equiv \frac{-E}{b^2} > 0 \tag{6.1.27}$$

and the connection between phase and time is given, for any negative value of the constant $E$, either by formula

$$\tan\frac{\phi}{2} = -\frac{\sqrt{-E}}{a} \cdot \tanh\frac{t\sqrt{-E}}{2} - \sqrt{1-\frac{E}{a^2}} \tag{6.1.28}$$

or by formula

$$\tan\frac{\phi}{2} = -\frac{\sqrt{-E}}{a} \cdot \coth\frac{t\sqrt{-E}}{2} - \sqrt{1-\frac{E}{a^2}} \tag{6.1.29}$$

depending on the relative sign of quadratic from equation (6.1.24) with respect to the sign of $v_0$: different or the same sign, respectively.

Enough, for now, about the mathematics of the surfaces of negative curvatures *per se*. Concluding on our intuitive image of the Kasner's two-blade cosmological solution, we must say that it is the best fit for the needs of the Yang-Mills fields theory, if we understand this theory in the Chen-Ning Yang's take (see §4.4 above). The instantaneity in such a universe, in general, can be expressed, indeed, by a two-blade: according to Einstein's and Yang's concepts the blades are overlapping each other, thus requiring a connection between them. While in the two original works we now have in view – of Einstein's from 1917, and of Yang's from 1974 – the connection is just *imagined*, we go a little further, on tying up this imagination with a logical fact: the two blades are connected by *the physical structure of the universe*. Imagine, indeed, two families of cycles (6.1.18), one for each blade. They constitute a typical dipole structure 'submerged', as it were, in a Maxwell fish-eye medium. The geodesics of this medium passing through the centers of the two families of cycles are the field lines of the dipole.

Nothing assures us that this figure can be a space figure: that is, nothing else than the fact that the two surfaces playing the parts of blades are different. But we have an indication: according to Poincaré, the motion of a charge in this configuration cannot be but helical mimicking a motion in magnetic field (Poincaré, 1896). In general, though, it appears that the condition of definition of holography by coherence can provide such a demonstration: *every phase of a given frequency contains cycles*. Let us show how.

### 6.2 Holography: the Phases and the Cycles

The critical problem occurring in an Einsteinian approach of the concept of phase, should be identification of the classical action $S$ with a phase. In the view promoted here, this involves explicitly the idea of holography in the construction of the theory. Then, if referred to a physical meaning at all, in view of the property of coherence leading to holography, the relation between the phases can only be described by a homography (see §4.5). Such a homography will be written here in the form once used by Élie Cartan in detailing a suggestive example for his theory of moving coframes [see (Cartan, 1951), the examples to §§102, 108, 112, 161, 214]:

$$\theta(\phi) = S + \frac{\phi}{u \cdot \phi + v} \tag{6.2.1}$$

where $\theta$ and $\phi$ are two phases corresponding to the same frequency, and $S$, $u$, and $v$ are the three parameters describing the homography. A possible reading, probably the right one, of this equation can be done as follows:

$\theta$ is the phase in matter, while $\phi$ is due exclusively to the field according to Schwarzschild's observation (Yang-Mills *y* or *z*, or both; in general, Ernst's *z*, or Barbilian's (*z*, *k*). *S* is the phase involving just particles from the interpretative constitution of the matter: it is the only one containing a genuine time, and that cannot appear but in a classical way, as an action!

In the interest of further analysis of this relation, let us notice the corresponding 'full homography', as it were, for which we have previously written the formulas of calculating the Cartan's coframe:

$$\theta(\phi) = \frac{(1+uS)\cdot\phi + vS}{u\cdot\phi + v} \tag{6.2.2}$$

One can notice right away that the phase $\theta$ reduces to a classic phase that can be identified with the action *S* in the cases where $\phi = 0$. Therefore, for the cases where the phase can be taken as time, we have unconditionally an identity between phase and action, like in Madelung's take for instance (Madelung, 1927):

$$\theta(0) = S \tag{6.2.3}$$

On the other hand if the time goes to infinity, we have, again, a known interpretation of the situation:

$$\theta(\infty) = S + u^{-1} = \theta\big|_{v=0}(\phi) \tag{6.2.4}$$

The parameter *u* is, therefore, itself a phase. If this phase is zero, we have a linear relationship between phases:

$$\theta\big|_{u=0}(\phi) = S - v^{-1}\cdot\phi \tag{6.2.5}$$

in which case *v* can be, incidentally, a period.

Assume that the phase $\theta$ is constant – locked, as the physicists like to characterize this situation – and is taken for reference, as in the case of the definition of the holographic phenomenon. Disregarding, for now, the description of a way in which this reference phase enters the expression of the other phases coherent with it, we want to find the circumstances in which this requirement is satisfied in general. In order to do this we have for the time a Riccati differential equation like (4.5.16), expressing the fact that the phase is constant

$$d\phi = \omega^1\phi^2 + \omega^2\phi + \omega^3 \tag{6.2.6}$$

Here $\omega^{1,2,3}$ are the following differential 1-forms, obtained by using the equation (6.2.2) for the phase $\theta$:

$$\omega^1 = \frac{u^2}{v}dS + \frac{du}{v}, \quad \omega^2 = 2udS + \frac{dv}{v}, \quad \omega^3 = vdS \tag{6.2.7}$$

Therefore the circumstances in which the phase $\theta$ is locked are given, in general, by an ensemble of phases that are solutions of equation (6.2.6). This is a three-dimensional ensemble, whereby each one of the phases is located by the three values (*S*, *u*, *v*).

We are able to say something about this ensemble, if we can describe the time moments as the values of some continuous function, which needs to be even differentially continuous, in view of the equation (6.2.6). According to a classical definition of such a function, this description comes down to associating a continuous parameter to phases, in order to transform the equation (6.2.6) into a system of ordinary differential equations. This, in turn, can be done if, and only if, the differential forms (6.2.7) are exact differentials proportional to the differential of that parameter, *i.e.* if we can write:

$$\frac{u^2}{v}dS + \frac{du}{v} = a^1(d\varphi), \quad 2udS + \frac{dv}{v} = 2a^2(d\varphi), \quad vdS = a^3(d\varphi) \tag{6.2.8}$$

where $a^{1,2,3}$ are constants, and $\varphi$ is the parameter in question. In this case the equation (6.2.6) becomes an ordinary differential equation of Riccati type, offering, by its solutions, the phases $\phi$ as the values of a function of the continuous parameter $\varphi$:

$$\frac{d\phi}{d\varphi} = a^1\phi^2 + 2a^2\phi + a^3 \tag{6.2.9}$$

There are three possibilities for the solutions of this equation, according to the sign of the discriminant of quadratic from its right hand side. The first of these is, by the way of example, the most interesting one, from a physical point of view, and we choose it for a meaningful illustration inasmuch as it is, geometrically speaking, the most suggestive one. Namely, in that case we have:

$$a^1\phi + a^2 = \sqrt{\Delta}\cdot\tan[\sqrt{\Delta}(\varphi-\varphi_0)], \quad \Delta \overset{def}{=} a^1a^3 - (a^2)^2 > 0 \tag{6.2.10}$$

with $\varphi_0$ a constant; this constant is, however, not entirely arbitrary with respect to the parameters of our problem. But, let us see the consequences of (6.2.10) in order to draw the right conclusions.

In the case given by (6.2.10), we shall also have a corresponding solution for the differential system (6.2.8). This shows how the equation $d\theta = 0$ needs to be interpreted in general. Specifically, the system can be solved to give the parameters as functions of the phase $\varphi$:

$$\begin{gathered} S(\varphi) = S_0 + A\cdot\tan[\sqrt{\Delta}(\varphi-\varphi_0)], \quad v(\varphi) = v_0\cdot\cos^2[\sqrt{\Delta}(\varphi-\varphi_0)] \\ u(\varphi) = u_0\cdot\cos[\sqrt{\Delta}(\varphi-\varphi_0)]\left\{a^2\cdot\cos[\sqrt{\Delta}(\varphi-\varphi_0)] + \sqrt{\Delta}\cdot\sin[\sqrt{\Delta}(\varphi-\varphi_0)]\right\} \end{gathered} \tag{6.2.11}$$

where $A$ is a new constant of integration. The situation has now a statistical interpretation: we have an ensemble of phases of mean $\phi$ given by (6.2.10), having a distribution function with *quadratic variance* given by

$$a^1\phi^2 + 2a^2\phi + a^3 = \frac{\Delta}{a^1\cos^2[\sqrt{\Delta}(\varphi-\varphi_0)]} \tag{6.2.12}$$

in terms of the ensemble mean (Morris, 1982). This is actually a one-parameter family of such distributions, indexed by parameter $\varphi$. The function $[v(\varphi)]^{-1}$ from (6.2.11) measures the variance of this ensemble.

The parameters $u$, $v$ from equation (6.2.11) can be transcribed *independently* of the parameter $S$, for instance as in the system:

$$\frac{u}{v} = \frac{u_0}{v_0}[a^2 + \sqrt{\Delta}\tan\sqrt{\Delta}(\varphi-\varphi_0)]; \quad \frac{1}{v} = \frac{1}{v_0}[1 + \tan^2\sqrt{\Delta}(\varphi-\varphi_0)] \tag{6.2.13}$$

This transcription is intended to make the fact obvious, that the parameters $u$ and $v$ are defined on some cycles, which, in the parameters (u, v) are, in fact, *ellipses*. To see this, notice that we can eliminate the parameter $\varphi$ from (6.2.13), in order to get the *parabolas*:

$$z = \frac{z_0}{x_0^2\Delta}(x^2 - 2a^2x_0\cdot x + a^1a^3x_0^2); \quad x \equiv \frac{u}{v}, \quad z \equiv \frac{1}{v} \tag{6.2.14}$$

where $x_0$ and $z_0$ are the constants from equation (6.2.13). Then, the implicit equation of the original cycles in parameters $(u,v)$ can be obtained by coming back to these variables, which gives:

$$(v_0u - a^2u_0v)^2 + \Delta\cdot u_0^2\cdot(v - v_0/2)^2 = u_0^2\cdot v_0^2/4 \tag{6.2.15}$$

thus proving our statement: this equation represents a quadratic form whose matrix has the eigenvalues of the same sign. Therefore, in the case the conic it represents is real, it cannot be but an ellipse. But this means that the parameters $u_0$ and $v_0$ are values that can be measured in contemporaneity. Let us show how this statement can be understood, by using ideas touching the *fundamental analogy* of our knowledge (see §2.5).

Assume the earthly situation of a vessel moving uniformly on the surface of a quiet sea. The dynamical state of this boat is decided by a certain intermingling of its two obvious properties: on one hand it dwells on the surface of Earth, while, on the other hand, it also dwells in the physical universe. The first of these aspects can be dynamically addressed using the closest source of gravitational force, which is the Earth, more precisely its center of force, while the second dynamical aspect can be addressed using the idea of remote material sources of the field. In other words, if the gravitational field is universal, it is universal in a 'double take', as it were: once as Earth's field and once *as a field created by the remote bodies*. Or, perhaps, a better idea would be that the field in question is not universal at all. Be it as it may, the truth of the matter is that the local state of our boat assumes a twofold mathematical description, just like the Yang-Mills fields in C.-N. Yang's take (see §4.4 above): first, through an *approximately parallel field of forces*, acting 'vertically' as it were, and generated by the closest center of force that can be rationally taken as such with respect to the boat, and, secondly, by the action of an ensemble of *many remote bodies*, acting 'in all directions in space', at least in the cases of a real sea, anyway. It is this combined action that can be deemed, indeed, as universal. One can imagine that the center of Earth and a fictitious remote point in space are forming a dipole of the kind naturally existing in a Maxwell fish-eye medium. As a matter of fact, the 'center of Earth' here is, practically speaking, just as fictitious as that 'fictitious remote' point in space, from the point of view of our experience: *we cannot but imagine them both*. This general situation can also be declared as existent in an Einstein elevator, thus allowing us a comparison between this mind invention and the Wien-Lummer enclosure from the case of light. The bottom line is that the principle of equivalence in Einstein's first take reveals a typical example of Yang-Mills fields (Yang, 1977). The theory from the previous section allows us to conclude that between the two states of gravitation – the one created by local matter, and the one created by global matter – there is a transformation, and that transformation has been discovered long ago by Paul Appell.

Indeed, the second kind of action – incidentally, the proper action at a distance – is the one chiefly addressed by the classical dynamics, and the main point of experience leading to its description is the Kepler's second law. We have expressed this law in the form given in equation (4.5.8), which was used as an incentive for deducing equation facilitating the definition of frequency based on the phenomenon of holography, as initially conceived by Dennis Gabor. In terms of the geometrical parameters, $(u,v)$ say, in a plane through the axis of the 'gravitational dipole' imagined as above, this law can be written in the form:

$$udv - vdu = \dot{a}\cdot(d\tau) \tag{6.2.16}$$

where $\dot{a}$ is the area constant as usual, and $d\tau$ is an appropriate differential of the time parameter in terms of which the area rate is calculated. This means that the ratio $(u/v)$ represents a uniform motion on the plane, in terms of a special time:

$$\frac{d^2}{dt^2}\left(\frac{u}{v}\right) = 0, \quad dt = \frac{d\tau}{v^2} \tag{6.2.17}$$

In the framework of Newtonian definition of the forces, the equation (6.2.16) is a consequence of the following equations of motion on the surface:

$$\frac{d^2u}{d\tau^2} = f(u,v)\cdot\frac{u}{\sqrt{u^2+v^2}}, \quad \frac{d^2v}{d\tau^2} = f(u,v)\cdot\frac{v}{\sqrt{u^2+v^2}} \tag{6.2.18}$$

where $f$ is the magnitude of the local action of the forces upon boat, assumed to depend only on the coordinates of the boat on surface. Thus the quantity ($u/v$) can be taken as a coordinate on our surface, which represents a uniform motion in a given direction on the surface, in the time defined by equation (6.2.14). By the same token, using the second of equations (6.2.14), we find right away

$$\frac{d}{dt}\left(\frac{1}{v}\right) = -\frac{dv}{d\tau} \quad \therefore \quad \frac{d^2}{dt^2}\left(\frac{1}{v}\right) = -f(u,v)\frac{v^3}{\sqrt{u^2+v^2}} \equiv Z(u,v) \tag{6.2.19}$$

Therefore, the coordinates ($x$, $z$) defined in equation (6.2.13) are what we would like to call the *Appell coordinates* in order to honor the memory of their illustrious promoter (Appell, 1891). They have the property that made the Galilean relativity possible, namely:

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2z}{dt^2} = Z \tag{6.2.20}$$

that is, in an arbitrary vertical plane on Earth, a horizontal uniform motion, and a vertical accelerated motion. The equations (6.2.20) suggest then that the corresponding conics in ($x$,$z$) coordinates are the well known Galilean parabolas, which further implies that $Z(u,v) = g$, a constant. The construction, therefore, is not possible without an Einstein elevator!

### 6.3 The Reverse Interpretation: Charges in a Coll Universal Deformation

An idea must be explored by the way of conclusion, which has, according to the principles of the first quantization, a high significance in describing 'qualitatively', so to speak, the fundamental physical unit of a Maxwell fish-eye, with surprising results. Namely, the idea that the fundamental material unit in the universe *must be a dipole*: inasmuch as the physical and geometrical contents of a model of this fundamental material unit can be quite complicated, we might be tempted to think of it in a global way, so to speak. And it is here the place to meet an old expedient of the classical electrodynamics 'in new clothes', as it were. The content and the results of this kind of reasoning constitute the object of the following last lines of this work.

In the three-dimensional case Bartolomé Coll's universal deformation (Coll, 1999) is always expressed *via* matrices that we would like to call as 'equivalent to a vector field'. We understand this equivalence in the following, quite explicit way: having a vector field $|v\rangle$, we can construct the following matrix using two arbitrary scalar functions $\alpha$ and $\beta$:

$$v_{ij} = \alpha\delta_{ij} + \beta v_i v_j \tag{6.3.1}$$

Now, it is clear that, because $v_k$ are the components of a vector, and supposing $\alpha$ and $\beta$ scalars, gives $v_{ij}$ as the components of a *tensor*. In the case of Coll's definition, this tensor represents a universal deformation 'generated by the vector $|v\rangle$', we might say. Incidentally, we use here the *ket* notation for vectors, insofar as these vectors are understood to be defined by components only, as it should be the case with the charge vectors (Zwanziger, 1968), with no reference to a space reference frame whatsoever. The bold face notation is then reserved for the corresponding matrices. One of the eigenvalues of the matrix $\boldsymbol{v}$, namely $\alpha$, is double. The other eigenvalue, different from $\alpha$, is given by

$$\alpha' = \alpha + \beta v^2 \tag{6.3.2}$$

Notice some interesting features of this kind of matrix. First of all, if either $\beta$ or $v_k$ is null, $\boldsymbol{v}$ is a purely spherical tensor. Thus, barring a scaling factor $\sqrt{(\beta)}$, the vector $|v\rangle$ determines the deformation of $\boldsymbol{v}$ with respect to its spherical state. Secondly, if we calculate the eigenvector of the *tensor* $\boldsymbol{v}$, corresponding to the eigenvalue (6.3.2), we find out that this eigenvector is just the vector $|v\rangle$, up to a normalization factor. This property is independent of the parameter $\alpha$, and this is what we mean by the above mentioned equivalence: given the vector $|v\rangle$ we can directly construct the tensor $\boldsymbol{v}$ as a family of two-parameters tensor matrices having it as an eigenvector. One can say that $\boldsymbol{v}$ represents a kind of action that points in the *general direction* of $|v\rangle$, as it were, however, not *exactly* in that direction. This property allows us to consider the tensor $\boldsymbol{v}$ as representative of the dipole, along the following guiding lines borrowed from classical electrodynamics.

The eigenvalue (6.3.2) may represent either a distance or a length, depending on the charge content of the dipole. Indeed, according to Katz's natural philosophy of charge, if the dipole is electric, then its charges are joined together by a *piece of vacuum*, therefore the eigenvalue (6.3.2) *must be a distance*. On the other hand, if the dipole is magnetic, the same natural philosophy says that its charges must be joined together by a *piece of matter*, therefore the eigenvalue (6.3.2) *must be a length*. The possibility cannot be *a priori* excluded that the dipole could have mixed poles, one electric and one magnetic: such a structure is known in physics as a *dyon*, but in the present context it should await for some further specifications. Anyway, excluding for the moment this 'mixed' case of dipole, the physical representative of it – vacuum or matter – connects two charges of the same kind, and no matter what it represents, it is established *instantaneously* at the scale of time allowed by the holographic principle based on the coherence property. This means that the propagation of signals already finds it established inside a Wien-Lummer cavity or an Einstein elevator, as it is normal with any preexisting physical construction perceived by humans. Our experience, in this respect, knows of what appear to be physical structures existing forever, *i.e. given*, as they say. However, as we have learned historically, the 'given' itself may have a limited life span: it just 'lives' at another time scale. And this time scale should be represented in the physical theory by some phases analogous to the *advanced* phases of the classical action at a distance.

One way to get the 'dipole characterization' of the ether inside a Wien-Lummer cavity is by simply entering into play *pairs of dipoles*, *i.e*. by admitting that the ether is described not by *one* tensor of the general type (6.3.1) but by *two*, with two characteristic vectors, $|u\rangle$ and $|v\rangle$ say. Assuming that a fundamental ephemeral structure that describes this ether can be represented by a linear combination of dipoles, the complete tensor describing a structure of connected with the pairs of poles would then have entries depending on three parameters:

$$w_{ij} = \alpha\delta_{ij} + \beta u_i u_j + \gamma v_i v_j \tag{6.3.3}$$

The calculations are 'more symmetric', if we may, in case we write this definition more… conveniently, namely in the form of a *Maxwell stress tensor* as in the case of classical electromagnetic fields [see (Stratton, 1941) for pertinent details of this construction]:

$$w_{ij} = \lambda u_{ij} + \mu v_{ij} \quad \therefore \quad \boldsymbol{w} \overset{def}{=} \lambda \boldsymbol{u} + \mu \boldsymbol{v} \tag{6.3.4}$$

where $\lambda$ and $\mu$ are two real parameters, describing the properties of connection, with the matrices $\boldsymbol{u}$ and $\boldsymbol{v}$ defined as the tensors

$$u_{ij} \overset{def}{=} u_i u_j - \frac{1}{2}\boldsymbol{u}^2 \delta_{ij}, \quad \boldsymbol{u}^2 \equiv \boldsymbol{u} \cdot \boldsymbol{u} = \sum\nolimits_i u_i^2, \quad v_{ij} \overset{def}{=} v_i v_j - \frac{1}{2}\boldsymbol{v}^2 \delta_{ij}, \quad \boldsymbol{v}^2 \equiv \boldsymbol{v} \cdot \boldsymbol{v} = \sum\nolimits_i v_i^2 \tag{6.3.5}$$

Incidentally we have here the same boldface notation for the tensors and the vectors generating them, but hope for no confusion at all. The tensor (6.3.4) contains *nine* measurable quantities: $\lambda$, $\mu$, and the two intrinsic vectors, and the definition (6.3.3) also contains *nine* arbitrary parameters. Written at length, the entries of this tensor (6.3.4) are of the form

$$w_{ij} = \lambda u_i u_j + \mu v_i v_j - \frac{1}{2}(\lambda \boldsymbol{u}^2 + \mu \boldsymbol{v}^2)\delta_{ij} \quad \therefore \quad \boldsymbol{w} = \lambda \boldsymbol{u} \otimes \boldsymbol{u} + \mu \boldsymbol{v} \otimes \boldsymbol{v} - e\boldsymbol{1} \tag{6.3.6}$$

where $\boldsymbol{1}$ in the second equality is the identity matrix as usual, and we shall use from now the following notations:

$$e \overset{def}{=} \frac{1}{2}(\lambda \boldsymbol{u}^2 + \mu \boldsymbol{v}^2) \quad and \quad \boldsymbol{g} \overset{def}{=} \sqrt{\lambda\mu} \cdot (\boldsymbol{u} \times \boldsymbol{v}) \tag{6.3.7}$$

It is easy to see that the tensor (6.3.6) has three real eigenvalues, in general manifestly distinct. Indeed, its orthogonal invariants (the coefficients of the secular equation) are:

$$I_1 = -e, \quad I_2 = -e^2 + \boldsymbol{g}^2, \quad I_3 = -e(e^2 - \boldsymbol{g}^2) \equiv -I_1 I_2 \tag{6.3.8}$$

so that the eigenvalues of tensor $\boldsymbol{w}$ can then be calculated as the roots of the corresponding characteristic equation – the cubic having the invariants (6.3.8) as coefficients – and they are

$$w_1 = e \quad and \quad w_{2,3} = \pm\sqrt{e^2 - \boldsymbol{g}^2} \tag{6.3.9}$$

It turns out that the pair from equation (6.3.7) gives an eigenvalue of $\boldsymbol{w}$ and its corresponding eigenvector. The other two eigenvectors of $\boldsymbol{w}$ are orthogonal, and located in the plane of the vectors $|u\rangle$ and $|v\rangle$.

Before going any further, at this juncture it is better to discuss the place of the 'mixed dipole structure' in this construction, which we purposively postponed before. It should be connected with the condition of 'ephemerality', if we may say so, of a physical structure determined by a Wien-Lummer cavity containing ether. To be more precise, two charges of different nature – electric and magnetic – can be connected instantaneously with each other in a Maxwell fish-eye medium, by a special type of Lorentz transformation: Fowles or Cook (see Chapter 2, §2.3 for the discussion of such transformations). Not only the resulting structure is *instantaneous* in time, but it should be also *located* in space: it is a material particle of Hertz's type, having gravitational mass, electric charge and magnetic charge. In other words, the fundamental particle serving for interpretation is a dipole of this special type: *having mixed charges*, *i.e.* a modern *dyon*. Mathematically, such a particle can be described as an instanton represented as a dipole of *null vector of Coll universal deformation*.

Just for the sake of offering an image of what this representation may mean, let us assume an Euclidean Coll deformation vector: its length can be expressed as a sum of squares. A null sum of squares – a null deformation vector – represents analytically the spin of particles (Yamamoto, 1952). Therefore the Hertz's material particle defined by the three physical magnitude in a static interpretative ensemble, is characterized by spin half in its most general acceptance; and this spin is generated by the charges. The idea is then aroused, that the spin is a *purely relativistic effect*, indeed, as occasionally discussed and stressed in the modern theoretical physics. We shall need to return to this topic on some other occasion.

Coming back to our streak of discussion, according to this theory, the eigenvalues of tensor ***w*** defined in equation (6.3.6), and given explicitly in equation (6.3.9) are already statistical expressions of measured quantities. They are based on the *Novozhilov*'*s averages*, which are estimates over the *orientations of the planes in space*, assumed uniformly distributed, which describe any physical tensorial quantity characterizing a matter continuum (Mazilu, Agop, & Mercheş, 2021):

$$w_n = \frac{1}{3}(w_1 + w_2 + w_3) \quad and \quad w_t^2 = \frac{1}{15}[(w_2 - w_3)^2 + (w_3 - w_1)^2 + (w_1 - w_2)^2] \tag{6.3.10}$$

Let us calculate these averages by using here the eigenvalues (6.3.9). To wit: we can construct the following two measurable statistical components of the tensor ***w*** which turn out to be

$$|w_n\rangle \equiv |n\rangle\langle w|n\rangle = -\frac{2}{\sqrt{3}}e\cdot|n\rangle, \quad |w_t\rangle = \frac{2}{3}\begin{pmatrix} -2e \\ e + 3\sqrt{e^2 - \mathbf{g}^2} \\ e - 3\sqrt{e^2 - \mathbf{g}^2} \end{pmatrix} \tag{6.3.11}$$

The second one of these vectors is located on one of the *octahedral planes* of a reference frame at the position where the measurement is performed, while the first one is *normal* to that octahedral plane, whose normal is supposed to be $|n\rangle$. As long as only the values (6.3.10) are accessible to measurement for a tensorial quantity in a continuum, the orientation of the second vector from (6.3.11), in his octahedral plane of residence, always remains undecided. This orientation is out of our control *per se*, but it can be measured anyway, provided a gauging exists. Indeed, it can be accounted for by an angle easy to measure in case we have a reference direction in the octahedral plane at our disposal, and this angle turns out to be just the arbitrary phase $\phi$ from §§5.4 and 5.5 above. Assume that we have such a reference, as given by a particular tensor of the form given in equation (6.3.5) with the characteristic vector $|\xi\rangle$ say. Then, for this tensor we have, with obvious notations:

$$\langle\xi|n\rangle = -\frac{1}{\sqrt{3}}\xi^2, \quad |\xi_t\rangle = \frac{2}{3}\xi^2\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \tag{6.3.12}$$

If the vector $|\xi\rangle$ is perpendicular on both $|u\rangle$ and $|v\rangle$ then the tensors ***w*** and ***ξ*** commute. Thus they have a common reference frame and it can be arranged that their octahedral planes coincide. In this case the direction of the vector from equation (6.3.12), which is fixed, can be appropriately chosen as a reference direction in the octahedral plane. Then the angle $\phi$ of the vector (6.3.11) with respect to this fixed direction in the common octahedral plane can be calculated from a well-known geometrical formula, which here amounts to:

$$\cos\phi = -\frac{e}{\sqrt{4e^2 - 3\mathbf{g}^2}} \tag{6.3.13}$$

This shows that, under specified conditions, the angle $\phi$ is independent of the reference vector. With a proper choice of sign for the square root, the origin $\phi = 0(\text{mod}2\pi)$ of this angle occurs only for the cases where $e = g$.

This condition means, in turn, that the angle, $\theta$ say, between the vectors $|u\rangle$ and $|v\rangle$, calculated on the basis of the quantities from equation (6.3.7), is given by equation

$$|\sin\theta| = \frac{1}{2}\left|\frac{\lambda u^2 + \mu v^2}{uv\sqrt{\lambda\mu}}\right| \tag{6.3.14}$$

As the quantity from the right hand side here is always greater than or equal to *1*, the real angle between vectors $|u\rangle$ and $|v\rangle$ cannot be but *90*°. Thus, the *initial condition* for the characteristic angle of tensor $\boldsymbol{w}$ in the octahedral plane takes place when the vector $|u\rangle$ is perpendicular to $|v\rangle$ and their plane is perpendicular to vector $|\xi\rangle$.

Regarding the problem of measurement, let us notice once again that a measurement here can actually produce just two quantities (6.3.10) and an angle $\phi$: as long as tensorial quantities are to be considered, anything else seems to be inference from these three quantities. In a word, in the case of measurement of tensors, there should be minimum three redundant theoretical quantities. The redundancy is due, as always in physics, to our geometrical models of reality: vectors and tensors. Mention should be made, however, of the important fact that the perpendicularity of the vectors $|u\rangle$ and $|v\rangle$ is not a purely geometrical property but, in the case of a material continuum, must be a consequence of some preexistent statistics. These statistics are usually associated with the different concepts of hidden parameters.

The tensor from equation (6.3.6) is, as already mentioned, a *classical Maxwell stress tensor*, if for the vectors $|u\rangle$ and $|v\rangle$ we take the classical electromagnetic fields $|e\rangle$ and respectively $|b\rangle$. Then the parameters $\lambda$ and $\mu$ can be taken to represent some measures which would indicate how much of this medium is space and how much is matter. This seems to be the conclusion of an exhaustive analysis of the electromagnetic quantities (Fessenden, 1900), showing that at least one of these two parameters *has to be taken as a density*. In a word, classically speaking, the equation (6.3.6) represents, indeed, a Maxwell fish-eye medium. However, this representation means more along the idea of properly understanding the old constructions related to the theory of light, which seems to be accepted, easier than any other mind constructions at that, as part of an Aristotelian environment in matters of education.

Indeed, according to equation (6.3.12) we have what properly can be termed as 'three quarks' which offer just a reference direction in the octahedral plane, serving for the construction of a Maxwell stress tensor. There are a total of eight possibilities of such a construction, incidentally depending on which octahedral plane that happen to be realized momentarily for the measurement purposes. This is plainly an *eightfold way*! In order to give it some classical meaning, recall the Fresnel's fundamental principle of the physics of light, which we reproduce here, just for the sake of transparency, expressed by the illustrious Henri Poincaré. After explaining at length the Fresnel's theory of light, Poincaré concludes:

> This is, in a nutshell, *the theory of Fresnel*. It is in every respect in conformity with the experimental laws; but we notice that it rests upon *two hypotheses* demanding a closer examination. These two hypotheses can be enunciated as:
>
> 1° The *elastic force aroused by the motion of a plane wave* is *independent of the direction of the plane of wave*, it *depends only on the direction of vibrations* of the molecules, and *is proportional to the force developed by an isolated molecule*, the *other molecules from the plane of the wave remaining at rest*.

> 2° The only *effectual component of the elastic force* is the *component parallel to the wave plane*.
>
> The first of these hypotheses, which Fresnel vainly tried to justify, *is entirely arbitrary*, but nothing precludes its acceptance … [(Poincaré, 1889), §151, pp. 229 – 230; *our translation and emphasis*, *n*/*a*]

In hindsight, this summary of the Fresnel's theory shows that it actually contains all the details Planck needed in developing his theory of quantization, plus something that it missed. That something banned, during the latter history, the quarks out of the 'Aristotelian environment', where the light – their ancestor along the line of physical optics – would have secured them a natural place. A short analysis of some details of the Poincaré's excerpt above will elucidate our statement.

Let us start with the general observation that a wave 'arouses a force' in an optical medium – a medium through which the light can pass: after all, we are talking here of the light waves! – and this wave is a 'plane wave', since Poincaré and, no doubt, Fresnel himself, would make reference to the 'direction of the plane of wave', as a first incentive in establishing a 'local topology', as it were. Now, while *locally* imposed as a concept – perhaps here it is the case to be more precise, by saying that it was established at an *infrafinite scale* of space – the idea of plane wave was, and still is, actually, even today, used *ad libitum*, so to speak, more to the point, at a *finite* and even a *transfinite* scale, with no restriction whatsoever: the physics has not, as yet, a concept of judging things in connection with an idea of space scale, simply because it lives under the illusion that its concepts should always refer to a reality of the kind revealed to man by senses. The present case in point is important in making this statement more comprehensible.

The concept of a plane wave was established in connection with the idea of a light ray, whose physical image was always close to the Louis de Broglie's model: *a kind of capillary tube* (de Broglie, 1926b,c). Thus, a physical ray would cut in any conceivable wave surface – 'conceivable' should be taken, of course, in the sense of the description of old Huygens for the concept of wave – a portion that can be reckoned as infinitesimal. Now, ever since the inception ideas of differential geometry got in the open, an infinitesimal portion of a *continuous* surface started being assimilated with a plane portion. And, of course, a plane cannot be conceived otherwise than its concept shows it. To wit: the wave origin has been forgotten here, because, obviously, it cannot be a predicate for the geometrical concept of plane. On the contrary, the 'plane' became a predicate for the concept of wave, so that, when the case occurred with Fresnel's theory of light, people have started to speak freely of the… *plane waves*. And this concept also became one of the main theme of the Planck's physics, inescapable, as it were.

In this respect, the observation can safely be made to the effect that the universal function exacted by the Wien's displacement law (1.1.1) must be established based on two levels of statistics: one of them which is dictated by the ultimate element of the ensemble representing the blackbody radiation, while the second is dictated by the density of this element within its ensemble. If not an observation *per se*, the way we usually understand it today, this dichotomy had nonetheless at least singular consequences, one of which, no matter how singular, has become quite remarkable. That one observation led Lord Rayleigh to the idea of using, on one hand, the classical theorem of the equipartition of energy in order to physically characterize *the element* of the ensemble representing the equilibrium radiation, while using, on the other hand, the uniform probability distribution regarding the number (*i.e. cardinal* in the previous parlance of the present work) density of that element within the ensemble (Rayleigh, 1900), which is plainly a probability density.

Importantly enough, the element of the ensemble characterizing the equilibrium radiation here is the *plane wave* we are just talking about. Again, notice the predicate: it is not the 'wave' for the 'plane', as it should have been by historical reference to the concept of ray; it is rather 'plane' for the 'wave', according to the idea that the geometry (*sic*!) is universal, not the physics. And universal, here, basically meant, according to existing human experience: liable to offer predicates for any of the concepts we create. Thus physics deals with 'plane waves', a wave plane being a plane like any other geometrical plane! Notice further that, when speaking of an ensemble of waves, we cannot mean an interpretation *per se*: the classical material point, or even the Hertz's material particle for that matter, is missing from the picture. However, it is 'implicitly' present, as it were, and this presence is to be recognized in the fact that, with the Fresnel's physical theory of light, the plane wave is regarded as a *harmonic oscillator*. This is a mathematically sound approach of the idea of interpretation in the natural philosophy, inasmuch as the harmonic oscillator is a free particle 'in the making', so to speak, just pending an Arnold transformation (see in the §4.5 above, the discussion and comments regarding the Arnold's theorem). Fact is that for an oscillator the equipartition of $(½)kT$ *per* degree of freedom is only natural, where $k$ is, this time, the Boltzmann's constant. Using this result and the equation provided by physics:

$$E = \nu \cdot g(\nu/T)$$

whereby $E$ must be taken as the average energy of an oscillator over the ensemble of oscillators of frequency $\nu$, we have the universal function $g(\ldots)$ from equation (1.1.1) in the form:

$$g(x) = k \cdot x^{-1} \quad where \quad x \equiv \nu/T \qquad (6.3.15)$$

leaving an explanation open for the presence of a quadratic term in frequency responsible for having the correct spectral density. For this explanation, everything comes down to a second part of statistics, involving the calculation of the number of the possible oscillators from an infinitesimal range of frequency, in order to calculate $dn(\nu)$ from an equation giving the differential of energy over the ensemble of oscillators:

$$dU_\nu = dn(\nu) \cdot E \qquad (6.3.16)$$

In order to do this, Jeans used an ingenious argument based on the concept of *plane wave* (Jeans, 1905), improved, just about the same time, by Rayleigh himself (Rayleigh, 1905) who took in consideration the polarization of the waves, which in this case was naturally described considering the electromagnetic nature of the plane waves representing the equilibrium radiation. The procedure of calculation rests upon the general idea that *the frequency should be taken as a space vector when calculating its density*.

In order to simplify the argument, while delivering the essential message unaltered, one usually admits that the Wien-Lummer enclosure containing the radiation in equilibrium is a classical cube – the everlasting geometrical shape of a reference frame since the times of Descartes – having the edge $L$ say. The enclosed equilibrium radiation is thereby physically represented by stationary plane waves, with the normals of the plane chaotically oriented inside the enclosure. This chaos is assumed to be characterized by an uniform distribution in volume. Let $\alpha_x$, $\alpha_y$, $\alpha_z$ be the direction cosines of the propagation direction of such a stationary wave. Without violating the generality of our argument and its conclusion, we can accept that the geometrical reference frame has the axes oriented along the edges of the enclosure containing the radiation. Therefore between the wavelength of the stationary vibration and its component along the $x$-axis, for instance, we must have the following relationship:

$$\lambda = \lambda_x \cdot \alpha_x$$

On the other hand, the optical condition of existence of stationary vibrations gives

$$\frac{\lambda_x}{2} \cdot n_x = L$$

where $n_x$ is an integer. The very same equations can, of course, be written for each and every one of the three edges of the cube representing the enclosure, so that finally we have the following three expressions for the direction cosines of the plane normal of a stationary plane wave in the enclosure:

$$\alpha_x = \frac{\lambda}{2L} \cdot n_x, \quad \alpha_y = \frac{\lambda}{2L} \cdot n_y, \quad \alpha_z = \frac{\lambda}{2L} \cdot n_z$$

Now, the sum of squares of these cosines must be, naturally, *1*. On the other hand, the relation between *wavelength* and *frequency* of the light waves amounts to: $\lambda \cdot \nu = c$, where $c$ is the universal constant representing the speed of propagation of the electromagnetic waves, among which the light can be counted, according to the precepts of the electromagnetic theory of light. Therefore, we shall have:

$$n_x^2 + n_x^2 + n_x^2 = R^2, \quad R \equiv 2\frac{L \cdot \nu}{c} \tag{6.3.17}$$

This is the equation of a sphere of radius $R$ in the space of integer numbers. One can read on it that to every triplet of integers ($n_x$, $n_y$, $n_z$) there corresponds a frequency to be calculated according to equation:

$$\nu = \frac{c}{2L}\sqrt{n_x^2 + n_x^2 + n_x^2}$$

This correspondence is one-to-one, and helps us in calculating the number of stationary waves from the enclosure. Indeed, a wave has just a frequency, and for this frequency we have a single triplet of integers representing coordinates of the wave in a three-dimensional space of frequencies. The total volume occupied by these points representing the frequencies between *0* and $\nu$, is just one eighth from the whole volume of the sphere of radius $R$, and by equation (6.3.17) it is given as:

$$N_\nu = \frac{4\pi}{3}\frac{(L\nu)^3}{c^3}$$

We can thus calculate the number of frequencies in the infinitesimal range ($\nu$, $\nu + d\nu$). This is simply the volume of a spherical shell of thickness $d\nu$ taken twice, in view of the fact that an electromagnetic wave, appropriated as oscillator, has two different polarizations: it counts as two oscillators, not just one. Therefore, we can write:

$$dN_\nu = \frac{8\pi\nu^2}{c^3}Vd\nu \tag{6.3.18}$$

where $V \equiv L^3$ is the volume of the enclosure of equilibrium radiation.

Now, in view of the equation (6.3.15), we can give a precise theoretical form to the infinitesimal energy from equation (6.3.16). We have

$$u_\nu d\nu = \frac{8\pi\nu^2}{c^3}kTd\nu$$

so that the spectral density satisfying the Wien's displacement law (1.1.1) will be exactly:

$$u_\nu(T) = \frac{8\pi k}{c^3}\nu^3\left(\frac{\nu}{T}\right)^{-1} \tag{6.3.19}$$

which is the algebraical form of the *Rayleigh-Jeans radiation law*.

This algebraical expression is the theoretical apogee, we might say, of the classical statistical mechanics in matters regarding the blackbody radiation. However, it seems to be somehow incomplete, insofar as it does not offer a finite total density for the radiation at a given temperature, to say nothing of the fact that, even if we do not extend the frequency indefinitely, it cannot possibly satisfy the Stefan-Boltzman law of proportionality with the fourth power of temperature. In spite of this important fact, it was nevertheless determined that the formula (6.3.19) should be considered good enough at low frequencies and/or relatively high temperatures. Since, in broad lines, its inference involves steps in accordance to the classical prescriptions, both physical and geometrical, it would be therefore to be expected that these prescriptions are incomplete when it comes to the blackbody radiation. This was the attitude of choice at the beginning of the last century, and one can say that it shaped the whole modern theoretical physics. Mention should be made that, while it does not touch the classical *geometrical* prescriptions – which, by the way, remained in their full power, as shown above – that attitude of choice was fully bent on changing the theoretical *physics* indeed. The changes, however, meant – at least in hindsight, as it were – just deepening the branches of theoretical physics into ones with apparently no chance of bridging between them.

Among the many attempts to fit the experimental data on equilibrium radiation, that of Wily Wien, dating from 1896 (Wien, 1896) [see also Wien's report on the problem of blackbody radiation at the Congress of the Physicists in Paris (Wien, 1900)], can also be explained by the previous philosophy. It retained the attention of theorists as a limit case too, for the laws of radiation, regarding the conditions of frequency and temperature, however, exactly opposite to those conditions where the Rayleigh-Jeans radiation law is valid. This is why it has often been tried to see if Wien's purely heuristic argument could not have a physical basis, the way this is understood within the limits of the classical mechanics and of classical statistical mechanics. Such attempts culminated with Einstein's work that laid the basis of the modern second quantization doctrine (Einstein, 1905b). There is an objective reason for this path of our knowledge, and we think it can be jotted down as follows.

To start with, the Wien's argument, extending the one leading to the Wien's displacement law, does contain an explicit hypothesis bearing resemblance to quantization. This hypothesis was well specified by Wien himself in a work that appeared long after the quantization in the old form of Planck was established [(Wien, 1915); see also §1.1 of the present work]. It is our opinion that this work of Wien's foreshadows, almost explicitly we should say, the connection wave-particle of de Broglie, which occurred within a decade from the work just cited, based pretty much on the same kind of arguments (De Broglie, 1922, 1923).

More importantly, though, Wien went directly for the *statistical analogy* between the electromagnetic waves and the molecules of a gas with which the blackbody radiation might be in thermal equilibrium (see §1.1). Today we are able to say that the fact is a mathematical consequence of the Arnold's theorem (see §4.5), but the fact remains that there is a formal identity between the physical idea of equilibrium and the statistical theory of ensembles liable to allow treating the frequency as a statisical variable. The impasse would then be overcome, of the definition of temperature as a statistic connected to translational degrees of freedom, just by noticing that there is a genuine statistic – the frequency – connected to the idea of phase. The Wien's idea was kind of structural, in the sense of connecting physically two conceptually disjunctive structures: classical material points and waves. More to the point, he assumed that the thermal equilibrium is manifested in such a way that the probability density of the molecular velocities of a Maxwellian gas, given, as known, by the Maxwell density function:

$$p_M(v) \propto v^2 \cdot \exp\left(-C\frac{v^2}{T}\right) \tag{6.3.20}$$

with $C$ a constant, must be somehow 'mirrored' by the spectral density of radiation. Indeed, the molecules of a gas can be conceived as bodies inside the enclosure containing thermal equilibrium radiation. Therefore, one can figure out that they absorb, emit, or reflect radiation just like any other body from Kirchhoff's phenomenological argument; we can even grant them the designation 'black'. In this case, at thermal equilibrium the Maxwellian probability density from equation (6.3.20) must be somehow expressed as a probability density characterizing a certain frequency from the spectrum of radiation at a given temperature. And, apparently aware of the fact that the frequency of light needs a statistics, just like the temperature in the case of molecules of a gas, Wien went directly for the transformation of the Maxwellian probability. The exponent from the right hand side of the equation (6.3.20) already contains the ratio between the kinetic energy of a molecule of gas and the temperature. If one assumes that the kinetic energy of the molecule is, in the conditions of thermal equilibrium, completely transmitted to the emitted radiation of frequency $v$ then, in order to satisfy the displacement law, the exponent from equation (6.3.20) should be proportional to ($v/T$). Therefore the kinetic energy of a gas molecule must be proportional to the frequency of the equilibrium radiation, so that the probability of radiation mirroring the Maxwellian one given by equation (6.3.20) should be

$$p_M(\nu) \propto \nu \cdot \mathrm{e}^{-C\frac{\nu}{T}}$$

which is exactly the physical form of the energy of the average energy of an oscillator over the ensemble, only with an exponential form of the function $g(\ldots)$. In view of the fact that the density of these oscillators in a frequency interval is given by equation (6.3.18), which we have so far no reasons to reject, it follows that the spectral density of radiation, considered as a probability density, must be something of the form:

$$u_\nu(T) = C_1\nu^3 \cdot \mathrm{e}^{-C_2\frac{\nu}{T}} \tag{6.3.21}$$

This is the radiation law proposed by Wien based just on the displacement law. The heuristic argument leading to it, as well as the conclusion that the spectral density of the blackbody radiation must be considered as a probability density of the frequency at a given temperature, or of the temperature at a given frequency, as it turned out later on, must have had some truth in them. Indeed, as we already mentioned, the formula (6.3.21) proved to be valid in conditions complementary to those where the Rayleigh-Jeans radiation law is valid. Specifically, the Wien's radiation law is valid at high frequencies and/or low temperatures. Practically speaking, Max Planck just interpolated between the two cases. However, the problem of deciding if the law of radiation can be considered a law of probability remained open: as a matter of fact this issue is not settled even today, simply because it does not even exist in physics. Based on the modern existing data on light – the NASA COBE FIRAS measurements on the cosmic background radiation – we are, however, inclined to think that this is, indeed, a case to be taken in consideration as such [(Priest, 1919); see our arguments in §1.1 of the first chapter of this work and the works cited by us there].

Wien's case shows the necessity of interpretation, involving the frequency as a statistic. That this is physically necessary was shown by Einstein himself in the work acknowledged in physics as a first example of the second quantization procedure (Einstein, 1905b), discussed by us in the §4.5 above. Wien himself, however, could not relate but to the statistical theories of Maxwell and Boltzmann, whereby the interpretation is implicit by the classical concept of material point: the molecule. The field is absent here, and could not be seen but through that into 'the only effectual component of the force', for which the reference is the 'wave plane'. As it turns out such a plane cannot be used in interpretation but only through the idea of particle, and this needs a reformulation of the fundamental principles of Fresnel at least in one respect, to the effect that 'the only *effectual component of the force arisen by the motion* is the *component parallel to a local octahedral plane*'. The Fresnel theory of light is part of the eightfold way in the universe, and this should be considered as a physical fact. The important point here, is that the eightfold way is just a natural thing, actually continuing the classical natural philosophy initiated by James Clerk Maxwell, which turns out to continue the Augustin Fresnel's physical theory of light, which turns out to continue in a Newtonian manner the old Hooke's physical theory of light and colors... The queer quarks, entering the theoretical physics out of a dream, are not so strange after all: considered as mind creations they are just as another mind creation that seems to appear as natural, namely the Fresnel's theory of light… in a Maxwell take, as it were. This is a lesson that can be learned only from a proper reading of the Planck's ideas leading to the first quantization. There is, however, another lesson, more important, that we need to learn along this line, from the Coll's concept of universal deformation.

### 6.4 The Fundamental Le Roux-Loedel-Amar Representation

Assuming, therefore, an interpretation by static ensembles made possible as ensembles of equilibrium with static Newtonian force fields, but considering, nevertheless, the precepts of Einsteinian general relativity (see the Israel & Wilson's observations quoted in §4.1), the suggestion presents itself that the motion of matter through ether brings a rotation acting upon these force fields. In §2.2 we pushed this suggestion even further, indicating that the charges would be the physical generators of such rotations [see equation (2.2.4)]. It is now the time to put the Lorentz's hypothesis itself – the one that introduced the Lorentz's transformation to the knowledge (see §2.2 above) – on a firmer basis. If the matter remains the same in its motion through vacuum, and we imagine that the rotation 'by charges' is due to the motion, we can describe this phenomenon by constraints on the tensor (6.3.6), which is a representative of both the matter and the vacuum. The problem then occur as to uniqueness of that tensor, for the question can be aroused: are the charges generating exclusively rotations of forces in all conditions? For instance, in vacuum the charges are obliterated, and only the electromagnetic field exists, while only in matter we can say that they are existent and generate rotations.

We can reformulate this problem by generalizing the first of the Fresnel's hypotheses in the following way: what is the most general linear transformation of the vector fields $|e\rangle$ and $|b\rangle$, representing the 'elastic forces aroused' by matter in its motion? A direct answer to this question is mathematically provided by a general homogeneous linear transformation of the vector fields in their plane:

$$|e'\rangle = \alpha|e\rangle + \beta|b\rangle, \quad |b'\rangle = \gamma|e\rangle + \delta|b\rangle \tag{6.4.1}$$

which preserves the Maxwell stress tensor representing the vacuum. This is, again, a suggestion coming from the methods of optics in the physical description of the propagation in layered optical mediums (De Micheli, Scorza, & Viano, 2006). And the rezultant transformation plainly generalizes the idea of rotation by charges, which thus appears as a special case of it. Then our problem would be just to characterize this transformation knowing only is limiting case of rotation. Rewriting the tensor of Maxwell stresses in this case, we have, as in equation (6.3.4) and (6.3.5):

$$t_{ij} = \lambda e_{ij} + \mu b_{ij}, \quad e_{ij} \overset{def}{=} e_i e_j - \frac{1}{2} e^2 \delta_{ij}, \quad b_{ij} \overset{def}{=} b_i b_j - \frac{1}{2} b^2 \delta_{ij} \tag{6.4.2}$$

Any kind of invariance of this tensor would necessarily lead to a connection between the parameters $\lambda$, $\mu$ and the entries of the matrix from equation (6.4.1), which allows us a concrete description – and a closed solution, hopefully – of the modern problems, like that of *vacuum tunneling* (Jackiw & Rebbi, 1976), for instance: the fields are changed by the presence of matter in space, in order to adapt themselves to the different *local* properties represented by the parameters $\lambda$ and $\mu$.

What remains to be decided is how do we define the invariance of the tensor (6.4.2), and a proposal presents itself just naturally. Namely, if the light carries the information within the universe we inhabit, then this information should be the same everywhere, in places where the light is. One of the most obvious way to express this is that the entries of the matrix ***t*** *have to remain unchanged*. Then, *a fortiori*, all of the invariants of this tensor remain the same and, therefore, what is measured or recorded out of it has the same value for the whole coordinate space of definition for this tensor. This proposal, apparently just like its classical counterpart (Gabor, 1961), comes out from the belief that what we are locally measuring is what has been happening far away in space and, therefore, our conclusions regarding the structure of the physical universe, based on this recording, are the right ones. So, if by the transformation (6.4.1) the fields ($\mathbf{e}'$, $\mathbf{b}'$) are in an environment described by ($\lambda'$, $\mu'$), then the conservation: $t_{ij} = t'_{ij}$ can be transcribed as:

$$\begin{gathered} (\alpha\delta - \beta\gamma)\sqrt{\lambda\mu} = \sqrt{\lambda'\mu'}, \qquad \alpha\beta\lambda + \gamma\delta\mu = 0 \\ \alpha^2\lambda + \gamma^2\mu = \lambda', \qquad \beta^2\lambda + \delta^2\mu = \mu' \end{gathered} \tag{6.4.3}$$

In order to learn how to use this equation, let us just find some particular solutions of this system.

The handiest case is that of *homogeneity of the vacuum* whereby the parameters $\lambda$ and $\mu$ are the same in the whole space covered by light, so that we have: $\lambda = \lambda'$, $\mu = \mu'$. The first of the equations (6.4.3) then shows that the 2×2 matrix giving the transformation (6.4.1) is a *matrix of unit determinant*. From an algebraic point of view, the last three remaining equations then form a separate homogeneous system, and thus the system (6.4.3) is equivalent to the following equations:

$$\alpha = \delta, \quad \beta = -\frac{\mu}{\lambda}\gamma, \quad \alpha^2 + \frac{\mu}{\lambda}\gamma^2 = 1$$

The last of these equations shows that we can express the two entries $\alpha$ and $\gamma$ trigonometrically, either in the usual trigonometry or in the hyperbolic trigonometry, by an arbitrary phase parameter, $\phi$ say:

$$\alpha = \cos\phi, \quad \gamma = \sqrt{\frac{\lambda}{\mu}}\sin\phi$$

In this case, the transformation that does not change the Maxwell stress tensor is described by the matrix of unit determinant:

$$\begin{pmatrix} \cos\phi & -\sqrt{\dfrac{\mu}{\lambda}}\sin\phi \\ \sqrt{\dfrac{\lambda}{\mu}}\sin\phi & \cos\phi \end{pmatrix} \tag{6.4.4}$$

Thus, we can say that the motion of matter, if described according to equations (6.4.1) for the fields of electric nature, generates more than a rotation for the particle used for its interpretation. Let us insist for a while with some geometrical details on the first of these special cases of this matrix.

Using the transformation of parameters

$$u = \xi\cot\phi, \quad v = \frac{\xi}{\sin\phi}, \quad \xi \equiv \sqrt{\frac{\mu}{\lambda}} \tag{6.4.5}$$

the first matrix from equation (6.4.4) can be cast into the form

$$\boldsymbol{m} \equiv \frac{1}{v}\begin{pmatrix} u & u^2 - v^2 \\ 1 & u \end{pmatrix} \tag{6.4.6}$$

and this matrix indicates that the parameters $u$ and $v$ defined by the equation (6.4.5) can be taken as coordinates on a certain surface in space, as we discussed in the §2.3, on the occasion of generalizations of the Fowles' and Cook's type of Lorentz transformations. Let us speculate around these results, using the same symbolics as in the §2.3. Notice first that the parameter $v$ from equation (6.4.5) can be taken as the coordinate of a uniform motion in the time defined by $\cot\phi$, according to Arnold's theorem, if $\xi$ counts as charge (see §4.5). On the other hand, by comparison with the results of Jean-Marie Le Roux, described by us in the §2.3, the two parameters $(u,v)$ of the optical medium offer the Loedel-Amar parametrization of the matrix (6.4.4) making a veritable Lorentz matrix out of it. That is to say that the Loedel-Amar parametrization thereby gains a fundamental significance: it is not just a representation of a spacetime transformation, but a clear indication of the fact that the universe can be treated as a Maxwell fish-eye optical medium. In other words, the Loedel-Amar parameterization is of essence for the theory of special relativity, showing that Lorentz's hypothesis of 'invariance of matter', as it were, with respect to its motion through the world, cannot be described but electrodynamically, indeed, even with no reference to the shrinking of lengths, or lengthening of times or anything like that.

Finally, let us play with the mathematical results just gotten. First, we should have to notice the factorization relation:

$$\boldsymbol{m} = -\boldsymbol{J}\cdot\boldsymbol{K}_0, \quad \boldsymbol{J} \equiv \frac{1}{v}\begin{pmatrix} u & -u^2 + v^2 \\ 1 & -u \end{pmatrix} \quad \boldsymbol{K}_0 \equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.4.7}$$

In other words, the matrix $\boldsymbol{m}$ thus constructed is the product of the involution $\boldsymbol{J}$ from the general reference frame (2.5.1), associated with a surface, and the involution $\boldsymbol{K}_0$ from the Shchepetilov reference frame (2.4.4) associated with the geometry of charges. This indicates a kind of mixed local action involving both the charges *per se* and a surface 'steadfastly attached to matter' as in the Lorentz's theory of electric matter. Any linear combination of the two involutions associated with the matrix $\boldsymbol{m}$ is also an involution. Indeed, we have

$$\boldsymbol{m}^{-1} \equiv -\boldsymbol{K}_0^{-1} \cdot \boldsymbol{J}^{-1} = -\boldsymbol{K}_0 \cdot \boldsymbol{J} \tag{6.4.8}$$

where, in calculating the inverses, we have used the multiplication table from equation (2.4.11). We can further calculate the square of the 'mixed Cook matrix' [see equation (2.3.12)]:

$$\boldsymbol{C} \stackrel{def}{=} a\boldsymbol{K}_0 + b\boldsymbol{J} \quad \therefore \quad \boldsymbol{C} \cdot \boldsymbol{C} = (a^2 + b^2)\boldsymbol{1} + ab(\boldsymbol{K}_0 \cdot \boldsymbol{J} + \boldsymbol{J} \cdot \boldsymbol{K}_0) \tag{6.4.9}$$

where, again, for calculations involved in reaching the last result, we have used the multiplication table from equation (2.4.11) for both types of matrices, for this multiplication table is the same in both cases. We find such an equation quite remarkable to the point that we need to show why, even if in just a few words.

The main point of this exercise is that the mixed Cook matrix $\boldsymbol{C}$ combines the location in space (the presence of $\boldsymbol{K}_0$), with the location on a surface (the presence of $\boldsymbol{J}$ in the linear combination). The matrix in round parantheses from the last expression of equation (6.4.9) is $-(\boldsymbol{m}^{-1}+\boldsymbol{m})$, on account of equations (6.4.7) and (6.4.8), and in order to calculate it we can use the Hamilton-Cayley equation for the matrix $\boldsymbol{m}$, which can be written right away using equation (6.4.6). Thus we have

$$\boldsymbol{m}^2 - 2(\frac{u}{v})\boldsymbol{m} + \boldsymbol{1} = \boldsymbol{0} \quad \therefore \quad \boldsymbol{m} + \boldsymbol{m}^{-1} = 2(\frac{u}{v})\boldsymbol{1}$$

and, with this last result, the square of matrix $\boldsymbol{C}$ from equation (6.4.9) becomes

$$\boldsymbol{C}^2 = \left( a^2 + b^2 - 2(\frac{u}{v})ab \right)\boldsymbol{1}$$

This relation proves the involution property of the matrix $\boldsymbol{C}$. The coefficient of identity matrix in this equation is a quite suggestive quadratic form. Indeed, using the parameters $u$ and $v$ as defined in equation (6.4.5) it can be written as:

$$a^2 + b^2 - 2\cos\phi \cdot ab$$

Therefore, assuming now that ($a$, $b$) are differential elements associated with the surface, this quadratic form defines a kind of *Chebyshev net* on the surface, of characteristic angle $\phi$. Importantly enough, this net is simply determined by the fact that 'the position in space is located on a surface', if it is to characterize the situation in geometrical terms. This seems quite natural in view of the problem defined here: in its description according to Lorentz theory, the charge induces nets on a surface of equilibrium, like, for instance, a wave surface. It would seem that the light phenomenon has this general effect on the matter, which can explain the idea of quantization.

Continuing, however, with just the mathematics of the differential geometry involved here, the matrix $\boldsymbol{m}^{-1} \cdot d\boldsymbol{m}$ is an involution. Indeed, using (6.4.7), we get:

$$\boldsymbol{m}^{-1} \cdot d\boldsymbol{m} = \begin{pmatrix} \frac{u}{v^2}du - \frac{dv}{v} & \frac{u^2+v^2}{v^2}du - 2\frac{u}{v}dv \\ -\frac{du}{v^2} & -\frac{u}{v^2}du + \frac{dv}{v} \end{pmatrix}$$

Consequently, the metric properties are decided by the determinant of this matrix. It is

$$\det(\boldsymbol{m}^{-1} \cdot d\boldsymbol{m}) = \frac{(du)^2 - (dv)^2}{v^2} \tag{6.4.10}$$

Thus we have

$$(\boldsymbol{m}^{-1}\cdot d\boldsymbol{m})^2 = -\frac{(du)^2-(dv)^2}{v^2}\cdot\boldsymbol{1} \quad \therefore \quad tr\left\{(\boldsymbol{m}^{-1}\cdot d\boldsymbol{m})^2\right\} = -2\frac{(du)^2-(dv)^2}{v^2}$$

It is worth mentioning that getting back the original parameters $\xi$ and $\phi$ *via* equations (6.4.5), the metric (6.4.10) can be cast into the notable form

$$\det(\boldsymbol{m}^{-1}\cdot d\boldsymbol{m}) = (d\phi)^2 - \sin^2\phi\left(\frac{d\xi}{\xi}\right)^2 \tag{6.4.11}$$

Most certainly the action of motion on matter is connected with the charge: therefore the phase here, may be taken as the phase of charge, while the parameter $\xi$ is connected with the magnitude of charges, as reflected by the parameters $\lambda$ and $\mu$ of the optical medium. In this kind of motion we can therefore have solitons, which are described *via* Ernst-type harmonic mapping of the metric given by equation (6.4.11). Probably the solitons also involving only the mass are simply static solitons.

These speculations encourage us to assume that in general we may accept a more relaxed condition for a kind of 'Lorentz vacuum', as it were, equivalent in a way with the fact that the 'refraction index' is constant. Such a condition amounts to

$$\frac{\mu}{\lambda} = \frac{\mu'}{\lambda'} \equiv n^2$$

It means matter nonhomogeneity in regards to physical properties, although when the physics is referred to the index of refraction $n$, the matter is in fact homogeneous. In this case, the matrix of transformation in equation (6.4.1), is given by

$$\sqrt{m}\cdot\begin{pmatrix} \cos\phi & -n\sin\phi \\ \sin\phi/n & \cos\phi \end{pmatrix}, \quad m \equiv \frac{\lambda'}{\lambda} \tag{6.4.12}$$

where $\phi$ is, again, an arbitrary phase. Therefore the differential geometry above does not change. However, a problem occurs since the matrix (6.4.12), or even (6.4.4) for that matter, cannot count as a pure rotation matrix: if we can cope with the prefactor $\sqrt{(m)}$, say by a rescaling of coordinates, the problem remains that the matrix is not orthogonal. So one can rightfully ask: what is the connection of such a general transformation with a rotation?

As the previous development shows, the '*rotation power*' *in matter*, as defined by equation (6.4.1) under constraints (6.4.3), canot be described by a matrix of pure rotation, as suggested taking guidance from the classical theory of charges (see §2.2) but by a matrix, $\boldsymbol{Q}(n,\phi)$ say, which being of unit determinant, is nevertheless not orthogonal:

$$\boldsymbol{Q}(n,\phi) \stackrel{def}{=} \begin{pmatrix} \cos\phi & -n\sin\phi \\ \sin\phi/n & \cos\phi \end{pmatrix} \tag{6.4.13}$$

According to the precepts of the matrix optics [see (Abe & Sheridan, 1994), equation (32)], this matrix represents, indeed, a rotation of angle $\phi$, but 'sandwiched', so to speak, between a magnification of amplitude $\ln\sqrt{n}$ and a reduction of $-\ln\sqrt{n}$, or vice versa, depending on the definition of magnification:

$$\boldsymbol{Q}(n,\phi) = \begin{pmatrix} \sqrt{n} & 0 \\ 0 & 1/\sqrt{n} \end{pmatrix}\cdot\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\cdot\begin{pmatrix} 1/\sqrt{n} & 0 \\ 0 & \sqrt{n} \end{pmatrix} \tag{6.4.14}$$

A magnification is given in optics by the matrix exponential:

$$\begin{pmatrix} \sqrt{n} & 0 \\ 0 & 1/\sqrt{n} \end{pmatrix} \equiv \exp(\xi \boldsymbol{K}_0) = (\cosh\xi)\boldsymbol{1} + (\sinh\xi)\boldsymbol{K}_0, \quad \xi = -\ln\sqrt{n} \tag{6.4.15}$$

where $\boldsymbol{K}_0$ is the corresponding matrix from §2.4; obviously, a reduction would have the parameter $-\xi$. Likewise, a rotation is also represented in optics by a matrix exponential:

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \equiv \exp(\phi \boldsymbol{I}_0) = (\cos\phi)\boldsymbol{1} + (\sin\phi)\boldsymbol{I}_0 \tag{6.4.16}$$

There is no optical *lens action* or *free-space propagation* involved in the structure of the matrix $\boldsymbol{Q}(n,\phi)$. In the language of matrix optics [(Abe & Sheridan, 1994), equations (4 – 7)] these actions would pe expressed by special lower triangular, respectively upper triangular 2×2 matrices like the matrix $\boldsymbol{R}$ involved in the construction of the Yang's *R-gauge*. One can say that Yang's gauge is manifested by a *magnification* followed by a *lens action*, as equation (4.4.13) shows. However, the matrix $\boldsymbol{Q}(n,\phi)$ has the important property of being unimodular, and this means that it can beneficiate of an Iwasawa decomposition that might be able to help in finding its optical structure. Let us elaborate a little on this aspect of the problem, as a final touch of our work.

### 6.5 Gravitation: a Moral of Yang's Concept of Field

Hopefully, no trouble will arise, in using the symbol $\boldsymbol{K}$ for a 2×2 rotation matrix, having, however, nothing to do with the matrix denoted by the same letter from the triad (2.5.1): we need to retain this notation here though, for historical reasons, connected to the so called *Iwasawa decomposition*, or the ***KAN*** *representation* of the 2×2 matrices, as they often call this decomposition. Just to make sure of no confusion, we will attach the angle of rotation in expressing the rotation: say $\boldsymbol{K}(\varphi)$. In a formulation pertinent to the description of propagation along the optical beams (Simon & Mukunda, 1993), the theorem of decomposition is: a general SL(2,R) matrix, like the one realizing the transformation (6.4.1), can assume an Iwasawa decomposition in the form of the product of matrices:

$$\boldsymbol{a} \stackrel{def}{=} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \boldsymbol{N}(x)\cdot\boldsymbol{A}(y)\cdot\boldsymbol{K}(\varphi) \tag{6.5.1}$$

involving the following three matrices:

$$\boldsymbol{N}(x) \stackrel{def}{=} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \quad \boldsymbol{A}(y) \stackrel{def}{=} \begin{pmatrix} y & 0 \\ 0 & 1/y \end{pmatrix}, \quad \boldsymbol{K}(\varphi) \stackrel{def}{=} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \tag{6.5.2}$$

The order of matrices in the product (6.5.1) can conveniently vary: it depends on the task one follows with decomposition. And this task, as the optical applications plainly show, derives from the action of the optical medium on light: the matrix $\boldsymbol{a}$ represents this medium in its relation to light. Its action is complex and can be decomposed in three simpler actions – rotation, magnification and lens action – effected in a certain order indicated by the product of the three matrices. In the decomposition (6.5.1) we have a rotation first, followed by a magnification, after which the medium acts like a lens. And we chose this order primarily because it is quite instructive on the manner to calculate the parameters ($x$, $y$, $\varphi$), in terms of the entries of the matrix $\boldsymbol{a}$, to be represented by the product. Case in point: the lines of the matrices are the components of two Euclidean vectors that can serve for quantifying charges in the sense of Katz [see (Mazilu, 2020), §4.4]. To wit, in the order from equation (6.5.1) we have:

$$x = \pm\frac{\alpha\cdot\gamma+\beta\cdot\delta}{\alpha^2+\beta^2},\quad y=\pm\sqrt{\alpha^2+\beta^2},\quad \tan\varphi=-\frac{\beta}{\alpha} \tag{6.5.3}$$

where the *condition of unit determinant* of the matrix group is used. Therefore, the decomposition (6.5.1) describes the propagation of light along a beam, as we just said, by three simple operations in sequence: a rotation of angle $\varphi$, followed by a magnification of factor $y$, and then by a lens action of power $x$. [see also (Gerrard & Burch, 1994), especially Chapter II, for appropriate explanations; (Simon & Mukunda, 1998) can be consulted for further mathematical details on beam propagation].

Let us notice a tell of equation (6.5.3), on the manner of realization of the action of a general unimodular matrix (6.4.1) along the beam: it is saying that the Euclidean vector having as components the entries of first line of the matrix (6.5.1) dominates the construction of the Iwasawa decomposition in this case. To wit: for once, the orientation of this vector provides the angle of rotation of the fields in the beam propagation, while its length gives the magnification factor. On the other hand, the projection of the vector given by the entries of the second line of matrix along the direction of the vector given by the first line, offers the parameter of the lens action of the medium in propagation. The vector given by the entries of the second line of the matrix seems to play here only a secondary role, which may appear as unnatural: we would expect that all the entries of the matrix should play 'equal parts' in this construction. Fact is that there is also a situation where the second line of the matrix dominates, while the first line is 'auxilliary', but in that situation the Iwasawa decomposition does not involve a 'lens'. We can say that it is rather referring to a wave or a particle, once it involves a *displacement*, that can even be a *propagation*. Let us briefly tell this story too.

If instead of the matrices (6.5.2) we use the matrices of decomposition of $\boldsymbol{a}$ as:

$$\boldsymbol{N}(x)\overset{def}{=}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},\quad \boldsymbol{A}(y)\overset{def}{=}\begin{pmatrix} y & 0 \\ 0 & 1/y \end{pmatrix},\quad \boldsymbol{K}(\varphi)\overset{def}{=}\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \tag{6.5.4}$$

we get the following parameters of decomposition as functions of the entries of the matrix:

$$x = \pm\frac{\alpha\cdot\gamma+\beta\cdot\delta}{\gamma^2+\delta^2},\quad y^{-1}=\pm\sqrt{\gamma^2+\delta^2},\quad \tan\varphi=\frac{\gamma}{\delta} \tag{6.5.5}$$

Obviously, this time the second line of the matrix $\boldsymbol{a}$ 'prevails', and we did not change but the matrix $\boldsymbol{N}$, representing the 'lens' action, into one representing a free propagation. In the language of the geometrical optics, this matrix now represents *a displacement of magnitude x* through the medium, asking therefore either for the propagation of a wave, or for the motion of a particle, as we said. In this setup, the parameters of magnification and rotation are decided by the vector having as components the entries of the second line of the matrix.

Besides its algebraical interest in showing how an Iwasawa decomposition works, the previous developments are mainly intended to show how the physics may work in the case of an optical medium. In the specific Maxwell fish-eye medium, the characteristic matrix can be reduced to the form (6.4.4) or (6.4.12), and their Iwasawa decomposition reveals the important parts played by *the lines* of the matrix in deciding the physical magnitudes describing either the propagation, or the motion through the medium. Mention should be made in this respect, just for the sake of incidental comparison, that in determining the Iwasawa decomposition, mathematicians usually take *the columns* of the matrix as vectors, not the lines as we did here. We also need to keep in mind that the previous conclusions are expression of an invariance law for the six components of the Maxwell tensor representing pairs of dipoles. A dipole imposes association of charges of the same kind: that is the theory is either valid *only for light* – involving purely electric dipoles – or *only for matter* – involving purely magnetic dipoles. Apparently, the case of mixed poles in the composition of a dipole – *the dyons*, as they call such a dipole – cannot be represented *via* this invariance law. In order to represent it we seem to need expressly a reference of the vacuum to matter and vice versa, which, however does not change the nature of the geometry involved in the description of the physical situation. Taking heed of the Iwasawa decomposition illustrated above, let us work on an alternative idea, closer, in fact, to the spirit of matrix optics.

As we have shown in the previous section, the 'initial conditions' for the two vector fields in one of the octahedral planes of a reference frame, are provided by fields perpendicular to one another [see equation (6.3.14) and discussion around it]. The fields perpendicular to each other must, therefore, provide a reference by themselves, and the classical Maxwellian electrodynamics indicates that these reference fields should be characteristic to *vacuum*. The problem of vacuum acquires thereby an important significance, and we propose here the following solution for introducing the gravitation within the Einsteinian concept (Einstein, 1919). This solution is suggested by the old de Broglie's treatment of the 'application of field on charges' [(de Broglie, 1935); see also (Mazilu, 2020), §2.4]. It is, perhaps, the case to frame the idea of de Broglie, in a natural philosophical line of concepts, started by the Fresnel's physical theory of light.

We take de Broglie's concept 'in duality', so to speak, whereby it reads: 'the application of the field on charges, and of charges on the field'. The second part of this duality statement involves an instantaneous kinematics of the charge over the Riemannian space of an instanton, *i.e*. a kinematics involving an infrafinite scale of time. Along this idea, and helped by the grand analogy, let us present a case of reverse interpretation, whereby the charge is 'engaged', as it were, by gravitation into forming a continuum of instanton type. The algebraical structure of such an instanton will be judged as a Riemannian space of the type presented by us in the §6.1, which can be used in order to illustrate the Schwarzschild statement on the essence of Einsteinian natural philosophy (see §§3.3 and 5.5).

The analogy basis is that with the atmosphere electricity: a kind of behavior of charges in gravitational field. The phenomenon of lightning can be thought as an instantaneous annihilation of charges, and we are entitled to think, in the case of vacuum, of a reverse effect: the creation of charges from vacuum into electric dipoles. In the real case of atmospheric electricity, the whirls of atmosphere can be assumed *to only help* building the electric dipoles, but in vacuum only the gravitation can be suspected of doing this job. This idea can even be extended for the matter, in which case the magnetic charges enter the stage, along the Katz's natural philosophy. In other words, the matter is thought as vacuum, the difference being in charges: the matter is a vacuum with magnetic charges in its characteristic dipoles along the rays. A vacuum with electric charges organized into dipoles in order to make a Planck medium, must obviously be taken as *ether*. The motion can then be judged as a transition between matter and ether. The de Sitter continuum comprises both kinds of vacua, in the geometrical form of Kasner's blades (see Chapter 5, especially §§5.3 and 5.4).

Assume, now, that the charges are seen as two statistical vectors, say $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ which make up a reference frame *in matter*, however, not in a standard Euclidean position and magnitude. However, with the Katz's type geometry at our disposal we can describe their relative position and magnitudes by equations of the form

$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = \epsilon\mu\cos\theta, \quad \boldsymbol{e}_1^2 = \epsilon^2, \quad \boldsymbol{e}_2^2 = \mu^2 \tag{6.5.6}$$

where $\epsilon$ and $\mu$ are the 'measured' magnetic charges of the dipoles in matter, as Katz defines them (Katz, 1965), that is, each one of them having in turn two components: electric and magnetic. The ether, though, is characterized in an Euclidean manner, with the aid of a local orthonormal vector frame (see §6.2):

$$\hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2 = 0, \quad \hat{\boldsymbol{e}}_1^2 = 1, \quad \hat{\boldsymbol{e}}_2^2 = 1 \tag{6.5.7}$$

where $\hat{\boldsymbol{e}}_{1,2}$ are assumed, for the moment at least, to be unit vectors with no significant loss of generality in helping to make up our mind.

Assume, further, that the transition between ether and matter is modeled by a transformation between the two bases as given by a nonsingular matrix, where *the motion* must somehow enter the stage:

$$|\boldsymbol{e}\rangle = \boldsymbol{M} \cdot |\hat{\boldsymbol{e}}\rangle, \quad \boldsymbol{M} \stackrel{def}{=} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{6.5.8}$$

Now, the entries ($A$, $B$) of the first line of the matrix $\boldsymbol{M}$ are the components of an Euclidean vector proper, just as the entries ($C$, $D$) of the second line of the matrix. If the matrix is normalized so as to have its determinant unity, it becomes prone to be represented by an Iwasawa decomposition. The four entries of $\boldsymbol{M}$ satisfy some constraints required by the equations (6.5.6), (6.5.7) and (6.5.8), which imply:

$$AC + BD = \epsilon\mu\cos\theta, \quad A^2 + B^2 = \epsilon^2, \quad C^2 + D^2 = \mu^2 \tag{6.5.9}$$

In the spirit of the previous construction of the Iwasawa decomposition, we intend to give a physical meaning to these symbols: thus, just by their relationship $\epsilon$ appears as the magnetic charge according to Katz, having the components $A$ and $B$, while $\mu$ is the 'associated' magnetic charge of the dipole, having the components $C$ and $D$. The relations (6.5.9) determine the matrix $\boldsymbol{a}$ up to an arbitrary phase. Indeed, it is quite obvious in these relations, that we can choose

$$A = \epsilon\cos\phi, \quad B = \epsilon\sin\phi; \quad C = \mu\cos\phi', \quad D = \mu\sin\phi' \tag{6.5.10}$$

so that the two phases, $\phi$ and $\phi'$, are given by the *angles of split* for the two kinds of charges, electric and magnetic, necessarily existing, according to our philosophy, in any process of interpretation. The area supported by the two vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ is

$$|\boldsymbol{e}_1 \times \boldsymbol{e}_2| = \epsilon\mu\sin\theta \quad \leftrightarrow \quad AD - BC = \epsilon\mu\sin(\phi' - \phi) \tag{6.5.11}$$

So, if we choose the *electric split angle* and the *phase lag* to describe the matrix $\boldsymbol{M}$, by setting $\phi' = \theta + \phi$, modulo $2\pi$, of course, we get the following matrix $\boldsymbol{M}$:

$$\boldsymbol{M} \equiv \begin{pmatrix} \epsilon\cos\phi & \epsilon\sin\phi \\ \mu\cos(\phi+\theta) & \mu\sin(\phi+\theta) \end{pmatrix} \tag{6.5.12}$$

On the other hand, if we choose the magnetic split angle and the phase lag between the two charges to describe it, by setting $\phi' - \theta = \phi$, the matrix $\boldsymbol{M}$ will appear as

$$\boldsymbol{M} \equiv \begin{pmatrix} \mu\cos(\phi-\theta) & \mu\sin(\phi-\theta) \\ \epsilon\cos\phi & \epsilon\sin\phi \end{pmatrix} \tag{6.5.13}$$

The geometric results that follow are independent of the description we choose for these two matrices, so that, in order to settle our ideas, we choose whichever may come in handy.

The idea here is that a matrix like (6.5.12) or (6.5.13) is fit to represent a Kasner blade and thus we have two blades, as required by the general relativity with cosmological term, just naturally, as it were. Our contention is that the two Kasner blades represent the stochastic processes of charge creation from the vacuum, according to Bartolomé Coll's concept of universal deformation, which can be proved just by exhibiting the Riemannian metric structure. Indeed, we already have learned that a Cayley-Klein metric can be presented as a metric 'universally deformed', as it were, in the Coll's manner, starting from a Maxwell fish-eye metric (see the discussion in §§3.4 and 3.5). On the other hand, the ensemble of matrices (6.5.12) or (6.5.13) can assume a Cayley-Klein metric with respect to an absolute representing the set of singular matrices (see §4.3 for the idea of construction of this geometry). In order to show that this geometry represents an instanton, according to our holographic definition by coherence (see §4.5 above), let us calculate the $\mathfrak{sl}(2,R)$ coframe for the matrix (6.5.13), according to equation (4.3.18), after performing the transformation:

$$\alpha \equiv \epsilon\cos\phi, \quad \beta \equiv \epsilon\sin\phi; \quad u \equiv \frac{\mu}{\epsilon}\cos\theta, \quad v \equiv \frac{\mu}{\epsilon}\sin\theta \tag{6.5.14}$$

In this case the matrix $\boldsymbol{M}$ from equation (6.5.13) can be written in the form

$$\boldsymbol{M} \equiv \begin{pmatrix} \alpha u - \beta v & \beta u + \alpha v \\ \alpha & \beta \end{pmatrix} \tag{6.5.15}$$

and the coframe (4.3.18) can be calculated to give:

$$\begin{aligned} \omega^1 &= \frac{\alpha^2}{\alpha^2+\beta^2}\frac{du}{v} - \frac{\alpha\beta}{\alpha^2+\beta^2}\frac{dv}{v} + \frac{\beta d\alpha - \alpha d\beta}{\alpha^2+\beta^2} \\ \omega^2 &= \frac{2\alpha\beta}{\alpha^2+\beta^2}\frac{du}{v} + \frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}\frac{dv}{v} \\ \omega^3 &= \frac{\beta^2}{\alpha^2+\beta^2}\frac{du}{v} + \frac{\alpha\beta}{\alpha^2+\beta^2}\frac{dv}{v} + \frac{\beta d\alpha - \alpha d\beta}{\alpha^2+\beta^2} \end{aligned} \tag{6.5.16}$$

As one can see right away, this coframe depends on just three variables

$$\begin{aligned} \omega^1 &= d\phi + \cos^2\phi\frac{du}{v} - \sin\phi\cos\phi\frac{dv}{v} \\ \omega^2 &= \sin 2\phi\frac{du}{v} + \cos 2\phi\frac{dv}{v} \\ \omega^3 &= d\phi + \sin^2\phi\frac{du}{v} + \sin\phi\cos\phi\frac{dv}{v} \end{aligned} \tag{6.5.17}$$

The same calculations can be done on the form (6.5.12) of the matrices, and the result is the same. The absolute metric of this expression of the coframe (see §§4.5 and 6.1) is given by the quadratic differential form:

$$(\omega^2)^2 - 4\omega^1\omega^3 = \left(\frac{dv}{v}\right)^2 - (2d\phi)^2 - 2(2d\phi)\left(\frac{du}{v}\right) \equiv \frac{(du)^2+(dv)^2}{v^2} - \left(2d\phi + \frac{du}{v}\right)^2 \tag{6.5.18}$$

which, up to sign and a redefinition of phase, is the metric (6.1.9), and can be taken as a universal Coll deformation of the Maxwell fish-eye metric, given by the Beltrami-Poincaré metric of the Lobachevsky plane. Therefore, one can, indeed, say that the matrix (6.5.15) and, implicitly, its counterpart deriving from (6.5.13), characterize the Maxwell fish-eye optical medium, more precisely a Kasner blade of it, which, among other functions, has to accomplish the important one of the correspondence between fields in vacuum and the charges from the de Sitter continuum.

Assume now that the transition from vacuum to matter or *vice versa*, in an Einsteinian natural-philosophical stand, *involves a Weiss molecular field proper*, as suggested by de Broglie, representing the gravitation, according to Boltyanskii's view (see §4.3 above), where the intervention of the motion is explicit. A Boltyanskii case can be made for the 'radial motion', which, in cases where the interpretation is involved, can be taken as that 'fall towards a center of force' of Newton. The interpretation is realized by instantaneous ensembles of particles in their 'fall to a center of force'. The pertinent result of Vladimir Boltyanskii, to be used here, is the one from equation (4.3.4), which can be written in matrix form, where the space coordinate represents a generic radial coordinate with respect to a center. Assume that we have an ensemble of such situations over the space of instanton represented by the Riemannian metric (6.5.18), so that we can write:

$$\begin{pmatrix} dt' \\ dx' \end{pmatrix} = \boldsymbol{A}\cdot\begin{pmatrix} dt \\ dx \end{pmatrix}, \quad \boldsymbol{A} \equiv \boldsymbol{1} + v\boldsymbol{B}, \quad \boldsymbol{B} \stackrel{def}{=} \begin{pmatrix} 0 & -(c_1^{-1}\cdot c_2^{-1}) \\ 1 & -(c_1^{-1}+c_2^{-1}) \end{pmatrix} \tag{6.5.19}$$

The matrix $\boldsymbol{B}$, which we would like to call the *Boltyanskii*'*s matrix*, must be assumed universal for the construction we have in mind. It has its entries defined by the two fundamental quantities representing an *invariant velocity* and the *Boltyanskii*'*s gravitation level*, respectively

$$c^{-2} \equiv c_1^{-1}\cdot c_2^{-1}, \quad g \equiv c_1 + c_2 \tag{6.5.20}$$

with a self-obvious notation. The universality means here 'independent of motion': the only specific parameter regarding the motion is the velocity of particles starting from rest, and it does not enter in any of the entries of $\boldsymbol{B}$. In terms of these invariants, we can exhibit a matrix $\boldsymbol{R}$ analogous to that of Yang from the case of Yang-Mills fields (§4.4). It can be written as

$$\boldsymbol{B} = \begin{pmatrix} 0 & -c^{-2} \\ 1 & -c^{-2}g \end{pmatrix} = c^{-1}\begin{pmatrix} 0 & -c^{-1} \\ c & -c^{-1}g \end{pmatrix} \equiv c^{-1}\boldsymbol{R} \tag{6.5.21}$$

Here $c$ is a fundamental velocity of the kind Clerk Maxwell once introduced from electromagnetic considerations. It is supposed to reduce to that Maxwell constant, in the case of *zero Boltyanskii gravitational level*, so that we maintained this suggestive notation here, in the hope of uncovering a possible connection between gravity and electromagnetism. The working principle in achieving such a result is based on the idea of Iwasawa decomposition, as delineated above [see equations (6.5.1 – 5)]. This being said, we proceed on to solve an important problem of the human knowledge: that of the construction of a *Weiss field*. Let us explain its terms and incentives, as we go on presenting the solution.

Suppose a Hertz material particle located in an ether described by a Maxwell stress tensor, as above. In each and every one of its points, this ether is described by a matrix acting on the two electromagnetic fields, in such a way that the entries of the Maxwell stress tensor are preserved, as described in §6.4 above, assuming, of course, that those results apply. We need to describe the situation of existence of our Hertz particle in the vacuum thus portrayed. This situation means motion, rest in particular, of the particle and all the vacuum phenomena it entails. A modern natural-philosophical reasoning tells us that the internal structure of the particle is also influenced by the vacuum in which it exists, so, naturally, one can assume that the situation of the particle in vacuum can be described by the mathematical connection of their two structures. Unfortunately, there is no structure for the particle as there seems to be for the ether, incarnated into the Maxwell stress tensor. Fortunately, on the other hand, we have the possibility of further hypotheses, and 'the hypotheses are nets: only those who cast them can catch the fish we call ideas', says an utterance attributed to the eternal young Novalis. And, continuing with this romantic streak of language, we can say that 'standing on the shoulders of giants like Einstein, de Sitter, and Mie – to say nothing of a few others – we are able to formulate a sound hypothesis as to the internal structure of the particle (see §5.1 above): it is Einstein electromagnetic structure in the Mie's take, as Einstein himself would certainly be entitled to deem it.

Thus, we can assume that the structure of the particle is described in the Mie's fashion by a Maxwellian tensor built of the two vectors from equation (6.5.6), in a general position in the instanton space of the particle's structure. This structure cannot be more than an interpretation structure, based on charges from a de Sitter background, attached to different positions inside the instanton, in a Feynman-type interpretation (see §5.1). However, in this case we can say more about this Einstein-Maxwell field: it is a field of the nature of the Weiss molecular field, once defined by Pierre Weiss to help describe the property of ferromagnetism [(Weiss, 1907); see §2.2 above]. The reason of the existence of this field seems quite clear: a Hertz particle cannot exist in matter but only in the structure of a Hertz material point, as Hertz himself defined it. The whole ensemble of material particles from the interpretative structure of the material point, exert influence upon each and every one of its component particles, and we claim that Weiss, with his molecular field, has defined just such a physical situation. One might even say, above and beyond the Hertz's definition, as it were, that this field is the very condition of existence of a Hertz material point: it is a material point composed of an ensemble of material particles having 'the same Weiss field'. This is, after all, the usual image of the matter in the case of the study of phase transition, anyway [(Kadanoff, 1976), Figures 1.3 and 1.4 of this fundamental work on the theory of scale transitions, are edifying in illustrating our standpoint on what we like to call *Weiss fields*].

Regarding this way of understanding the physical issues, it may help to observe that we place the general Weiss field as ranking equal to *the Fresnel field of light*, as defined by Poincaré (see §6.3 above): for once, the two fields are defined by the *local action* of the nonlocal fields. Significantly enough, the *Thomson field* is such a Maxwellian case as we have shown in the §2.2 [see equation (2.2.11) and the discussion around it]. We can even say that Thomson's theory is one of the most instructive *examples of interpretation*. But the defining instance of such a field is given by a matrix that we choose in the form (6.5.13), which represents the general situation of a Weiss field: this turns out to be described by *two averages* not just by one, as in the original case of Weiss. These are the *Novozhilov averages* of a tensorial field, given by us in equation (6.3.10) for the case of Maxwellian tensor: the first of them can count, indeed, as *a pressure* satisfying Einstein's original idea in eliminating the cosmological constant (see introduction to §5.1). However, the second average is just as important in the definition of a complete Weiss field, and if it could be neglected in the case of pure charges, it certainly becomes significant in the case of action at distance of those charges: that is just the Fresnel's case of light, which thus explains the Fresnel hypotheses. Therefore, the correct definition of a Weiss field appears to involve more than one average for the local action of matter. We shall need to insist on this point, but with another occasion.

Now, the situation of a particle in vacuum can be seen as a permanent transformation of the vacuum fields into matter fields and vice versa. So we can construct a generic field based on the Yang's scheme (4.4.13), where instead of Yang's $\boldsymbol{R}$, we use the matrix (6.5.21), together with the structural rotation matrix from equation (2.2.4). This matrix is intended to represent the continuous field generated by particles in their 'moments of rotation', according to Newton's view. The manner of construction is a 'polarly decomposed matrix', having the factors:

$$\boldsymbol{R} = \begin{pmatrix} 0 & -c^{-1} \\ c & -c^{-1}g \end{pmatrix}, \quad \begin{aligned} \boldsymbol{K}(\phi) &\equiv \exp(\boldsymbol{I}_0\phi) \\ &= (\cos\phi)\boldsymbol{1} + (\sin\phi)\boldsymbol{I}_0, \end{aligned} \quad \boldsymbol{K}(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{6.5.22}$$

whereby the transition between matter and vacuum, or vice versa is given by the matrix having the structure of the complex numbers in terms of modulus and phase:

$$\boldsymbol{M} \overset{def}{=} \boldsymbol{R} \cdot \boldsymbol{K}(\phi) \quad \therefore \quad \boldsymbol{M} = \begin{pmatrix} -c^{-1}\sin\phi & -c^{-1}\cos\phi \\ c\cos\phi - c^{-1}g\sin\phi & -c\sin\phi - c^{-1}g\cos\phi \end{pmatrix} \tag{6.5.23}$$

wherein Boltyanskii's gravitation level plays an explicit part. According to equation (6.5.3) this represents an already Iwasawa decomposed matrix: its factors are those from equation (6.5.2), having, up to sign, the following parameters:

$$x = g, \quad y = c^{-1}, \quad \varphi = \phi \tag{6.5.24}$$

In terms of the parameters thus discovered, the Yang's $R$-gauge matrix from equation (4.4.13) would be given by the product:

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix} \cdot \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix}, \quad R \equiv g, \quad \phi \equiv c^2 \tag{6.5.25}$$

This makes out of the parameter $R$ of C.-N. Yang *a Boltyanskii gravitation level*, but also tells us something about the parameter $n$ from the equation (6.4.13), representing the matrix of a de Sitter background: it turns out to be the logarithmic potential generating the Yang-Mills fields [see equation (4.4.10)]. With such a refurbishment of the Yang-Mills fields, we are able to turn back to Maxwell, for a reevaluation. These Yang-Mills fields have more to do with the *constitutive constants* of the Maxwellian fields, rather than electromagnetic forces!

## Conclusion: a Profession of Faith

In order to close this work on the note we opened it, let us just say that it certifies an important fact: there are solutions of the scientific problems out of the reach of *human experience*. And as far as the Aristotelian environment is concerned, only that experience can be part of it. In which case the quarks have no place in an Aristotelian atmosphere, to come back to the example of Paul de Haas from the beginning of this work. In such cases the solutions sought for only have to satisfy just some logical requirements, and their reality is not *experiential*, but *ideal*, conceptual at best. The case of interpretation is epitome here: there cannot exist, *in the reality of our experience*, ensembles of classical material points interpreting the continua, but we can think of them logically. The two Einsteinian relativities are enough proof for this statement. However, they are not a singular case, for the history is full of such examples.

Speaking of the Aristotelian environment, it seems that the man, even in his manifestation as a social man, was much closer to the Creator in the old times of the dawn of civilizations, when the gods were able to spur him on to a right judgments just by affecting his defining existential structure. Those were the times when the word would not have progressed yet into serving the social purpose of misleading. To wit: we can think especially of the quintessential episode of Delos plague, sent to Delians by the god Apollo in order to temper some of their social verve. This social verve, the *turba* of old Seneca, was the generator, just like today for that matter, of rude but rather vain passions, estranging the man from himself. The Delian episode simply shows that there is something beyond experience to be necessarily accepted. This episode is an epitome for the actual situation of the theoretical physics with one big difference: while in the Delian plague case there was no hope whatsoever, in the case of modern physics there is plenty of hope for straightening the things, given enough time! So let us expound a little that old story…

The tale, like any tale, is shrouded in legend, and varies from an author to another, since the existing physical data is scarce, to say the least, and as such mostly unreliable. To some, it is not even about plague; to others, it is indeed about plague, but to the Athenians not Delians. Fact is that plagues are documented archaeologically to Athens, – to Attica in general, due, of course, to its maritime opening which facilitated the presence of germs coming from afar – the most violent of these being the one from the times of the Peloponnesian War (431 BC – 404 BC), waged between the city-states of the Delian League led by Athens, and those of the Peloponnesian League led by Sparta. Pericles, the Olympian, is known to have died during that plague, thus causing, implicitly of course, an end of the Athenian hegemony among Greeks. Still another fact is that the island of Delos, which hosted one of the Apollo's temples of high mark, and even sheltered the Delian League's treasury for a while – according to the existing legend Apollo was born on the island – was by then under the control of Athens. It seems, however, that at the historical times we are talking about here, the temple of Apollo from Delos, no matter how significant, was not quite as high in the consideration of Greeks, especially in the matters of supplying oracles, at least not as high as it was the temple of Apollo from Delphi.

No matter what the case may have been in reality, though, it is not worth attaching to it here an exaggerated importance regarding the details. For, what interests us in the present argument seems to be certain: plagues infested Attica just about the times we are talking about. So it does not disarrange unfolding our argument if we admit – just for settling the ideas illustrated by that argument – that they also infested the Delians, compelling them into soliciting the oracle of Apollo of Delphi, not that of Apollo from Delos, say just in order to make sure the oracle they get was a reliable one, indeed! What stands out from this legend, as a truth beyond those physical facts, is that in those times the man had the firm belief that he can address the Creator directly, and that this one even had the duty, as it were, to answer to these requests, if they were to be answered orderly, by oracles: *the Creator belonged to the defining existential structure of man*. That is, the Creator was subordinated to the process of realization and completion of that structure, just as He is considered today, for that matter, by the overwhelming majority of people.

So it comes that a Delian mission sent to Delphi in order to bring the oracle regarding an end of the plague crisis, brought back an apparently very clear answer of the God: *the altar* of Delos temple *must be doubled*! The Delians set immediately on to work, rejoicing over such a simple remedy: the altar being a cubic shape stone, they simply made another identical stone and set it on the top of the existing one, or sideways, does not really matter, for the oracle did not specify the situation of such a 'double'. The result was that the plague grew fiercer! The community judgment probably corrected immediately the first move: wait a minute! *The altar must remain still an… altar*, *therefore a cubic shape*! As a matter of fact, this would explain the missing specification of the position of 'doubling' from the oracle: it was unnecessary by the very definition of the 'altar'! Therefore the Delians built a cubic solid having a *double edge*, but Apollo still has not relented, so that people would continue to die wholesale.

A new mission sent to Delphi, probably bearing gifts commensurate with the fear and panic inherent to the situation, brought an oracle apparently untranslatable, but especially baffled by a laconism which, we have to admit, had nothing to do with Sparta, the moment's enemy of Delians: "Learn geometry!" Only later on, and consulting many sages – Plato himself was allegedly approached, who at that time would just come back from a visit in Egypt, and happened to be somewhere close by – would the Delians understand that the problem of duplication of the cube was wrongly solved, thus naturally offending their patron god. The first time they would make a lucky strike with doubling, but not for an altar proper, while the second time they would luckily hit the idea of altar, but unfortunately did nothing of a duplication: the new altar was *eight times* the initial one. In fact, they needed to have a cube of the volume *twice* the initial one!

"Learn geometry!" A subtle commandment, showing first and foremost that the gods never forgive, since the request made to Apollo was hopeless: the Delians were convicted with no appeal, but the hateful god did not even bother himself into telling them. This is a fact that we have only learned along centuries, even milennia: the problem that allegedly generated the crisis has no solution that would be accessible from a social point of view, *i.e.* technologically. For once, perhaps the gods got smarter, after the hoax of Mecone, where Prometheus proved that, although in the heights of Olympus, the gods are still among men, for the mountains themselves are from the world of men. So, they may have reached the conclusion that making it too easy for man, does not bring over their creation the welfare anticipated by the creator. Or, perhaps, knowing that a plague cannot be stopped at all, the gods would have wished to give the man something to think about for a few milennia, in order to reach later on, when he would become more appreciative, the status of… appreciating that a plague cannot be stopped but by the structure that hosts it: the mankind! No matter how we take it, fact is that the laconic commandment took the anticipated effect: – beginning with the Greeks the geometry started being eagerly cultivated – and the problem of duplication of the cube bears today witness for the necessity of an elaborated mathematics to control a right thinking. Unfortunately – perhaps as a revenge of old gods! – it does not witness at all for a simple thinking, commensurate with the Laconian… laconism of the old commandment!

The history of the dualism wave-particle illustrates the old legend in the contemporary actuality: the physics is not exclusively a technological matter, and perhaps it would be worth, but mostly instructive for us, to place it, in the conclusions of the present work, along the lines of construction of a thinking emanating from that old legend. So much the more as we were in fact forced, so to speak, to conceive this work the way we presented it, according to an observation springing, as if quite naturally, from the studies necessary for its elaboration. Namely, nothing from the old cogitation has been lost through the milennia! This fact is fully illustrated by the mathematics associated with the natural philosophy of the last three or four centuries of thinking, that got through to us along socially recorded pathways.

Historically speaking, the wave-particle dualism started in the old times with the idea of wave: if this idea would not exist, we would not have the Fermat's principle whose gnoseological role is most aptly illustrated by the concept of ray of the physical optics (see §1.2). Then, the Huygens' principle must have taken existence, that should have sanctioned, *more geometrico* just like in the old legend, the concept of material particle associated with the Fermat's kinematical principle. In geometrical formulation the Huygens' principle relates to a global wave surface, as the envelope of local wave surfaces. In this form, the idea is appropriate even for matter: the matter in revolution, of the Kepler classical problem, is representable by a family of centered closed surfaces, whose envelope is, in this case not a sphere or an ellipsoid, but a canal surface of the most general species one can imagine. Therefore, the geometrical idea of canal surfaces actually represents a Huygens' principle for the matter, provided, obviously, we accept some natural facts, as in the old legend: *the matter has space expansion*. From this point of view, the wave-particle dualism is as natural as it gets: both the corpuscle and the wave are just two particular aspects of the same physical property, namely that of space expansion. This dualism came to being in the form stated by Louis de Broglie (Mazilu, 2020), who brought the physical idea of capillary tube 'forestage', as it were, in order to generalize the Huygens' principle along the concept of optical ray (de Broglie, 1926). The canal surface generated by the limit of the matter of the body in revolution in the classical Kepler problem is, in fact, such a capillary tube! Then everything in physics can be arranged in a logical order, starting from this geometrical idea (see §5.1).

However, the physics cannot be restricted to just geometry: it goes always further on, to causes. So it comes that, initially, the Huygens' principle had to be interpreted in that *the light acts upon space* (producing the sources of light in any position in space). From a natural-philosophical point of view, in the latter times this action is explained by the fact that the light, being electromagnetic by its nature, acts upon ether, which, in this instance is just space. Thus, the electromagnetism allows us to dispense with – if only to a certain extent, in fact – the idea of materiality of the ether. At a definite historical moment, the existence of ether was even denied, based on the reason that the special relativity would warrant us to dispense even with ether *per se*, not just its substantiality. Soon enough, though, one took notice of the fact that our way to see the world through physics does not give us the right to such a view: the ether must be maintained in the image of man for the world he inhabits! Our conclusion is that the ether is an expression of the limit of materiality of our world: the matter *per se is not directly accessible to us*, *but only ponderably*, *through inertia*, and the human experience just proves that the ether cannot be cogitated *but only as matter of null density*. Therefore, taking the density as a measure of ponderability of matter within a physical structure, the ether is only a particular matter. Thus we can relegate the description of the action of light to the inherent matter 'kneading', by declaring that, really, the light acts only on the *structured matter* in general – which is, we have to admit, an observation of current experience.

However, at this stage, the *structured matter* can not be *a physical structure* yet. The old plague's legend also contains an usually disparaged teaching: in the construction of an independent world, the man cannot involve the Creator! The communication with the Creator is *always mediated*: the Creator belongs to a structure that can only be imagined but never effectively reached, like would be, for instance, the top of Olympus. Exactly like the altar of the god Apollo, which cannot be but cogitated, along with the cubic root of the number 2 staying at the basis of its technological realization! Like in the old legend, a structure in matter cannot be a physical structure yet, but only a defining existential structure, as we have called it before. It is 'defining', mostly because it 'levies', as it were, the physical structure, through its rational definition in the exterior of matter. This 'exterior' is the place and the manner in which the matter lends itself to be penetrated by space, or reciprocally: the space lends itself to be penetrated by matter, no importance. This is why, along the whole present work, when we talked of the matter *per se*, we attached only the determinative '*structure*', having none other at our disposal, but we have stopped to only this word, not qualifying further by '*physical*', for it is not the case: *the matter cannot have a physical structure*. It is only its knowledge that is mediated by a physical structure, and this knowledge is by no means reduced to our senses! That is why we think that the fundamental law of this world is that of quantization: it is imposed on us by the necessity of scale transition!

The fact that the *analogy*, as a fundamental way of knowledge, must explain away rather the details of the standard concept first, should be law and guide in thinking: the standard of an analogy is actually the only one among the physical structures *directly accessible to human senses*. Momentarily, however, as always in fact, the natural philosophy is almost exclusively tributary to that old materiality concept that makes a law out of the technology, so that it cannot see the celestial but only as a physical structure, exactly like in the old plague tale. It might be perhaps the case to recount again Plato, who is said to have been elaborating much on the meaning of the short message of the god Apollo to the Delians, showing that it would represent actually the general urge to cultivate the arts leading to happiness, not the art of war leading to destruction.

It is even plausible that the contingency would have served to the great sage in order to illustrate a lesson in the open, as it were: with the destruction, the nature itself is occcupied, according to the laws of eternal change; the duty of society is to promote the idea of Man as individual, not that of war, which is actually the demolisher of the physical structure within which that very idea is coming to fruition. It is, indeed, plausible! It goes with the personality of Plato, as a matter of fact. For, in his *Republic*, often cited as a source on the role of the geometry, mostly in connection with the Delian problem, Plato records the following dialog involving Socrates, his great teacher, the one who has had presented to Athens the right measure of *the word* through the *idea of Man*:

> Well then, on one point at any rate we shall encounter no opposition from those who are even slightly acquainted with geometry, when we assert that this *science* holds a position which flatly contradicts the language employed by those who handle it.
>
> How so?
>
> They talk, I believe, in a *very ridiculous and poverty stricken style*. For they speak invariably of squaring, and producing, and adding, and so on, as if they were engaged in some business, *and as if all their propositions had a practical end in view*: whereas in reality I conceive that *the science is pursued wholly for the sake of knowledge*.
>
> Assuredly it is.
>
> There is still a point about which we must be agreed, is there not?
>
> What is it?
>
> That the *science is pursued for the sake of the knowledge of what eternally exists*, *and not of what comes for a moment into existence*, *and then perishes*.
>
> We shall soon be agreed about that. *Geometry*, *no doubt*, *is a knowledge of what eternally exists*.
>
> If that be so, my excellent friend, *geometry must tend to draw the soul towards truth*, and to give the finishing stroke to the philosophic spirit, thus contributing to raise up what, at present, *we so wrongly* keep down. (Plato, *Republic*, Book VII, 527; *our Italics*)

To all appearances, the stimulus contained – at all levels! – in the Plato's dialogue was of no avail. As a matter of fact, it is for such ideas that the city of Athens forced Socrates to drink hemlock! Fact is that the incentive still remained to no avail over centuries and in our times, when the society at large is referring, more often by the day, to the wisdom of … warriors and politicians, if we may be allowed for such a paradoxical expression. The society, especially as we have it today, with democracy and all it brings along, is precisely antagonistic to *any wisdom* (Le Bon, 1906). There is no conscience in it; we cannot see how a society might be able take heed of the observation of Constantin Noica from his 'accommodating word' to the Romanian version of the *Republic* of Plato:

> … we all became accustomed to read faultily this dialog. And we continue to do so in our time, when harsh words fell upon the political and aesthetic vision of Plato. Only, a society which grants *too much to the politics*, abiding by the ridiculous words of Napoleon to Goethe: «Le destin c'est la politique», and which, at the same time, grants *too little* to the art, stimulating it into being quite often sheer pleasure, and sometimes even regular imposture, would do better to tend to its own sorrows. Not daring to read Plato's *Republic* if it is not able to read it right, but leave it to other times that would not settle into judging the great masterpieces by their own desolation (*our translation*; *original Italics*)

We take the liberty to correct the great philosopher in two essential points of this quote, if for nothing else, just in order to update it: today we do not deal anymore with 'too much' but rather with 'all of it', and surely we do not have to deal with 'too little', but with a flat 'nothing'. This much we need to realize, at any level, especially scientific!

The scientific world makes a point of honor from searching for figments of imagination in the details of experimental data. However, the search remains vain if a positive conclusion is sought for, as long as we do not accept that a mystery exists, that is: a thing that is not accessible but only for *acknowledgment* and *acceptance*. And, maybe, it would be more appropriate for science to frankly admit the *concept of mystery*, as represented by the untouchable. Without this, it seems that we cannot have a unitary idea of the universe: it appears to be the true data-binding agent! In physics this can be already acknowledged: we call it *matter per se* and, as we have shown in the present work, it serves only for *interpretation*. However, when it comes to its acceptance as such, the case is harder: physics has always strived to offer an explanation to interpretation in terms of physical structures, therefore accessible to our senses. The present work submits the essentials of this endeavor, insofar as the physics of relativity is involved. The explanation it suggests remains, obviously, tributary to the old legendary manner of existence: not only it must be done – otherwise the plague is getting worse… – but it must be done exclusively *more geometrico* in order to be in line with our very existence!

It is only through the lack of such an explanation that we can… explain, for instance, how come that after an otherwise extremely penetrant analysis, however based *exclusively* on apparently logical combinations of experimental conclusions, Malcolm Mac Gregor – *may he rest in peace*! – was compelled, we must admit, into concluding that …

> … if the electron is truly large, and if it always operates in such a way as to cancel out its finite size, then we may legitimately inquire if there is any point in paying attention to its size. If the electron is always contained inside an inscrutable black box, we may as well work with just the properties of the box. Hopefully, the *Creator has not been that unkind* to us, and has given us an electron we can eventually understand. [(Mac Gregor, 1992); *our Italics*]

The present work shows that, according to physics, such a statement ignores the fundamental fact that, in a black box, the 'legal' fundamental matter structure *is a dipole*. The existing conception, taken from experience, that the elementary particles must have a physical structure, just proves this statement.

However, if it is to involve a Creator at any rate, we dare take the above quote from the specific angle it suggests, that is, by placing ourselves within a kind of religious ambient: we must confess being Christian. In this capacity we need to acknowledge *especially the kindness* of the Creator we recognize, a benevolence extended so far as to show us how to explain the world He provided for us: *only through a mediation*! In this respect there is no room for doubt, for the conclusion is quite clearly established for the mankind, even historically and especially religiously.

Indeed, it is highly significant for us that the *Creator we acknowledge* even sent His Son, embodied in the *physical structure* of Man, to live among us, in order to make us understand, once and for all, that great sin – maybe even the so much discussed original sin! – is the one of trying to… understand the world exclusively by the intermediation of the concept of a *physical structure*. We have to give up that exclusive attempt of understanding the world, in spite of the fact that the earthly life insistently enforces it upon us… perhaps just in order to deceive us, indeed! Nevertheless, taking a positive view, that enforcement might have a good reason after all: perhaps it was intended to clue us into a right thinking, like in the old legend of the Delian plague. For, being humans, it seems that only deceived can we go along a way, be it even the right way. In this respect, we are to consider ourselves lucky enough that the ways in which the ideas stream into eternity, defying any logical explanation, still remain *a mystery* for us! Ending our tale with the words of Steven Weinberg, *may he rest in peace*! *... let us not retreat from this accomplishment*!

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