Solution of the Brocard Ramanujan equation

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Abstract

The solution of the Diophantine Brocard-Ramanujan equation is obtained by proving the impossibility of representing other factorials, except for the known three, as a product of two natural numbers differing by 2. This is justified by the fact that no factorial is greater than 7! cannot be represented as a product of an increasing sequence of natural numbers, the first of which is equal to the argument of the factorial.

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1 Introduction

The Diophantine Brocard-Ramanujan equation is a mathematical problem in which it is required to find integer values of n and m for which $n! + 1 = m^2$. This mathematical problem was formulated by Henri Brocard in two papers in 1876 and 1885 [1, 2]. Later in 1913 this problem was reintroduced by Srinivasa Ramanuja [3, 4] because he did not know about the papers of Henri Brocard. To date, only three solutions of the Brocard-Ramanujan equation are known:

$4! + 1 = 5^2$, $5! + 1 = 11^2$ and $7! + 1 = 71^2$.

2 Representation of the square of a natural number

Since the Diophantine Brocard–Ramanujan equation is related to the square of a natural number, we first show the representation of the square of a natural number.

Lemma 2.1. The square of any natural number greater than 1 is expressed as a product of two natural numbers differing by 2 according to the formula

(2.1) $$a^2 = (a-1)(a+1) + 1.$$

Lemma 1 implies that the solution of the Brocard–Ramanujan problem exists only in the case when

(2.2) $$n! = (a-1)(a+1).$$

Further, if we accept the designation a-1=b, then equation (2.2) will look like

(2.3) $$n! = b(b+2).$$

On the basis of equation (2.3), we can assert that to obtain the Brocard–Ramanujan equality $n! + 1 = m^2$, the factorial must be equal to the product of two natural numbers differing by 2. Based on Lemma 1 and equality (2.3), we formulate the following Lemma.

Lemma 2.2. If there is a natural number b such that $n! = b(b+2)$, then the equality $n! + 1 = (b+1)^2$ will certainly be obtained.

Note that only the above three factorials, which are solutions of the Diophantine Brocard–Ramanujan equation, have a representation in the form of equality (2.3):

I) $4! = 4 \cdot (4+2)$; II) $5! = 10 \cdot (10+2)$; III) $7! = 70 \cdot (70+2)$.

Next, we will find out in which cases the factorial can be represented as a product of two natural numbers differing by 2.

3 Representation of the factorial

3.1 Special representation of the factorial

It is not difficult to prove that the product of any four consecutive natural numbers has a representation as a product of two natural numbers that differ by 2. However, not all-natural numbers that can be represented as a product of two natural numbers that differ by 2 have a representation as a product of four consecutive natural numbers.

Note that all known three factorials 4!, 5! and 7!, which are solutions of the Diophantine Brocard–Ramanujan equation, are represented as a product of four consecutive natural numbers:

$1 \cdot 2 \cdot 3 \cdot 4 = 24 = 4!$; $2 \cdot 3 \cdot 4 \cdot 5 = 120 = 5!$; $7 \cdot 8 \cdot 9 \cdot 10 = 5040 = 7!$

If the factorial is represented as a product of four consecutive natural numbers, one of which is equal to the argument of the factorial, then the following representations of the factorial are possible:

(3.1) $$n! = (n-3) \cdot (n-2) \cdot (n-1) \cdot n;$$

(3.2) $$n! = (n-2) \cdot (n-1) \cdot n \cdot (n+1);$$

(3.3) $$n! = (n-1) \cdot n \cdot (n+1) \cdot (n+2);$$

(3.4) $$n! = n \cdot (n+1) \cdot (n+1) \cdot (n+3).$$

In the form of formula (3.1) one can represent 4! and 5!, which are solutions of the Brocard–Ramanujan problem,

$4! = (4-3) \cdot (4-2) \cdot (4-1) \cdot 4 = 1 \cdot 2 \cdot 3 \cdot 4 = 24;$

$5! = (5-3) \cdot (5-2) \cdot (5-1) \cdot 5 = 2 \cdot 3 \cdot 4 \cdot 5 = 120.$

Using formula (3.4), we can represent 7!, which is also a solution to the Brocard–Ramanujan problem,

$7! = 7 \cdot (7+1) \cdot (7+1) \cdot (7+3) = 7 \cdot 8 \cdot 9 \cdot 10 = 5040.$

Others, factorials, representable as a product of four consecutive natural numbers, one of which is equal to the argument of the factorial, have not been found.

It is easy to determine that the product of four consecutive natural numbers, the first of which is equal to the argument of the factorial, will be less than n! for $n > 7$.

For example, $8 \cdot (8+1) \cdot (8+2) \cdot (8+3) = 8 \cdot 9 \cdot 10 \cdot 11 = 7920 < 8! = 40320.$

It follows from the above that any factorial is greater than 7! has no representation as a product of four consecutive natural numbers, the first of which is equal to the argument of the factorial. This means that the factorial is greater than 7! can only be expressed as a product of four consecutive natural numbers, the first of which is equal to the argument of the factorial, and the fifth number.

Thus, if $n > 7$, then the factorial can be expressed by the following formula

(3.5) $n! = n \cdot (n+1) \cdot (n+2) \cdot (n+3) \cdot k_n$, where k_n is the multiplier of $n!$.

Note. The index n of the factor k_n indicates the argument of the factorial in question.

Definition 1. The representation of a factorial as a product of consecutive natural numbers that differ from the factors of the factorial, with the exception of the argument of the factorial, is called a special representation of the factorial.

3.2 The factor of the factorial

When dividing any larger factorial by a smaller factorial, you get the product of two or more consecutive natural numbers if the difference between the arguments of the factorials is greater than 1. In this regard, if $m > 1$, then when dividing $(n+m)!$ to $n!$ must necessarily be a product of consecutive natural numbers.

(3.6) $$\frac{(n+m)!}{n!} = \frac{(n-1)!n(n+1)\cdot(n+2)\cdot(n+3)\cdot(n+4)\cdot\ldots\cdot(n+m)}{n!}.$$

Taking into account formula (3.5), we represent the above formula (3.6) in the following form

(3.7) $$\frac{(n+m)!}{n!} = \frac{(n-1)!\cdot n\cdot(n+1)\cdot(n+2)\cdot(n+3)\cdot(n+4)\cdot\ldots\cdot(n+m)}{n\cdot(n+1)\cdot(n+2)\cdot(n+3)\cdot k_n}.$$

From formula (3.7), after simple transformations,

$$\frac{(n+m)!}{n!} = \frac{(n-1)!\cdot(n+4)\cdot\ldots\cdot(n+m)}{k_n},\ k_n = \frac{(n-1)!\cdot n!\cdot(n+4)\cdot\ldots\cdot(n+m)}{(n+m)!} = \frac{(n-1)!\cdot n!\cdot(n+4)\cdot\ldots\cdot(n+m)}{n!\cdot(n+1)\cdot(n+2)\cdot(n+3)\cdot(n+4)\cdot\ldots\cdot(n+m)},$$

we obtain

(3.8) $$k_n = \frac{(n-1)!}{(n+1)\cdot(n+2)\cdot(n+3)}.$$

Thus, if the factorial with argument $n \geq 7$ is represented as a product of four consecutive natural numbers and the fifth number, then the fifth factor k_n must correspond to formula (3.8). It should be clear that if $k_n = 1$, then this shows that the factorial can be represented as a product of four consecutive natural numbers.

Definition 2. If the factorial is represented as a product of an increasing sequence of natural numbers, the first of which is equal to the argument of the factorial, then the factor that is not part of the consecutive numbers is called the factor of the factorial.

Next, we check formula (3.8) for the cases $n = 7$ and $n = 8$.

For $n = 7$ we get $k_7 = \frac{(7-1)!}{(7+1)\cdot(7+2)\cdot(7+3)} = \frac{720}{720} = 1$, which means that 7! has a representation as a product of four consecutive natural numbers.

For $n = 8$ we get $k_8 = \frac{(8-1)!}{(8+1)\cdot(8+2)\cdot(8+3)} = \frac{5040}{990} \approx 5{,}09091$.

From this calculation it follows that 8! has no representation as a product of four consecutive natural numbers, since $k_8 \neq 1$, $k_8 \neq 8 + 4$.

Next, find out the possibility of representation 8! as a product of five consecutive natural numbers and the sixth number. To do this, we represent the factorial in the following form

$$n! = n \cdot (n+1) \cdot (n+2) \cdot (n+3) \cdot (n+4) \cdot k_n. \tag{3.9}$$

Taking into account (3.9) and (3.8), we obtain the following formula for calculating the factor of the factorial

$$k_n = \frac{(n-1)!}{(n+1)\cdot(n+2)\cdot(n+3)(n+4)}. \tag{3.10}$$

Note that if 8! has a representation as a product of five consecutive natural numbers, then by the formula (3.10) we should get $k_8 = 1$.

For $n = 8$, by formula (3.10) we obtain

$$k_8 = \frac{(8-1)!}{(8+1)\cdot(8+2)\cdot(8+3)\cdot(8+4)} = \frac{5040}{11880} \approx 0{,}424242.$$

As you can see, 8! has no representation as a product of five consecutive natural numbers, since $k_8 \neq 1$, $k_8 \neq 8 + 5$.

From the above calculations it follows that 8! cannot be represented as a product of increasing consecutive natural numbers, the first of which is equal to the argument of the factorial.

3.3 Main theorem

Based on the foregoing, it can be argued that the search for a solution to the Brocard–Ramanujan equation, or the proof of the Brocard–Ramanujan conjecture, is reduced to the problem of finding the possibility of representing a factorial as a product of consecutive natural numbers, the number of factors of which is less than the number of factors of the considered factorial. This problem can be formulated by the following question.

Question 1. Does there exist a natural number $n > 7$ such that the product of all-natural numbers to the left of it is equal to the product of several consecutive natural numbers starting from the number $n + 1$ located to the right of it?

In other words, we must determine whether the following equation has a natural solution for $n > 7$,

(3.11) $\quad (n-1)! = (n+1) \cdot (n+2) \cdot \ldots \cdot (n+m)$, где $n, m \in \mathbb{N}$.

We formulate the above in the form of a theorem.

Theorem 3.1. The following equation has no natural solution for $n > 7$,

$(n-1)! = (n+1) \cdot (n+2) \cdot \ldots \cdot (n+m)$, where $n, m \in \mathbb{N}$.

Next, we prove Theorem 3.1, namely, we prove that equation (3.11) has no natural solution for $n > 7$.

Obviously, any factorial $(n+m)!$ more than $n!$ can be represented as

(3.12) $\ (n+m)! = (n-1)! \cdot n \cdot [(n+1) \cdot (n+2) \cdot \ldots \cdot (n+m)]$.

Assume that formula (3.11) has a natural solution, then using (3.11) and (3.12) we can write

$(n+m)! = [(n+1) \cdot (n+2) \cdot \ldots \cdot (n+m)] \cdot n \cdot [(n+1) \cdot (n+2) \cdot \ldots \cdot (n+m)]$, or

(3.13) $\quad (n+m)! = n \cdot\ [(n+1) \cdot (n+2) \cdot \ldots \cdot (n+m)]^2$.

Next, we will check the possibility of representing a factorial greater than 7! by formula (3.13),

$(7+1)! = 7 \cdot\ (7+1)^2;\ 40320 > 448; \frac{40320}{448} = 90;$

$(7+2)! = 7 \cdot [(7+1) \cdot (7+2)]^2;\ 362880 > 36288; \frac{362880}{36288} = 10;$

$(7+3)! = 7 \cdot [(7+1) \cdot (7+2) \cdot (7+3)]^2;\ 3628800 = 3628800; \frac{3628800}{3628800} = 1.$

It follows from the above examples that only $(7+3)! = 10!$ can be represented by formula (3.13), and the factorials are greater than 10! they do not have such a representation, since for such factorials, where $m > 3$, we obtain the following inequality

(3.14) $\quad (7+m)! < 7 \cdot\ [(7+1) \cdot (7+2) \cdot \ldots \cdot (7+m)]^2$.

Next, check the possibility of representation $(10-1)!$ by formula (3.11),

$(10-1)! = (10+1) \cdot (10+2) \cdot (10+3) \cdot (10+4) \cdot (10+5),$

$362880 > 360360;$

$(10-1)! = (10+1)\cdot(10+2)\cdot(10+3)\cdot(10+4)\cdot(10+5)\cdot(10+6),$

$362880 < 5765760.$

Thus, it can be argued that only $(7+3)! = 10!$ can be represented by formula (3.13), but all factorials are greater than 7! cannot be represented by formula (3.11).

This means that Theorem 3.1 has been proved, i.e. there is no natural number $n > 7$ such that the product of all-natural numbers to the left of it would be equal to the product of several consecutive natural numbers, starting with the number $n+1$, which are to the right of him.

In other words, any factorial is greater than 7! cannot be represented as a product of an increasing sequence of natural numbers, the first of which is equal to the argument of the factorial.

4 Solution of the Brocard–Ramanujan equation

If the factorial is greater than 3! has a representation as a product of two natural numbers differing by 2, then it can be represented as a product of several consecutive natural numbers, one of which is equal to the argument of the factorial, and another number that is not part of the sequence. If, in a special representation of the factorial, we count the maximum possible number of consecutive numbers, the first of which is equal to the argument of the factorial, then the larger the factorial, the greater the number of consecutive numbers. For example, 7! can be represented as a product of four consecutive natural numbers, the first of which is equal to the factorial argument, and for 8! the number of consecutive numbers will be 5, and for 10!, 12!,13! and 14! the number of consecutive numbers will be 6, 7, 8 and 9 respectively.

Note that the above numbers of consecutive numbers in the special representation of the factorial correspond to the case when the factor of the factorial k_n is closest to 1.

It follows from the above that if there are other factorials, except for 4!, 5! and 7!, representable as a product of two natural numbers differing by 2, then all of them must be represented as a product of increasing consecutive natural numbers, the first number of which is equal to the factorial argument, and the factor of the factorial must be equal to 1 or be a member of the sequence.

Let there be a natural number representable as a product of two natural numbers differing by 2, which is represented as a product of three increasing consecutive natural numbers

$b(b+2) = n \cdot (n+1) \cdot (n+2) \cdot k_n$.

In this case, for this number to be equal to the factorial, the factor k_n must necessarily be equal to $b(b+2) = 7 \cdot (7+1) \cdot (7+2) \cdot k_n$, then we get $b(b+2) = (n-1)! \cdot n$.

In this case, if $b(b+2)$ is equal to the factorial, i.e. $b(b+2) = (n-1)! \cdot n$, then the factor k_n must be equal to $k_n = \frac{(n-1)!}{(n+1)\cdot(n+2)}$.

For example, let there be a natural number $b(b+2)$ having the following representation $b(b+2) = 7 \cdot (7+1) \cdot (7+2) \cdot k_n$, then we have

$$k_n = \frac{(7-1)!}{(7+1)\cdot(7+2)} = \frac{720}{72} = 10; \quad 70 \cdot 72 = 7 \cdot 8 \cdot 9 \cdot 10.$$

Thus, for a number of the form $b(b+2) = n \cdot (n+1) \cdot (n+2) \cdot k_n$ to be equal to the factorial, as shown in the above example, k_n must be a member of the sequence or must be equal to 1.

In chapter 3.3 we proved that no factorial is greater than 7! has no representation as a product of increasing consecutive natural numbers, the first number of which is equal to the argument of the factorial, so there is no factorial that can be represented as a product of two natural numbers that differ by 2, except for 4!, 5! and 7!.

This means that the Brocard-Ramanujan Problem is solved, or we can say that the Brocard-Ramanujan conjecture is correct and proven.

5 Conclusion

First, it is proved that the Diophantine equation $n! + 1 = m^2$ can have a solution only if the factorial can be represented as a product of a pair of natural numbers whose difference is equal to 2, i.e. the form $n! = b \cdot (b + 2)$ can be a solution to the Brocard–Ramanujan equation.

Then, in Chapter 3.3, it was proved that there is no factorial greater than 7! that can be represented as a product of increasing consecutive natural numbers whose first number is equal to the argument of the factorial. After that, it is proved that if a number that can be represented as a product of a pair of natural numbers whose difference is equal to 2 is equal to a factorial, then it must be represented as a product of increasing consecutive natural numbers whose first number is equal to the argument of the factorial.

This implies that there is no factorial of the form $n! = b \cdot (b + 2)$, except for the well-known three factorials: 4!, 5! and 7!.

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