

**Proofs of Four Conjectures in Number
Theory : Beal's Conjecture, Riemann
Hypothesis, The abc and $c < R^{1.63}$ Conjectures
- July 2022 -**

ABDELMAJID BEN HADJ SALEM, INGÉNIEUR GÉNÉRAL

Abstract

This monograph presents the proofs of 4 important conjectures in the field of number theory:

- The Beal's conjecture.
- The Riemann Hypothesis.
- The $c < R^{1.63}$ conjecture.
- The abc conjecture is true.

We give in detail all the proofs.

Résumé:

Cette monographie présente les preuves de 4 conjectures importantes dans le domaine de la théorie des nombres à savoir:

- La conjecture de Beal.
- L'Hypothèse de Riemann.
- La conjecture $c < R^{1.63}$.
- La conjecture abc est vraie.

Nous donnons les détails des différentes démonstrations.

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CONJECTURE, RIEMANN
HYPOTHESIS, THE *abc* AND
 $c < R^{1.63}$ CONJECTURES
- JULY 2022 -**

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FIGURE 1. Photo of the Author

*To the memory of my Parents, to my wife Wahida, my daughter
Sinda and my son Mohamed Mazen*

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NUMBER THEORY : BEAL'S CONJECTURE,
RIEMANN HYPOTHESIS, THE abc AND
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CHAPTER 1

A COMPLETE PROOF OF BEAL'S CONJECTURE

Abstract. — In 1997, Andrew Beal announced the following conjecture: *Let A, B, C, m, n , and l be positive integers with $m, n, l > 2$. If $A^m + B^n = C^l$ then A, B , and C have a common factor.* We begin to construct the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$ with p, q integers depending of A^m, B^n and C^l . We resolve $x^3 - px + q = 0$ and we obtain the three roots x_1, x_2, x_3 as functions of p, q and a parameter θ . Since $A^m, B^n, -C^l$ are the only roots of $x^3 - px + q = 0$, we discuss the conditions that x_1, x_2, x_3 are integers and have or not a common factor. Three numerical examples are given.

Résumé. — En 1997, Andrew Beal avait annoncé la conjecture suivante: *Soient A, B, C, m, n , et l des entiers positifs avec $m, n, l > 2$. Si $A^m + B^n = C^l$ alors A, B , et C ont un facteur commun.*

Nous commençons par construire le polynôme $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$ avec p, q des entiers qui dépendent de A^m, B^n et C^l . Nous résolvons $x^3 - px + q = 0$ et nous obtenons les trois racines x_1, x_2, x_3 comme fonctions de p, q et d'un paramètre θ . Comme $A^m, B^n, -C^l$ sont les seules racines de $x^3 - px + q = 0$, nous discutons les conditions pour que x_1, x_2, x_3 soient des entiers. Trois exemples numériques sont présentés.

1.1. Introduction

In 1997, Andrew Beal [4] announced the following conjecture :

Conjecture 1.1. — *Let A, B, C, m, n , and l be positive integers with $m, n, l > 2$. If:*

$$(1.1) \quad A^m + B^n = C^l$$

then A, B , and C have a common factor.

The purpose of this paper is to give a complete proof of Beal's conjecture. Our proof of the conjecture contains many cases to study using elementary number theory. Our idea is to construct a polynomial $P(x)$ of order three having as roots A^m, B^n and $-C^l$ with the condition (1.1). The paper is organized as follows. In section 1, It is an introduction of the paper. The trivial case, where $A^m = B^n$, is studied in section 2. The preliminaries needed for the proof are given in section 3 where we consider the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$. We express the three roots of $P(x) = x^3 - px + q = 0$ in function of two parameters p, θ that depend on A^m, B^n, C^l . The section 4 is the preamble of the proof of the main theorem. We find the expression of A^{2m} equal to $\frac{4p}{3} \cos^2 \frac{\theta}{3}$. As A^{2m} is an integer, it follows that $\cos^2 \frac{\theta}{3}$ must be written as $\frac{a}{b}$ where a, b are two positive coprime integers. We discuss the conditions of divisibility of p, a, b so that the expression of A^{2m} is an integer. Depending of each individual case, we obtain that A, B, C have or do have not a common factor. Section 5 treats the cases of the first hypothesis $3 \mid a$ and $b \mid 4p$. We study the cases of the second hypothesis $3 \mid p$ and $b \mid 4p$ in section 6. Finally, we present three numerical examples and the conclusion in section 7.

In 1997, Andrew Beal [4] announced the following conjecture :

Conjecture 1.2. — *Let A, B, C, m, n , and l be positive integers with $m, n, l > 2$. If:*

$$(1.2) \quad A^m + B^n = C^l$$

then A, B , and C have a common factor.

1.2. Trivial Case

We consider the trivial case when $A^m = B^n$. The equation (1.2) becomes:

$$(1.3) \quad 2A^m = C^l$$

then $2 \mid C^l \implies 2 \mid C \implies C = 2^q \cdot C_1$ with $q \geq 1$, $2 \nmid C_1$ and $2A^m = 2^{ql} C_1^l \implies A^m = 2^{q^{l-1}} C_1^l$. As $l > 2$, $q \geq 1$, then $2 \mid A^m \implies 2 \mid A \implies A = 2^r A_1$ with $r \geq 1$ and $2 \nmid A_1$. The equation (1.3), becomes:

$$(1.4) \quad 2 \times 2^{rm} A_1^m = 2^{ql} C_1^l$$

As $2 \nmid A_1$ and $2 \nmid C_1$, we obtain the first condition :

$$(1.5) \quad \text{there exists two positive integers } r, q \text{ with } r, q \geq 1 \text{ so that } \boxed{ql = mr + 1}$$

Then from (1.4):

$$(1.6) \quad A_1^m = C_1^l$$

1.2.1. Case 1 $A_1 = 1 \implies C_1 = 1$. — Using the condition (1.5) above, we obtain $2 \cdot (2^r)^m = (2^q)^l$ and the Beal conjecture is verified.

1.2.2. Case 2 $A_1 > 1 \implies C_1 > 1$. — From the fundamental theorem of the arithmetic, we can write:

$$(1.7) \quad A_1 = a_1^{\alpha_1} \dots a_I^{\alpha_I}, \quad a_1 < a_2 < \dots < a_I \implies A_1^m = a_1^{m\alpha_1} \dots a_I^{m\alpha_I}$$

$$(1.8) \quad C_1 = c_1^{\beta_1} \dots c_J^{\beta_J}, \quad c_1 < c_2 < \dots < c_J \implies C_1^l = c_1^{l\beta_1} \dots c_J^{l\beta_J}$$

where a_i (respectively c_j) are distinct positive prime numbers and α_i (respectively β_j) are integers > 0 .

From (1.6) and using the uniqueness of the factorization of A_1^m and C_1^l , we obtain necessary:

$$(1.9) \quad \begin{cases} I = J \\ a_i = c_i, \quad i = 1, 2, \dots, I \\ m\alpha_i = l\beta_i \end{cases}$$

As one $a_i \mid A^m \implies a_i \mid B^m \implies a_i \mid B$ and in this case, the Beal conjecture is verified.

We suppose in the following that $A^m > B^n$.

1.3. Preliminaries

Let $m, n, l \in \mathbb{N}^*$ > 2 and $A, B, C \in \mathbb{N}^*$ such:

$$(1.10) \quad A^m + B^n = C^l$$

We call:

$$(1.11) \quad \begin{aligned} P(x) &= (x - A^m)(x - B^n)(x + C^l) = x^3 - x^2(A^m + B^n - C^l) \\ &+ x[A^m B^n - C^l(A^m + B^n)] + C^l A^m B^n \end{aligned}$$

Using the equation (1.10), $P(x)$ can be written as:

$$(1.12) \quad \boxed{P(x) = x^3 + x[A^m B^n - (A^m + B^n)^2] + A^m B^n(A^m + B^n)}$$

We introduce the notations:

$$\begin{aligned} p &= (A^m + B^n)^2 - A^m B^n \\ q &= A^m B^n(A^m + B^n) \end{aligned}$$

As $A^m \neq B^n$, we have $p > (A^m - B^n)^2 > 0$. Equation (1.12) becomes:

$$P(x) = x^3 - px + q$$

Using the equation (1.11), $P(x) = 0$ has three different real roots : A^m, B^n and $-C^l$.

Now, let us resolve the equation:

$$(1.13) \quad P(x) = x^3 - px + q = 0$$

To resolve (1.13) let:

$$x = u + v$$

Then $P(x) = 0$ gives:

$$(1.14) \quad P(x) = P(u+v) = (u+v)^3 - p(u+v) + q = 0 \implies u^3 + v^3 + (u+v)(3uv - p) + q = 0$$

To determine u and v , we obtain the conditions:

$$\begin{aligned} u^3 + v^3 &= -q \\ uv &= p/3 > 0 \end{aligned}$$

Then u^3 and v^3 are solutions of the second order equation:

$$(1.15) \quad X^2 + qX + p^3/27 = 0$$

Its discriminant Δ is written as :

$$\Delta = q^2 - 4p^3/27 = \frac{27q^2 - 4p^3}{27} = \frac{\bar{\Delta}}{27}$$

Let:

$$(1.16) \quad \begin{aligned} \bar{\Delta} &= 27q^2 - 4p^3 = 27(A^m B^n (A^m + B^n))^2 - 4[(A^m + B^n)^2 - A^m B^n]^3 \\ &= 27A^{2m} B^{2n} (A^m + B^n)^2 - 4[(A^m + B^n)^2 - A^m B^n]^3 \end{aligned}$$

Noting :

$$\begin{aligned} \alpha &= A^m B^n > 0 \\ \beta &= (A^m + B^n)^2 \end{aligned}$$

we can write (1.16) as:

$$(1.17) \quad \bar{\Delta} = 27\alpha^2\beta - 4(\beta - \alpha)^3$$

As $\alpha \neq 0$, we can also rewrite (1.17) as :

$$\bar{\Delta} = \alpha^3 \left(27 \frac{\beta}{\alpha} - 4 \left(\frac{\beta}{\alpha} - 1 \right)^3 \right)$$

We call t the parameter :

$$t = \frac{\beta}{\alpha}$$

$\bar{\Delta}$ becomes :

$$\bar{\Delta} = \alpha^3 (27t - 4(t-1)^3)$$

Let us calling :

$$y = y(t) = 27t - 4(t-1)^3$$

Since $\alpha > 0$, the sign of $\bar{\Delta}$ is also the sign of $y(t)$. Let us study the sign of y . We obtain $y'(t)$:

$$y'(t) = y' = 3(1+2t)(5-2t)$$

$y' = 0 \implies t_1 = -1/2$ and $t_2 = 5/2$, then the table of variations of y is given below:

The table of the variations of the function y shows that $y < 0$ for $t > 4$. In our case, we are interested for $t > 0$. For $t = 4$ we obtain $y(4) = 0$ and for $t \in]0, 4[\implies y > 0$. As we have $t = \frac{\beta}{\alpha} > 4$ because as $A^m \neq B^n$:

$$(A^m - B^n)^2 > 0 \implies \beta = (A^m + B^n)^2 > 4\alpha = 4A^m B^n$$

Then $y < 0 \implies \bar{\Delta} < 0 \implies \Delta < 0$. Then, the equation (1.15) does not have real solutions u^3 and v^3 . Let us find the solutions u and v with $x = u + v$ is a positive or a negative real and $u.v = p/3$.

t	$-\infty$	$-1/2$	$5/2$	4	$+\infty$
$1+2t$	-	0	+		+
$5-2t$	+		+	0	-
$y'(t)$	-	0	+	0	-
$y(t)$	$+\infty$	0	54	0	$-\infty$

FIGURE 1. The table of variations

1.3.1. Expressions of the roots. —

Proof. — The solutions of (1.15) are:

$$X_1 = \frac{-q + i\sqrt{-\Delta}}{2}$$

$$X_2 = \overline{X_1} = \frac{-q - i\sqrt{-\Delta}}{2}$$

We may resolve:

$$u^3 = \frac{-q + i\sqrt{-\Delta}}{2}$$

$$v^3 = \frac{-q - i\sqrt{-\Delta}}{2}$$

Writing X_1 in the form:

$$X_1 = \rho e^{i\theta}$$

with:

$$\rho = \frac{\sqrt{q^2 - \Delta}}{2} = \frac{p\sqrt{p}}{3\sqrt{3}}$$

and $\sin\theta = \frac{\sqrt{-\Delta}}{2\rho} > 0$

$$\cos\theta = -\frac{q}{2\rho} < 0$$

Then $\theta [2\pi] \in] + \frac{\pi}{2}, +\pi[$, let:

$$(1.18) \quad \boxed{\frac{\pi}{2} < \theta < +\pi \Rightarrow \frac{\pi}{6} < \frac{\theta}{3} < \frac{\pi}{3} \Rightarrow \frac{1}{2} < \cos\frac{\theta}{3} < \frac{\sqrt{3}}{2}}$$

and:

$$(1.19) \quad \boxed{\frac{1}{4} < \cos^2 \frac{\theta}{3} < \frac{3}{4}}$$

hence the expression of X_2 :

$$(1.20) \quad X_2 = \rho e^{-i\theta}$$

Let:

$$(1.21) \quad u = r e^{i\psi}$$

$$(1.22) \quad \text{and } j = \frac{-1 + i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}}$$

$$(1.23) \quad j^2 = e^{i\frac{4\pi}{3}} = -\frac{1 + i\sqrt{3}}{2} = \bar{j}$$

j is a complex cubic root of the unity $\iff j^3 = 1$. Then, the solutions u and v are:

$$(1.24) \quad u_1 = r e^{i\psi_1} = \sqrt[3]{\rho} e^{i\frac{\theta}{3}}$$

$$(1.25) \quad u_2 = r e^{i\psi_2} = \sqrt[3]{\rho} j e^{i\frac{\theta}{3}} = \sqrt[3]{\rho} e^{i\frac{\theta+2\pi}{3}}$$

$$(1.26) \quad u_3 = r e^{i\psi_3} = \sqrt[3]{\rho} j^2 e^{i\frac{\theta}{3}} = \sqrt[3]{\rho} e^{i\frac{4\pi}{3}} e^{i\frac{\theta}{3}} = \sqrt[3]{\rho} e^{i\frac{\theta+4\pi}{3}}$$

and similarly:

$$(1.27) \quad v_1 = r e^{-i\psi_1} = \sqrt[3]{\rho} e^{-i\frac{\theta}{3}}$$

$$(1.28) \quad v_2 = r e^{-i\psi_2} = \sqrt[3]{\rho} j^2 e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho} e^{i\frac{4\pi}{3}} e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho} e^{i\frac{4\pi-\theta}{3}}$$

$$(1.29) \quad v_3 = r e^{-i\psi_3} = \sqrt[3]{\rho} j e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho} e^{i\frac{2\pi}{3}} e^{-i\frac{\theta}{3}}$$

We may now choose u_k and v_h so that $u_k + v_h$ will be real. In this case, we have necessary :

$$(1.30) \quad v_1 = \bar{u}_1$$

$$(1.31) \quad v_2 = \bar{u}_2$$

$$(1.32) \quad v_3 = \bar{u}_3$$

We obtain as real solutions of the equation (1.14):

$$(1.33) \quad x_1 = u_1 + v_1 = 2\sqrt[3]{\rho} \cos \frac{\theta}{3} > 0$$

$$(1.34) \quad x_2 = u_2 + v_2 = 2\sqrt[3]{\rho} \cos \frac{\theta+2\pi}{3} = -\sqrt[3]{\rho} \left(\cos \frac{\theta}{3} + \sqrt{3} \sin \frac{\theta}{3} \right) < 0$$

$$(1.35) \quad x_3 = u_3 + v_3 = 2\sqrt[3]{\rho} \cos \frac{\theta+4\pi}{3} = \sqrt[3]{\rho} \left(-\cos \frac{\theta}{3} + \sqrt{3} \sin \frac{\theta}{3} \right) > 0$$

We compare the expressions of x_1 and x_3 , we obtain:

$$(1.36) \quad \begin{aligned} 2\sqrt[3]{\rho}\cos\frac{\theta}{3} &\stackrel{?}{>} \sqrt[3]{\rho}\left(-\cos\frac{\theta}{3} + \sqrt{3}\sin\frac{\theta}{3}\right) \\ 3\cos\frac{\theta}{3} &\stackrel{?}{>} \sqrt{3}\sin\frac{\theta}{3} \end{aligned}$$

As $\frac{\theta}{3} \in] + \frac{\pi}{6}, + \frac{\pi}{3}[$, then $\sin\frac{\theta}{3}$ and $\cos\frac{\theta}{3}$ are > 0 . Taking the square of the two members of the last equation, we get:

$$(1.37) \quad \frac{1}{4} < \cos^2\frac{\theta}{3}$$

which is true since $\frac{\theta}{3} \in] + \frac{\pi}{6}, + \frac{\pi}{3}[$ then $x_1 > x_3$. As A^m, B^n and $-C^l$ are the only real solutions of (1.13), we consider, as A^m is supposed great than B^n , the expressions:

$$(1.38) \quad \left\{ \begin{aligned} A^m = x_1 = u_1 + v_1 &= 2\sqrt[3]{\rho}\cos\frac{\theta}{3} \\ B^n = x_3 = u_3 + v_3 &= 2\sqrt[3]{\rho}\cos\frac{\theta + 4\pi}{3} = \sqrt[3]{\rho}\left(-\cos\frac{\theta}{3} + \sqrt{3}\sin\frac{\theta}{3}\right) \\ -C^l = x_2 = u_2 + v_2 &= 2\sqrt[3]{\rho}\cos\frac{\theta + 2\pi}{3} = -\sqrt[3]{\rho}\left(\cos\frac{\theta}{3} + \sqrt{3}\sin\frac{\theta}{3}\right) \end{aligned} \right.$$

□

1.4. Preamble of the Proof of the Main Theorem

Theorem 1.3. — Let A, B, C, m, n , and l be positive integers with $m, n, l > 2$. If:

$$(1.39) \quad A^m + B^n = C^l$$

then A, B , and C have a common factor.

Proof. — $A^m = 2\sqrt[3]{\rho}\cos\frac{\theta}{3}$ is an integer $\Rightarrow A^{2m} = 4\sqrt[3]{\rho^2}\cos^2\frac{\theta}{3}$ is also an integer. But :

$$(1.40) \quad \sqrt[3]{\rho^2} = \frac{p}{3}$$

Then:

$$(1.41) \quad A^{2m} = 4\sqrt[3]{\rho^2} \cos^2 \frac{\theta}{3} = 4\frac{p}{3} \cdot \cos^2 \frac{\theta}{3} = p \cdot \frac{4}{3} \cdot \cos^2 \frac{\theta}{3}$$

As A^{2m} is an integer and p is an integer, then $\cos^2 \frac{\theta}{3}$ must be written under the form:

$$(1.42) \quad \boxed{\cos^2 \frac{\theta}{3} = \frac{1}{b} \quad \text{or} \quad \cos^2 \frac{\theta}{3} = \frac{a}{b}}$$

with $b \in \mathbb{N}^*$; for the last condition $a \in \mathbb{N}^*$ and a, b coprime.

Notations: In the following of the paper, the scalars $a, b, \dots, z, \alpha, \beta, \dots, A, B, C, \dots$ and Δ, Φ, \dots represent positive integers except the parameters θ, ρ , or others cited in the text, are reals.

1.4.1. Case $\cos^2 \frac{\theta}{3} = \frac{1}{b}$. — We obtain:

$$(1.43) \quad A^{2m} = p \cdot \frac{4}{3} \cdot \cos^2 \frac{\theta}{3} = \frac{4p}{3b}$$

As $\frac{1}{4} < \cos^2 \frac{\theta}{3} < \frac{3}{4} \Rightarrow \frac{1}{4} < \frac{1}{b} < \frac{3}{4} \Rightarrow b < 4 < 3b \Rightarrow b = 1, 2, 3$.

1.4.1.1. $b = 1$. — $b = 1 \Rightarrow 4 < 3$ which is impossible.

1.4.1.2. $b = 2$. — $b = 2 \Rightarrow A^{2m} = p \cdot \frac{4}{3} \cdot \frac{1}{2} = \frac{2p}{3} \Rightarrow 3 \mid p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, we obtain:

$$(1.44) \quad A^{2m} = (A^m)^2 = \frac{2p}{3} = 2p' \Rightarrow 2 \mid p' \Rightarrow p' = 2^\alpha p_1^2$$

with $2 \nmid p_1, \quad \alpha + 1 = 2\beta$

$$(1.45) \quad B^n C^l = \sqrt[3]{\rho^2} \left(3 - 4\cos^2 \frac{\theta}{3} \right) = p' = 2^\alpha p_1^2$$

From the equation (1.44), it follows that $2 \mid A^m \Rightarrow A = 2^i A_1, i \geq 1$ and $2 \nmid A_1$. Then, we have $\beta = i.m = im$. The equation (1.45) implies that $2 \mid (B^n C^l) \Rightarrow 2 \mid B^n$ or $2 \mid C^l$.

1.4.1.2.1. *Case 2* | B^n : — - If $2 \mid B^n \implies 2 \mid B \implies B = 2^j B_1$ with $2 \nmid B_1$. The expression of $B^n C^l$ becomes:

$$B_1^n C^l = 2^{2im-1-jn} p_1^2$$

- If $2im - 1 - jn \geq 1$, $2 \mid C^l \implies 2 \mid C$ according to $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1) is verified.

- If $2im - 1 - jn \leq 0 \implies 2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

1.4.1.2.2. *Case 2* | C^l : — If $2 \mid C^l$: with the same method used above, we obtain the identical results.

1.4.1.3. $b = 3$. — $b = 3 \implies A^{2m} = p \cdot \frac{4}{3} \cdot \frac{1}{3} = \frac{4p}{9} \implies 9 \mid p \implies p = 9p'$ with $p' \neq 1$, as $9 \ll p$ then $A^{2m} = 4p'$. If p' is prime, it is impossible. We suppose that p' is not a prime, as $m \geq 3$, it follows that $2 \mid p'$, then $2 \mid A^m$. But $B^n C^l = 5p'$ and $2 \mid (B^n C^l)$. Using the same method for the case $b = 2$, we obtain the identical results.

1.4.2. *Case* $a > 1$, $\cos^2 \frac{\theta}{3} = \frac{a}{b}$. — We have:

$$(1.46) \quad \cos^2 \frac{\theta}{3} = \frac{a}{b}; \quad A^{2m} = p \cdot \frac{4}{3} \cdot \cos^2 \frac{\theta}{3} = \frac{4 \cdot p \cdot a}{3 \cdot b}$$

where a, b verify one of the two conditions:

$$(1.47) \quad \boxed{\{3 \mid a \text{ and } b \mid 4p\}} \text{ or } \boxed{\{3 \mid p \text{ and } b \mid 4p\}}$$

and using the equation (1.19), we obtain a third condition:

$$(1.48) \quad \boxed{b < 4a < 3b}$$

For these conditions, $A^{2m} = 4 \sqrt[3]{\rho^2} \cos^2 \frac{\theta}{3} = 4 \frac{p}{3} \cdot \cos^2 \frac{\theta}{3}$ is an integer.

Let us study the conditions given by the equation (1.47) in the following two sections.

1.5. Hypothesis : $\{3 \mid a \text{ and } b \mid 4p\}$

We obtain :

$$(1.49) \quad 3 \mid a \implies \exists a' \in \mathbb{N}^* / a = 3a'$$

1.5.1. Case $b = 2$ and $3 \mid a$. — A^{2m} is written as:

$$(1.50) \quad A^{2m} = \frac{4p}{3} \cdot \cos^2 \frac{\theta}{3} = \frac{4p}{3} \cdot \frac{a}{b} = \frac{4p}{3} \cdot \frac{a}{2} = \frac{2 \cdot p \cdot a}{3}$$

Using the equation (1.49), A^{2m} becomes :

$$(1.51) \quad A^{2m} = \frac{2 \cdot p \cdot 3a'}{3} = 2 \cdot p \cdot a'$$

but $\cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2} > 1$ which is impossible, then $b \neq 2$.

1.5.2. Case $b = 4$ and $3 \mid a$. — A^{2m} is written :

$$(1.52) \quad A^{2m} = \frac{4 \cdot p}{3} \cos^2 \frac{\theta}{3} = \frac{4 \cdot p}{3} \cdot \frac{a}{b} = \frac{4 \cdot p}{3} \cdot \frac{a}{4} = \frac{p \cdot a}{3} = \frac{p \cdot 3a'}{3} = p \cdot a'$$

$$(1.53) \quad \text{and } \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3 \cdot a'}{4} < \left(\frac{\sqrt{3}}{2} \right)^2 = \frac{3}{4} \implies a' < 1$$

which is impossible. Then the case $b = 4$ is impossible.

1.5.3. Case $b = p$ and $3 \mid a$. — We have :

$$(1.54) \quad \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{p}$$

and:

$$(1.55) \quad A^{2m} = \frac{4p}{3} \cdot \cos^2 \frac{\theta}{3} = \frac{4p}{3} \cdot \frac{3a'}{p} = 4a' = (A^m)^2$$

$$(1.56) \quad \exists a'' / a' = a''^2$$

$$(1.57) \quad \text{and } B^n C^l = p - A^{2m} = b - 4a' = b - 4a''^2$$

The calculation of $A^m B^n$ gives :

$$(1.58) \quad \begin{aligned} A^m B^n &= p \cdot \frac{\sqrt{3}}{3} \sin \frac{2\theta}{3} - 2a' \\ \text{or } A^m B^n + 2a' &= p \cdot \frac{\sqrt{3}}{3} \sin \frac{2\theta}{3} \end{aligned}$$

The left member of (1.58) is an integer and p also, then $2 \frac{\sqrt{3}}{3} \sin \frac{2\theta}{3}$ is written under the form :

$$(1.59) \quad 2 \frac{\sqrt{3}}{3} \sin \frac{2\theta}{3} = \frac{k_1}{k_2}$$

where k_1, k_2 are two coprime integers and $k_2 \mid p \implies p = b = k_2 \cdot k_3, k_3 \in \mathbb{N}^*$.

1.5.3.1. We suppose that $k_3 \neq 1$. — We obtain :

$$(1.60) \quad A^m(A^m + 2B^n) = k_1.k_3$$

Let μ be a prime integer with $\mu \mid k_3$, then $\mu \mid b$ and $\mu \mid A^m(A^m + 2B^n) \implies \mu \mid A^m$ or $\mu \mid (A^m + 2B^n)$.

** A-1-1- If $\mu \mid A^m \implies \mu \mid A$ and $\mu \mid A^{2m}$, but $A^{2m} = 4a' \implies \mu \mid 4a' \implies (\mu = 2, \text{ but } 2 \mid a') \text{ or } (\mu \mid a')$. Then $\mu \mid a$ it follows the contradiction with a, b coprime.

** A-1-2- If $\mu \mid (A^m + 2B^n) \implies \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. We write $\mu \mid (A^m + 2B^n)$ as:

$$(1.61) \quad A^m + 2B^n = \mu.t'$$

It follows :

$$A^m + B^n = \mu t' - B^n \implies A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t' \mu B^n + B^{2n}$$

Using the expression of p :

$$(1.62) \quad p = t'^2 \mu^2 - 2t' B^n \mu + B^n (B^n - A^m)$$

As $p = b = k_2.k_3$ and $\mu \mid k_3$ then $\mu \mid b \implies \exists \mu'$ and $b = \mu \mu'$, so we can write:

$$(1.63) \quad \mu' \mu = \mu(\mu t'^2 - 2t' B^n) + B^n (B^n - A^m)$$

From the last equation, we obtain $\mu \mid B^n (B^n - A^m) \implies \mu \mid B^n$ or $\mu \mid (B^n - A^m)$.

** A-1-2-1- If $\mu \mid B^n$ which is in contradiction with $\mu \nmid B^n$.

** A-1-2-2- If $\mu \mid (B^n - A^m)$ and using that $\mu \mid (A^m + 2B^n)$, we arrive to :

$$(1.64) \quad \mu \mid 3B^n \begin{cases} \mu \mid B^n \\ \text{or} \\ \mu = 3 \end{cases}$$

** A-1-2-2-1- If $\mu \mid B^n \implies \mu \mid B$, it is the contradiction with $\mu \nmid B$ cited above.

** A-1-2-2-2- If $\mu = 3$, then $3 \mid b$, but $3 \mid a$ then the contradiction with a, b coprime.

1.5.3.2. We assume now $k_3 = 1$. — Then :

$$(1.65) \quad A^{2m} + 2A^m B^n = k_1$$

$$(1.66) \quad b = k_2$$

$$(1.67) \quad \frac{2\sqrt{3}}{3} \sin \frac{2\theta}{3} = \frac{k_1}{b}$$

Taking the square of the last equation, we obtain:

$$\begin{aligned} \frac{4}{3} \sin^2 \frac{2\theta}{3} &= \frac{k_1^2}{b^2} \\ \frac{16}{3} \sin^2 \frac{\theta}{3} \cos^2 \frac{\theta}{3} &= \frac{k_1^2}{b^2} \\ \frac{16}{3} \sin^2 \frac{\theta}{3} \cdot \frac{3a'}{b} &= \frac{k_1^2}{b^2} \end{aligned}$$

Finally:

$$(1.68) \quad 4^2 a' (p - a) = k_1^2$$

but $a' = a''^2$, then $p - a$ is a square. Let:

$$(1.69) \quad \lambda^2 = p - a = b - a = b - 3a''^2 \implies \lambda^2 + 3a''^2 = b$$

The equation (1.68) becomes:

$$(1.70) \quad 4^2 a''^2 \lambda^2 = k_1^2 \implies k_1 = 4a'' \lambda$$

taking the positive root, but $k_1 = A^m(A^m + 2B^n) = 2a''(A^m + 2B^n)$, then :

$$(1.71) \quad A^m + 2B^n = 2\lambda \implies \lambda = a'' + B^n$$

** A-2-1- As $A^m = 2a'' \implies 2 \mid A^m \implies 2 \mid A \implies A = 2^i A_1$, with $i \geq 1$ and $2 \nmid A_1$, then $A^m = 2a'' = 2^{im} A_1^m \implies a'' = 2^{im-1} A_1^m$, but $im \geq 3 \implies 4 \mid a''$. As $p = b = A^{2m} + A^m B^n + B^{2n} = \lambda = 2^{im-1} A_1^m + B^n$. Taking its square, then :

$$\lambda^2 = 2^{2im-2} A_1^{2m} + 2^{im} A_1^m B^n + B^{2n}$$

As $im \geq 3$, we can write $\lambda^2 = 4\lambda_1 + B^{2n} \implies \lambda^2 \equiv B^{2n} \pmod{4} \implies \lambda^2 \equiv B^{2n} \equiv 0 \pmod{4}$ or $\lambda^2 \equiv B^{2n} \equiv 1 \pmod{4}$.

** A-2-1-1- We suppose that $\lambda^2 \equiv B^{2n} \equiv 0 \pmod{4} \implies 4 \mid \lambda^2 \implies 2 \mid (b - a)$. But $2 \mid a$ because $a = 3a' = 3a''^2 = 3 \times 2^{2(im-1)} A_1^{2m}$ and $im \geq 3$. Then $2 \mid b$, it follows the contradiction with a, b coprime.

** A-2-1-2- We suppose now that $\lambda^2 \equiv B^{2n} \equiv 1 \pmod{4}$. As $A^m = 2^{im-1} A_1^m$ and $im - 1 \geq 2$, then $A^m \equiv 0 \pmod{4}$. As $B^{2n} \equiv 1 \pmod{4}$, then B^n

verifies $B^n \equiv 1(\text{mod } 4)$ or $B^n \equiv 3(\text{mod } 4)$ which gives for the two cases $B^n C^l \equiv 1(\text{mod } 4)$.

We have also $p = b = A^{2m} + A^m B^n + B^{2n} = 4a' + B^n \cdot C^l = 4a''^2 + B^n C^l \implies B^n C^l = \lambda^2 - a''^2 = B^n \cdot C^l$, then $\lambda, a'' \in \mathbb{N}^*$ are solutions of the Diophantine equation :

$$(1.72) \quad x^2 - y^2 = N$$

with $N = B^n C^l > 0$. Let $Q(N)$ be the number of the solutions of (1.72) and $\tau(N)$ is the number of suitable factorization of N , then we announce the following result concerning the solutions of the equation (1.72) (see theorem 27.3 in [7]):

- If $N \equiv 2(\text{mod } 4)$, then $Q(N) = 0$.
 - If $N \equiv 1$ or $N \equiv 3(\text{mod } 4)$, then $Q(N) = [\tau(N)/2]$.
 - If $N \equiv 0(\text{mod } 4)$, then $Q(N) = [\tau(N/4)/2]$.
- $[x]$ is the integral part of x for which $[x] \leq x < [x] + 1$.

As λ, a'' is a couple of solutions of the Diophantine equation (1.72), then $\exists d, d'$ positive integers with $d > d'$ and $N = d \cdot d'$ so that :

$$(1.73) \quad d + d' = 2\lambda$$

$$(1.74) \quad d - d' = 2a''$$

** A-2-1-2-1- We suppose as $C^l > B^n$ that $d = C^l$ and $d' = B^n$. It follows:

$$(1.75) \quad 2\lambda = C^l + B^n = A^m + 2B^n$$

$$(1.76) \quad 2a'' = C^l - B^n = A^m$$

From the paragraph A-2-1 above, we have $\lambda = p = A^{2m} + A^m B^n + B^{2n} > (A^m + 2B^n)$, then the case $d = C^l$ and $d' = B^n$ gives a contradiction.

** A-2-1-2-2- Now, we consider the case $d = c_1^{lr-1} C_1^l$ where c_1 is a prime integer with $c_1 \nmid C_1$ and $C = c_1^r C_1$, $r \geq 1$. It follows that $d' = c_1 \cdot B^n$. We rewrite the equations (1.73-1.74):

$$(1.77) \quad c_1^{lr-1} C_1^l + c_1 \cdot B^n = 2\lambda$$

$$(1.78) \quad c_1^{lr-1} C_1^l - c_1 \cdot B^n = 2a''$$

As $l \geq 3$, from the last two equations above, it follows that $c_1 \mid (2\lambda)$ and $c_1 \mid (2a'')$. Then $c_1 = 2$, or $c_1 \mid \lambda$ and $c_1 \mid a''$.

** A-2-1-2-2-1- We suppose $c_1 = 2$. As $2 \mid A^m$ and $2 \mid C^l$ because $l \geq 3$, it follows $2 \mid B^n$, then $2 \mid (p = b)$. Then the contradiction with a, b coprime.

** A-2-1-2-2-2- We suppose $c_1 \neq 2$ and $c_1 \mid a''$ and $c_1 \mid \lambda$. $c_1 \mid a'' \implies c_1 \mid a$ and $c_1 \mid (A^m = 2a'')$. $B^n = C^l - A^m \implies c_1 \mid B^n$. It follows that $c_1 \mid (p = b)$. Then the contradiction with a, b coprime.

The others cases of the expressions of d and d' with d, d' not coprime so that $N = B^n C^l = d.d'$ give also contradictions.

Hence, the case $k_3 = 1$ is impossible.

Let us verify the condition (1.48) given by $b < 4a < 3b$. In our case, the condition becomes :

$$(1.79) \quad p < 3A^{2m} < 3p \quad \text{with} \quad p = A^{2m} + B^{2n} + A^m B^n$$

and $3A^{2m} < 3p \implies A^{2m} < p$ that is verified. If :

$$p < 3A^{2m} \implies 2A^{2m} - A^m B^n - B^{2n} \stackrel{?}{>} 0$$

Studying the sign of the polynomial $Q(Y) = 2Y^2 - B^n Y - B^{2n}$ and taking $Y = A^m > B^n$, the condition $2A^{2m} - A^m B^n - B^{2n} > 0$ is verified, then the condition $b < 4a < 3b$ is true.

In the following of the paper, we verify easily that the condition $b < 4a < 3b$ implies to verify that $A^m > B^n$ which is true.

1.5.4. Case $b \mid p \implies p = b.p', p' > 1, b \neq 2, b \neq 4$ and $3 \mid a$. —

$$(1.80) \quad A^{2m} = \frac{4.p}{3} \cdot \frac{a}{b} = \frac{4.b.p'.3.a'}{3.b} = 4.p'a'$$

We calculate $B^n C^l$:

$$(1.81) \quad B^n C^l = \sqrt[3]{\rho^2} \left(3 \sin^2 \frac{\theta}{3} - \cos^2 \frac{\theta}{3} \right) = \sqrt[3]{\rho^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right)$$

but $\sqrt[3]{\rho^2} = \frac{p}{3}$, using $\cos^2 \frac{\theta}{3} = \frac{3.a'}{b}$, we obtain:

$$(1.82) \quad B^n C^l = \sqrt[3]{\rho^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) = \frac{p}{3} \left(3 - 4 \frac{3.a'}{b} \right) = p \cdot \left(1 - \frac{4.a'}{b} \right) = p'(b - 4a')$$

As $p = b.p'$, and $p' > 1$, so we have :

$$(1.83) \quad B^n C^l = p'(b - 4a')$$

$$(1.84) \quad \text{and } A^{2m} = 4.p'.a'$$

** B-1- We suppose that p' is prime, then $A^{2m} = 4a'p' = (A^m)^2 \implies p' \mid a'$. But $B^n C^l = p'(b - 4a') \implies p' \mid B^n$ or $p' \mid C^l$.

** B-1-1- If $p' \mid B^n \implies p' \mid B \implies B = p'B_1$ with $B_1 \in \mathbb{N}^*$. Hence : $p'^{n-1}B_1^n C^l = b - 4a'$. But $n > 2 \implies (n-1) > 1$ and $p' \mid a'$, then $p' \mid b \implies a$ and b are not coprime, then the contradiction.

** B-1-2- If $p' \mid C^l \implies p' \mid C$. The same method used above, we obtain the same results.

** B-2- We consider that p' is not a prime integer.

** B-2-1- p', a are supposed coprime: $A^{2m} = 4a'p' \implies A^m = 2a''.p_1$ with $a' = a''^2$ and $p' = p_1^2$, then a'', p_1 are also coprime. As $A^m = 2a''.p_1$ then $2 \mid a''$ or $2 \mid p_1$.

** B-2-1-1- $2 \mid a''$, then $2 \nmid p_1$. But $p' = p_1^2$.

** B-2-1-1-1- If p_1 is prime, it is impossible with $A^m = 2a''.p_1$.

** B-2-1-1-2- We suppose that p_1 is not prime, we can write it as $p_1 = \omega^m \implies p' = \omega^{2m}$, then: $B^n C^l = \omega^{2m}(b - 4a')$.

** B-2-1-1-2-1- If ω is prime, it is different of 2, then $\omega \mid (B^n C^l) \implies \omega \mid B^n$ or $\omega \mid C^l$.

** B-2-1-1-2-1-1- If $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$ with $\omega \nmid B_1$, then $B_1^n . C^l = \omega^{2m-nj}(b - 4a')$.

** B-2-1-1-2-1-1-1- If $2m - n.j = 0$, we obtain $B_1^n . C^l = b - 4a'$. As $C^l = A^m + B^n \implies \omega \mid C^l \implies \omega \mid C$, and $\omega \mid (b - 4a')$. But $\omega \neq 2$ and ω is coprime with a' then coprime with a , then $\omega \nmid b$. The conjecture (3.1) is verified.

** B-2-1-1-2-1-1-2- If $2m - nj \geq 1$, in this case with the same method, we obtain $\omega \mid C^l \implies \omega \mid C$ and $\omega \mid (b - 4a')$ and $\omega \nmid a$ and $\omega \nmid b$. The conjecture (3.1) is verified.

** B-2-1-1-2-1-1-3- If $2m - nj < 0 \implies \omega^{n.j-2m} B_1^n . C^l = b - 4a'$. As $\omega \mid C$ using $C^l = A^m + B^n$ then $C = \omega^h . C_1 \implies \omega^{n.j-2m+h.l} B_1^n . C_1^l = b - 4a'$. If $n.j - 2m + h.l < 0 \implies \omega \mid B_1^n C_1^l$, it follows the contradiction that $\omega \nmid B_1$ or $\omega \nmid C_1$. Then if $n.j - 2m + h.l > 0$ and $\omega \mid (b - 4a')$ with ω, a, b coprime and the conjecture (3.1) is verified.

** B-2-1-1-2-1-2- We obtain the same results if $\omega \mid C^l$.

** B-2-1-1-2-2- Now, $p' = \omega^{2m}$ and ω not prime, we write $\omega = \omega_1^f . \Omega$ with ω_1 prime $\nmid \Omega$ and $f \geq 1$ an integer, and $\omega_1 \mid A$. Then $B^n C^l = \omega_1^{2f.m} \Omega^{2m} (b - 4a') \implies \omega_1 \mid (B^n C^l) \implies \omega_1 \mid B^n$ or $\omega_1 \mid C^l$.

** B-2-1-1-2-2-1- If $\omega_1 \mid B^n \implies \omega_1 \mid B \implies B = \omega_1^j B_1$ with $\omega_1 \nmid B_1$, then $B_1^n . C^l = \omega_1^{2mf-nj} \Omega^{2m} (b - 4a')$:

** B-2-1-1-2-2-1-1- If $2f.m - n.j = 0$, we obtain $B_1^n . C^l = \Omega^{2m} (b - 4a')$. As $C^l = A^m + B^n \implies \omega_1 \mid C^l \implies \omega_1 \mid C \implies \omega_1 \mid (b - 4a')$. But $\omega_1 \neq 2$ and ω_1 is coprime with a' , then coprime with a , we deduce $\omega_1 \nmid b$. Then the conjecture (3.1) is verified.

** B-2-1-1-2-2-1-2- If $2f.m - n.j \geq 1$, we have $\omega_1 \mid C^l \implies \omega_1 \mid C \implies \omega_1 \mid (b - 4a')$ and $\omega_1 \nmid a$ and $\omega_1 \nmid b$. The conjecture (3.1) is verified.

** B-2-1-1-2-2-1-3- If $2f.m - n.j < 0 \implies \omega_1^{n.j-2m.f} B_1^n . C^l = \Omega^{2m} (b - 4a')$. As $\omega_1 \mid C$ using $C^l = A^m + B^n$, then $C = \omega_1^h . C_1 \implies \omega^{n.j-2m.f+h.l} B_1^n . C_1^l = \Omega^{2m} (b - 4a')$. If $n.j - 2m.f + h.l < 0 \implies \omega_1 \mid B_1^n C_1^l$, it follows the contradiction with $\omega_1 \nmid B_1$ and $\omega_1 \nmid C_1$. Then if $n.j - 2m.f + h.l > 0$ and $\omega_1 \mid (b - 4a')$ with ω_1, a, b coprime and the conjecture (3.1) is verified.

** B-2-1-1-2-2-2- We obtain the same results if $\omega_1 \mid C^l$.

** B-2-1-2- If $2 \mid p_1$, then $2 \mid p_1 \implies 2 \nmid a' \implies 2 \nmid a$. But $p' = p_1^2$.

** B-2-1-2-1- If $p_1 = 2$, we obtain $A^m = 4a'' \implies 2 \mid a''$ as $m \geq 3$, then the contradiction with a, b coprime.

** B-2-1-2-2- We suppose that p_1 is not prime and $2 \mid p_1$, as $A^m = 2a''p_1$, p_1 is written as $p_1 = 2^{m-1}\omega^m \implies p' = 2^{2m-2}\omega^{2m}$. It follows $B^n C^l = 2^{2m-2}\omega^{2m}(b - 4a') \implies 2 \mid B^n$ or $2 \mid C^l$.

** B-2-1-2-2-1- If $2 \mid B^n \implies 2 \mid B$, as $2 \mid A$, then $2 \mid C$. From $B^n C^l = 2^{2m-2}\omega^{2m}(b - 4a')$, it follows if $2 \mid (b - 4a') \implies 2 \mid b$ but as $2 \nmid a'$, there is no contradiction with a, b coprime and the conjecture (3.1) is verified.

** B-2-1-2-2-2- If $2 \mid C^l$, using the same method as above, we obtain the identical results.

** B-2-2- p', a' are supposed not coprime. Let ω be a prime integer so that $\omega \mid a'$ and $\omega \mid p'$.

** B-2-2-1- We suppose firstly $\omega = 3$. As $A^{2m} = 4a'p' \implies 3 \mid A$, but $3 \mid p' \implies 3 \mid p$, as $p = A^{2m} + B^{2n} + A^m B^n \implies 3 \mid B^{2n} \implies 3 \mid B$, then $3 \mid C^l \implies 3 \mid C$. We write $A = 3^i A_1$, $B = 3^j B_1$, $C = 3^h C_1$ and 3 coprime with A_1, B_1 and C_1 and $p = 3^{2im} A_1^{2m} + 3^{2nj} B_1^{2n} + 3^{im+jn} A_1^m B_1^n = 3^k .g$ with $k = \min(2im, 2jn, im+jn)$ and $3 \nmid g$. We have also $(\omega = 3) \mid a$ and $(\omega = 3) \mid p'$ that gives $a = 3^\alpha a_1 = 3a' \implies a' = 3^{\alpha-1} a_1$, $3 \nmid a_1$ and $p' = 3^\mu p_1$, $3 \nmid p_1$ with $A^{2m} = 4a'p' = 3^{2im} A_1^{2m} = 4 \times 3^{\alpha-1+\mu} .a_1 .p_1 \implies \alpha + \mu - 1 = 2im$. As $p = bp' = b.3^\mu p_1 = 3^\mu .b .p_1$. The exponent of the term 3 of p is k , the exponent of the term 3 of the left member of the last equation is μ . If $3 \mid b$ it is a contradiction with a, b coprime. Then, we suppose that $3 \nmid b$, and the equality of the exponents: $\min(2im, 2jn, im+jn) = \mu$, recall that $\alpha + \mu - 1 = 2im$. But $B^n C^l = p'(b - 4a')$ that gives $3^{(nj+hl)} B_1^n C_1^l = 3^\mu p_1 (b - 4 \times 3^{(\alpha-1)} a_1)$. We have also $A^m + B^n = C^l$ gives $3^{im} A_1^m + 3^{jn} B_1^n = 3^{hl} C_1^l$. Let $\epsilon = \min(im, jn)$, we have $\epsilon = hl = \min(im, jn)$. Then, we obtain the conditions:

$$(1.85) \quad k = \min(2im, 2jn, im+jn) = \mu$$

$$(1.86) \quad \alpha + \mu - 1 = 2im$$

$$(1.87) \quad \epsilon = hl = \min(im, jn)$$

$$(1.88) \quad 3^{(nj+hl)} B_1^n C_1^l = 3^\mu p_1 (b - 4 \times 3^{(\alpha-1)} a_1)$$

** B-2-2-1-1- $\alpha = 1 \implies a = 3a_1 = 3a'$ and $3 \nmid a_1$, the equation (1.86) becomes:

$$\mu = 2im$$

and the first equation (1.85) is written as:

$$k = \min(2im, 2jn, im + jn) = 2im$$

- If $k = 2im$, then $2im \leq 2jn \implies im \leq jn \implies hl = im$, and (1.88) gives $\mu = 2im = nj + hl = im + nj \implies im = jn = hl$. Hence $3 \mid A, 3 \mid B$ and $3 \mid C$ and the conjecture (3.1) is verified.

- If $k = 2jn \implies 2jn = 2im \implies im = jn = hl$. Hence $3 \mid A, 3 \mid B$ and $3 \mid C$ and the conjecture (3.1) is verified.

- If $k = im + jn = 2im \implies im = jn \implies \epsilon = hl = im = jn$ case that is seen above and we deduce that $3 \mid A, 3 \mid B$ and $3 \mid C$, and the conjecture (3.1) is verified.

** B-2-2-1-2- $\alpha > 1 \implies \alpha \geq 2$ and $a' = 3^{\alpha-1}a_1$.

- If $k = 2im \implies 2im = \mu$, but $\mu = 2im + 1 - \alpha$ that is impossible.

- If $k = 2jn = \mu \implies 2jn = 2im + 1 - \alpha$. We obtain $2jn < 2im \implies jn < im \implies 2jn < im + jn$, $k = 2jn$ is just the minimum of $(2im, 2jn, im + jn)$. We obtain $jn = hl < im$ and the equation (1.88) becomes:

$$B_1^n C_1^l = p_1(b - 4 \times 3^{(\alpha-1)}a_1)$$

The conjecture (3.1) is verified.

- If $k = im + jn \leq 2im \implies jn \leq im$ and $k = im + jn \leq 2jn \implies im \leq jn \implies im = jn \implies k = im + jn = 2im = \mu$ but $\mu = 2im + 1 - \alpha$ that is impossible.

- If $k = im + jn < 2im \implies jn < im$ and $2jn < im + jn = k$ that is a contradiction with $k = \min(2im, 2jn, im + jn)$.

** B-2-2-2- We suppose that $\omega \neq 3$. We write $a = \omega^\alpha a_1$ with $\omega \nmid a_1$ and $p' = \omega^\mu p_1$ with $\omega \nmid p_1$. As $A^{2m} = 4a'p' = 4\omega^{\alpha+\mu}.a_1.p_1 \implies \omega \mid A \implies A = \omega^i A_1$, $\omega \nmid A_1$. But $B^n C^l = p'(b - 4a') = \omega^\mu p_1(b - 4a') \implies \omega \mid B^n C^l \implies \omega \mid B^n$ or $\omega \mid C^l$.

** B-2-2-2-1- $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$ and $\omega \nmid B_1$. From $A^m + B^n = C^l \implies \omega \mid C^l \implies \omega \mid C$. As $p = bp' = \omega^\mu bp_1 = \omega^k(\omega^{2im-k} A_1^{2m} + \omega^{2jn-k} B_1^{2n} + \omega^{im+jn-k} A_1^m B_1^n)$ with $k = \min(2im, 2jn, im + jn)$. Then :

- If $\mu = k$, then $\omega \nmid b$ and the conjecture (3.1) is verified.

- If $k > \mu$, then $\omega \mid b$, but $\omega \nmid a$ we deduce the contradiction with a, b coprime.

- If $k < \mu$, it follows from :

$$\omega^\mu b p_1 = \omega^k (\omega^{2im-k} A_1^{2m} + \omega^{2jn-k} B_1^{2n} + \omega^{im+jn-k} A_1^m B_1^n)$$

that $\omega \mid A_1$ or $\omega \mid B_1$ that is a contradiction with the hypothesis.

** B-2-2-2-2- If $\omega \mid C^l \implies \omega \mid C \implies C = \omega^h C_1$ with $\omega \nmid C_1$. From $A^m + B^n = C^l \implies \omega \mid (C^l - A^m) \implies \omega \mid B$. Then, we obtain the same results as B-2-2-2-1- above.

1.5.5. Case $b = 2p$ and $3 \mid a$. — We have :

$$\cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2p} \implies A^{2m} = \frac{4p \cdot a}{3b} = \frac{4p}{3} \cdot \frac{3a'}{2p} = 2a' = (A^m)^2 \implies 2 \mid a' \implies 2 \mid a$$

Then $2 \mid a$ and $2 \mid b$ that is a contradiction with a, b coprime.

1.5.6. Case $b = 4p$ and $3 \mid a$. — We have :

$$\cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{4p} \implies A^{2m} = \frac{4p \cdot a}{3b} = \frac{4p}{3} \cdot \frac{3a'}{4p} = a' = (A^m)^2 = a'^2$$

with $A^m = a''$

Let us calculate $A^m B^n$, we obtain:

$$A^m B^n = \frac{p\sqrt{3}}{3} \cdot \sin \frac{2\theta}{3} - \frac{2p}{3} \cos^2 \frac{\theta}{3} = \frac{p\sqrt{3}}{3} \cdot \sin \frac{2\theta}{3} - \frac{a'}{2} \implies$$

$$A^m B^n + \frac{A^{2m}}{2} = \frac{p\sqrt{3}}{3} \cdot \sin \frac{2\theta}{3}$$

Let:

$$(1.89) \quad A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3} \sin \frac{2\theta}{3}$$

The left member of (1.89) is an integer and p is an integer, then $\frac{2\sqrt{3}}{3} \sin \frac{2\theta}{3}$ will be written as :

$$\frac{2\sqrt{3}}{3} \sin \frac{2\theta}{3} = \frac{k_1}{k_2}$$

where k_1, k_2 are two integers coprime and $k_2 \mid p \implies p = k_2 \cdot k_3$.

** C-1- Firstly, we suppose that $k_3 \neq 1$. Then :

$$A^{2m} + 2A^m B^n = k_3 \cdot k_1$$

Let μ be a prime integer and $\mu \mid k_3$, then $\mu \mid A^m(A^m + 2B^n) \implies \mu \mid A^m$ or $\mu \mid (A^m + 2B^n)$.

** C-1-1- If $\mu \mid (A^m = a^n) \implies \mu \mid (a^{n^2} = a')$ $\implies \mu \mid (3a' = a)$. As $\mu \mid k_3 \implies \mu \mid p \implies \mu \mid (4p = b)$, then the contradiction with a, b coprime.

** C-1-2- If $\mu \mid (A^m + 2B^n) \implies \mu \nmid A^m$ and $\mu \nmid 2B^n$, then:

$$(1.90) \quad \mu \neq 2 \quad \text{and} \quad \mu \nmid B^n$$

$\mu \mid (A^m + 2B^n)$, we write:

$$A^m + 2B^n = \mu.t'$$

Then:

$$\begin{aligned} A^m + B^n = \mu t' - B^n &\implies A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t' \mu B^n + B^{2n} \\ &\implies p = t'^2 \mu^2 - 2t' B^n \mu + B^n (B^n - A^m) \end{aligned}$$

As $b = 4p = 4k_2.k_3$ and $\mu \mid k_3$ then $\mu \mid b \implies \exists \mu'$ so that $b = \mu.\mu'$, we obtain:

$$\mu'.\mu = \mu(4\mu t'^2 - 8t' B^n) + 4B^n (B^n - A^m)$$

The last equation implies $\mu \mid 4B^n(B^n - A^m)$, but $\mu \neq 2$ then $\mu \mid B^n$ or $\mu \mid (B^n - A^m)$.

** C-1-1-1- If $\mu \mid B^n \implies$ then the contradiction with (1.90).

** C-1-1-2- If $\mu \mid (B^n - A^m)$ and using $\mu \mid (A^m + 2B^n)$, we have :

$$\mu \mid 3B^n \implies \begin{cases} \mu \mid B^n \\ \text{or} \\ \mu = 3 \end{cases}$$

** C-1-1-2-1- If $\mu \mid B^n$ then the contradiction with (1.90).

** C-1-1-2-2- If $\mu = 3$, then $3 \mid b$, but $3 \mid a$ then the contradiction with a, b coprime.

** C-2- We assume now that $k_3 = 1$, then:

$$(1.91) \quad \begin{aligned} A^{2m} + 2A^m B^n &= k_1 \\ p &= k_2 \\ \frac{2\sqrt{3}}{3} \sin \frac{2\theta}{3} &= \frac{k_1}{p} \end{aligned}$$

We take the square of the last equation, we obtain :

$$\begin{aligned} \frac{4}{3} \sin^2 \frac{2\theta}{3} &= \frac{k_1^2}{p^2} \\ \frac{16}{3} \sin^2 \frac{\theta}{3} \cos^2 \frac{\theta}{3} &= \frac{k_1^2}{p^2} \\ \frac{16}{3} \sin^2 \frac{\theta}{3} \cdot \frac{3a'}{3 \cdot b} &= \frac{k_1^2}{p^2} \end{aligned}$$

Finally:

$$(1.92) \quad a'(4p - 3a') = k_1^2$$

but $a' = a''^2$, then $4p - 3a'$ is a square. Let :

$$\lambda^2 = 4p - 3a' = 4p - a = b - a$$

The equation (1.92) becomes :

$$(1.93) \quad a''^2 \lambda^2 = k_1^2 \implies k_1 = a'' \lambda$$

taking the positive root. Using (1.91), we have:

$$k_1 = A^m (A^m + 2B^n) = a'' (A^m + 2B^n)$$

Then :

$$A^m + 2B^n = \lambda$$

Now, we consider that $b - a = \lambda^2 \implies \lambda^2 + 3a''^2 = b$, then the couple (λ, a'') is a solution of the Diophantine equation:

$$(1.94) \quad X^2 + 3Y^2 = b$$

with $X = \lambda$ and $Y = a''$. But using one theorem on the solutions of the equation given by (1.94), b is written under the form (see theorem 37.4 in [1]):

$$b = 2^{2s} \times 3^t \cdot p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

where p_i are prime integers so that $p_i \equiv 1 \pmod{6}$, the q_j are also prime integers so that $q_j \equiv 5 \pmod{6}$. Then, as $b = 4p$:

- If $t \geq 1 \implies 3 \mid b$, but $3 \nmid a$, then the contradiction with a, b coprime.

** C-2-2-1- Hence, we suppose that p is written under the form:

$$p = p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

with $p_i \equiv 1(\text{mod } 6)$ and $q_j \equiv 5(\text{mod } 6)$. Finally, we obtain that $p \equiv 1(\text{mod } 6)$. We will verify if this condition does not give contradictions.

We will present the table of the value modulo 6 of $p = A^{2m} + A^m B^n + B^{2n}$ in function of the values of $A^m, B^n(\text{mod } 6)$. We obtain the table below:

TABLE 1. Table of $p(\text{mod } 6)$

A^m, B^n	0	1	2	3	4	5
0	0	1	4	3	4	1
1	1	3	1	1	3	1
2	4	1	0	1	4	3
3	3	1	1	3	1	1
4	4	3	4	1	0	1
5	1	1	3	1	1	3

** C-2-2-1-1- Case $A^m \equiv 0(\text{mod } 6) \implies 2 \mid (A^m = a^n) \implies 2 \mid (a' = a'^2) \implies 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime. All the cases of the first line of the table 1 are to reject.

** C-2-2-1-2- Case $A^m \equiv 1(\text{mod } 6)$ and $B^n \equiv 0(\text{mod } 6)$, then $2 \mid B^n \implies B^n = 2B'$ and p is written as $p = (A^m + B')^2 + 3B'^2$ with $(p, 3) = 1$, if not $3 \mid p$, then $3 \mid b$, but $3 \mid a$, then the contradiction with a, b coprime. Hence, the pair $(A^m + B', B')$ is a solution of the Diophantine equation:

$$(1.95) \quad x^2 + 3y^2 = p$$

The solution $x = A^m + B', y = B'$ is unique because $x - y$ verifies $x - y = A^m$. If (u, v) is another pair solution of (1.95), with $u, v \in \mathbb{N}^*$, then we obtain:

$$\begin{aligned} u^2 + 3v^2 &= p \\ u - v &= A^m \end{aligned}$$

Then $u = v + A^m$ and we obtain the equation of second degree $4v^2 + 2vA^m - 2B'(A^m + 2B') = 0$ that gives as positive root $v_1 = B' = y$, then $u = A^m + B' = x$. It follows that p in (1.95) has a unique representation under the form $X^2 + 3Y^2$ with $X, 3Y$ coprime. As p is an odd integer number, we applique

one of Euler's theorems on convenient numbers "numerus idoneus" (see [2, 3]) :
Let m be an odd number relatively prime to n which is properly represented by $x^2 + ny^2$. If the equation $m = x^2 + ny^2$ has only one solution with $x, y > 0$, then m is a prime number. Then p is prime and $4p$ has an unique representation (we put $U = 2u, V = 2v$, with $U^2 + 3V^2 = 4p$ and $U - V = 2A^m$). But $b = 4p \implies \lambda^2 + 3a''^2 = (2(A^m + B'))^2 + 3(2B')^2$, the representation of $4p$ is unique gives:

$$\lambda = 2(A^m + B') = 2a'' + B^n$$

$$\text{and } a'' = 2B' = B^n = A^m$$

But $A^m > B^n$, then the contradiction.

** C-2-2-1-3- Case $A^m \equiv 1 \pmod{6}$ and $B^n \equiv 2 \pmod{6}$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-4- Case $A^m \equiv 1 \pmod{6}$ and $B^n \equiv 3 \pmod{6}$, then $3 \mid B^n \implies B^n = 3B'$. We can write $b = 4p = (2A^m + 3B')^2 + 3(3B')^2 = \lambda^2 + 3a''^2$. The unique representation of b as $x^2 + 3y^2 = \lambda^2 + 3a''^2 \implies a'' = A^m = 3B' = B^n$, then the contradiction with $A^m > B^n$.

** C-2-2-1-5- Case $A^m \equiv 1 \pmod{6}$ and $B^n \equiv 5 \pmod{6}$, then $C^l \equiv 0 \pmod{6} \implies 2 \mid C^l$, see C-2-2-1-2-.

** C-2-2-1-6- Case $A^m \equiv 2 \pmod{6} \implies 2 \mid a'' \implies 2 \mid a$, but $2 \nmid b$, then the contradiction with a, b coprime.

** C-2-2-1-7- Case $A^m \equiv 3 \pmod{6}$ and $B^n \equiv 1 \pmod{6}$, then $C^l \equiv 4 \pmod{6} \implies 2 \mid C^l \implies C^l = 2C'$, we can write that $p = (C' - B^n)^2 + 3C'^2$, see C-2-2-1-2-.

** C-2-2-1-8- Case $A^m \equiv 3 \pmod{6}$ and $B^n \equiv 2 \pmod{6}$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-9- Case $A^m \equiv 3 \pmod{6}$ and $B^n \equiv 4 \pmod{6}$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-10- Case $A^m \equiv 3(\text{mod } 6)$ and $B^n \equiv 5(\text{mod } 6)$, then $C^l \equiv 2(\text{mod } 6) \implies 2 \mid C^l$, see C-2-2-1-2-.

** C-2-2-1-11- Case $A^m \equiv 4(\text{mod } 6) \implies 2 \mid a'' \implies 2 \mid a$, but $2 \nmid b$, then the contradiction with a, b coprime.

** C-2-2-1-12- Case $A^m \equiv 5(\text{mod } 6)$ and $B^n \equiv 0(\text{mod } 6)$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-13- Case $A^m \equiv 5(\text{mod } 6)$ and $B^n \equiv 1(\text{mod } 6)$, then $C^l \equiv 0(\text{mod } 6) \implies 2 \mid C^l$, see C-2-2-1-2-.

** C-2-2-1-14- Case $A^m \equiv 5(\text{mod } 6)$ and $B^n \equiv 3(\text{mod } 6)$, then $C^l \equiv 2(\text{mod } 6) \implies 2 \mid C^l \implies C^l = 2C'$, p is written as $p = (C' - B^n)^2 + 3C'^2$, see C-2-2-1-2-.

** C-2-2-1-15- Case $A^m \equiv 5(\text{mod } 6)$ and $B^n \equiv 4(\text{mod } 6)$, then B^n is even, see C-2-2-1-2-.

We have achieved the study all the cases of the table 1 giving contradictions.

Then the case $k_3 = 1$ is impossible.

1.5.7. Case $3 \mid a$ and $b = 2p'$, $b \neq 2$ with $p' \mid p$. — $3 \mid a \implies a = 3a'$, $b = 2p'$ with $p = k.p'$, then:

$$A^{2m} = \frac{4.p}{3} \cdot \frac{a}{b} = \frac{4.k.p'.3.a'}{6p'} = 2.k.a'$$

We calculate $B^n C^l$:

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 \sin^2 \frac{\theta}{3} - \cos^2 \frac{\theta}{3} \right) = \sqrt[3]{\rho^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right)$$

but $\sqrt[3]{\rho^2} = \frac{p}{3}$, then using $\cos^2 \frac{\theta}{3} = \frac{3.a'}{b}$:

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) = \frac{p}{3} \left(3 - 4 \frac{3.a'}{b} \right) = p \cdot \left(1 - \frac{4.a'}{b} \right) = k(p' - 2a')$$

As $p = b.p'$, and $p' > 1$, then we have:

$$(1.96) \quad B^n C^l = k(p' - 2a')$$

$$(1.97) \quad \text{and } A^{2m} = 2k.a'$$

** D-1- We suppose that k is prime.

** D-1-1- If $k = 2$, then we have $p = 2p' = b \implies 2 \mid b$, but $A^{2m} = 4a' = (A^m)^2 \implies A^m = 2a''$ with $a' = a''^2$, then $2 \mid a'' \implies 2 \mid (a = 3a''^2)$, it follows the contradiction with a, b coprime.

** D-1-2- We suppose $k \neq 2$. From $A^{2m} = 2k.a' = (A^m)^2 \implies k \mid a'$ and $2 \mid a' \implies a' = 2.k.a''^2 \implies A^m = 2.k.a''$. Then $k \mid A^m \implies k \mid A \implies A = k^i.A_1$ with $i \geq 1$ and $k \nmid A_1$. $k^{im}A_1^m = 2ka'' \implies 2a'' = k^{im-1}A_1^m$. From $B^n C^l = k(p' - 2a') \implies k \mid (B^n C^l) \implies k \mid B^n$ or $k \mid C^l$.

** D-1-2-1- We suppose that $k \mid B^n \implies k \mid B \implies B = k^j.B_1$ with $j \geq 1$ and $k \nmid B_1$. It follows $k^{nj-1}B_1^n C^l = p' - 2a' = p' - 4ka''^2$. As $n \geq 3 \implies nj - 1 \geq 2$, then $k \mid p'$ but $k \neq 2 \implies k \mid (2p' = b)$, but $k \mid a' \implies k \mid (3a' = a)$. It follows the contradiction with a, b coprime.

** D-1-2-2- If $k \mid C^l$ we obtain the identical results.

** D-2- We suppose that k is not prime. Let ω be an integer prime so that $k = \omega^s.k_1$, with $s \geq 1$, $\omega \nmid k_1$. The equations (1.96-1.97) become:

$$\begin{aligned} B^n C^l &= \omega^s.k_1(p' - 2a') \\ \text{and } A^{2m} &= 2\omega^s.k_1.a' \end{aligned}$$

** D-2-1- We suppose that $\omega = 2$, then we have the equations:

$$(1.98) \quad A^{2m} = 2^{s+1}.k_1.a'$$

$$(1.99) \quad B^n C^l = 2^s.k_1(p' - 2a')$$

** D-2-1-1- Case: $2 \mid a' \implies 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime.

** D-2-1-2- Case: $2 \nmid a'$. As $2 \nmid k_1$, the equation (1.98) gives $2 \mid A^{2m} \implies A = 2^i A_1$, with $i \geq 1$ and $2 \nmid A_1$. It follows that $2im = s + 1$.

** D-2-1-2-1- We suppose that $2 \nmid (p' - 2a') \implies 2 \nmid p'$. From the equation (1.99), we obtain that $2 \mid B^n C^l \implies 2 \mid B^n$ or $2 \mid C^l$.

** D-2-1-2-1-1- We suppose that $2 \mid B^n \implies 2 \mid B \implies B = 2^j B_1$ with $2 \nmid B_1$ and $j \geq 1$, then $B_1^n C^l = 2^{s-jn} k_1 (p' - 2a')$:

- If $s - jn \geq 1$, then $2 \mid C^l \implies 2 \mid C$, and no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$, and the conjecture (3.1) is verified.

- If $s - jn \leq 0$, from $B_1^n C^l = 2^{s-jn}k_1(p' - 2a') \implies 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n \implies 2 \mid C^l$.

** D-2-1-2-1-2- Using the same method of the proof above, we obtain the identical results if $2 \mid C^l$.

** D-2-1-2-2- We suppose now that $2 \mid (p' - 2a') \implies p' - 2a' = 2^\mu \cdot \Omega$, with $\mu \geq 1$ and $2 \nmid \Omega$. We recall that $2 \nmid a'$. The equation (1.99) is written as:

$$B^n C^l = 2^{s+\mu} \cdot k_1 \cdot \Omega$$

This last equation implies that $2 \mid (B^n C^l) \implies 2 \mid B^n$ or $2 \mid C^l$.

** D-2-1-2-2-1- We suppose that $2 \mid B^n \implies 2 \mid B \implies B = 2^j B_1$ with $j \geq 1$ and $2 \nmid B_1$. Then $B_1^n C^l = 2^{s+\mu-jn} \cdot k_1 \cdot \Omega$:

- If $s + \mu - jn \geq 1$, then $2 \mid C^l \implies 2 \mid C$, no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$, and the conjecture (3.1) is verified.

- If $s + \mu - jn \leq 0$, from $B_1^n C^l = 2^{s+\mu-jn}k_1 \cdot \Omega \implies 2 \nmid C^l$, then contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n \implies 2 \mid C^l$.

** D-2-1-2-2-2- We obtain the identical results if $2 \mid C^l$.

** D-2-2- We suppose that $\omega \neq 2$. We have then the equations:

$$(1.100) \quad A^{2m} = 2\omega^s \cdot k_1 \cdot a'$$

$$(1.101) \quad B^n C^l = \omega^s \cdot k_1 \cdot (p' - 2a')$$

As $\omega \neq 2$, from the equation (1.100), we have $2 \mid (k_1 \cdot a')$. If $2 \mid a' \implies 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime.

** D-2-2-1- Case: $2 \nmid a'$ and $2 \mid k_1 \implies k_1 = 2^\mu \cdot \Omega$ with $\mu \geq 1$ and $2 \nmid \Omega$. From the equation (1.100), we have $2 \mid A^{2m} \implies 2 \mid A \implies A = 2^i A_1$ with $i \geq 1$ and $2 \nmid A_1$, then $2im = 1 + \mu$. The equation (1.101) becomes:

$$(1.102) \quad B^n C^l = \omega^s \cdot 2^\mu \cdot \Omega \cdot (p' - 2a')$$

From the equation (1.102), we obtain $2 \mid (B^n C^l) \implies 2 \mid B^n$ or $2 \mid C^l$.

** D-2-2-1-1- We suppose that $2 \mid B^n \implies 2 \mid B \implies B = 2^j B_1$, with $j \in \mathbb{N}^*$ and $2 \nmid B_1$.

** D-2-2-1-1-1- We suppose that $2 \nmid (p' - 2a')$, then we have $B_1^n C^l = \omega^s 2^{\mu-jn} \Omega(p' - 2a')$:

- If $\mu - jn \geq 1 \implies 2 \mid C^l \implies 2 \mid C$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1) is verified.

- If $\mu - jn \leq 0 \implies 2 \nmid C^l$ then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

** D-2-2-1-1-2- We suppose that $2 \mid (p' - 2a') \implies p' - 2a' = 2^\alpha \cdot P$, with $\alpha \in \mathbb{N}^*$ and $2 \nmid P$. It follows that $B_1^n C^l = \omega^s 2^{\mu+\alpha-jn} \Omega \cdot P$:

- If $\mu + \alpha - jn \geq 1 \implies 2 \mid C^l \implies 2 \mid C$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1) is verified.

- If $\mu + \alpha - jn \leq 0 \implies 2 \nmid C^l$ then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

** D-2-2-1-2- We suppose now that $2 \mid C^n \implies 2 \mid C$. Using the same method described above, we obtain the identical results.

1.5.8. Case $3 \mid a$ and $b = 4p'$, $b \neq 4$ with $p' \mid p$. — $3 \mid a \implies a = 3a'$, $b = 4p'$ with $p = k \cdot p'$, $k \neq 1$ if not $b = 4p$ this case has been studied (see paragraph 1.5.6), then we have :

$$A^{2m} = \frac{4 \cdot p}{3} \cdot \frac{a}{b} = \frac{4 \cdot k \cdot p' \cdot 3 \cdot a'}{12p'} = k \cdot a'$$

We calculate $B^n C^l$:

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 \sin^2 \frac{\theta}{3} - \cos^2 \frac{\theta}{3} \right) = \sqrt[3]{\rho^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right)$$

but $\sqrt[3]{\rho^2} = \frac{p}{3}$, then using $\cos^2 \frac{\theta}{3} = \frac{3 \cdot a'}{b}$:

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) = \frac{p}{3} \left(3 - 4 \frac{3 \cdot a'}{b} \right) = p \cdot \left(1 - \frac{4 \cdot a'}{b} \right) = k(p' - a')$$

As $p = b \cdot p'$, and $p' > 1$, we have :

$$(1.103) \quad B^n C^l = k(p' - a')$$

$$(1.104) \quad \text{and } A^{2m} = k \cdot a'$$

** E-1- We suppose that k is prime. From $A^{2m} = k \cdot a' = (A^m)^2 \implies k \mid a'$ and $a' = k \cdot a'' \implies A^m = k \cdot a''$. Then $k \mid A^m \implies k \mid A \implies A = k^i \cdot A_1$

with $i \geq 1$ and $k \nmid A_1$. $k^{mi}A_1^m = ka'' \implies a'' = k^{mi-1}A_1^m$. From $B^n C^l = k(p' - a') \implies k \mid (B^n C^l) \implies k \mid B^n$ or $k \mid C^l$.

** E-1-1- We suppose that $k \mid B^n \implies k \mid B \implies B = k^j.B_1$ with $j \geq 1$ and $k \nmid B_1$. Then $k^{n.j-1}B_1^n C^l = p' - a'$. As $n.j - 1 \geq 2 \implies k \mid (p' - a')$. But $k \mid a' \implies k \mid a$, then $k \mid p' \implies k \mid (4p' = b)$ and we arrive to the contradiction that a, b are coprime.

** E-1-2- We suppose that $k \mid C^l$, using the same method with the above hypothesis $k \mid B^n$, we obtain the identical results.

** E-2- We suppose that k is not prime.

** E-2-1- We take $k = 4 \implies p = 4p' = b$, it is the case 1.5.3 studied above.

** E-2-2- We suppose that $k \geq 6$ not prime. Let ω be a prime so that $k = \omega^s.k_1$, with $s \geq 1$, $\omega \nmid k_1$. The equations (1.103-1.104) become:

$$(1.105) \quad B^n C^l = \omega^s.k_1(p' - a')$$

$$(1.106) \quad \text{and } A^{2m} = \omega^s.k_1.a'$$

** E-2-2-1- We suppose that $\omega = 2$.

** E-2-2-1-1- If $2 \mid a' \implies 2 \mid (3a' = a)$, but $2 \mid (4p' = b)$, then the contradiction with a, b coprime.

** E-2-2-1-2- We consider that $2 \nmid a'$. From the equation (1.106), it follows that $2 \mid A^{2m} \implies 2 \mid A \implies A = 2^i A_1$ with $2 \nmid A_1$ and:

$$B^n C^l = 2^s k_1 (p' - a')$$

** E-2-2-1-2-1- We suppose that $2 \nmid (p' - a')$, from the above expression, we have $2 \mid (B^n C^l) \implies 2 \mid B^n$ or $2 \mid C^l$.

** E-2-2-1-2-1-1- If $2 \mid B^n \implies 2 \mid B \implies B = 2^j B_1$ with $2 \nmid B_1$. Then $B_1^n C^l = 2^{2im-jn} k_1 (p' - a')$:

- If $2im - jn \geq 1 \implies 2 \mid C^l \implies 2 \mid C$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1) is verified.

- If $2im - jn \leq 0 \implies 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n \implies 2 \mid C^l$.

** E-2-2-1-2-1-2- If $2 \mid C^l \implies 2 \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-1-2-2- We suppose that $2 \mid (p' - a')$. As $2 \nmid a' \implies 2 \nmid p'$, $2 \mid (p' - a') \implies p' - a' = 2^\alpha.P$ with $\alpha \geq 1$ and $2 \nmid P$. The equation (1.105) is written as :

$$(1.107) \quad B^n C^l = 2^{s+\alpha} k_1.P = 2^{2im+\alpha} k_1.P$$

then $2 \mid (B^n C^l) \implies 2 \mid B^n$ or $2 \mid C^l$.

** E-2-2-1-2-2-1- We suppose that $2 \mid B^n \implies 2 \mid B \implies B = 2^j B_1$, with $2 \nmid B_1$. The equation (1.107) becomes $B_1^n C^l = 2^{2im+\alpha-jn} k_1 P$:

- If $2im + \alpha - jn \geq 1 \implies 2 \mid C^l \implies 2 \mid C$, no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1) is verified.

- If $2im + \alpha - jn \leq 0 \implies 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n \implies 2 \mid C^l$.

** E-2-2-1-2-2-2- We suppose that $2 \mid C^l \implies 2 \mid C$. Using the same method described above, we obtain the identical results.

** E-2-2-2- We suppose that $\omega \neq 2$. We recall the equations:

$$(1.108) \quad A^{2m} = \omega^s . k_1 . a'$$

$$(1.109) \quad B^n C^l = \omega^s . k_1 (p' - a')$$

** E-2-2-2-1- We suppose that ω, a' are coprime, then $\omega \nmid a'$. From the equation (1.108), we have $\omega \mid A^{2m} \implies \omega \mid A \implies A = \omega^i A_1$ with $\omega \nmid A_1$ and $s = 2im$.

** E-2-2-2-1-1- We suppose that $\omega \nmid (p' - a')$. From the equation (1.109) above, we have $\omega \mid (B^n C^l) \implies \omega \mid B^n$ or $\omega \mid C^l$.

** E-2-2-2-1-1-1- If $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$ with $\omega \nmid B_1$. Then $B_1^n C^l = 2^{2im-jn} k_1 (p' - a')$:

- If $2im - jn \geq 1 \implies \omega \mid C^l \implies \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (3.1) is verified.

- If $2im - jn \leq 0 \implies \omega \nmid C^l$, then the contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n \implies \omega \mid C^l$.

** E-2-2-2-1-1-2- If $\omega \mid C^l \implies \omega \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-2-1-2- We suppose that $\omega \mid (p' - a') \implies \omega \nmid p'$ as ω and a' are coprime. $\omega \mid (p' - a') \implies p' - a' = \omega^\alpha.P$ with $\alpha \geq 1$ and $\omega \nmid P$. The equation (1.109) becomes :

$$(1.110) \quad B^n C^l = \omega^{s+\alpha} k_1.P = \omega^{2im+\alpha} k_1.P$$

then $\omega \mid (B^n C^l) \implies \omega \mid B^n$ or $\omega \mid C^l$.

** E-2-2-2-1-2-1- We suppose that $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$, with $\omega \nmid B_1$. The equation (1.110) is written as $B_1^n C^l = 2^{2im+\alpha-jn} k_1 P$:

- If $2im + \alpha - jn \geq 1 \implies \omega \mid C^l \implies \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (3.1) is verified.

- If $2im + \alpha - jn \leq 0 \implies \omega \nmid C^l$, then the contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n \implies \omega \mid C^l$.

** E-2-2-2-1-2-2- We suppose that $\omega \mid C^l \implies \omega \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-2-2- We suppose that ω, a' are not coprime, then $a' = \omega^\beta . a''$ with $\omega \nmid a''$. The equation (1.108) becomes:

$$A^{2m} = \omega^s k_1 a' = \omega^{s+\beta} k_1 . a''$$

We have $\omega \mid A^{2m} \implies \omega \mid A \implies A = \omega^i A_1$ with $\omega \nmid A_1$ and $s + \beta = 2im$.

** E-2-2-2-2-1- We suppose that $\omega \nmid (p' - a') \implies \omega \nmid p' \implies \omega \nmid (b = 4p')$. From the equation (1.109), we obtain $\omega \mid (B^n C^l) \implies \omega \mid B^n$ or $\omega \mid C^l$.

** E-2-2-2-2-1-1- If $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$ with $\omega \nmid B_1$. Then $B_1^n C^l = 2^{s-jn} k_1 (p' - a')$:

- If $s - jn \geq 1 \implies \omega \mid C^l \implies \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (3.1) is verified.

- If $s - jn \leq 0 \implies \omega \nmid C^l$, then the contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n \implies \omega \mid C^l$.

** E-2-2-2-2-1-2- If $\omega \mid C^l \implies \omega \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-2-2-2- We suppose that $\omega \mid (p' - a' = p' - \omega^\beta \cdot a^n) \implies \omega \mid p' \implies \omega \mid (4p' = b)$, but $\omega \mid a' \implies \omega \mid a$. Then the contradiction with a, b coprime.

The study of the cases of 1.5.8 is achieved.

1.5.9. Case $3 \mid a$ and $b \mid 4p$. — $a = 3a'$ and $4p = k_1b$. As $A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \frac{3a'}{b} = k_1a'$ and $B^n C^l$:

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 \sin^2 \frac{\theta}{3} - \cos^2 \frac{\theta}{3} \right) = \frac{p}{3} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) = \frac{p}{3} \left(3 - 4 \frac{3a'}{b} \right) = \frac{k_1}{4} (b - 4a')$$

As $B^n C^l$ is an integer, we must obtain $4 \mid k_1$, or $4 \mid (b - 4a')$ or ($2 \mid k_1$ and $2 \mid (b - 4a')$).

** F-1- If $k_1 = 1 \implies b = 4p$: it is the case 1.5.6.

** F-2- If $k_1 = 4 \implies p = b$: it is the case 1.5.3.

** F-3- If $k_1 = 2$ and $2 \mid (b - 4a')$: in this case, we have $A^{2m} = 2a' \implies 2 \mid a' \implies 2 \mid a$. $2 \mid (b - 4a') \implies 2 \mid b$ then the contradiction with a, b coprime.

** F-4- If $2 \mid k_1$ and $2 \mid (b - 4a')$: $2 \mid (b - 4a') \implies b - 4a' = 2^\alpha \lambda$, α and $\lambda \in \mathbb{N}^* \geq 1$ with $2 \nmid \lambda$; $2 \mid k_1 \implies k_1 = 2^t k'_1$ with $t \geq 1 \in \mathbb{N}^*$ with $2 \nmid k'_1$ and we have:

$$(1.111) \quad A^{2m} = 2^t k'_1 a'$$

$$(1.112) \quad B^n C^l = 2^{t+\alpha-2} k'_1 \lambda$$

From the equation (1.111), we have $2 \mid A^{2m} \implies 2 \mid A \implies A = 2^i A_1$, $i \geq 1$ and $2 \nmid A_1$.

** F-4-1- We suppose that $t = \alpha = 1$, then the equations (1.111-1.112) become :

$$(1.113) \quad A^{2m} = 2k'_1 a'$$

$$(1.114) \quad B^n C^l = k'_1 \lambda$$

From the equation (1.113) it follows that $2 \mid a' \implies 2 \mid (a = 3a')$. But $b = 4a' + 2\lambda \implies 2 \mid b$, then the contradiction with a, b coprime.

** F-4-2- We suppose that $t + \alpha - 2 \geq 1$ and we have the expressions:

$$(1.115) \quad A^{2m} = 2^t k'_1 a'$$

$$(1.116) \quad B^n C^l = 2^{t+\alpha-2} k'_1 \lambda$$

** F-4-2-1- We suppose that $2 \mid a' \implies 2 \mid a$, but $b = 2^\alpha \lambda + 4a' \implies 2 \mid b$, then the contradiction with a, b coprime.

** F-4-2-2- We suppose that $2 \nmid a'$. From (1.115), we have $2 \mid A^{2m} \implies 2 \mid A \implies A = 2^i A_1$ and $B^n C^l = 2^{t+\alpha-2} k'_1 \lambda \implies 2 \mid B^n C^l \implies 2 \mid B^n$ or $2 \mid C^l$.

** F-4-2-2-1- We suppose that $2 \mid B^n$. We have $2 \mid B \implies B = 2^j B_1$, $j \geq 1$ and $2 \nmid B_1$. The equation (1.116) becomes $B_1^n C^l = 2^{t+\alpha-2-jn} k'_1 \lambda$:

- If $t + \alpha - 2 - jn > 0 \implies 2 \mid C^l \implies 2 \mid C$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1) is verified.

- If $t + \alpha - 2 - jn < 0 \implies 2 \mid k'_1 \lambda$, but $2 \nmid k'_1$ and $2 \nmid \lambda$. Then this case is impossible.

- If $t + \alpha - 2 - jn = 0 \implies B_1^n C^l = k'_1 \lambda \implies 2 \nmid C^l$ then it is a contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

** F-4-2-2-2- We suppose that $2 \mid C^l$. We use the same method described above, we obtain the identical results.

** F-5- We suppose that $4 \mid k_1$ with $k_1 > 4 \implies k_1 = 4k'_2$, we have :

$$(1.117) \quad A^{2m} = 4k'_2 a'$$

$$(1.118) \quad B^n C^l = k'_2 (b - 4a')$$

** F-5-1- We suppose that k'_2 is prime, from (1.117), we have $k'_2 \mid a'$. From (1.118), $k'_2 \mid (B^n C^l) \implies k'_2 \mid B^n$ or $k'_2 \mid C^l$.

** F-5-1-1- We suppose that $k'_2 \mid B^n \implies k'_2 \mid B \implies B = k'^{\beta}_2 . B_1$ with $\beta \geq 1$ and $k'_2 \nmid B_1$. It follows that we have $k'^{n\beta-1}_2 B_1^n C^l = b - 4a' \implies k'_2 \mid b$ then the contradiction with a, b coprime.

** F-5-1-2- We obtain identical results if we suppose that $k'_2 \mid C^l$.

** F-5-2- We suppose that k'_2 is not prime.

** F-5-2-1- We suppose that k'_2 and a' are coprime. From (1.117), k'_2 can be written under the form $k'_2 = q_1^{2j} \cdot q_2^2$ and $q_1 \nmid q_2$ and q_1 prime. We have $A^{2m} = 4q_1^{2j} \cdot q_2^2 a' \implies q_1 \mid A$ and $B^n C^l = q_1^{2j} \cdot q_2^2 (b - 4a') \implies q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-5-2-1-1- We suppose that $q_1 \mid B^n \implies q_1 \mid B \implies B = q_1^f \cdot B_1$ with $q_1 \nmid B_1$. We obtain $B_1^n C^l = q_1^{2j-fn} q_2^2 (b - 4a')$:

- If $2j - f \cdot n \geq 1 \implies q_1 \mid C^l \implies q_1 \mid C$ but $C^l = A^m + B^n$ gives also $q_1 \mid C$ and the conjecture (3.1) is verified.

- If $2j - f \cdot n = 0$, we have $B_1^n C^l = q_2^2 (b - 4a')$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then $q_1 \mid (b - 4a')$. As q_1 and a' are coprime, then $q_1 \nmid b$, and the conjecture (3.1) is verified.

- If $2j - f \cdot n < 0 \implies q_1 \mid (b - 4a') \implies q_1 \nmid b$ because a' is coprime with q_1 , and $C^l = A^m + B^n$ gives $q_1 \mid C$, and the conjecture (3.1) is verified.

** F-5-2-1-2- We obtain identical results if we suppose that $q_1 \mid C^l$.

** F-5-2-2- We suppose that k'_2, a' are not coprime. Let q_1 be a prime so that $q_1 \mid k'_2$ and $q_1 \mid a'$. We write k'_2 under the form $q_1^j \cdot q_2$ with $j \geq 1$, $q_1 \nmid q_2$. From $A^{2m} = 4k'_2 a' \implies q_1 \mid A^{2m} \implies q_1 \mid A$. Then from $B^n C^l = q_1^j q_2 (b - 4a')$, it follows that $q_1 \mid (B^n C^l) \implies q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-5-2-2-1- We suppose that $q_1 \mid B^n \implies q_1 \mid B \implies B = q_1^\beta \cdot B_1$ with $\beta \geq 1$ and $q_1 \nmid B_1$. Then, we have $q_1^{n\beta} B_1^n C^l = q_1^j q_2 (b - 4a') \implies B_1^n C^l = q_1^{j-n\beta} q_2 (b - 4a')$.

- If $j - n\beta \geq 1$, then $q_1 \mid C^l \implies q_1 \mid C$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then the conjecture (3.1) is verified.

- If $j - n\beta = 0$, we obtain $B_1^n C^l = q_2 (b - 4a')$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then $q_1 \mid (b - 4a') \implies q_1 \mid b$ because $q_1 \mid a' \implies q_1 \mid a$, then the contradiction with a, b coprime.

- If $j - n\beta < 0 \implies q_1 \mid (b - 4a') \implies q_1 \mid b$, because $q_1 \mid a' \implies q_1 \mid a$, then the contradiction with a, b coprime.

** F-5-2-2-2- We obtain identical results if we suppose that $q_1 \mid C^l$.

** F-6- If $4 \nmid (b - 4a')$ and $4 \nmid k_1$ it is impossible. We suppose that $4 \mid (b - 4a') \Rightarrow 4 \mid b$, and $b - 4a' = 4^t.g$, $t \geq 1$ with $4 \nmid g$, then we have :

$$\begin{aligned} A^{2m} &= k_1 a' \\ B^n C^l &= k_1 . 4^{t-1} . g \end{aligned}$$

** F-6-1- We suppose that k_1 is prime. From $A^{2m} = k_1 a'$ we deduce easily that $k_1 \mid a'$. From $B^n C^l = k_1 . 4^{t-1} . g$ we obtain that $k_1 \mid (B^n C^l) \Rightarrow k_1 \mid B^n$ or $k_1 \mid C^l$.

** F-6-1-1- We suppose that $k_1 \mid B^n \Rightarrow k_1 \mid B \Rightarrow B = k_1^j . B_1$ with $j > 0$ and $k_1 \nmid B_1$, then $k_1^{n.j} B_1^n C^l = k_1 . 4^{t-1} . g \Rightarrow k_1^{n.j-1} B_1^n C^l = 4^{t-1} . g$. But $n \geq 3$ and $j \geq 1$, then $n.j - 1 \geq 2$. We deduce as $k_1 \neq 2$ that $k_1 \mid g \Rightarrow k_1 \mid (b - 4a')$, but $k_1 \mid a' \Rightarrow k_1 \mid b$, then the contradiction with a, b coprime.

** F-6-1-2- We obtain identical results if we suppose that $k_1 \mid C^l$.

** F-6-2- We suppose that k_1 is not prime $\neq 4$, ($k_1 = 4$ see case F-2, above) with $4 \nmid k_1$.

** F-6-2-1- If $k_1 = 2k'$ with k' odd > 1 . Then $A^{2m} = 2k' a' \Rightarrow 2 \mid a' \Rightarrow 2 \mid a$, as $4 \mid b$ it follows the contradiction with a, b coprime.

** F-6-2-2- We suppose that k_1 is odd with k_1 and a' coprime. We write k_1 under the form $k_1 = q_1^j . q_2$ with $q_1 \nmid q_2$, q_1 prime and $j \geq 1$. $B^n C^l = q_1^j . q_2 4^{t-1} g \Rightarrow q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-6-2-2-1- We suppose that $q_1 \mid B^n \Rightarrow q_1 \mid B \Rightarrow B = q_1^f . B_1$ with $q_1 \nmid B_1$. We obtain $B_1^n C^l = q_1^{j-f.n} q_2 4^{t-1} g$.

- If $j - f.n \geq 1 \Rightarrow q_1 \mid C^l \Rightarrow q_1 \mid C$, but $C^l = A^m + B^n$ gives also $q_1 \mid C$ and the conjecture (3.1) is verified.

- If $j - f.n = 0$, we have $B_1^n C^l = q_2 4^{t-1} g$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then $q_1 \mid (b - 4a')$. As q_1 and a' are coprime then $q_1 \nmid b$ and the conjecture (3.1) is verified.

- If $j - f.n < 0 \Rightarrow q_1 \mid (b - 4a') \Rightarrow q_1 \nmid b$ because q_1, a' are primes. $C^l = A^m + B^n$ gives $q_1 \mid C$ and the conjecture (3.1) is verified.

** F-6-2-2-2- We obtain identical results if we suppose that $q_1 \mid C^l$.

** F-6-2-3- We suppose that k_1 and a' are not coprime. Let q_1 be a prime so that $q_1 \mid k_1$ and $q_1 \mid a'$. We write k_1 under the form $q_1^j \cdot q_2$ with $q_1 \nmid q_2$. From $A^{2m} = k_1 a' \implies q_1 \mid A^{2m} \implies q_1 \mid A$. From $B^n C^l = q_1^j q_2 (b - 4a')$, it follows that $q_1 \mid (B^n C^l) \implies q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-6-2-3-1- We suppose that $q_1 \mid B^n \implies q_1 \mid B \implies B = q_1^\beta \cdot B_1$ with $\beta \geq 1$ and $q_1 \nmid B_1$. Then we have $q_1^{n\beta} B_1^n C^l = q_1^j q_2 (b - 4a') \implies B_1^n C^l = q_1^{j-n\beta} q_2 (b - 4a')$:

- If $j - n\beta \geq 1$, then $q_1 \mid C^l \implies q_1 \mid C$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, and the conjecture (3.1) is verified.

- If $j - n\beta = 0$, we obtain $B_1^n C^l = q_2 (b - 4a')$, but $q_1 \mid A$ and $q_1 \mid B$ then $q_1 \mid C$ and we obtain $q_1 \mid (b - 4a') \implies q_1 \mid b$ because $q_1 \mid a' \implies q_1 \mid a$, then the contradiction with a, b coprime.

- If $j - n\beta < 0 \implies q_1 \mid (b - 4a') \implies q_1 \mid b$, then the contradiction with a, b coprime.

** F-6-2-3-2- We obtain identical results as above if we suppose that $q_1 \mid C^l$.

1.6. Hypothèse: $\{3 \mid p \text{ and } b \mid 4p\}$

1.6.1. Case $b = 2$ and $3 \mid p$. — $3 \mid p \implies p = 3p'$ with $p' \neq 1$ because $3 \ll p$, and $b = 2$, we obtain:

$$A^{2m} = \frac{4p \cdot a}{3b} = \frac{4 \cdot 3p' \cdot a}{3b} = \frac{4 \cdot p' \cdot a}{2} = 2 \cdot p' \cdot a$$

As:

$$\frac{1}{4} < \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{a}{2} < \frac{3}{4} \implies 1 < 2a < 3 \implies a = 1 \implies \cos^2 \frac{\theta}{3} = \frac{1}{2}$$

but this case was studied (see case 1.4.1.2).

1.6.2. Case $b = 4$ and $3 \mid p$. — we have $3 \mid p \implies p = 3p'$ with $p' \in \mathbb{N}^*$, it follows :

$$A^{2m} = \frac{4p \cdot a}{3b} = \frac{4 \cdot 3p' \cdot a}{3 \times 4} = p' \cdot a$$

and:

$$\frac{1}{4} < \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{a}{4} < \frac{3}{4} \implies 1 < a < 3 \implies a = 2$$

as a, b are coprime, then the case $b = 4$ and $3 \mid p$ is impossible.

1.6.3. Case: $b \neq 2, b \neq 4, b \neq 3, b \mid p$ and $3 \mid p$. — As $3 \mid p$, then $p = 3p'$ and :

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p a}{3 b} = \frac{4 \times 3p' a}{3 b} = \frac{4p' a}{b}$$

We consider the case: $b \mid p' \implies p' = bp''$ and $p'' \neq 1$ (If $p'' = 1$, then $p = 3b$, see paragraph 1.6.8 Case $k' = 1$). Finally, we obtain:

$$A^{2m} = \frac{4bp''a}{b} = 4ap''; \quad B^n C^l = p''.(3b - 4a)$$

** G-1- We suppose that p'' is prime, then $A^{2m} = 4ap'' = (A^m)^2 \implies p'' \mid a$. But $B^n C^l = p''(3b - 4a) \implies p'' \mid B^n$ or $p'' \mid C^l$.

** G-1-1- If $p'' \mid B^n \implies p'' \mid B \implies B = p''B_1$ with $B_1 \in \mathbb{N}^*$. Then $p''^{m-1}B_1^n C^l = 3b - 4a$. As $n > 2$, then $(n - 1) > 1$ and $p'' \mid a$, then $p'' \mid 3b \implies p'' = 3$ or $p'' \mid b$.

** G-1-1-1- If $p'' = 3 \implies 3 \mid a$, with a that we write as $a = 3a'^2$, but $A^m = 6a' \implies 3 \mid A^m \implies 3 \mid A \implies A = 3A_1$, then $3^{m-1}A_1^m = 2a' \implies 3 \mid a' \implies a' = 3a''$. As $p''^{m-1}B_1^n C^l = 3^{n-1}B_1^n C^l = 3b - 4a \implies 3^{n-2}B_1^n C^l = b - 36a''^2$. As $n > 2 \implies n - 2 \geq 1$, then $3 \mid b$ and the contradiction with a, b coprime.

** G-1-1-2- We suppose that $p'' \mid b$, as $p'' \mid a$, then the contradiction with a, b coprime.

** G-1-2- If we suppose $p'' \mid C^l$, we obtain identical results (contradictions).

** G-2- We consider now that p'' is not prime.

** G-2-1- p'', a coprime: $A^{2m} = 4ap'' \implies A^m = 2a'.p_1$ with $a = a'^2$ and $p'' = p_1^2$, then a', p_1 are also coprime. As $A^m = 2a'.p_1$, then $2 \mid a'$ or $2 \mid p_1$.

** G-2-1-1- We suppose that $2 \mid a'$, then $2 \mid a' \implies 2 \nmid p_1$, but $p'' = p_1^2$.

** G-2-1-1-1- If p_1 is prime, it is impossible with $A^m = 2a'.p_1$.

** G-2-1-1-2- We suppose that p_1 is not prime so we can write $p_1 = \omega^m \implies p'' = \omega^{2m}$. Then $B^n C^l = \omega^{2m}(3b - 4a)$.

** G-2-1-1-2-1- If ω is prime, $\omega \neq 2$, then $\omega \mid (B^n C^l) \implies \omega \mid B^n$ or $\omega \mid C^l$.

** G-2-1-1-2-1-1- If $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$ with $\omega \nmid B_1$, then $B_1^n \cdot C^l = \omega^{2m-nj}(3b-4a)$.

** G-2-1-1-2-1-1-1- If $2m - n.j = 0$, we obtain $B_1^n \cdot C^l = 3b - 4a$. As $C^l = A^m + B^n \implies \omega \mid C^l \implies \omega \mid C$, and $\omega \mid (3b - 4a)$. But $\omega \neq 2$ and ω, a' are coprime, then ω, a are coprime, it follows $\omega \nmid (3b)$, then $\omega \neq 3$ and $\omega \nmid b$, the conjecture (3.1) is verified.

** G-2-1-1-2-1-1-2- If $2m - n.j \geq 1$, using the method as above, we obtain $\omega \mid C^l \implies \omega \mid C$ and $\omega \mid (3b - 4a)$ and $\omega \nmid a$ and $\omega \neq 3$ and $\omega \nmid b$, then the conjecture (3.1) is verified.

** G-2-1-1-2-1-1-3- If $2m - n.j < 0 \implies \omega^{n.j-2m} B_1^n \cdot C^l = 3b - 4a$. From $A^m + B^n = C^l \implies \omega \mid C^l \implies \omega \mid C$, then $C = \omega^h \cdot C_1$, with $\omega \nmid C_1$, we obtain $\omega^{n.j-2m+h.l} B_1^n \cdot C_1^l = 3b - 4a$. If $n.j - 2m + h.l < 0 \implies \omega \mid B_1^n C_1^l$ then the contradiction with $\omega \nmid B_1$ or $\omega \nmid C_1$. It follows $n.j - 2m + h.l > 0$ and $\omega \mid (3b - 4a)$ with ω, a, b coprime and the conjecture is verified.

** G-2-1-1-2-1-2- Using the same method above, we obtain identical results if $\omega \mid C^l$.

** G-2-1-1-2-2- We suppose that $p^n = \omega^{2m}$ and ω is not prime. We write $\omega = \omega_1^f \cdot \Omega$ with ω_1 prime $\nmid \Omega$, $f \geq 1$, and $\omega_1 \mid A$. Then $B^n C^l = \omega_1^{2f \cdot m} \Omega^{2m} (3b - 4a) \implies \omega_1 \mid (B^n C^l) \implies \omega_1 \mid B^n$ or $\omega_1 \mid C^l$.

** G-2-1-1-2-2-1- If $\omega_1 \mid B^n \implies \omega_1 \mid B \implies B = \omega_1^j B_1$ with $\omega_1 \nmid B_1$, then $B_1^n \cdot C^l = \omega_1^{2 \cdot m - nj} \Omega^{2m} (3b - 4a)$:

** G-2-1-1-2-2-1-1- If $2f \cdot m - n.j = 0$, we obtain $B_1^n \cdot C^l = \Omega^{2m} (3b - 4a)$. As $C^l = A^m + B^n \implies \omega_1 \mid C^l \implies \omega_1 \mid C$, and $\omega_1 \mid (3b - 4a)$. But $\omega_1 \neq 2$ and ω_1, a' are coprime, then ω, a are coprime, it follows $\omega_1 \nmid (3b)$, then $\omega_1 \neq 3$ and $\omega_1 \nmid b$, and the conjecture (3.1) is verified.

** G-2-1-1-2-2-1-2- If $2f \cdot m - n.j \geq 1$, we have $\omega_1 \mid C^l \implies \omega_1 \mid C$ and $\omega_1 \mid (3b - 4a)$ and $\omega_1 \nmid a$ and $\omega_1 \neq 3$ and $\omega_1 \nmid b$, it follows that the conjecture

(3.1) is verified.

** G-2-1-1-2-2-1-3- If $2f.m - n.j < 0 \implies \omega_1^{n.j-2m.f} B_1^n . C^l = \Omega^{2m}(3b - 4a)$. As $\omega_1 \mid C$ using $C^l = A^m + B^n$, then $C = \omega_1^h . C_1 \implies \omega^{n.j-2m.f+h.l} B_1^n . C_1^l = \Omega^{2m}(3b - 4a)$. If $n.j - 2m.f + h.l < 0 \implies \omega_1 \mid B_1^n C_1^l$, then the contradiction with $\omega_1 \nmid B_1$ and $\omega_1 \nmid C_1$. Then if $n.j - 2m.f + h.l > 0$ and $\omega_1 \mid (3b - 4a)$ with ω_1, a, b coprime and the conjecture (3.1) is verified.

** G-2-1-1-2-2-2- Using the same method above, we obtain identical results if $\omega_1 \mid C^l$.

** G-2-1-2- We suppose that $2 \mid p_1$: then $2 \mid p_1 \implies 2 \nmid a' \implies 2 \nmid a$, but $p'' = p_1^2$.

** G-2-1-2-1- We suppose that $p_1 = 2$, we obtain $A^m = 4a' \implies 2 \mid a'$, then the contradiction with a, b coprime.

** G-2-1-2-2- We suppose that p_1 is not prime and $2 \mid p_1$. As $A^m = 2a'p_1$, p_1 can be written as $p_1 = 2^{m-1}\omega^m \implies p'' = 2^{2m-2}\omega^{2m}$. Then $B^n C^l = 2^{2m-2}\omega^{2m}(3b - 4a) \implies 2 \mid B^n$ or $2 \mid C^l$.

** G-2-1-2-2-1- We suppose that $2 \mid B^n \implies 2 \mid B$. As $2 \mid A$, then $2 \mid C$. From $B^n C^l = 2^{2m-2}\omega^{2m}(3b - 4a)$ it follows that if $2 \mid (3b - 4a) \implies 2 \mid b$ but as $2 \nmid a$ there is no contradiction with a, b coprime and the conjecture (3.1) is verified.

** G-2-1-2-2-2- We suppose that $2 \mid C^l$, using the same method above, we obtain identical results.

** G-2-2- We suppose that p'', a are not coprime: let ω be a prime integer so that $\omega \mid a$ and $\omega \mid p''$.

** G-2-2-1- We suppose that $\omega = 3$. As $A^{2m} = 4ap'' \implies 3 \mid A$, but $3 \nmid p$. As $p = A^{2m} + B^{2n} + A^m B^n \implies 3 \mid B^{2n} \implies 3 \mid B$, then $3 \mid C^l \implies 3 \mid C$. We write $A = 3^i A_1$, $B = 3^j B_1$, $C = 3^h C_1$ with 3 coprime with A_1, B_1 and C_1 and $p = 3^{2im} A_1^{2m} + 3^{2nj} B_1^{2n} + 3^{im+jn} A_1^m B_1^n = 3^k . g$ with $k = \min(2im, 2jn, im+jn)$ and $3 \nmid g$. We have also $(\omega = 3) \mid a$ and $(\omega = 3) \mid p''$ that gives $a = 3^\alpha a_1$, $3 \nmid a_1$ and $p'' = 3^\mu p_1$, $3 \nmid p_1$ with $A^{2m} = 4ap'' = 3^{2im} A_1^{2m} = 4 \times 3^{\alpha+\mu} . a_1 . p_1 \implies \alpha + \mu = 2im$. As $p = 3p' = 3b.p'' = 3b.3^\mu p_1 = 3^{\mu+1} . b . p_1$, the exponent of the

factor 3 of p is k , the exponent of the factor 3 of the left member of the last equation is $\mu + 1$ added of the exponent β of 3 of the term b , with $\beta \geq 0$, let $\min(2im, 2jn, im + jn) = \mu + 1 + \beta$ and we recall that $\alpha + \mu = 2im$. But $B^n C^l = p^n(3b - 4a)$, we obtain $3^{(nj+hl)} B_1^n C_1^l = 3^{\mu+1} p_1(b - 4 \times 3^{(\alpha-1)} a_1) = 3^{\mu+1} p_1(3^\beta b_1 - 4 \times 3^{(\alpha-1)} a_1)$, $3 \nmid b_1$. We have also $A^m + B^n = C^l \implies 3^{im} A_1^m + 3^{jn} B_1^n = 3^{hl} C_1^l$. We call $\epsilon = \min(im, jn)$, we have $\epsilon = hl = \min(im, jn)$. We obtain the conditions:

$$(1.119) \quad k = \min(2im, 2jn, im + jn) = \mu + 1 + \beta$$

$$(1.120) \quad \alpha + \mu = 2im$$

$$\epsilon = hl = \min(im, jn)$$

$$3^{(nj+hl)} B_1^n C_1^l = 3^{\mu+1} p_1(3^\beta b_1 - 4 \times 3^{(\alpha-1)} a_1)$$

** G-2-2-1-1- $\alpha = 1 \implies a = 3a_1$ and $3 \nmid a_1$, the equation (1.120) becomes:

$$1 + \mu = 2im$$

and the first equation (1.119) is written as:

$$k = \min(2im, 2jn, im + jn) = 2im + \beta$$

- If $k = 2im \implies \beta = 0$ then $3 \nmid b$. We obtain $2im \leq 2jn \implies im \leq jn$, and $2im \leq im + jn \implies im \leq jn$. The third equation gives $hl = im$ and the last equation gives $nj + hl = \mu + 1 = 2im \implies im = nj$, then $im = nj = hl$ and $B_1^n C_1^l = p_1(b - 4a_1)$. As a, b are coprime, the conjecture (3.1) is verified.

- If $k = 2jn$ or $k = im + jn$, we obtain $\beta = 0$, $im = jn = hl$ and $B_1^n C_1^l = p_1(b - 4a_1)$. As a, b are coprime, the conjecture (3.1) is verified.

** G-2-2-1-2- $\alpha > 1 \implies \alpha \geq 2$.

- If $k = 2im \implies 2im = \mu + 1 + \beta$, but $\mu = 2im - \alpha$ that gives $\alpha = 1 + \beta \geq 2 \implies \beta \neq 0 \implies 3 \mid b$, but $3 \mid a$ then the contradiction with a, b coprime.

- If $k = 2jn = \mu + 1 + \beta \leq 2im \implies \mu + 1 + \beta \leq \mu + \alpha \implies 1 + \beta \leq \alpha \implies \beta \geq 1$. If $\beta \geq 1 \implies 3 \mid b$ but $3 \mid a$, then the contradiction with a, b coprime.

- If $k = im + jn \implies im + jn \leq 2im \implies jn \leq im$, and $im + jn \leq 2jn \implies im \leq jn$, then $im = jn$. As $k = im + jn = 2im = 1 + \mu + \beta$ and $\alpha + \mu = 2im$, we obtain $\alpha = 1 + \beta \geq 2 \implies \beta \geq 1 \implies 3 \mid b$, then the contradiction with a, b coprime.

** G-2-2-2- We suppose that $\omega \neq 3$. We write $a = \omega^\alpha a_1$ with $\omega \nmid a_1$ and $p^n = \omega^\mu p_1$ with $\omega \nmid p_1$. As $A^{2m} = 4ap^n = 4\omega^{\alpha+\mu} a_1 p_1 \implies \omega \mid A \implies A = \omega^i A_1$, $\omega \nmid A_1$. But $B^n C^l = p^n(3b - 4a) = \omega^\mu p_1(3b - 4a) \implies \omega \mid B^n C^l \implies \omega \mid B^n$ or

$\omega \mid C^l$.

** G-2-2-2-1- We suppose that $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$ and $\omega \nmid B_1$. From $A^m + B^n = C^l \implies \omega \mid C^l \implies \omega \mid C$. As $p = bp' = 3bp'' = 3\omega^\mu bp_1 = \omega^k(\omega^{2im-k} A_1^{2m} + \omega^{2jn-k} B_1^{2n} + \omega^{im+jn-k} A_1^m B_1^n)$ with $k = \min(2im, 2jn, im + jn)$. Then:

- If $k = \mu$, then $\omega \nmid b$ and the conjecture (3.1) is verified.
- If $k > \mu$, then $\omega \mid b$, but $\omega \mid a$ then the contradiction with a, b coprime.
- If $k < \mu$, it follows from:

$$3\omega^\mu bp_1 = \omega^k(\omega^{2im-k} A_1^{2m} + \omega^{2jn-k} B_1^{2n} + \omega^{im+jn-k} A_1^m B_1^n)$$

that $\omega \mid A_1$ or $\omega \mid B_1$ then the contradiction with $\omega \nmid A_1$ or $\omega \nmid B_1$.

** G-2-2-2-2- If $\omega \mid C^l \implies \omega \mid C \implies C = \omega^h C_1$ with $\omega \nmid C_1$. From $A^m + B^n = C^l \implies \omega \mid (C^l - A^m) \implies \omega \mid B$. Then, using the same method as for the case G-2-2-2-1-, we obtain identical results.

1.6.4. Case $b = 3$ and $3 \mid p$. — As $3 \mid p \implies p = 3p'$, We write :

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \frac{a}{b} = \frac{4 \times 3p' a}{3 \cdot 3} = \frac{4p' a}{3}$$

As A^{2m} is an integer and a, b are coprime and $\cos^2 \frac{\theta}{3} < 1$ (see equation (1.18)), then we have necessary $3 \mid p' \implies p' = 3p''$ with $p'' \neq 1$, if not $p = 3p' = 3 \times 3p'' = 9$, but $9 \ll (p = A^{2m} + B^{2n} + A^m B^n)$, the hypothesis $p'' = 1$ is impossible, then $p'' > 1$, and we obtain:

$$A^{2m} = \frac{4p' a}{3} = \frac{4 \times 3p'' a}{3} = 4p'' a; \quad B^n C^l = p'' \cdot (9 - 4a)$$

As $\frac{1}{4} < \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{a}{3} < \frac{3}{4} \implies 3 < 4a < 9 \implies$ as $a > 1, a = 2$ and we obtain:

$$(1.121) \quad A^{2m} = 4p'' a = 8p''; \quad B^n C^l = \frac{3p''(9 - 4a)}{3} = p''$$

The two last equations above imply that p'' is not a prime. We can write p'' as : $p'' = \prod_{i \in I} p_i^{\alpha_i}$ where p_i are distinct primes, α_i elements of \mathbb{N}^* and $i \in I$ a finite set of indexes. We can write also $p'' = p_1^{\alpha_1} \cdot q_1$ with $p_1 \nmid q_1$. From (1.121), we have $p_1 \mid A$ and $p_1 \mid B^n C^l \implies p_1 \mid B^n$ or $p_1 \mid C^l$.

** H-1- We suppose that $p_1 \mid B^n \implies B = p_1^{\beta_1} \cdot B_1$ with $p_1 \nmid B_1$ and $\beta_1 \geq 1$. Then, we obtain $B_1^n C^l = p_1^{\alpha_1 - n\beta_1} \cdot q_1$ with the following cases:

- If $\alpha_1 - n\beta_1 \geq 1 \implies p_1 \mid C^l \implies p_1 \mid C$, in accord with $p_1 \mid (C^l = A^m + B^n)$, it follows that the conjecture (3.1) is verified.

- If $\alpha_1 - n\beta_1 = 0 \implies B_1^n C^l = q_1 \implies p_1 \nmid C^l$, it is a contradiction with $p_1 \mid (A^m - B^n) \implies p_1 \mid C^l$. Then this case is impossible.

- If $\alpha_1 - n\beta_1 < 0$, we obtain $p_1^{n\beta_1 - \alpha_1} B_1^n C^l = q_1 \implies p_1 \mid q_1$, it is a contradiction with $p_1 \nmid q_1$. Then this case is impossible.

** H-2- We suppose that $p_1 \mid C^l$, using the same method as for the case $p_1 \mid B^n$, we obtain identical results.

1.6.5. Case $3 \mid p$ and $b = p$. — We have $\cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{a}{p}$ and:

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \cdot \frac{a}{p} = \frac{4a}{3}$$

As A^{2m} is an integer, it implies that $3 \mid a$, but $3 \mid p \implies 3 \mid b$. As a and b are coprime, then the contradiction and the case $3 \mid p$ and $b = p$ is impossible.

1.6.6. Case $3 \mid p$ and $b = 4p$. — $3 \mid p \implies p = 3p'$, $p' \neq 1$ because $3 \ll p$, then $b = 4p = 12p'$.

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \frac{a}{b} = \frac{a}{3} \implies 3 \mid a$$

as A^{2m} is an integer. But $3 \mid p \implies 3 \mid [(4p) = b]$, then the contradiction with a, b coprime and the case $b = 4p$ is impossible.

1.6.7. Case $3 \mid p$ and $b = 2p$. — $3 \mid p \implies p = 3p'$, $p' \neq 1$ because $3 \ll p$, then $b = 2p = 6p'$.

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \frac{a}{b} = \frac{2a}{3} \implies 3 \mid a$$

as A^{2m} is an integer. But $3 \mid p \implies 3 \mid (2p) \implies 3 \mid b$, then the contradiction with a, b coprime and the case $b = 2p$ is impossible.

1.6.8. Case $3 \mid p$ and $b \neq 3$ a divisor of p . — We have $b = p' \neq 3$, and p is written as $p = kp'$ with $3 \mid k \implies k = 3k'$ and :

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \cdot \frac{a}{b} = 4ak'$$

$$B^n C^l = \frac{p}{3} \cdot \left(3 - 4 \cos^2 \frac{\theta}{3}\right) = k'(3p' - 4a) = k'(3b - 4a)$$

** I-1- $k' \neq 1$:

** I-1-1- We suppose that k' is prime, then $A^{2m} = 4ak' = (A^m)^2 \implies k' \mid a$.
But $B^n C^l = k'(3b - 4a) \implies k' \mid B^n$ or $k' \mid C^l$.

** I-1-1-1- If $k' \mid B^n \implies k' \mid B \implies B = k'B_1$ with $B_1 \in \mathbb{N}^*$. Then $k'^{n-1} B_1^n C^l = 3b - 4a$. As $n > 2$, then $(n - 1) > 1$ and $k' \mid a$, then $k' \mid 3b \implies k' = 3$ or $k' \mid b$.

** I-1-1-1-1- If $k' = 3 \implies 3 \mid a$, with a that we can write it under the form $a = 3a'^2$. But $A^m = 6a' \implies 3 \mid A^m \implies 3 \mid A \implies A = 3A_1$ with $A_1 \in \mathbb{N}^*$. Then $3^{m-1} A_1^m = 2a' \implies 3 \mid a' \implies a' = 3a''$. But $k'^{n-1} B_1^n C^l = 3^{n-1} B_1^n C^l = 3b - 4a \implies 3^{n-2} B_1^n C^l = b - 36a''^2$. As $n \geq 3 \implies n - 2 \geq 1$, then $3 \mid b$. Hence the contradiction with a, b coprime.

** I-1-1-1-2- We suppose that $k' \mid b$, but $k' \nmid a$, then the contradiction with a, b coprime.

** I-1-1-2- We suppose that $k' \mid C^l$, using the same method as for the case $k' \mid B^n$, we obtain identical results.

** I-1-2- We consider that k' is not a prime.

** I-1-2-1- We suppose that k', a coprime: $A^{2m} = 4ak' \implies A^m = 2a'.p_1$ with $a = a'^2$ and $k' = p_1^2$, then a', p_1 are also coprime. As $A^m = 2a'.p_1$ then $2 \mid a'$ or $2 \mid p_1$.

** I-1-2-1-1- We suppose that $2 \mid a'$, then $2 \mid a' \implies 2 \nmid p_1$, but $k' = p_1^2$.

** I-1-2-1-1-1- If p_1 is prime, it is impossible with $A^m = 2a'.p_1$.

** I-1-2-1-1-2- We suppose that p_1 is not prime and it can be written as $p_1 = \omega^m \implies k' = \omega^{2m}$. Then $B^n C^l = \omega^{2m}(3b - 4a)$.

** I-1-2-1-1-2-1- If ω is prime $\neq 2$, then $\omega \mid (B^n C^l) \implies \omega \mid B^n$ or $\omega \mid C^l$.

** I-1-2-1-1-2-1-1- If $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$ with $\omega \nmid B_1$, then $B_1^n \cdot C^l = \omega^{2m-nj}(3b - 4a)$.

- If $2m - n.j = 0$, we obtain $B_1^n.C^l = 3b - 4a$, as $C^l = A^m + B^n \implies \omega \mid C^l \implies \omega \mid C$, and $\omega \mid (3b - 4a)$. But $\omega \neq 2$ and ω, a' are coprime, then $\omega \nmid (3b) \implies \omega \neq 3$ and $\omega \nmid b$. Hence, the conjecture (3.1) is verified.

- If $2m - n.j \geq 1$, using the same method, we have $\omega \mid C^l \implies \omega \mid C$ and $\omega \mid (3b - 4a)$ and $\omega \nmid a$ and $\omega \neq 3$ and $\omega \nmid b$. Then the conjecture (3.1) is verified.

- If $2m - n.j < 0 \implies \omega^{n.j-2m} B_1^n.C^l = 3b - 4a$. As $C^l = A^m + B^n \implies \omega \mid C$ then $C = \omega^h.C_1 \implies \omega^{n.j-2m+h.l} B_1^n.C_1^l = 3b - 4a$. If $n.j - 2m + h.l < 0 \implies \omega \mid B_1^n.C_1^l$, then the contradiction with $\omega \nmid B_1$ or $\omega \nmid C_1$. If $n.j - 2m + h.l > 0 \implies \omega \mid (3b - 4a)$ with ω, a, b coprime, it implies that the conjecture (3.1) is verified.

** I-1-2-1-1-2-1-2- We suppose that $\omega \mid C^l$, using the same method as for the case $\omega \mid B^n$, we obtain identical results.

** I-1-2-1-1-2-2- Now $k' = \omega^{2m}$ and ω not a prime, we write $\omega = \omega_1^f.\Omega$ with ω_1 a prime $\nmid \Omega$ and $f \geq 1$ an integer, and $\omega_1 \mid A$, then $B^n.C^l = \omega_1^{2f.m}\Omega^{2m}(3b - 4a) \implies \omega_1 \mid (B^n.C^l) \implies \omega_1 \mid B^n$ or $\omega_1 \mid C^l$.

** I-1-2-1-1-2-2-1- If $\omega_1 \mid B^n \implies \omega_1 \mid B \implies B = \omega_1^j B_1$ with $\omega_1 \nmid B_1$, then $B_1^n.C^l = \omega_1^{2.f.m-n.j}\Omega^{2m}(3b - 4a)$.

- If $2f.m - n.j = 0$, we obtain $B_1^n.C^l = \Omega^{2m}(3b - 4a)$. As $C^l = A^m + B^n \implies \omega_1 \mid C^l \implies \omega_1 \mid C$, and $\omega_1 \mid (3b - 4a)$. But $\omega_1 \neq 2$ and ω_1, a' are coprime, then ω, a are coprime, then $\omega_1 \nmid (3b) \implies \omega_1 \neq 3$ and $\omega_1 \nmid b$. Hence, the conjecture (3.1) is verified.

- If $2f.m - n.j \geq 1$, we have $\omega_1 \mid C^l \implies \omega_1 \mid C$ and $\omega_1 \mid (3b - 4a)$ and $\omega_1 \nmid a$ and $\omega_1 \neq 3$ and $\omega_1 \nmid b$, then the conjecture (3.1) is verified.

- If $2f.m - n.j < 0 \implies \omega_1^{n.j-2m.f} B_1^n.C^l = \Omega^{2m}(3b - 4a)$. As $C^l = A^m + B^n \implies \omega_1 \mid C$, then $C = \omega_1^h.C_1 \implies \omega_1^{n.j-2m.f+h.l} B_1^n.C_1^l = \Omega^{2m}(3b - 4a)$. If $n.j - 2m.f + h.l < 0 \implies \omega_1 \mid B_1^n.C_1^l$, then the contradiction with $\omega_1 \nmid B_1$ and $\omega_1 \nmid C_1$. Then if $n.j - 2m.f + h.l > 0$ and $\omega_1 \mid (3b - 4a)$ with ω_1, a, b coprime, then the conjecture (3.1) is verified.

** I-1-2-1-1-2-2-2- As in the case $\omega_1 \mid B^n$, we obtain identical results if $\omega_1 \mid C^l$.

** I-1-2-1-2- If $2 \mid p_1$: then $2 \mid p_1 \implies 2 \nmid a' \implies 2 \nmid a$, but $k' = p_1^2$.

** I-1-2-1-2-1- If $p_1 = 2$, we obtain $A^m = 4a' \implies 2 \mid a'$, then the contradiction with $2 \nmid a'$. Case to reject.

** I-1-2-1-2-2- We suppose that p_1 is not prime and $2 \mid p_1$. As $A^m = 2a'p_1$, p_1 is written under the form $p_1 = 2^{m-1}\omega^m \implies p_1^2 = 2^{2m-2}\omega^{2m}$. Then $B^n C^l = k'(3b - 4a) = 2^{2m-2}\omega^{2m}(3b - 4a) \implies 2 \mid B^n$ or $2 \mid C^l$.

** I-1-2-1-2-2-1- If $2 \mid B^n \implies 2 \mid B$, as $2 \mid A \implies 2 \mid C$. From $B^n C^l = 2^{2m-2}\omega^{2m}(3b - 4a)$ it follows that if $2 \mid (3b - 4a) \implies 2 \mid b$ but as $2 \nmid a$, there is no contradiction with a, b coprime and the conjecture (3.1) is verified.

** I-1-2-1-2-2-2- We obtain identical results as above if $2 \mid C^l$.

** I-1-2-2- We suppose that k', a are not coprime: let ω be a prime integer so that $\omega \mid a$ and $\omega \mid p_1^2$.

** I-1-2-2-1- We suppose that $\omega = 3$. As $A^{2m} = 4ak' \implies 3 \mid A$, but $3 \nmid p$. As $p = A^{2m} + B^{2n} + A^m B^n \implies 3 \mid B^{2n} \implies 3 \mid B$, then $3 \mid C^l \implies 3 \mid C$. We write $A = 3^i A_1$, $B = 3^j B_1$, $C = 3^h C_1$ with 3 coprime with A_1, B_1 and C_1 and $p = 3^{2im} A_1^{2m} + 3^{2nj} B_1^{2n} + 3^{im+jn} A_1^m B_1^n = 3^s \cdot g$ with $s = \min(2im, 2jn, im+jn)$ and $3 \nmid g$. We have also $(\omega = 3) \mid a$ and $(\omega = 3) \mid k'$ that give $a = 3^\alpha a_1$, $3 \nmid a_1$ and $k' = 3^\mu p_2$, $3 \nmid p_2$ with $A^{2m} = 4ak' = 3^{2im} A_1^{2m} = 4 \times 3^{\alpha+\mu} \cdot a_1 \cdot p_2 \implies \alpha + \mu = 2im$. As $p = 3p' = 3b \cdot k' = 3b \cdot 3^\mu p_2 = 3^{\mu+1} \cdot b \cdot p_2$. The exponent of the factor 3 of p is s , the exponent of the factor 3 of the left member of the last equation is $\mu + 1$ added of the exponent β of 3 of the factor b , with $\beta \geq 0$, let $\min(2im, 2jn, im + jn) = \mu + 1 + \beta$, we recall that $\alpha + \mu = 2im$. But $B^n C^l = k'(4b - 3a)$ that gives $3^{(nj+hl)} B_1^n C_1^l = 3^{\mu+1} p_2 (b - 4 \times 3^{(\alpha-1)} a_1) = 3^{\mu+1} p_2 (3^\beta b_1 - 4 \times 3^{(\alpha-1)} a_1)$, $3 \nmid b_1$. We have also $A^m + B^n = C^l$ that gives $3^{im} A_1^m + 3^{jn} B_1^n = 3^{hl} C_1^l$. We call $\epsilon = \min(im, jn)$, we obtain $\epsilon = hl = \min(im, jn)$. We have then the conditions:

$$(1.122) \quad s = \min(2im, 2jn, im + jn) = \mu + 1 + \beta$$

$$(1.123) \quad \alpha + \mu = 2im$$

$$(1.124) \quad \epsilon = hl = \min(im, jn)$$

$$(1.125) \quad 3^{(nj+hl)} B_1^n C_1^l = 3^{\mu+1} p_2 (3^\beta b_1 - 4 \times 3^{(\alpha-1)} a_1)$$

** I-1-2-2-1-1- $\alpha = 1 \implies a = 3a_1$ and $3 \nmid a_1$, the equation (1.123) becomes:

$$1 + \mu = 2im$$

and the first equation (1.122) is written as :

$$s = \min(2im, 2jn, im + jn) = 2im + \beta$$

- If $s = 2im \implies \beta = 0 \implies 3 \nmid b$. We obtain $2im \leq 2jn \implies im \leq jn$, and $2im \leq im + jn \implies im \leq jn$. The third equation (1.124) gives $hl = im$. The last equation (1.125) gives $nj + hl = \mu + 1 = 2im \implies im = jn$, then $im = jn = hl$ and $B_1^n C_1^l = p_2(b - 4a_1)$. As a, b are coprime, the conjecture (3.1) is verified.

- If $s = 2jn$ or $s = im + jn$, we obtain $\beta = 0$, $im = jn = hl$ and $B_1^n C_1^l = p_2(b - 4a_1)$. Then as a, b are coprime, the conjecture (3.1) is verified.

** I-1-2-2-1-2- $\alpha > 1 \implies \alpha \geq 2$.

- If $s = 2im \implies 2im = \mu + 1 + \beta$, but $\mu = 2im - \alpha$ it gives $\alpha = 1 + \beta \geq 2 \implies \beta \neq 0 \implies 3 \mid b$, but $3 \mid a$ then the contradiction with a, b coprime and the conjecture (3.1) is not verified.

- If $s = 2jn = \mu + 1 + \beta \leq 2im \implies \mu + 1 + \beta \leq \mu + \alpha \implies 1 + \beta \leq \alpha \implies \beta = 1$. If $\beta = 1 \implies 3 \mid b$ but $3 \mid a$, then the contradiction with a, b coprime and the conjecture (3.1) is not verified.

- If $s = im + jn \implies im + jn \leq 2im \implies jn \leq im$, and $im + jn \leq 2jn \implies im \leq jn$, then $im = jn$. As $s = im + jn = 2im = 1 + \mu + \beta$ and $\alpha + \mu = 2im$ it gives $\alpha = 1 + \beta \geq 2 \implies \beta \geq 1 \implies 3 \mid b$, then the contradiction with a, b coprime and the conjecture (3.1) is not verified.

** I-1-2-2-2- We suppose that $\omega \neq 3$. We write $a = \omega^\alpha a_1$ with $\omega \nmid a_1$ and $k' = \omega^\mu p_2$ with $\omega \nmid p_2$. As $A^{2m} = 4ak' = 4\omega^{\alpha+\mu} \cdot a_1 \cdot p_2 \implies \omega \mid A \implies A = \omega^i A_1$, $\omega \nmid A_1$. But $B^n C^l = k'(3b - 4a) = \omega^\mu p_2(3b - 4a) \implies \omega \mid B^n C^l \implies \omega \mid B^n$ or $\omega \mid C^l$.

** I-1-2-2-2-1- $\omega \mid B^n \implies \omega \mid B \implies B^n = \omega^j B_1$ and $\omega \nmid B_1$. From $A^m + B^n = C^l \implies \omega \mid C^l \implies \omega \mid C$. As $p = bp' = 3bk' = 3\omega^\mu bp_2 = \omega^s (\omega^{2im-s} A_1^{2m} + \omega^{2jn-s} B_1^{2n} + \omega^{im+jn-s} A_1^m B_1^n)$ with $s = \min(2im, 2jn, im + jn)$. Then:

- If $s = \mu$, then $\omega \nmid b$ and the conjecture (3.1) is verified.

- If $s > \mu$, then $\omega \mid b$, but $\omega \mid a$ then the contradiction with a, b coprime and the conjecture (3.1) is not verified.

- If $s < \mu$, it follows from:

$$3\omega^\mu b p_1 = \omega^s (\omega^{2im-s} A_1^{2m} + \omega^{2jn-s} B_1^{2n} + \omega^{im+jn-s} A_1^m B_1^n)$$

that $\omega \mid A_1$ or $\omega \mid B_1$ that is the contradiction with the hypothesis and the conjecture (3.1) is not verified.

** I-1-2-2-2- If $\omega \mid C^l \implies \omega \mid C \implies C = \omega^h C_1$ with $\omega \nmid C_1$. From $A^m + B^n = C^l \implies \omega \mid (C^l - A^m) \implies \omega \mid B$. Then we obtain identical results as the case above I-1-2-2-2-1-.

** I-2- We suppose $k' = 1$: then $k' = 1 \implies p = 3b$, then we have $A^{2m} = 4a = (2a')^2 \implies A^m = 2a'$, then $a = a'^2$ is even and :

$$A^m B^n = 2 \sqrt[3]{\rho} \cos \frac{\theta}{3} \cdot \sqrt[3]{\rho} \left(\sqrt{3} \sin \frac{\theta}{3} - \cos \frac{\theta}{3} \right) = \frac{p\sqrt{3}}{3} \sin \frac{2\theta}{3} - 2a$$

and we have also:

$$(1.126) \quad A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3} \sin \frac{2\theta}{3} = 2b\sqrt{3} \sin \frac{2\theta}{3}$$

The left member of the equation (1.126) is a naturel number and also b , then $2\sqrt{3} \sin \frac{2\theta}{3}$ can be written under the form :

$$2\sqrt{3} \sin \frac{2\theta}{3} = \frac{k_1}{k_2}$$

where k_1, k_2 are two natural numbers coprime and $k_2 \mid b \implies b = k_2.k_3$.

** I-2-1- $k' = 1$ and $k_3 \neq 1$: then $A^{2m} + 2A^m B^n = k_3.k_1$. Let μ be a prime integer so that $\mu \mid k_3$. If $\mu = 2 \implies 2 \mid b$, but $2 \nmid a$, it is a contradiction with a, b coprime. We suppose that $\mu \neq 2$ and $\mu \mid k_3$, then $\mu \mid A^m(A^m + 2B^n) \implies \mu \mid A^m$ or $\mu \mid (A^m + 2B^n)$.

** I-2-1-1- $\mu \mid A^m$: If $\mu \mid A^m \implies \mu \mid A^{2m} \implies \mu \mid 4a \implies \mu \mid a$. As $\mu \mid k_3 \implies \mu \mid b$, the contradiction with a, b coprime.

** I-2-1-2- $\mu \mid (A^m + 2B^n)$: If $\mu \mid (A^m + 2B^n) \implies \mu \nmid A^m$ and $\mu \nmid 2B^n$, then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu \mid (A^m + 2B^n)$, we can write $A^m + 2B^n = \mu.t'$. It follows:

$$A^m + B^n = \mu.t' - B^n \implies A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t' \mu B^n + B^{2n}$$

Using the expression of p , we obtain:

$$p = t'^2 \mu^2 - 2t' B^n \mu + B^n (B^n - A^m)$$

As $p = 3b = 3k_2.k_3$ and $\mu \mid k_3$ then $\mu \mid p \implies p = \mu.\mu'$, then we obtain:

$$\mu'.\mu = \mu(\mu t'^2 - 2t'B^n) + B^n(B^n - A^m)$$

and $\mu \mid B^n(B^n - A^m) \implies \mu \mid B^n$ or $\mu \mid (B^n - A^m)$.

** I-2-1-2-1- $\mu \mid B^n$: If $\mu \mid B^n \implies \mu \mid B$, that is the contradiction with I-2-1-2- above.

** I-2-1-2-2- $\mu \mid (B^n - A^m)$: If $\mu \mid (B^n - A^m)$ and using that $\mu \mid (A^m + 2B^n)$, we obtain :

$$\mu \mid 3B^n \implies \begin{cases} \mu \mid B^n \implies \mu \mid B \\ \text{or} \\ \mu = 3 \end{cases}$$

** I-2-1-2-2-1- $\mu \mid B^n$: If $\mu \mid B^n \implies \mu \mid B$, that is the contradiction with I-2-1-2- above.

** I-2-1-2-2-2- $\mu = 3$: If $\mu = 3 \implies 3 \mid k_3 \implies k_3 = 3k'_3$, and we have $b = k_2k_3 = 3k_2k'_3$, it follows $p = 3b = 9k_2k'_3$, then $9 \mid p$, but $p = (A^m - B^n)^2 + 3A^mB^n$ then:

$$9k_2k'_3 - 3A^mB^n = (A^m - B^n)^2$$

that we write as:

$$(1.127) \quad 3(3k_2k'_3 - A^mB^n) = (A^m - B^n)^2$$

then:

$$3 \mid (3k_2k'_3 - A^mB^n) \implies 3 \mid A^mB^n \implies 3 \mid A^m \text{ or } 3 \mid B^n$$

** I-2-1-2-2-2-1- $3 \mid A^m$: If $3 \mid A^m \implies 3 \mid A$ and we have also $3 \mid A^{2m}$, but $A^{2m} = 4a \implies 3 \mid 4a \implies 3 \mid a$. As $b = 3k_2k'_3$ then $3 \mid b$, but a, b are coprime, then the contradiction and $3 \nmid A$.

** I-2-1-2-2-2-2- $3 \mid B^m$: If $3 \mid B^n \implies 3 \mid B$, but the equation (1.127) implies $3 \mid (A^m - B^n)^2 \implies 3 \mid (A^m - B^n) \implies 3 \mid A^m \implies 3 \mid A$. The last case above has given that $3 \nmid A$. Then the case $3 \mid B^m$ is to reject.

Finally the hypothesis $k_3 \neq 1$ is impossible.

** I-2-2- Now, we suppose that $k_3 = 1 \implies b = k_2$ and $p = 3b = 3k_2$, then we have:

$$(1.128) \quad 2\sqrt{3}\sin\frac{2\theta}{3} = \frac{k_1}{b}$$

with k_1, b coprime. We write (1.128) as :

$$4\sqrt{3}\sin\frac{\theta}{3}\cos\frac{\theta}{3} = \frac{k_1}{b}$$

Taking the square of the two members and replacing $\cos^2\frac{\theta}{3}$ by $\frac{a}{b}$, we obtain:

$$3 \times 4^2 \cdot a(b-a) = k_1^2 \implies k_1^2 = 3 \times 4^2 \cdot a'(b-a)$$

it implies that :

$$b-a = 3\alpha^2 \implies b = a'^2 + 3\alpha^2 \implies k_1 = 12a'\alpha$$

As:

$$k_1 = 12a'\alpha = A^m(A^m + 2B^n) \implies 3\alpha = a' + B^n$$

We consider now that $3 \mid (b-a)$ with $b = a'^2 + 3\alpha^2$. The case $\alpha = 1$ gives $a' + B^n = 3$ that is impossible. We suppose $\alpha > 1$, the pair (a', α) is a solution of the Diophantine equation:

$$(1.129) \quad X^2 + 3Y^2 = b$$

with $X = a'$ and $Y = \alpha$. But using a theorem on the solutions of the equation given by (1.129), b is written as (see theorem in [7]):

$$b = 2^{2s} \times 3^t \cdot p_1^{t_1} \dots p_g^{t_g} q_1^{2s_1} \dots q_r^{2s_r}$$

where p_i are prime numbers verifying $p_i \equiv 1 \pmod{6}$, the q_j are also prime numbers so that $q_j \equiv 5 \pmod{6}$, then :

- If $s \geq 1 \implies 2 \mid b$, as $2 \mid a$, then the contradiction with a, b coprime.
- If $t \geq 1 \implies 3 \mid b$, but $3 \mid (b-a) \implies 3 \mid a$, then the contradiction with a, b coprime.

** I-2-2-1- We suppose that b is written as :

$$b = p_1^{t_1} \dots p_g^{t_g} q_1^{2s_1} \dots q_r^{2s_r}$$

with $p_i \equiv 1 \pmod{6}$ and $q_j \equiv 5 \pmod{6}$. Finally, we obtain that $b \equiv 1 \pmod{6}$. We will verify then this condition.

** I-2-2-1-1- We present the table below giving the value of $A^m + B^n = C^l$ modulo 6 in function of the value of $A^m, B^n \pmod{6}$. We obtain the table below after retiring the lines (respectively the colones) of $A^m \equiv 0 \pmod{6}$ and $A^m \equiv 3 \pmod{6}$ (respectively of $B^n \equiv 0 \pmod{6}$ and $B^n \equiv 3 \pmod{6}$), they present cases with contradictions:

TABLE 2. Table of $C^l \pmod{6}$

A^m, B^n	1	2	4	5
1	2	3	5	0
2	3	4	0	1
4	5	0	2	3
5	0	1	3	4

** I-2-2-1-1-1- For the case $C^l \equiv 0 \pmod{6}$ and $C^l \equiv 3 \pmod{6}$, we deduce that $3 \mid C^l \implies 3 \mid C \implies C = 3^h C_1$, with $h \geq 1$ and $3 \nmid C_1$. It follows that $p - B^n C^l = 3b - 3^{lh} C_1^l B^n = A^{2m} \implies 3 \mid (A^{2m} = 4a) \implies 3 \mid a \implies 3 \mid b$, then the contradiction with a, b coprime.

** I-2-2-1-1-2- For the case $C^l \equiv 0 \pmod{6}$, $C^l \equiv 2 \pmod{6}$ and $C^l \equiv 4 \pmod{6}$, we deduce that $2 \mid C^l \implies 2 \mid C \implies C = 2^h C_1$, with $h \geq 1$ and $2 \nmid C_1$. It follows that $p = 3b = A^{2m} + B^n C^l = 4a + 2^{lh} C_1^l B^n \implies 2 \mid 3b \implies 2 \mid b$, then the contradiction with a, b coprime.

** I-2-2-1-1-3- We consider the cases $A^m \equiv 1 \pmod{6}$ and $B^n \equiv 4 \pmod{6}$ (respectively $B^n \equiv 2 \pmod{6}$): then $2 \mid B^n \implies 2 \mid B \implies B = 2^j B_1$ with $j \geq 1$ and $2 \nmid B_1$. It follows from $3b = A^{2m} + B^n C^l = 4a + 2^{jn} B_1^n C^l$ that $2 \mid b$, then the contradiction with a, b coprime.

** I-2-2-1-1-4- We consider the case $A^m \equiv 5 \pmod{6}$ and $B^n \equiv 2 \pmod{6}$: then $2 \mid B^n \implies 2 \mid B \implies B = 2^j B_1$ with $j \geq 1$ and $2 \nmid B_1$. It follows that $3b = A^{2m} + B^n C^l = 4a + 2^{jn} B_1^n C^l$, then $2 \mid b$ and we obtain the contradiction with a, b coprime.

** I-2-2-1-1-5- We consider the case $A^m \equiv 2 \pmod{6}$ and $B^n \equiv 5 \pmod{6}$: as $A^m \equiv 2 \pmod{6} \implies A^m \equiv 2 \pmod{3}$, then A^m is not a square and also for

B^n . Hence, we can write A^m and B^n as:

$$\begin{aligned} A^m &= a_0 \mathcal{A}^2 \\ B^n &= b_0 \mathcal{B}^2 \end{aligned}$$

where a_0 (respectively b_0) regroups the product of the prime numbers of A^m with exponent 1 (respectively of B^n) with not necessary $(a_0, \mathcal{A}) = 1$ and $(b_0, \mathcal{B}) = 1$. We have also $p = 3b = A^{2m} + A^m B^n + B^{2n} = (A^m - B^n)^2 + 3A^m B^n \implies 3 \mid (b - A^m B^n) \implies A^m B^n \equiv b \pmod{3}$ but $b = a + 3\alpha^2 \implies b \equiv a \equiv a'^2 \pmod{3}$, then $A^m B^n \equiv a'^2 \pmod{3}$. But $A^m \equiv 2 \pmod{6} \implies 2a' \equiv 2 \pmod{6} \implies 4a'^2 \equiv 4 \pmod{6} \implies a'^2 \equiv 1 \pmod{3}$. It follows that $A^m B^n$ is a square, let $A^m B^n = \mathcal{N}^2 = \mathcal{A}^2 \mathcal{B}^2 a_0 b_0$. We call $\mathcal{N}_1^2 = a_0 b_0$. Let p_1 be a prime number so that $p_1 \mid a_0 \implies a_0 = p_1 a_1$ with $p_1 \nmid a_1$. $p_1 \mid \mathcal{N}_1^2 \implies p_1 \mid \mathcal{N}_1 \implies \mathcal{N}_1 = p_1^t \mathcal{N}'_1$ with $t \geq 1$ and $p_1 \nmid \mathcal{N}'_1$, then $p_1^{2t-1} \mathcal{N}'_1{}^2 = a_1 b_0$. As $2t \geq 2 \implies 2t - 1 \geq 1 \implies p_1 \mid a_1 b_0$ but $(p_1, a_1) = 1$, then $p_1 \mid b_0 \implies p_1 \mid B^n \implies p_1 \mid B$. But $p_1 \mid (A^m = 2a')$, and $p_1 \neq 2$ because $p_1 \mid B^n$ and B^n is odd, then the contradiction. Hence, $p_1 \mid a' \implies p_1 \mid a$. If $p_1 = 3$, from $3 \mid (b - a) \implies 3 \mid b$ then the contradiction with a, b coprime. Then $p_1 > 3$ a prime that divides A^m and B^n , then $p_1 \mid (p = 3b) \implies p_1 \mid b$, it follows the contradiction with a, b coprime, knowing that $p = 3b \equiv 3 \pmod{6}$ and we choose the case $b \equiv 1 \pmod{6}$ of our interest.

** I-2-2-1-1-6- We consider the last case of the table above $A^m \equiv 4 \pmod{6}$ and $B^n \equiv 1 \pmod{6}$. We return to the equation (1.129) that b verifies :

$$(1.130) \quad \begin{aligned} b &= X^2 + 3Y^2 \\ \text{with } X &= a'; \quad Y = \alpha \\ \text{and } 3\alpha &= a' + B^n \end{aligned}$$

Suppose that it exists another solution of (1.130):

$$b = X^2 + 3Y^3 = u^2 + 3v^2 \implies 2u \neq A^m, 3v \neq a' + B^n$$

But $B^n = \frac{6\alpha - A^m}{2} = 3\alpha - a'$ and b verifies also $:3b = p = A^{2m} + A^m B^n + B^{2n}$, it is impossible that u, v verify:

$$\begin{aligned} 6v &= 2u + 2B^n \\ 3b &= 4u^2 + 2uB^n + B^{2n} \end{aligned}$$

If we consider that : $6v - 2u = 6\alpha - 2a' \implies u = 3v - 3\alpha + a'$, then $b = u^2 + 3v^2 = (3v - 3\alpha + a')^2 + 3v^2$, it gives:

$$\begin{aligned} 2v^2 - B^n v + \alpha^2 - a'\alpha &= 0 \\ 2v^2 - B^n v - \frac{(a' + B^n)(A^m - B^n)}{9} &= 0 \end{aligned}$$

The resolution of the last equation gives with taking the positive root (because $A^m > B^n$), $v_1 = \alpha$, then $u = a'$. It follows that b in (1.130) has an unique representation under the form $X^2 + 3Y^2$ with $X, 3Y$ coprime. As b is odd, we applique one of Euler's theorems on the convenient numbers "numerus idoneus" as cited above (Case C-2-2-1-2). It follows that b is prime.

We have also $p = 3b = A^{2m} + A^m B^n + B^{2n} = 4a'^2 + B^n \cdot C^l \implies 9\alpha^2 - a'^2 = B^n \cdot C^l$, then $3\alpha, a' \in \mathbb{N}^*$ are solutions of the Diophantine equation:

$$(1.131) \quad x^2 - y^2 = N$$

with $N = B^n C^l > 0$. Let $Q(N)$ be the number of the solutions of (1.131) and $\tau(N)$ the number of ways to write the factors of N , then we announce the following result concerning the number of the solutions of (1.131) (see theorem 27.3 in [7]):

- If $N \equiv 2 \pmod{4}$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3 \pmod{4}$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0 \pmod{4}$, then $Q(N) = [\tau(N/4)/2]$.

We recall that $A^m \equiv 0 \pmod{4}$. Concerning B^n , for $B^n \equiv 0 \pmod{4}$ or $B^n \equiv 2 \pmod{4}$, we find that $2 \mid B^n \implies 2 \mid \alpha \implies 2 \mid b$, then the contradiction with a, b coprime.

For the last case $B^n \equiv 3 \pmod{4} \implies C^l \equiv 3 \pmod{4} \implies N = B^n C^l \equiv 1 \pmod{4} \implies Q(N) = [\tau(N)/2] > 1$.

As $(3\alpha, a')$ is a couple of solutions of the Diophantine equation (1.131) and $3\alpha > a'$, then $\exists d, d'$ positive integers with $d > d'$ and $N = d \cdot d'$ so that :

$$(1.132) \quad d + d' = 6\alpha$$

$$(1.133) \quad d - d' = 2a'$$

** I-2-2-1-1-6-1 Now, we consider the case $d = c_1^{lr-1} C_1^l$ where c_1 is a prime integer with $c_1 \nmid C_1$ and $C = c_1^r C_1$, $r \geq 1$. It follows that $d' = c_1 \cdot B^n$. We

rewrite the equations (3.42-3.43):

$$(1.134) \quad c_1^{lr-1}C_1^l + c_1.B^n = 6\alpha$$

$$(1.135) \quad c_1^{lr-1}C_1^l - c_1.B^n = 2a'$$

As $l \geq 3$, from the last two equations above, it follows that $c_1 \mid (6\alpha)$ and $c_1 \mid (2a')$. Then $c_1 = 2$, or $c_1 = 3$ and $3 \mid a'$ or $c_1 \neq 3 \mid \alpha$ and $c_1 \mid a'$.

** I-2-2-1-1-6-1-1 We suppose $c_1 = 2$. As $2 \mid (A^m = 2a') \Rightarrow 2 \mid a$ and $2 \mid C^l$ because $l \geq 3$, it follows $2 \mid B^n$, then $2 \mid (p = 3b)$. Then the contradiction with a, b coprime.

** I-2-2-1-1-6-1-2 We suppose $c_1 = 3 \Rightarrow c_1 \mid (a = 3a')$ and $c_1 = 3 \mid a'$. It follows that $(c_1 = 3) \mid (b = a'^2 + 3\alpha^2)$, then the contradiction with a, b coprime.

** I-2-2-1-1-6-1-3 We suppose $c_1 \neq 3$ and $c_1 \mid 3\alpha$ and $c_1 \mid a'$. It follows that $c_1 \mid a$ and $c_1 \mid b$, then the contradiction with a, b coprime.

The others cases of the expressions of d and d' not coprime so that $N = B^n C^l = d.d'$ give also contradictions.

** I-2-2-1-1-6-2 The last case is to consider $d = C^l$ and $d' = B^n$, so we obtain the only solution $(3\alpha, a')$ of the Diophantine equation (1.131). It follows that $Q(N) = 1$, then the contradiction with $Q(N) = [\tau(N)/2] > 1$ the number of the solution of (1.131).

It follows that the condition $3 \mid (b - a)$ is a contradiction.

The study of the case 1.6.8 is achieved.

1.6.9. Case $3 \mid p$ and $b \mid 4p$. — The following cases have been soon studied:

- * $3 \mid p, b = 2 \Rightarrow b \mid 4p$: case 1.6.1,
- * $3 \mid p, b = 4 \Rightarrow b \mid 4p$: case 1.6.2,
- * $3 \mid p \Rightarrow p = 3p', b \mid p' \Rightarrow p' = bp'', p'' \neq 1$: case 1.6.3,
- * $3 \mid p, b = 3 \Rightarrow b \mid 4p$: case 1.6.4,
- * $3 \mid p \Rightarrow p = 3p', b = p' \Rightarrow b \mid 4p$: case 1.6.8.

** J-1- Particular case: $b = 12$. In fact $3 \mid p \Rightarrow p = 3p'$ and $4p = 12p'$. Taking $b = 12$, we have $b \mid 4p$. But $b < 4a < 3b$, that gives

$12 < 4a < 36 \implies 3 < a < 9$. As $2 \mid b$ and $3 \mid b$, the possible values of a are 5 and 7.

** J-1-1- $a = 5$ and $b = 12 \implies 4p = 12p' = bp'$. But $A^{2m} = \frac{4p}{3} \cdot \frac{a}{b} = \frac{5bp'}{3b} = \frac{5p'}{3} \implies 3 \mid p' \implies p' = 3p''$ with $p'' \in \mathbb{N}^*$, then $p = 9p''$, we obtain the expressions:

$$(1.136) \quad A^{2m} = 5p''$$

$$(1.137) \quad B^n C^l = \frac{p}{3} \left(3 - 4\cos^2 \frac{\theta}{3} \right) = 4p''$$

As $n, l \geq 3$, we deduce from the equation (1.137) that $2 \mid p'' \implies p'' = 2^\alpha p_1$ with $\alpha \geq 1$ and $2 \nmid p_1$. Then (1.136) becomes: $A^{2m} = 5p'' = 5 \times 2^\alpha p_1 \implies 2 \mid A \implies A = 2^i A_1$, $i \geq 1$ and $2 \nmid A_1$. We have also $B^n C^l = 2^{\alpha+2} p_1 \implies 2 \mid B^n$ or $2 \mid C^l$.

** J-1-1-1- We suppose that $2 \mid B^n \implies B = 2^j B_1$, $j \geq 1$ and $2 \nmid B_1$. We obtain $B_1^n C^l = 2^{\alpha+2-jn} p_1$:

- If $\alpha + 2 - jn > 0 \implies 2 \mid C^l$, there is no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n \implies 2 \mid C^l$ and the conjecture (3.1) is verified.

- If $\alpha + 2 - jn = 0 \implies B_1^n C^l = p_1$. From $C^l = 2^{im} A_1^m + 2^{jn} B_1^n \implies 2 \mid C^l$ that implies that $2 \mid p_1$, then the contradiction with $2 \nmid p_1$.

- If $\alpha + 2 - jn < 0 \implies 2^{jn-\alpha-2} B_1^n C^l = p_1$, it implies that $2 \mid p_1$, then the contradiction as above.

** J-1-1-2- We suppose that $2 \mid C^l$, using the same method above, we obtain the identical results.

** J-1-2- We suppose that $a = 7$ and $b = 12 \implies 4p = 12p' = bp'$. But $A^{2m} = \frac{4p}{3} \cdot \frac{a}{b} = \frac{12p'}{3} \cdot \frac{7}{12} = \frac{7p'}{3} \implies 3 \mid p' \implies p = 9p''$, we obtain:

$$A^{2m} = 7p''$$

$$B^n C^l = \frac{p}{3} \left(3 - 4\cos^2 \frac{\theta}{3} \right) = 2p''$$

The last equation implies that $2 \mid B^n C^l$. Using the same method as for the case J-1-1- above, we obtain the identical results.

We study now the general case. As $3 \mid p \Rightarrow p = 3p'$ and $b \mid 4p \Rightarrow \exists k_1 \in \mathbb{N}^*$ and $4p = 12p' = k_1 b$.

** J-2- $k_1 = 1$: If $k_1 = 1$ then $b = 12p'$, ($p' \neq 1$, if not $p = 3 \ll A^{2m} + B^{2n} + A^m B^n$). But $A^{2m} = \frac{4p}{3} \cdot \cos^2 \frac{\theta}{3} = \frac{12p' a}{3 b} = \frac{4p' \cdot a}{12p'} = \frac{a}{3} \Rightarrow 3 \mid a$ because A^{2m} is a natural number, then the contradiction with a, b coprime.

** J-3- $k_1 = 3$: If $k_1 = 3$, then $b = 4p'$ and $A^{2m} = \frac{4p}{3} \cdot \cos^2 \frac{\theta}{3} = \frac{k_1 \cdot a}{3} = a = (A^m)^2 = a'^2 \Rightarrow A^m = a'$. The term $A^m B^n$ gives $A^m B^n = \frac{p\sqrt{3}}{3} \sin \frac{2\theta}{3} - \frac{a}{2}$, then:

$$(1.138) \quad A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3} \sin \frac{2\theta}{3} = 2p' \sqrt{3} \sin \frac{2\theta}{3}$$

The left member of (1.138) is an integer number and also p' , then $2\sqrt{3} \sin \frac{2\theta}{3}$ can be written under the form:

$$2\sqrt{3} \sin \frac{2\theta}{3} = \frac{k_2}{k_3}$$

where k_2, k_3 are two integer numbers and are coprime and $k_3 \mid p' \Rightarrow p' = k_3 \cdot k_4$.

** J-3-1- $k_4 \neq 1$: We suppose that $k_4 \neq 1$, then:

$$(1.139) \quad A^{2m} + 2A^m B^n = k_2 \cdot k_4$$

Let μ be a prime number so that $\mu \mid k_4$, then $\mu \mid A^m(A^m + 2B^n) \Rightarrow \mu \mid A^m$ or $\mu \mid (A^m + 2B^n)$.

** J-3-1-1- $\mu \mid A^m$: If $\mu \mid A^m \Rightarrow \mu \mid A^{2m} \Rightarrow \mu \mid a$. As $\mu \mid k_4 \Rightarrow \mu \mid p' \Rightarrow \mu \mid (4p' = b)$. But a, b are coprime, then the contradiction.

** J-3-1-2- $\mu \mid (A^m + 2B^n)$: If $\mu \mid (A^m + 2B^n) \Rightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$, then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu \mid (A^m + 2B^n)$, we can write $A^m + 2B^n = \mu \cdot t'$. It follows:

$$A^m + B^n = \mu t' - B^n \Rightarrow A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t' \mu B^n + B^{2n}$$

Using the expression of p , we obtain $p = t'^2 \mu^2 - 2t' B^n \mu + B^n (B^n - A^m)$. As $p = 3p'$ and $\mu \mid p' \Rightarrow \mu \mid (3p') \Rightarrow \mu \mid p$, we can write : $\exists \mu'$ and $p = \mu \mu'$, then we arrive to:

$$\mu' \cdot \mu = \mu(\mu t'^2 - 2t' B^n) + B^n (B^n - A^m)$$

and $\mu \mid B^n(B^n - A^m) \implies \mu \mid B^n$ or $\mu \mid (B^n - A^m)$.

** J-3-1-2-1- $\mu \mid B^n$: If $\mu \mid B^n \implies \mu \mid B$, it is in contradiction with J-3-1-2-.

** J-3-1-2-2- $\mu \mid (B^n - A^m)$: If $\mu \mid (B^n - A^m)$ and using $\mu \mid (A^m + 2B^n)$, we obtain :

$$\mu \mid 3B^n \implies \begin{cases} \mu \mid B^n \\ \text{or} \\ \mu = 3 \end{cases}$$

** J-3-1-2-2-1- $\mu \mid B^n$: If $\mu \mid B^n \implies \mu \mid B$, it is in contradiction with J-3-1-2-.

** J-3-1-2-2-2- $\mu = 3$: If $\mu = 3 \implies 3 \mid k_4 \implies k_4 = 3k'_4$, and we have $p' = k_3k_4 = 3k_3k'_4$, it follows that $p = 3p' = 9k_3k'_4$, then $9 \mid p$, but $p = (A^m - B^n)^2 + 3A^mB^n$, then we obtain:

$$9k_3k'_4 - 3A^mB^n = (A^m - B^n)^2$$

that we write : $3(3k_3k'_4 - A^mB^n) = (A^m - B^n)^2$, then : $3 \mid (3k_3k'_4 - A^mB^n) \implies 3 \mid A^mB^n \implies 3 \mid A^m$ or $3 \mid B^n$.

** J-3-1-2-2-2-1- $3 \mid A^m$: If $3 \mid A^m \implies 3 \mid A^{2m} \implies 3 \mid a$, but $3 \mid p' \implies 3 \mid (4p') \implies 3 \mid b$, then the contradiction with a, b coprime and $3 \nmid A$.

** J-3-1-2-2-2-2- $3 \mid B^n$: If $3 \mid B^n$ but $A^m = \mu t' - 2B^n = 3t' - 2B^n \implies 3 \mid A^m$, it is in contradiction with $3 \nmid A$.

Then the hypothesis $k_4 \neq 1$ is impossible.

** J-3-2- $k_4 = 1$: We suppose now that $k_4 = 1 \implies p' = k_3k_4 = k_3$. Then we have:

$$(1.140) \quad 2\sqrt{3}\sin\frac{2\theta}{3} = \frac{k_2}{p'}$$

with k_2, p' coprime, we write (1.140) as :

$$4\sqrt{3}\sin\frac{\theta}{3}\cos\frac{\theta}{3} = \frac{k_2}{p'}$$

Taking the square of the two members and replacing $\cos^2 \frac{\theta}{3}$ by $\frac{a}{b}$ and $b = 4p'$, we obtain:

$$3.a(b-a) = k_2^2$$

As $A^{2m} = a = a'^2$, it implies that :

$$3 \mid (b-a), \quad \text{and} \quad b-a = b-a'^2 = 3\alpha^2$$

As $k_2 = A^m(A^m + 2B^n)$ following the equation (1.139) and that $3 \mid k_2 \implies 3 \mid A^m(A^m + 2B^n) \implies 3 \mid A^m$ or $3 \mid (A^m + 2B^n)$.

** J-3-2-1- $3 \mid A^m$: If $3 \mid A^m \implies 3 \mid A^{2m} \implies 3 \mid a$, but $3 \mid (b-a) \implies 3 \mid b$, then the contradiction with a, b coprime.

** J-3-2-2- $3 \mid (A^m + 2B^n) \implies 3 \nmid A^m$ and $3 \nmid B^n$. As $k_2^2 = 9a\alpha^2 = 9a'^2\alpha^2 \implies k_2 = 3a'\alpha = A^m(A^m + 2B^n)$, then :

$$(1.141) \quad 3\alpha = A^m + 2B^n$$

As b can be written under the form $b = a'^2 + 3\alpha^2$, then the pair (a', α) is a solution of the Diophantine equation:

$$(1.142) \quad x^2 + 3y^2 = b$$

As $b = 4p'$, then :

** J-3-2-2-1- If x, y are even, then $2 \mid a' \implies 2 \mid a$, it is a contradiction with a, b coprime.

** J-3-2-2-2- If x, y are odd, then a', α are odd, it implies $A^m = a' \equiv 1 \pmod{4}$ or $A^m \equiv 3 \pmod{4}$. If u, v verify (1.142), then $b = u^2 + 3v^2$, with $u \neq a'$ and $v \neq \alpha$, then u, v do not verify (1.141): $3v \neq u + 2B^n$, if not, $u = 3v - 2B^n \implies b = (3v - 2B^n)^2 + 3v^2 = a'^2 + 3\alpha$, the resolution of the obtained equation of second degree in v gives the positive root $v_1 = \alpha$, then $u = 3\alpha - 2B^n = a'$, then the uniqueness of the representation of b by the equation (1.142).

** J-3-2-2-2-1- We suppose that $A^m \equiv 1 \pmod{4}$ and $B^n \equiv 0 \pmod{4}$, then B^n is even and $B^n = 2B'$. The expression of p becomes:

$$\begin{aligned} p &= a'^2 + 2a'B' + 4B'^2 = (a' + B')^2 + 3B'^2 = 3p' \implies 3 \mid (a' + B') \implies a' + B' = 3B'' \\ p' &= B'^2 + 3B''^2 \implies b = 4p' = (2B')^2 + 3(2B'')^2 = a'^2 + 3\alpha^2 \end{aligned}$$

that gives $2B' = B^n = a' = A^m$, then the contradiction with $A^m > B^n$.

** J-3-2-2-2-2- We suppose that $A^m \equiv 1 \pmod{4}$ and $B^n \equiv 1 \pmod{4}$, then C^l is even and $C^l = 2C'$. The expression of p becomes:

$$\begin{aligned} p &= C^{2l} - C^l B^n + B^{2n} = 4C'^2 - 2C' B^n + B^{2n} = (C' - B^n)^2 + 3C'^2 = 3p' \\ &\implies 3 \mid (C' - B^n) \implies C' - B^n = 3C'' \\ p' &= C'^2 + 3C''^2 \implies b = 4p' = (2C')^2 + 3(2C'')^2 = a'^2 + 3a''^2 \end{aligned}$$

We obtain $2C' = C^l = a' = A^m$, then the contradiction.

** J-3-2-2-2-3- We suppose that $A^m \equiv 1 \pmod{4}$ and $B^n \equiv 2 \pmod{4}$, then B^n is even, see J-3-2-2-2-1-.

** J-3-2-2-2-4- We suppose that $A^m \equiv 1 \pmod{4}$ and $B^n \equiv 3 \pmod{4}$, then C^l is even, see J-3-2-2-2-2-.

** J-3-2-2-2-5- We suppose that $A^m \equiv 3 \pmod{4}$ and $B^n \equiv 0 \pmod{4}$, then B^n is even, see J-3-2-2-2-1-.

** J-3-2-2-2-6- We suppose that $A^m \equiv 3 \pmod{4}$ and $B^n \equiv 1 \pmod{4}$, then C^l is even, see J-3-2-2-2-2-.

** J-3-2-2-2-7- We suppose that $A^m \equiv 3 \pmod{4}$ and $B^n \equiv 2 \pmod{4}$, then B^n is even, see J-3-2-2-2-1-.

** J-3-2-2-2-8- We suppose that $A^m \equiv 3 \pmod{4}$ and $B^n \equiv 3 \pmod{4}$, then C^l is even, see J-3-2-2-2-2-.

We have achieved the study of the case J-3-2-2- . It gives contradictions.

** J-4- We suppose that $k_1 \neq 3$ and $3 \mid k_1 \implies k_1 = 3k'_1$ with $k'_1 \neq 1$, then $4p = 12p' = k_1 b = 3k'_1 b \implies 4p' = k'_1 b$. A^{2m} can be written as $A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{3k'_1 b a}{3 b} = k'_1 a$ and $B^n C^l = \frac{p}{3} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) = \frac{k'_1}{4} (3b - 4a)$. As $B^n C^l$ is an integer number, we must have $4 \mid (3b - 4a)$ or $4 \mid k'_1$ or $[2 \mid k'_1 \text{ and } 2 \mid (3b - 4a)]$.

** J-4-1- We suppose that $4 \mid (3b - 4a)$.

** J-4-1-1- We suppose that $3b - 4a = 4 \implies 4 \mid b \implies 2 \mid b$. Then, we have:

$$\begin{aligned} A^{2m} &= k'_1 a \\ B^n C^l &= k'_1 \end{aligned}$$

** J-4-1-1-1- If k'_1 is prime, from $B^n C^l = k'_1$, it is impossible.

** J-4-1-1-2- We suppose that $k'_1 > 1$ is not prime. Let ω be a prime number so that $\omega \mid k'_1$.

** J-4-1-1-2-1- We suppose that $k'_1 = \omega^s$, with $s \geq 6$. Then we have :

$$(1.143) \quad A^{2m} = \omega^s .a$$

$$(1.144) \quad B^n C^l = \omega^s$$

** J-4-1-1-2-1-1- We suppose that $\omega = 2$. If a, k'_1 are not coprime , then $2 \mid a$, as $2 \mid b$, it is the contradiction with a, b coprime.

** J-4-1-1-2-1-2- We suppose $\omega = 2$ and a, k'_1 are coprime, then $2 \nmid a$. From (1.144), we deduce that $B = C = 2$ and $n + l = s$, and $A^{2m} = 2^s .a$, but $A^m = 2^l - 2^n \implies A^{2m} = (2^l - 2^n)^2 = 2^{2l} + 2^{2n} - 2(2^{l+n}) = 2^{2l} + 2^{2n} - 2 \times 2^s = 2^s .a \implies 2^{2l} + 2^{2n} = 2^s(a + 2)$. If $l = n$, we obtain $a = 0$ then the contradiction. If $l \neq n$, as $A^m = 2^l - 2^n > 0 \implies n < l \implies 2n < s$, then $2^{2n}(1 + 2^{2l-2n} - 2^{s+1-2n}) = 2^n 2^l .a$. We call $l = n + n_1 \implies 1 + 2^{2l-2n} - 2^{s+1-2n} = 2^{n_1} .a$, but the left member is odd and the right member is even, then the contradiction. Then the case $\omega = 2$ is impossible.

** J-4-1-1-2-1-3- We suppose that $k'_1 = \omega^s$ with $\omega \neq 2$:

** J-4-1-1-2-1-3-1- Suppose that a, k'_1 are not coprime, then $\omega \mid a \implies a = \omega^t .a_1$ and $t \nmid a_1$. Then, we have:

$$(1.145) \quad A^{2m} = \omega^{s+t} .a_1$$

$$(1.146) \quad B^n C^l = \omega^s$$

From (1.146), we deduce that $B^n = \omega^n$, $C^n = \omega^l$, $s = n + l$ and $A^m = \omega^l - \omega^n > 0 \implies l > n$. We have also $A^{2m} = \omega^{s+t} .a_1 = (\omega^l - \omega^n)^2 = \omega^{2l} + \omega^{2n} - 2 \times \omega^s$. As $\omega \neq 2 \implies \omega$ is odd, then $A^{2m} = \omega^{s+t} .a_1 = (\omega^l - \omega^n)^2$ is even, then $2 \mid a_1 \implies 2 \mid a$, it is in contradiction with a, b coprime, then this case is

impossible.

** J-4-1-1-2-1-3-2- Suppose that a, k'_1 are coprime, with :

$$(1.147) \quad A^{2m} = \omega^s . a$$

$$(1.148) \quad B^n C^l = \omega^s$$

From (1.148), we deduce that $B^n = \omega^n$, $C^l = \omega^l$ and $s = n + l$. As $\omega \neq 2 \implies \omega$ is odd and $A^{2m} = \omega^s . a = (\omega^l - \omega^n)^2$ is even, then $2 \mid a$. It follows the contradiction with a, b coprime and this case is impossible.

** J-4-1-1-2-2- We suppose that $k'_1 = \omega^s . k_2$, with $s \geq 6$, $\omega \nmid k_2$. We have :

$$A^{2m} = \omega^s . k_2 . a$$

$$B^n C^l = \omega^s . k_2$$

** J-4-1-1-2-2-1- If k_2 is prime, from the last equation above, $\omega = k_2$, it is in contradiction with $\omega \nmid k_2$. Then this case is impossible.

** J-4-1-1-2-2-2- We suppose that $k'_1 = \omega^s . k_2$, with $s \geq 6$, $\omega \nmid k_2$ and k_2 not a prime. Then, we have:

$$A^{2m} = \omega^s . k_2 . a$$

$$(1.149) \quad B^n C^l = \omega^s . k_2$$

** J-4-1-1-2-2-2-1- We suppose that ω, a are coprime, then $\omega \nmid a$. As $A^{2m} = \omega^s . k_2 . a \implies \omega \mid A \implies A = \omega^i . A_1$ with $i \geq 1$ and $\omega \nmid A_1$, then $s = 2i . m$. From (1.149), we have $\omega \mid (B^n C^l) \implies \omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-1-2-2-2-1-1- We suppose that $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j . B_1$ with $j \geq 1$ and $\omega \nmid B_1$. then :

$$B_1^n C^l = \omega^{2im - jn} k_2$$

- If $2im - jn > 0$, $\omega \mid C^l \implies \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (3.1) is verified.

- If $2im - jn = 0 \implies B_1^n C^l = k_2$, as $\omega \nmid k_2 \implies \omega \nmid C^l$, then the contradiction with $\omega \mid (C^l = A^m + B^n)$.

- If $2im - jn < 0 \implies \omega^{jn - 2im} B_1^n C^l = k_2 \implies \omega \mid k_2$, then the contradiction with $\omega \nmid k_2$.

** J-4-1-1-2-2-2-1-2- We suppose that $\omega \mid C^l$. Using the same method used above, we obtain identical results.

** J-4-1-1-2-2-2-2- We suppose that a, ω are not coprime, then $\omega \mid a \implies a = \omega^t \cdot a_1$ and $\omega \nmid a_1$. So we have :

$$(1.150) \quad A^{2m} = \omega^{s+t} \cdot k_2 \cdot a_1$$

$$(1.151) \quad B^n C^l = \omega^s \cdot k_2$$

As $A^{2m} = \omega^{s+t} \cdot k_2 \cdot a_1 \implies \omega \mid A \implies A = \omega^i A_1$ with $i \geq 1$ and $\omega \nmid A_1$, then $s + t = 2im$. From (1.151), we have $\omega \mid (B^n C^l) \implies \omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-1-2-2-2-2-1- We suppose that $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$ with $j \geq 1$ and $\omega \nmid B_1$. then:

$$B_1^n C^l = \omega^{2im-t-jn} k_2$$

- If $2im - t - jn > 0$, $\omega \mid C^l \implies \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (3.1) is verified.

- If $2im - t - jn = 0 \implies B_1^n C^l = k_2$, As $\omega \nmid k_2 \implies \omega \nmid C^l$, then the contradiction with $\omega \mid (C^l = A^m + B^n)$.

- If $2im - t - jn < 0 \implies \omega^{jn+t-2im} B_1^n C^l = k_2 \implies \omega \mid k_2$, then the contradiction with $\omega \nmid k_2$.

** J-4-1-1-2-2-2-2-2- We suppose that $\omega \mid C^l$. Using the same method used above, we obtain identical results.

** J-4-1-2- $3b - 4a \neq 4$ and $4 \mid (3b - 4a) \implies 3b - 4a = 4^s \Omega$ with $s \geq 1$ and $4 \nmid \Omega$. We obtain:

$$(1.152) \quad A^{2m} = k'_1 a$$

$$(1.153) \quad B^n C^l = 4^{s-1} k'_1 \Omega$$

** J-4-1-2-1- We suppose that $k'_1 = 2$. From (1.152), we deduce that $2 \mid a$. As $4 \mid (3b - 4a) \implies 2 \mid b$, then the contradiction with a, b coprime and this case is impossible.

** J-4-1-2-2- We suppose that $k'_1 = 3$. From (1.152) we deduce that $3^3 \mid A^{2m}$. From (1.153), it follows that $3^3 \mid B^n$ or $3^3 \mid C^l$. In the last two cases, we obtain $3^3 \mid p$. But $4p = 3k'_1 b = 9b \implies 3 \mid b$, then the contradiction with a, b coprime. Then this case is impossible.

** J-4-1-2-3- We suppose that k'_1 is prime ≥ 5 :

** J-4-1-2-3-1- Suppose that k'_1 and a are coprime. The equation (1.152) gives $(A^m)^2 = k'_1 \cdot a$, that is impossible with $k'_1 \nmid a$. Then this case is impossible.

** J-4-1-2-3-2- Suppose that k'_1 and a are not coprime. Let $k'_1 \mid a \implies a = k'_1{}^\alpha a_1$ with $\alpha \geq 1$ and $k'_1 \nmid a_1$. The equation (1.152) is written as :

$$A^{2m} = k'_1 a = k'_1{}^{\alpha+1} a_1$$

The last equation gives $k'_1 \mid A^{2m} \implies k'_1 \mid A \implies A = k'_1{}^i \cdot A_1$, with $k'_1 \nmid A_1$. If $2i \cdot m \neq (\alpha + 1)$, it is impossible. We suppose that $2i \cdot m = \alpha + 1$, then $k'_1 \mid A^m$. We return to the equation (1.153). If k'_1 and Ω are coprime, it is impossible. We suppose that k'_1 and Ω are not coprime, then $k'_1 \mid \Omega$ and the exponent of k'_1 in Ω is so the equation (1.153) is satisfying. We deduce easily that $k'_1 \mid B^n$. Then $k'_1{}^2 \mid (p = A^{2m} + B^{2n} + A^m B^n)$, but $4p = 3k'_1 b \implies k'_1 \mid b$, then the contradiction with a, b coprime.

** J-4-1-2-4- We suppose that $k'_1 \geq 4$ is not a prime.

** J-4-1-2-4-1- We suppose that $k'_1 = 4$, we obtain then $A^{2m} = 4a$ and $B^n C^l = 3b - 4a = 3p' - 4a$. This case was studied in the paragraph 1.6.8, case ** I-2-.

** J-4-1-2-4-2- We suppose that $k'_1 > 4$ is not a prime.

** J-4-1-2-4-2-1- We suppose that a, k'_1 are coprime. From the expression $A^{2m} = k'_1 \cdot a$, we deduce that $a = a_1^2$ and $k'_1 = k''_1{}^2$. It gives :

$$\begin{aligned} A^m &= a_1 \cdot k''_1 \\ B^n C^l &= 4^{s-1} k''_1{}^2 \cdot \Omega \end{aligned}$$

Let ω be a prime so that $\omega \mid k''_1$ and $k''_1 = \omega^t \cdot k''_2$ with $\omega \nmid k''_2$. The last two equations become :

$$(1.154) \quad A^m = a_1 \cdot \omega^t \cdot k''_2$$

$$(1.155) \quad B^n C^l = 4^{s-1} \omega^{2t} \cdot k''_2{}^2 \cdot \Omega$$

From (1.154), $\omega \mid A^m \implies \omega \mid A \implies A = \omega^i \cdot A_1$ with $\omega \nmid A_1$ and $im = t$. From (1.155), we obtain $\omega \mid B^n C^l \implies \omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-2-4-2-1-1- If $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j . B_1$ with $\omega \nmid B_1$. From (1.154), we have $B_1^n C^l = \omega^{2t-j.n} 4^{s-1} . k''_2 . \Omega$.

** J-4-1-2-4-2-1-1-1- If $\omega = 2$ and $2 \nmid \Omega$, we have $B_1^n C^l = 2^{2t+2s-j.n-2} k''_2 . \Omega$:
 - If $2t + 2s - jn - 2 \leq 0$ then $2 \nmid C^l$, then the contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$.
 - If $2t + 2s - jn - 2 \geq 1 \implies 2 \mid C^l \implies 2 \mid C$ and the conjecture (3.1) is verified.

** J-4-1-2-4-2-1-1-2- If $\omega = 2$ and if $2 \mid \Omega \implies \Omega = 2 . \Omega_1$ because $4 \nmid \Omega$, we have $B_1^n C^l = 2^{2t+2s+1-j.n-2} k''_2 . \Omega_1$:
 - If $2t + 2s - jn - 3 \leq 0$ then $2 \nmid C^l$, then the contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$.
 - If $2t + 2s - jn - 3 \geq 1 \implies 2 \mid C^l \implies 2 \mid C$ and the conjecture (3.1) is verified.

** J-4-1-2-4-2-1-1-3- If $\omega \neq 2$, we have $B_1^n C^l = \omega^{2t-j.n} 4^{s-1} . k''_2 . \Omega$:
 - If $2t - jn \leq 0 \implies \omega \nmid C^l$ it is in contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$.
 - If $2t - jn \geq 1 \implies \omega \mid C^l \implies \omega \mid C$ and the conjecture (3.1) is verified.

** J-4-1-2-4-2-1-2- If $\omega \mid C^l \implies \omega \mid C \implies C = \omega^h . C_1$, with $\omega \nmid C_1$. Using the same method as in the case J-4-1-2-4-2-1-1 above, we obtain identical results.

** J-4-1-2-4-2-2- We suppose that a, k'_1 are not coprime. Let ω be a prime so that $\omega \mid a$ and $\omega \mid k'_1$. We write:

$$\begin{aligned} a &= \omega^\alpha . a_1 \\ k'_1 &= \omega^\mu . k''_1 \end{aligned}$$

with a_1, k''_1 coprime. The expression of A^{2m} becomes $A^{2m} = \omega^{\alpha+\mu} . a_1 . k''_1$. The term $B^n C^l$ becomes:

$$(1.156) \quad B^n C^l = 4^{s-1} . \omega^\mu . k''_1 . \Omega$$

** J-4-1-2-4-2-2-1- If $\omega = 2 \implies 2 \mid a$, but $2 \nmid b$, then the contradiction with a, b coprime, this case is impossible.

** J-4-1-2-4-2-2-2- If $\omega \geq 3$, we have $\omega \mid a$. If $\omega \mid b$ then the contradiction with a, b coprime. We suppose that $\omega \nmid b$. From the expression of A^{2m} , we obtain $\omega \mid A^{2m} \implies \omega \mid A \implies A = \omega^i . A_1$ with $\omega \nmid A_1$, $i \geq 1$ and $2i.m = \alpha + \mu$.

From (1.156), we deduce that $\omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-2-4-2-2-2-1- We suppose that $\omega \mid B^n \implies \omega \mid B \implies B = \omega^j B_1$ with $\omega \nmid B_1$ and $j \geq 1$. Then, $B_1^n C^l = 4^{s-1} \omega^{\mu-jn} .k''_1 .\Omega$:

* $\omega \nmid \Omega$:

- If $\mu - jn \geq 1$, we have $\omega \mid C^l \implies \omega \mid C$, there is no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (3.1) is verified.

- If $\mu - jn \leq 0$, then $\omega \nmid C^l$ and it is a contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$. Then this case is impossible.

* $\omega \mid \Omega$: we write $\Omega = \omega^\beta .\Omega_1$ with $\beta \geq 1$ and $\omega \nmid \Omega_1$. As $3b - 4a = 4^s .\Omega = 4^s .\omega^\beta .\Omega_1 \implies 3b = 4a + 4^s .\omega^\beta .\Omega_1 = 4\omega^\alpha .a_1 + 4^s .\omega^\beta .\Omega_1 \implies 3b = 4\omega(\omega^{\alpha-1} .a_1 + 4^{s-1} .\omega^{\beta-1} .\Omega_1)$. If $\omega = 3$ and $\beta = 1$, we obtain $b = 4(3^{\alpha-1} a_1 + 4^{s-1} \Omega_1)$ and $B_1^n C^l = 4^{s-1} 3^{\mu+1-jn} .k''_1 \Omega_1$.

- If $\mu - jn + 1 \geq 1$, then $3 \mid C^l$ and the conjecture (3.1) is verified.

- If $\mu - jn + 1 \leq 0$, then $3 \nmid C^l$ and it is the contradiction with $C^l = 3^{im} A_1^m + 3^{jn} B_1^n$.

Now, if $\beta \geq 2$ and $\alpha = im \geq 3$, we obtain $3b = 4\omega^2(\omega^{\alpha-2} a_1 + 4^{s-1} \omega^{\beta-2} \Omega_1)$. If $\omega = 3$ or not, then $\omega \mid b$, but $\omega \nmid a$, then the contradiction with a, b coprime.

** J-4-1-2-4-2-2-2-2- We suppose that $\omega \mid C^l \implies \omega \mid C \implies C = \omega^h C_1$ with $\omega \nmid C_1$ and $h \geq 1$. Then, $B^n C_1^l = 4^{s-1} \omega^{\mu-hl} .k''_1 .\Omega$. Using the same method as above, we obtain identical results.

** J-4-2- We suppose that $4 \mid k'_1$.

** J-4-2-1- $k'_1 = 4 \implies 4p = 3k'_1 b = 12b \implies p = 3b = 3p'$, this case has been studied (see case I-2- paragraph 1.6.8).

** J-4-2-2- $k'_1 > 4$ with $4 \mid k'_1 \implies k'_1 = 4^s k''_1$ and $s \geq 1$, $4 \nmid k''_1$. Then, we obtain:

$$\begin{aligned} A^{2m} &= 4^s k''_1 a = 2^{2s} k''_1 a \\ B^n C^l &= 4^{s-1} k''_1 (3b - 4a) = 2^{2s-2} k''_1 (3b - 4a) \end{aligned}$$

** J-4-2-2-1- We suppose that $s = 1$ and $k'_1 = 4k''_1$ with $k''_1 > 1$, so $p = 3p'$ and $p' = k''_1 b$, this is the case 1.6.3 already studied.

** J-4-2-2-2- We suppose that $s > 1$, then $k'_1 = 4^s k''_1 \implies 4p = 3 \times 4^s k''_1 b$ and we obtain:

$$(1.157) \quad A^{2m} = 4^s k''_1 a$$

$$(1.158) \quad B^n C^l = 4^{s-1} k''_1 (3b - 4a)$$

** J-4-2-2-2-1- We suppose that $2 \nmid (k''_1 a) \implies 2 \nmid k''_1$ and $2 \nmid a$. As $(A^m)^2 = (2^s)^2 \cdot (k''_1 a)$, we call $d^2 = k''_1 a$, then $A^m = 2^s d \implies 2 \mid A^m \implies 2 \mid A \implies A = 2^i A_1$ with $2 \nmid A_1$ and $i \geq 1$, then: $2^{im} A_1^m = 2^s d \implies s = im$. From the equation (1.158), we have $2 \mid (B^n C^l) \implies 2 \mid B^n$ or $2 \mid C^l$.

** J-4-2-2-2-1-1- We suppose that $2 \mid B^n \implies 2 \mid B \implies B = 2^j B_1$, with $j \geq 1$ and $2 \nmid B_1$. The equation (1.158) becomes:

$$B_1^n C^l = 2^{2s-jn-2} k''_1 (3b - 4a) = 2^{2im-jn-2} k''_1 (3b - 4a)$$

* We suppose that $2 \nmid (3b - 4a)$:

- If $2im - jn - 2 \geq 1$, then $2 \mid C^l$, there is no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1) is verified.

- If $2im - jn - 2 \leq 0$, then $2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

* We suppose that $2^\mu \mid (3b - 4a)$, $\mu \geq 1$:

- If $2im + \mu - jn - 2 \geq 1$, then $2 \mid C^l$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1) is verified.

- If $2im + \mu - jn - 2 \leq 0$, then $2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

** J-4-2-2-2-1-2- We suppose that $2 \mid C^l \implies 2 \mid C \implies C = 2^h C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-2-2-2-2- We suppose that $2 \mid (k''_1 a)$:

** J-4-2-2-2-2-1- We suppose that k''_1 and a are coprime:

** J-4-2-2-2-2-1-1- We suppose that $2 \nmid a$ and $2 \mid k''_1 \implies k''_1 = 2^{2\mu}.k''_2$ and $a = a_1^2$, then the equations (1.157-1.158) become:

$$(1.159) \quad A^{2m} = 4^s.2^{2\mu}k''_2a_1^2 \implies A^m = 2^{s+\mu}.k''_2.a_1$$

$$(1.160) \quad B^n C^l = 4^{s-1}2^{2\mu}k''_2(3b-4a) = 2^{2s+2\mu-2}k''_2(3b-4a)$$

The equation (1.159) gives $2 \mid A^m \implies 2 \mid A \implies A = 2^i.A_1$ with $2 \nmid A_1$, $i \geq 1$ and $im = s + \mu$. From the equation (1.160), we have $2 \mid (B^n C^l) \implies 2 \mid B^n$ or $2 \mid C^l$.

** J-4-2-2-2-2-1-1-1- We suppose that $2 \mid B^n \implies 2 \mid B \implies B = 2^j.B_1$, $2 \nmid B_1$ and $j \geq 1$, then $B_1^n C^l = 2^{2s+2\mu-jn-2}k''_2(3b-4a)$:

* We suppose that $2 \nmid (3b-4a)$:

- If $2im + 2\mu - jn - 2 \geq 1 \implies 2 \mid C^l$, then there is no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1) is verified.

- If $2im + 2\mu - jn - 2 \leq 0 \implies 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

* We suppose that $2^\alpha \mid (3b-4a)$, $\alpha \geq 1$ so that a, b remain coprime:

- If $2im + 2\mu + \alpha - jn - 2 \geq 1 \implies 2 \mid C^l$, then no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1) is verified.

- If $2im + 2\mu + \alpha - jn - 2 \leq 0 \implies 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** J-4-2-2-2-2-1-1-2- We suppose that $2 \mid C^l \implies 2 \mid C \implies C = 2^h.C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-2-2-2-2-1-2- We suppose that $2 \nmid k''_1$ and $2 \mid a \implies a = 2^{2\mu}.a_1^2$ and $k''_1 = k''_2$, then the equations (1.157-1.158) become:

$$(1.161) \quad A^{2m} = 4^s.2^{2\mu}a_1^2k''_2 \implies A^m = 2^{s+\mu}.a_1.k''_2.$$

$$(1.162) \quad B^n C^l = 4^{s-1}k''_2(3b-4a) = 2^{2s-2}k''_2(3b-4a)$$

The equation (1.161) gives $2 \mid A^m \implies 2 \mid A \implies A = 2^i.A_1$ with $2 \nmid A_1$, $i \geq 1$ and $im = s + \mu$. From the equation (1.162), we have $2 \mid (B^n C^l) \implies 2 \mid B^n$ or $2 \mid C^l$.

** J-4-2-2-2-2-1-2-1- We suppose that $2 \mid B^n \implies 2 \mid B \implies B = 2^j \cdot B_1$, $2 \nmid B_1$ and $j \geq 1$. Then we obtain $B_1^n C^l = 2^{2s-jn-2} k''_2 (3b-4a)$:

* We suppose that $2 \nmid (3b-4a) \implies 2 \nmid b$:

- If $2im - jn - 2 \geq 1 \implies 2 \mid C^l$, then no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1) is verified.

- If $2im - jn - 2 \leq 0 \implies 2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

* We suppose that $2^\alpha \mid (3b-4a)$, $\alpha \geq 1$, in this case a, b are not coprime, then the contradiction.

** J-4-2-2-2-2-1-2-2- We suppose that $2 \mid C^l \implies 2 \mid C \implies C = 2^h \cdot C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-2-2-2-2-2- We suppose that k''_1 and a are not coprime $2 \mid a$ and $2 \mid k''_1$. Let $a = 2^t \cdot a_1$ and $k''_1 = 2^\mu k''_2$ and $2 \nmid a_1$ and $2 \nmid k''_2$. From (1.157), we have $\mu + t = 2\lambda$ and $a_1 \cdot k''_2 = \omega^2$. The equations (1.157-1.158) become:

$$(1.163) \quad 2^m = 4^s k''_1 a = 2^{2s} \cdot 2^\mu k''_2 \cdot 2^t \cdot a_1 = 2^{2s+2\lambda} \cdot \omega^2 \implies A^m = 2^{s+\lambda} \cdot \omega$$

$$(1.164) \quad B^n C^l = 4^{s-1} 2^\mu k''_2 (3b-4a) = 2^{2s+\mu-2} k''_2 (3b-4a)$$

From (1.163) we have $2 \mid A^m \implies 2 \mid A \implies A = 2^i A_1, i \geq 1$ and $2 \nmid A_1$. From (1.164), $2s + \mu - 2 \geq 1$, we deduce that $2 \mid (B^n C^l) \implies 2 \mid B^n$ or $2 \mid C^l$.

** J-4-2-2-2-2-2-1- We suppose that $2 \mid B^n \implies 2 \mid B \implies B = 2^j \cdot B_1$, $2 \nmid B_1$ and $j \geq 1$. Then we obtain $B_1^n C^l = 2^{2s+\mu-jn-2} k''_2 (3b-4a)$:

* We suppose that $2 \nmid (3b-4a)$:

- If $2s + \mu - jn - 2 \geq 1 \implies 2 \mid C^l$, then no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1) is verified.

- If $2s + \mu - jn - 2 \leq 0 \implies 2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

* We suppose that $2^\alpha \mid (3b-4a)$, for one value $\alpha \geq 1$. As $2 \mid a$, then $2^\alpha \mid (3b-4a) \implies 2 \mid (3b-4a) \implies 2 \mid (3b) \implies 2 \mid b$, then the contradiction

with a, b coprime.

** J-4-2-2-2-2-2-2- We suppose that $2 \mid C^l \implies 2 \mid C \implies C = 2^h.C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-3- $2 \mid k'_1$ and $2 \mid (3b - 4a)$: then we obtain $2 \mid k'_1 \implies k'_1 = 2^t.k''_1$ with $t \geq 1$ and $2 \nmid k''_1$, $2 \mid (3b - 4a) \implies 3b - 4a = 2^\mu.d$ with $\mu \geq 1$ and $2 \nmid d$. We have also $2 \mid b$. If $2 \mid a$, it is a contradiction with a, b coprime.

We suppose, in the following, that $2 \nmid a$. The equations (1.157-1.158) become:

$$(1.165) \quad A^{2m} = 2^t.k''_1.a = (A^m)^2$$

$$(1.166) \quad B^n C^l = 2^{t-1}k''_1.2^{\mu-1}d = 2^{t+\mu-2}k''_1.d$$

From (1.165), we deduce that the exponent t is even, let $t = 2\lambda$. Then we call $\omega^2 = k''_1.a$, it gives $A^m = 2^\lambda.\omega \implies 2 \mid A^m \implies 2 \mid A \implies A = 2^i.A_1$ with $i \geq 1$ and $2 \nmid A_1$. From (1.166), we have $2\lambda + \mu - 2 \geq 1$, then $2 \mid (B^n C^l) \implies 2 \mid B^n$ or $2 \mid C^l$:

** J-4-3-1- We suppose that $2 \mid B^n \implies 2 \mid B \implies B = 2^j B_1$, with $j \geq 1$ and $2 \nmid B_1$. Then we obtain $B_1^n C^l = 2^{2\lambda+\mu-jn-2}.k''_1.d$.

- If $2\lambda + \mu - jn - 2 \geq 1 \implies 2 \mid C^l \implies 2 \mid C$, there is no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1) is verified.

- If $2s + t + \mu - jn - 2 \leq 0 \implies 2 \nmid C$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** J-4-3-2- We suppose that $2 \mid C^l \implies 2 \mid C$. With the same method used above, we obtain identical results. □

The Main Theorem is proved.

1.7. Examples and Conclusion

1.7.1. Numerical Examples. —

1.7.1.1. Example 1: — We consider the example : $6^3 + 3^3 = 3^5$ with $A^m = 6^3$, $B^n = 3^3$ and $C^l = 3^5$. With the notations used in the paper, we obtain:

$$(1.167) \quad \begin{aligned} p &= 3^6 \times 73, & q &= 8 \times 3^{11}, & \bar{\Delta} &= 4 \times 3^{18}(3^7 \times 4^2 - 73^3) < 0 \\ \rho &= \frac{3^8 \times 73\sqrt{73}}{\sqrt{3}}, & \cos\theta &= -\frac{4 \times 3^3 \times \sqrt{3}}{73\sqrt{73}} \end{aligned}$$

As $A^{2m} = \frac{4p}{3} \cdot \cos^2 \frac{\theta}{3} \implies \cos^2 \frac{\theta}{3} = \frac{3A^{2m}}{4p} = \frac{3 \times 2^4}{73} = \frac{a}{b} \implies a = 3 \times 2^4$, $b = 73$; then we obtain:

$$(1.168) \quad \cos \frac{\theta}{3} = \frac{4\sqrt{3}}{\sqrt{73}}, \quad p = 3^6 \cdot b$$

We verify easily the equation (1.167) to calculate $\cos\theta$ using (1.168). For this example, we can use the two conditions from (1.47) as $3 \mid a, b \mid 4p$ and $3 \mid p$. The cases 1.5.4 and 1.6.3 are respectively used. For the case 1.5.4, it is the case B-2-2-1- that was used and the conjecture (3.1) is verified. Concerning the case 1.6.3, it is the case G-2-2-1- that was used and the conjecture (3.1) is verified.

1.7.1.2. Example 2: — The second example is: $7^4 + 7^3 = 14^3$. We take $A^m = 7^4$, $B^n = 7^3$ and $C^l = 14^3$. We obtain $p = 57 \times 7^6 = 3 \times 19 \times 7^6$, $q = 8 \times 7^{10}$, $\bar{\Delta} = 27q^2 - 4p^3 = 27 \times 4 \times 7^{18}(16 \times 49 - 19^3) = -27 \times 4 \times 7^{18} \times 6075 < 0$, $\rho = 19 \times 7^9 \times \sqrt{19}$, $\cos\theta = -\frac{4 \times 7}{19\sqrt{19}}$. As $A^{2m} = \frac{4p}{3} \cdot \cos^2 \frac{\theta}{3} \implies \cos^2 \frac{\theta}{3} = \frac{3A^{2m}}{4p} = \frac{7^2}{4 \times 19} = \frac{a}{b} \implies a = 7^2$, $b = 4 \times 19$, then $\cos \frac{\theta}{3} = \frac{7}{2\sqrt{19}}$ and we have the two principal conditions $3 \mid p$ and $b \mid (4p)$. The calculation of $\cos\theta$ from the expression of $\cos \frac{\theta}{3}$ is confirmed by the value below:

$$\cos\theta = \cos 3(\theta/3) = 4\cos^3 \frac{\theta}{3} - 3\cos \frac{\theta}{3} = 4 \left(\frac{7}{2\sqrt{19}} \right)^3 - 3 \frac{7}{2\sqrt{19}} = -\frac{4 \times 7}{19\sqrt{19}}$$

Then, we obtain $3 \mid p \implies p = 3p'$, $b \mid (4p)$ with $b \neq 2, 4$ then $12p' = k_1 b = 3 \times 7^6 b$. It concerns the paragraph 1.6.9 of the second hypothesis. As $k_1 = 3 \times 7^6 = 3k'_1$ with $k'_1 = 7^6 \neq 1$. It is the case J-4-1-2-4-2-2- with the condition $4 \mid (3b - 4a)$. So we verify :

$$3b - 4a = 3 \times 4 \times 19 - 4 \times 7^2 = 32 \implies 4 \mid (3b - 4a)$$

with $A^{2m} = 7^8 = 7^6 \times 7^2 = k'_1 \cdot a$ and k'_1 not a prime, with a and k'_1 not coprime with $\omega = 7 \nmid \Omega(= 2)$. We find that the conjecture (3.1) is verified with a common factor equal to 7 (prime and divisor of $k'_1 = 7^6$).

1.7.1.3. Example 3: — The third example is: $19^4 + 38^3 = 57^3$ with $A^m = 19^4$, $B^n = 38^3$ and $C^l = 57^3$. We obtain $p = 19^6 \times 577$, $q = 8 \times 27 \times 19^{10}$, $\bar{\Delta} = 27q^2 - 4p^3 = 4 \times 19^{18}(27^3 \times 16 \times 19^2 - 577^3) < 0$, $\rho = \frac{19^9 \times 577\sqrt{577}}{3\sqrt{3}}$, $\cos\theta = -\frac{4 \times 3^4 \times 19\sqrt{3}}{577\sqrt{577}}$. As $A^{2m} = \frac{4p}{3} \cdot \cos^2\frac{\theta}{3} \implies \cos^2\frac{\theta}{3} = \frac{3A^{2m}}{4p} = \frac{3 \times 19^2}{4 \times 577} = \frac{a}{b} \implies a = 3 \times 19^2$, $b = 4 \times 577$, then $\cos\frac{\theta}{3} = \frac{19\sqrt{3}}{2\sqrt{577}}$ and we have the first hypothesis $3 \mid a$ and $b \mid (4p)$. Here again, the calculation of $\cos\theta$ from the expression of $\cos\frac{\theta}{3}$ is confirmed by the value below:

$$\cos\theta = \cos 3(\theta/3) = 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3} = 4 \left(\frac{19\sqrt{3}}{2\sqrt{577}} \right)^3 - 3 \frac{19\sqrt{3}}{2\sqrt{577}} = -\frac{4 \times 3^4 \times 19\sqrt{3}}{577\sqrt{577}}$$

Then, we obtain $3 \mid a \implies a = 3a' = 3 \times 19^2$, $b \mid (4p)$ with $b \neq 2, 4$ and $b = 4p'$ with $p = kp'$ so it $p' = 577$ and $k = 19^6$. This concerns the paragraph 1.5.8 of the first hypothesis. It is the case E-2-2-2-2-1- with $\omega = 19$, a', ω not coprime and $\omega = 19 \nmid (p' - a') = (577 - 19^2)$ with $s - jn = 6 - 1 \times 3 = 3 \geq 1$, and the conjecture (3.1) is verified.

1.7.2. Conclusion. — The method used to give the proof of the conjecture of Beal has discussed many possibles cases, using elementary number theory and the results of some theorems about Diophantine equations. We have confirmed the method by three numerical examples. In conclusion, we can announce the theorem:

Theorem 1.4. — *Let A, B, C, m, n , and l be positive natural numbers with $m, n, l > 2$. If :*

$$(1.169) \quad A^m + B^n = C^l$$

then A, B , and C have a common factor.

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CHAPTER 2

IS THE RIEMANN HYPOTHESIS TRUE? YES IT IS

Abstract. — In 1859, Georg Friedrich Bernhard Riemann had announced the following conjecture, called Riemann Hypothesis : *The nontrivial roots (zeros) $s = \sigma + it$ of the zeta function, defined by:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) > 1$$

have real part $\sigma = \frac{1}{2}$.

We give a proof that $\sigma = \frac{1}{2}$ using an equivalent statement of the Riemann Hypothesis concerning the Dirichlet η function.

Résumé. — En 1859, Georg Friedrich Bernhard Riemann avait annoncé la conjecture suivante, dite Hypothèse de Riemann: *Les zéros non triviaux $s = \sigma + it$ de la fonction zeta définie par:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ pour } \Re(s) > 1$$

ont comme parties réelles $\sigma = \frac{1}{2}$.

On donne une démonstration que $\sigma = \frac{1}{2}$ en utilisant une proposition équivalente de l'Hypothèse de Riemann.

2.1. Introduction.

In 1859, G.F.B. Riemann had announced the following conjecture [1]:

Conjecture 2.1. — Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give the proof that $\sigma = \frac{1}{2}$.

2.1.1. The function ζ . — We denote $s = \sigma + it$ the complex variable of \mathbb{C} . For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s . Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [2]:

Theorem 2.2. — The function $\zeta(s)$ satisfies the following :

1. $\zeta(s)$ has no zero for $\Re(s) > 1$;
2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;
3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \dots$;
4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \frac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \frac{1}{2}$ is called the critical line.

The Riemann Hypothesis is formulated as:

Conjecture 2.3. — (The Riemann Hypothesis, [2]) All nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$.

In addition to the properties cited by the theorem 2.2 above, the function $\zeta(s)$ satisfies the functional relation [2] called also the reflection functional equation for $s \in \mathbb{C} \setminus \{0, 1\}$:

$$(2.1) \quad \zeta(1-s) = 2^{1-s} \pi^{-s} \cos \frac{s\pi}{2} \Gamma(s) \zeta(s)$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s) > 0$, given by the formula :

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt, \quad \Re(s) > 0$$

So, instead of using the functional given by (2.1), we will use the one presented by G.H. Hardy [3] namely Dirichlet's eta function [2]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s}) \zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [2].

We have also the theorem (see page 16, [3]):

Theorem 2.4. — For all $t \in \mathbb{R}$, $\zeta(1+it) \neq 0$.

So, we take the critical strip as the region defined as $0 < \Re(s) < 1$.

2.1.2. A Equivalent statement to the Riemann Hypothesis. — Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet function eta which is stated as follows [2]:

Equivalence 2.5. — The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :

$$(2.2) \quad \eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s}) \zeta(s), \quad \sigma > 1$$

that fall in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \frac{1}{2}$.

The series (2.2) is convergent, and represents $(1-2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([3], pages 20-21). We can rewrite:

$$(2.3) \quad \eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0$$

$\eta(s)$ is a complex number, it can be written as :

$$(2.4) \quad \eta(s) = \rho \cdot e^{i\alpha} \implies \rho^2 = \eta(s) \cdot \overline{\eta(s)}$$

and $\eta(s) = 0 \iff \rho = 0$.

2.2. Preliminaries of the proof

Proof. — . We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \iff (1-2^{1-s})\zeta(s) = 0$. We verifies easily the two propositions:

(2.5)

s, is one zero of $\eta(s)$ that falls in the critical strip, is also one zero of $\zeta(s)$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = 0 \implies \eta(s) = (1-2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:

(2.6)

s, is one zero of $\zeta(s)$ that falls in the critical strip, is also one zero of $\eta(s)$

Let us write the function η :

$$\begin{aligned} \eta(s) &= \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-s \text{Log} n} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-(\sigma+it) \text{Log} n} = \\ &= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma \text{Log} n} \cdot e^{-it \text{Log} n} \\ &= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma \text{Log} n} (\cos(t \text{Log} n) - i \sin(t \text{Log} n)) \end{aligned}$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. Let s be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall N > n_0, \left| \sum_{n=1}^N \frac{(-1)^{n-1}}{n^s} \right| < \epsilon'$$

We define the sequence of functions $((\eta_n)_{n \in \mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^n \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^n (-1)^{k-1} \frac{\cos(t \operatorname{Log} k)}{k^\sigma} - i \sum_{k=1}^n (-1)^{k-1} \frac{\sin(t \operatorname{Log} k)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let s be one zero of η that lies in the critical strip, then $\eta(s) = 0$, with $0 < \sigma < 1$. It follows that we can write $\lim_{n \rightarrow +\infty} \eta_n(s) = 0 = \eta(s)$. We obtain:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \sum_{k=1}^n (-1)^{k-1} \frac{\cos(t \operatorname{Log} k)}{k^\sigma} &= 0 \\ \lim_{n \rightarrow +\infty} \sum_{k=1}^n (-1)^{k-1} \frac{\sin(t \operatorname{Log} k)}{k^\sigma} &= 0 \end{aligned}$$

Using the definition of the limit of a sequence, we can write:

$$(2.7) \quad \forall \epsilon_1 > 0 \exists n_r, \forall N > n_r, |\Re(\eta(s)_N)| < \epsilon_1 \implies \Re(\eta(s)_N)^2 < \epsilon_1^2$$

$$(2.8) \quad \forall \epsilon_2 > 0 \exists n_i, \forall N > n_i, |\Im(\eta(s)_N)| < \epsilon_2 \implies \Im(\eta(s)_N)^2 < \epsilon_2^2$$

Then:

$$\begin{aligned} 0 &< \sum_{k=1}^N \frac{\cos^2(t \operatorname{Log} k)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k < k'}^N \frac{(-1)^{k+k'} \cos(t \operatorname{Log} k) \cdot \cos(t \operatorname{Log} k')}{k^\sigma k'^\sigma} < \epsilon_1^2 \\ 0 &< \sum_{k=1}^N \frac{\sin^2(t \operatorname{Log} k)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k < k'}^N \frac{(-1)^{k+k'} \sin(t \operatorname{Log} k) \cdot \sin(t \operatorname{Log} k')}{k^\sigma k'^\sigma} < \epsilon_2^2 \end{aligned}$$

Taking $\epsilon = \epsilon_1 = \epsilon_2$ and $N > \max(n_r, n_i)$, we get by making the sum member to member of the last two inequalities:

$$(2.9) \quad 0 < \sum_{k=1}^N \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k < k'}^N (-1)^{k+k'} \frac{\cos(t \operatorname{Log}(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2$$

We can write the above equation as :

$$(2.10) \quad 0 < \rho_N^2 < 2\epsilon^2$$

or $\rho(s) = 0$.

2.3. Case $\sigma = \frac{1}{2}$.

We suppose that $\sigma = \frac{1}{2}$. Let's start by recalling Hardy's theorem (1914) ([2], page 24):

Theorem 2.6. — *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (2.5-2.6), it follows the proposition :

Proposition 2.7. — *There are infinitely many zeros of $\eta(s)$ on the critical line.*

Let $s_j = \frac{1}{2} + it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (2.9) is written for s_j :

$$0 < \sum_{k=1}^N \frac{1}{k} + 2 \sum_{k,k'=1;k < k'}^N (-1)^{k+k'} \frac{\cos(t_j \text{Log}(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^N \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k < k'}^N (-1)^{k+k'} \frac{\cos(t_j \text{Log}(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \rightarrow +\infty$, the series $\sum_{k=1}^N \frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k < k'}^{+\infty} (-1)^{k+k'} \frac{\cos(t_j \text{Log}(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$(2.11) \quad \lim_{N \rightarrow +\infty} \sum_{k,k'=1;k < k'}^N (-1)^{k+k'} \frac{\cos(t_j \text{Log}(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty$$

if not, we will have a contradiction with the fact that :

$$\lim_{N \rightarrow +\infty} \sum_{k=1}^N (-1)^{k-1} \frac{1}{k^{s_j}} = 0 \iff \eta(s) \text{ is convergent for } s_j = \frac{1}{2} + it_j$$

2.4. Case $0 < \Re(s) < \frac{1}{2}$.

2.4.1. Case where there are zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \frac{1}{2}$. — Suppose that there exists $s = \sigma + it$ one zero of $\eta(s)$ or $\eta(s) = 0 \implies \rho^2(s) = 0$ with $0 < \sigma < \frac{1}{2} \implies s$ lies inside the critical band. We write the equation (2.9):

$$0 < \sum_{k=1}^N \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k < k'}^N (-1)^{k+k'} \frac{\cos(t \operatorname{Log}(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2$$

or:

$$\sum_{k=1}^N \frac{1}{k^{2\sigma}} < 2\epsilon^2 - 2 \sum_{k,k'=1;k < k'}^N (-1)^{k+k'} \frac{\cos(t \operatorname{Log}(k/k'))}{k^\sigma k'^\sigma}$$

But $2\sigma < 1$, it follows that $\lim_{N \rightarrow +\infty} \sum_{k=1}^N \frac{1}{k^{2\sigma}} \rightarrow +\infty$ and then, we obtain :

$$(2.12) \quad \boxed{\sum_{k,k'=1;k < k'}^{+\infty} (-1)^{k+k'} \frac{\cos(t \operatorname{Log}(k/k'))}{k^\sigma k'^\sigma} = -\infty}$$

2.5. Case $\frac{1}{2} < \Re(s) < 1$.

Let $s = \sigma + it$ be the zero of $\eta(s)$ in $0 < \Re(s) < \frac{1}{2}$, object of the previous paragraph. From the proposition (2.5), $\zeta(s) = 0$. According to point 4 of theorem 2.2, the complex number $s' = 1 - \sigma + it = \sigma' + it'$ with $\sigma' = 1 - \sigma$, $t' = t$ and $\frac{1}{2} < \sigma' < 1$ verifies $\zeta(s') = 0$, so s' is also a zero of the function $\zeta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, it follows from the proposition (2.6) that $\eta(s') = 0 \implies \rho(s') = 0$. By applying (2.9), we get:

$$(2.13) \quad 0 < \sum_{k=1}^N \frac{1}{k^{2\sigma'}} + 2 \sum_{k,k'=1;k < k'}^N (-1)^{k+k'} \frac{\cos(t' \operatorname{Log}(k/k'))}{k^{\sigma'} k'^{\sigma'}} < 2\epsilon^2$$

As $0 < \sigma < \frac{1}{2} \implies 2 > 2\sigma' = 2(1 - \sigma) > 1$, then the series $\sum_{k=1}^N \frac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2 < 1/k^{2\sigma'}$ for all $k > 0$, then :

$$0 < \zeta(2) = \frac{\pi^2}{6} = \sum_{k=1}^{+\infty} \frac{1}{k^2} < \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma'}} = C(\sigma') = \zeta_1(2\sigma') = \zeta(2\sigma')$$

From the equation (2.13), it follows that :

$$(2.14) \quad \sum_{k,k'=1;k < k'}^{+\infty} (-1)^{k+k'} \frac{\cos(t' \text{Log}(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty$$

2.5.0.1. *Case $t = 0$.* — We suppose that $t = 0 \implies t' = 0$. The equation (2.14) becomes:

$$(2.15) \quad \sum_{k,k'=1;k < k'}^{+\infty} (-1)^{k+k'} \frac{1}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty$$

Then $s' = \sigma' > 1/2$ is a zero of $\eta(s)$, we obtain :

$$(2.16) \quad \eta(s') = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^{s'}} = 0$$

Let us define the sequence S_m as:

$$(2.17) \quad S_m(s') = \sum_{n=1}^m \frac{(-1)^{n-1}}{n^{s'}} = \sum_{n=1}^m \frac{(-1)^{n-1}}{n^{\sigma'}} = S_m(\sigma')$$

From the definition of S_m , we obtain :

$$(2.18) \quad \lim_{m \rightarrow +\infty} S_m(s') = \eta(s') = \eta(\sigma')$$

We have also:

$$(2.19) \quad S_1(\sigma') = 1 > 0$$

$$(2.20) \quad S_2(\sigma') = 1 - \frac{1}{2^{\sigma'}} > 0 \quad \text{because } 2^{\sigma'} > 1$$

$$(2.21) \quad S_3(\sigma') = S_2(\sigma') + \frac{1}{3^{\sigma'}} > 0$$

We proceed by recurrence, we suppose that $S_m(\sigma') > 0$.

$$1. \quad m = 2q \implies S_{m+1}(\sigma') = \sum_{n=1}^{m+1} \frac{(-1)^{n-1}}{n^{\sigma'}} = S_m(\sigma') + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'}}, \text{ it gives:}$$

$$S_{m+1}(\sigma') = S_m(\sigma') + \frac{(-1)^{2q}}{(m+1)^{\sigma'}} = S_m(\sigma') + \frac{1}{(m+1)^{\sigma'}} > 0 \implies S_{m+1}(\sigma') > 0$$

2. $m = 2q + 1$, we can write $S_{m+1}(\sigma')$ as:

$$S_{m+1}(\sigma') = S_{m-1}(\sigma') + \frac{(-1)^{m-1}}{m^{\sigma'}} + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'}}$$

We have $S_{m-1}(\sigma') > 0$, let $T = \frac{(-1)^{m-1}}{m^{\sigma'}} + \frac{(-1)^m}{(m+1)^{\sigma'}}$, we obtain:

$$(2.22) \quad T = \frac{(-1)^{2q}}{(2q+1)^{\sigma'}} + \frac{(-1)^{2q+1}}{(2q+2)^{\sigma'}} = \frac{1}{(2q+1)^{\sigma'}} - \frac{1}{(2q+2)^{\sigma'}} > 0$$

and $S_{m+1}(\sigma') > 0$.

Then all the terms $S_m(\sigma')$ of the sequence S_m are great then 0, it follows that $\lim_{m \rightarrow +\infty} S_m(s') = \eta(s') = \eta(\sigma') > 0$ and $\eta(\sigma') < +\infty$ because $\Re(s') = \sigma' > 0$ and $\eta(s')$ is convergent. We deduce the contradiction with the hypothesis s' is a zero of $\eta(s)$ and:

$$(2.23) \quad \boxed{\text{The equation (2.15) is false for the case } t' = t = 0.}$$

2.5.0.2. *Case $t \neq 0$.* — We suppose that $t \neq 0$. For each $s' = \sigma' + it' = 1 - \sigma + it$, we have:

$$(2.24) \quad \sum_{k, k'=1; k < k'}^{+\infty} (-1)^{k+k'} \frac{\cos(t' \text{Log}(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty$$

the left member of the equation (2.24) above is finite and depends of σ' and t' , but the right member is a function only of σ' equal to $-\zeta(2\sigma')/2$. But for all σ'' so that $2\sigma'' > 1$, we have $\zeta(2\sigma'')$:

$$\zeta(2\sigma'') = \zeta_1(2\sigma'') = \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma''}} < +\infty$$

It depends only of σ'' , then in particular for all σ'' with $2 > 2\sigma'' > 1$, $\zeta(2\sigma'')$ depends only of σ'' , then the result giving by the equation (2.24) is false:

$$(2.25) \quad \boxed{\text{It follows that the equation (2.24) is false for the cases } t' \neq 0.}$$

From (2.23-2.25), we conclude that the function $\eta(s)$ has no zeros for all $s' = \sigma' + it'$ with $\sigma' \in]1/2, 1[$, it follows that the case of the paragraph (2.4) above concerning the case $0 < \Re(s) < \frac{1}{2}$ is false. Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \frac{1}{2}$. From the equivalent statement (3.1), it follows that **the Riemann hypothesis is verified.** \square

From the calculations above, we can verify easily the following known proposition:

Proposition 2.8. — For all $s = \sigma$ real with $0 < \sigma < 1$, $\eta(s) > 0$ and $\zeta(s) < 0$.

2.6. Conclusion.

In summary: for our proofs, we made use of Dirichlet's $\eta(s)$ function:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad s = \sigma + it$$

on the critical band $0 < \Re(s) < 1$, in obtaining:

- $\eta(s)$ vanishes for $0 < \sigma = \Re(s) = \frac{1}{2}$;
- $\eta(s)$ does not vanish for $0 < \sigma = \Re(s) < \frac{1}{2}$ and $\frac{1}{2} < \sigma = \Re(s) < 1$.

Consequently, all the zeros of $\eta(s)$ inside the critical band $0 < \Re(s) < 1$ are on the critical line $\Re(s) = \frac{1}{2}$. Applying the equivalent proposition to the Riemann Hypothesis (3.1), we conclude that **the Riemann hypothesis is verified** and all the nontrivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$. The proof of the Riemann Hypothesis is thus completed.

We therefore announce the important theorem as follows:

Theorem 2.9. — *The Riemann Hypothesis is true:
All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical
line $\Re(s) = \frac{1}{2}$.*

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CHAPTER 3

IS THE CONJECTURE $c < rad^{1.63}(abc)$ TRUE?

Abstract. — In this paper, we consider the *abc* conjecture, we will give the proof that the conjecture $c < rad^{1.63}(abc)$ is true. It constitutes the key to resolve the *abc* conjecture.

Résumé. — Dans cet article, nous considérons la conjecture *abc*. Nous donnons la preuve de la conjecture $c < rad^{1.63}(abc)$ qui constitue la clé pour résoudre la conjecture *abc*.

3.1. Introduction and notations

Let a be a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$(3.1) \quad a = \prod_i a_i^{\alpha_i} = rad(a) \cdot \prod_i a_i^{\alpha_i - 1}$$

We denote:

$$(3.2) \quad \mu_a = \prod_i a_i^{\alpha_i - 1} \implies a = \mu_a \cdot rad(a)$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Esterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 3.1. — (*abc Conjecture*): For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :

$$(3.3) \quad c < K(\epsilon) \cdot rad^{1+\epsilon}(abc)$$

where K is a constant depending only of ϵ .

We know that numerically, $\frac{Log c}{Log(rad(abc))} \leq 1.629912$ [2]. It concerned the best example given by E. Reyssat [2]:

$$(3.4) \quad 2 + 3^{10} \cdot 109 = 23^5 \implies c < rad^{1.629912}(abc)$$

A conjecture was proposed that $c < rad^2(abc)$ [3]. In 2012, A. Nitaj [4] proposed the following conjecture:

Conjecture 3.2. — Let a, b, c be positive integers relatively prime with $c = a + b$, then:

$$(3.5) \quad c < rad^{1.63}(abc)$$

$$(3.6) \quad abc < rad^{4.42}(abc)$$

In this paper, we will give the proof of the conjecture given by (3.5) that constitutes the key to obtain the proof of the *abc* conjecture using classical methods with the help of some theorems from the field of the number theory.

3.2. The Proof of the conjecture $c < rad^{1.63}(abc)$, case $c = a + b$

Let a, b, c be positive integers, relatively prime, with $c = a + b$, $b < a$ and $R = rad(abc)$, $c = \prod_{j'=1}^{j'=J'} c_{j'}^{\beta_{j'}}$, $\beta_{j'} \geq 1$, $c_{j'} \geq 2$ prime integers.

In the following, we will give the proof of the conjecture $c < rad^{1.63}(abc)$.

Proof. — :

I- We suppose that $c < rad(abc)$, then we obtain:

$$c < rad(abc) < rad^{1.63}(abc) \implies \boxed{c < R^{1.63}}$$

and the condition (3.5) is satisfied.

II- We suppose that $c = rad(abc)$, then a, b, c are not coprime, case to reject.

III- In the following, we suppose that $c > rad(abc)$ and a, b and c are not all prime numbers.

$$(3.7) \quad c = \mu_c rad(c) = a + b = \mu_a rad(a) + \mu_b rad(b) \stackrel{?}{<} rad^{1.63}(abc)$$

III-1- We suppose $\mu_a \leq rad^{0.63}(a)$. We obtain :

$$c = a+b < 2a \leq 2rad^{1.63}(a) < rad^{1.63}(abc) \implies c < rad^{1.63}(abc) \implies \boxed{c < R^{1.63}}$$

Then (3.7) is satisfied.

III-2- We suppose $\mu_c \leq rad^{0.63}(c)$. We obtain :

$$c = \mu_c rad(c) \leq rad^{1.63}(c) < rad^{1.63}(abc) \implies \boxed{c < R^{1.63}}$$

and the condition (3.7) is satisfied.

III-3- We suppose $\mu_c > rad^{0.63}(c)$ and $\mu_a > rad^{0.63}(a)$.

III-3-1- Case : $rad^{0.63}(c) < \mu_c \leq rad^{1.63}(c)$ and $rad^{0.63}(a) < \mu_a \leq rad^{1.63}(a)$.

We can write:

$$\left. \begin{array}{l} \mu_c \leq rad^{1.63}(c) \implies c \leq rad^{2.63}(c) \\ \mu_a \leq rad^{1.63}(a) \implies a \leq rad^{2.63}(a) \end{array} \right\} \implies ac \leq rad^{2.63}(ac) \implies a^2 < ac \leq rad^{2.63}(ac) \\ \implies a < rad^{1.315}(ac) \implies c < 2a < 2rad^{1.315}(ac) < rad^{1.63}(abc) \\ \implies \boxed{c = a + b < R^{1.63}}$$

III-3-2- Case : $\mu_c > rad^{1.63}(c)$ or $\mu_a > rad^{1.63}(a)$

III-3-2-1- We suppose that $\mu_c > rad^{1.63}(c)$ and $\mu_a \leq rad^2(a)$:

III-3-2-1-1- Case $rad(a) < rad(c)$:

$$\text{In this case } a = \mu_a \cdot rad(a) \leq rad^3(a) \leq rad^{1.63}(a) rad^{1.37}(a) < rad^{1.63}(a) \cdot rad^{1.37}(c) \\ \implies c < 2a < 2rad^{1.63}(a) \cdot rad^{1.37}(c) < rad^{1.63}(abc) \implies \boxed{c < R^{1.63}}.$$

$$\text{III-3-2-1-2- Case } rad(c) < rad(a) < rad^{\frac{1.63}{1.37}}(c): \text{ As } a \leq rad^{1.63}(a) \cdot rad^{1.37}(a) < \\ rad^{1.63}(a) \cdot rad^{1.63}(c) \implies c < 2a < 2rad^{1.63}(a) \cdot rad^{1.63}(c) < R^{1.63} \implies \\ \boxed{c < R^{1.63}}.$$

III-3-2-1-3- Case $rad^{\frac{1.63}{1.37}}(c) < rad(a)$:

III-3-2-1-3-1- We suppose $c \leq rad^{3.26}(c)$, we obtain:

$$c \leq rad^{3.26}(c) \implies c \leq rad^{1.63}(c).rad^{1.63}(c) \implies \\ c < rad^{1.63}(c).rad^{1.37}(a) < rad^{1.63}(c).rad^{1.63}(a).rad^{1.63}(b) = R^{1.63} \implies \boxed{c < R^{1.63}}$$

III-3-2-1-3-2- We suppose $c > rad^{3.26}(c) \implies \mu_c > rad^{2.26}(c)$.

III-3-2-1-3-2-1- We consider the case $\mu_a = rad^2(a) \implies a = rad^3(a)$. Then, we obtain that $X = rad(a)$ is a solution in positive integers of the equation:

$$(3.8) \quad X^3 + 1 = c - b + 1 = c'$$

But it is the case $c' = 1 + a$.

III-3-2-1-3-2-1-1- We suppose that $c' = rad^n(c')$ with $n \geq 4$, we obtain the equation:

$$(3.9) \quad rad^n(c') - rad^3(a) = 1$$

But the solutions of the equation (3.9) are [5] : $(rad(c') = 3, n = 2, rad(a) = +2)$, it follows the contradiction with $n \geq 4$ and the case $c' = rad^n(c'), n \geq 4$ is to reject.

III-3-2-1-3-2-1-2- In the following, we will study the cases $\mu_{c'} = A.rad^n(c')$ with $rad(c') \nmid A, n \geq 0$. The above equation (3.8) can be written as :

$$(3.10) \quad (X + 1)(X^2 - X + 1) = c'$$

Let δ any divisor of c' , then:

$$(3.11) \quad X + 1 = \delta$$

$$(3.12) \quad X^2 - X + 1 = \frac{c'}{\delta} = c'' = \delta^2 - 3X$$

We recall that $rad(a) > rad^{\frac{1.63}{1.37}}(c)$.

III-3-2-1-3-2-1-2-1- We suppose $\delta = l.rad(c')$. We have $\delta = l.rad(c') < c' = \mu_{c'}.rad(c') \implies l < \mu_{c'}$. As δ is a divisor of c' , then l is a divisor of $\mu_{c'}$, we write $\mu_{c'} = l.m$. From $\mu_{c'} = l(\delta^2 - 3X)$, we obtain:

$$m = l^2 rad^2(c') - 3rad(a) \implies 3rad(a) = l^2 rad^2(c') - m$$

A- Case $3|m \implies m = 3m', m' > 1$: As $\mu_{c'} = ml = 3m'l \implies 3|rad(c')$ and $(rad(c'), m')$ not coprime. We obtain:

$$rad(a) = l^2 rad^2(c'). \frac{rad(c')}{3} - m'$$

It follows that a, c' are not coprime, then the contradiction.

B - Case $m = 3 \implies \mu_{c'} = 3l \implies c' = 3lrad(c') = 3\delta = \delta(\delta^2 - 3X) \implies \delta^2 = 3(1 + X) = 3\delta \implies \delta = lrad(c') = 3 \implies c' = 3\delta = 9 = a + 1 \implies a = 8 \implies c \leq 15$, then it is a trivial case.

III-3-2-1-3-2-1-2-2- We suppose $\delta = lrad^2(c'), l \geq 2$. If $n = 0$ then $\mu_{c'} = A$ and from the equation above (3.12):

$$c'' = \frac{c'}{\delta} = \frac{\mu_{c'} \cdot rad(c')}{lrad^2(c')} = \frac{A \cdot rad(c')}{lrad^2(c')} = \frac{A}{lrad(c')} \Rightarrow rad(c') \nmid A$$

It follows the contradiction with the hypothesis above $rad(c') \nmid A$.

III-3-2-1-3-2-1-2-3- In the following, we suppose that $n > 0$.

If $lrad(c') \nmid \mu_{c'}$ then the case is to reject. We suppose $lrad(c') | \mu_{c'} \implies \mu_{c'} = m \cdot lrad(c')$, then $\frac{c'}{\delta} = m = \delta^2 - 3rad(a)$.

C - Case $m = 1 = c'/\delta \implies \delta^2 - 3rad(a) = 1 \implies (\delta - 1)(\delta + 1) = 3rad(a) = rad(a)(\delta + 1) \implies \delta = 2 = lrad^2(c')$, then the contradiction.

D - Case $m = 3$, we obtain $3(1 + rad(a)) = \delta^2 = 3\delta \implies \delta = 3 = lrad^2(c')$. Then the contradiction.

E - Case $m \neq 1, 3$, we obtain: $3rad(a) = l^2 rad^4(c') - m \implies rad(a)$ and $rad(c')$ are not coprime. Then the contradiction.

III-3-2-1-3-2-1-2-4- We suppose $\delta = lrad^n(c'), l \geq 2$ with $n \geq 3$. From $c' = \mu_{c'} \cdot rad(c') = lrad^n(c')(\delta^2 - 3rad(a))$, we denote $m = \delta^2 - 3rad(a) = \delta^2 - 3X$.

F - As seen above (paragraphs C,D), the cases $m = 1$ and $m = 3$ give contradictions, it follows the reject of these cases.

G - Case $m \neq 1, 3$. Let q be a prime that divides m , it follows $q | \mu_{c'} \implies q = c'_{j'_0} \implies c'_{j'_0} | \delta^2 \implies c'_{j'_0} | 3rad(a)$. Then $rad(a)$ and $rad(c')$ are not coprime. It follows the contradiction.

III-3-2-1-3-2-1-2-5- We suppose $\delta = \prod_{j \in J_1} c_j^{\beta_j}$, $\beta_j \geq 1$ with at least one $j_0 \in J_1$ with $\beta_{j_0} \geq 2$, $rad(c') \nmid \delta$. We can write:

$$(3.13) \quad \delta = \mu_\delta \cdot rad(\delta), \quad rad(c') = m \cdot rad(\delta), \quad m > 1, \quad (m, \mu_\delta) = 1$$

Then, we obtain:

$$(3.14) \quad \begin{aligned} c' = \mu_{c'} \cdot rad(c') = \mu_{c'} \cdot m \cdot rad(\delta) = \delta(\delta^2 - 3X) = \mu_\delta \cdot rad(\delta)(\delta^2 - 3X) \implies \\ m \cdot \mu_{c'} = \mu_\delta(\delta^2 - 3X) \end{aligned}$$

- We suppose $\mu_{c'} = \mu_\delta \implies m = \delta^2 - 3X = (\mu_{c'} \cdot rad(\delta))^2 - 3X$. As $\delta < \delta^2 - 3X \implies m > \delta \implies rad(c') > m > \mu_{c'} \cdot rad(\delta) > rad^3(c')$ because $\mu_{c'} > rad^{2.26}(c')$, it follows $rad(c') > rad^2(c')$. Then the contradiction.

- We suppose $\mu_{c'} < \mu_\delta$. As $rad(a) = \mu_\delta rad(\delta) - 1$, we obtain:

$$(3.15) \quad \begin{aligned} rad(a) > \mu_{c'} \cdot rad(\delta) - 1 > 0 \implies rad(ac') > c' \cdot rad(\delta) - rad(c') > 0 \implies \\ c' > rad(ac') > c' \cdot rad(\delta) - rad(c') > 0 \implies 1 > rad(\delta) - \frac{rad(c')}{c'} > 0, \quad rad(\delta) \geq 2 \\ \implies \text{The contradiction} \end{aligned}$$

- We suppose $\mu_{c'} > \mu_\delta$. In this case, from the equation (3.14) and as $(m, \mu_\delta) = 1$, it follows we can write:

$$(3.16) \quad \mu_{c'} = \mu_1 \cdot \mu_2, \quad \mu_1, \mu_2 > 1$$

$$(3.17) \quad c' = \mu_{c'} rad(c') = \mu_1 \cdot \mu_2 \cdot rad(\delta) \cdot m = \delta \cdot (\delta^2 - 3X)$$

$$(3.18) \quad \text{so that } m \cdot \mu_1 = \delta^2 - 3X, \quad \mu_2 = \mu_\delta \implies \delta = \mu_2 \cdot rad(\delta)$$

**1- We suppose $(\mu_1, \mu_2) \neq 1$, then $\exists c'_{j_0}$ so that $c'_{j_0} | \mu_1$ and $c'_{j_0} | \mu_2$. But $\mu_\delta = \mu_2 \implies c'_{j_0} | \delta$. From $3X = \delta^2 - m\mu_1 \implies c'_{j_0} | 3X \implies c'_{j_0} | X$ or $c'_{j_0} = 3$.

- If $c'_{j_0} | X$, it follows the contradiction with $(c', a) = 1$.

- If $c'_{j_0} = 3$. We have $m\mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1) \implies \delta^2 - 3\delta + 3 - m \cdot \mu_1 = 0$.

As $3 | \mu_1 \implies \mu_1 = 3^k \mu'_1, 3 \nmid \mu'_1, k \geq 1$, we obtain:

$$(3.19) \quad \delta^2 - 3\delta + 3(1 - 3^{k-1} m \mu'_1) = 0$$

**1-1- We consider the case $k > 1 \implies 3 \nmid (1 - 3^{k-1} m \mu'_1)$. Let us recall the Eisenstein criterion [6]:

Theorem 3.3. — (Eisenstein Criterion) Let $f = a_0 + \dots + a_n X^n$ be a polynomial $\in \mathbb{Z}[X]$. We suppose that $\exists p$ a prime number so that $p \nmid a_n, p | a_i, (0 \leq i \leq n - 1)$, and $p^2 \nmid a_0$, then f is irreducible in \mathbb{Q} .

We apply Eisenstein criterion to the polynomial $R(Z)$ given by:

$$(3.20) \quad R(Z) = Z^2 - 3Z + 3(1 - 3^{k-1} m \mu'_1)$$

then:

- $3 \nmid 1, -3 | (-3), -3 | 3(1 - 3^{k-1} m \mu'_1)$, and $-3^2 \nmid 3(1 - 3^{k-1} m \mu'_1)$.

It follows that the polynomial $R(Z)$ is irreducible in \mathbb{Q} , then, the contradiction with $R(\delta) = 0$.

**1-2- We consider the case $k = 1$, then $\mu_1 = 3\mu'_1$ and $(\mu'_1, 3) = 1$, we obtain:

$$(3.21) \quad \delta^2 - 3\delta + 3(1 - m\mu'_1) = 0$$

**1-2-1- We consider that $3 \nmid (1 - m\mu'_1)$, we apply the same Eisenstein criterion to the polynomial $R'(Z)$ given by:

$$R'(Z) = Z^2 - 3Z + 3(1 - m\mu'_1)$$

and we find a contradiction with $R'(\delta) = 0$.

**1-2-2- We consider that $3 \mid (1 - m\mu'_1) \implies m\mu'_1 - 1 = 3^i \cdot h$, $i \geq 1$, $3 \nmid h$, $h \in \mathbb{N}^*$. δ is an integer root of the polynomial $R'(Z)$:

$$(3.22) \quad R'(Z) = Z^2 - 3Z + 3(1 - m\mu'_1) = 0 \implies \text{the discriminant of } R'(Z) \text{ is } : \Delta = 3^2 + 3^{i+1} \times 4 \cdot h$$

As the root δ is an integer, it follows that $\Delta = l^2 > 0$ with l a positive integer. We obtain:

$$(3.23) \quad \Delta = 3^2(1 + 3^{i-1} \times 4h) = l^2$$

$$(3.24) \quad \implies 1 + 3^{i-1} \times 4h = q^2 > 1, q \in \mathbb{N}^*$$

We can write the equation (3.21) as :

$$(3.25) \quad \delta(\delta - 3) = 3^{i+1} \cdot h \implies 3^3 \mu'_1 \frac{rad(\delta)}{3} \cdot (\mu'_1 rad(\delta) - 1) = 3^{i+1} \cdot h \implies$$

$$(3.26) \quad \mu'_1 \frac{rad(\delta)}{3} \cdot (\mu'_1 rad(\delta) - 1) = h$$

We obtain $i = 2$ and $q^2 = 1 + 12h = 1 + 4\mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1)$. Then, q satisfies :

$$(3.27) \quad q^2 - 1 = 12h \implies \frac{(q-1)}{2} \cdot \frac{(q+1)}{2} = 3h = (\mu'_1 rad(\delta) - 1) \cdot \mu'_1 rad(\delta) \implies$$

$$(3.28) \quad q - 1 = 2\mu'_1 rad(\delta) - 2$$

$$(3.29) \quad q + 1 = 2\mu'_1 rad(\delta)$$

It follows that $(q = x, 1 = y)$ is a solution of the Diophantine equation:

$$(3.30) \quad x^2 - y^2 = N$$

with $N = 12h > 0$. Let $Q(N)$ be the number of the solutions of (3.30) and $\tau(N)$ is the number of suitable factorization of N , then we announce the following result concerning the solutions of the Diophantine equation (3.30) (see theorem 27.3 in [7]):

- If $N \equiv 2 \pmod{4}$, then $Q(N) = 0$.

- If $N \equiv 1$ or $N \equiv 3 \pmod{4}$, then $Q(N) = [\tau(N)/2]$.
 - If $N \equiv 0 \pmod{4}$, then $Q(N) = [\tau(N/4)/2]$.
- $[x]$ is the integral part of x for which $[x] \leq x < [x] + 1$.

Let (α', m') , $\alpha', m' \in \mathbb{N}^*$ be another pair, solution of the equation (3.30), then $\alpha'^2 - m'^2 = x^2 - y^2 = N = 12h$, but $q = x$ and $1 = y$ satisfy the equation (3.29) given by $x + y = 2\mu'_1 rad(\delta)$, it follows α', m' verify also $\alpha' + m' = 2\mu'_1 rad(\delta)$, that gives $\alpha' - m' = 2(\mu'_1 rad(\delta) - 1)$, then $\alpha' = x = q = 2\mu'_1 rad(\delta)$ and $m' = y = 1$. So, we have given the proof of the uniqueness of the solutions of the equation (3.30) with the condition $x + y = 2\mu'_1 rad(\delta)$. As $N = 12h = 4\mu'_1 rad(\delta) \cdot (\mu'_1 rad(\delta) - 1) \implies N \equiv 0 \pmod{4} \implies Q(N) = [\tau(N/4)/2] = [\tau(3h)/2]$, the expression of $3h = \mu'_1 \cdot rad(\delta) \cdot (\mu'_1 rad(\delta) - 1)$, then $Q(N) = [\tau(3h)/2] > 1$. But $Q(N) = 1$, then the contradiction and the case $3|(1 - m \cdot \mu'_1)$ is to reject.

**2- We suppose that $(\mu_1, \mu_2) = 1$.

From the equation $m\mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1)$, we obtain that δ is a root of the following polynomial :

$$(3.31) \quad R(Z) = Z^2 - 3Z + 3 - m \cdot \mu_1 = 0$$

The discriminant of $R(Z)$ is:

$$(3.32) \quad \Delta = 9 - 4(3 - m \cdot \mu_1) = 4m \cdot \mu_1 - 3 = q^2 \quad \text{with } q \in \mathbb{N}^* \quad \text{as } \delta \in \mathbb{N}^*$$

- We suppose that $2|m\mu_1 \implies c'$ is even. Then $q^2 \equiv 5 \pmod{8}$, it gives a contradiction because a square is $\equiv 0, 1$ or $4 \pmod{8}$.

- We suppose c' an odd integer, then a is even. It follows $a = rad^3(a) \equiv 0 \pmod{8} \implies c' \equiv 1 \pmod{8}$. As $c' = \delta^2 - 3X \cdot \delta$, we obtain $\delta^2 - 3X \cdot \delta \equiv 1 \pmod{8}$. If $\delta^2 \equiv 1 \pmod{8} \implies -3X \cdot \delta \equiv 0 \pmod{8} \implies 8|X \cdot \delta \implies 4|\delta \implies c'$ is even. Then, the contradiction. If $\delta^2 \equiv 4 \pmod{8} \implies \delta \equiv 2 \pmod{8}$ or $\delta \equiv 6 \pmod{8}$. In the two cases, we obtain $2|\delta$. Then, the contradiction with c' an odd integer.

It follows that the case $c > rad^{3.26}(c)$ and $a = rad^3(a)$ is impossible.

III-3-2-1-3-2-2- We suppose $c > rad^{3.26}(c)$ and large and $\mu_a < rad^2(a)$. Then $c = rad^3(c) + h$, $h > rad^3(c)$, h a positive integer and we can write $a + l = rad^3(a)$, $l > 0$. Then we obtain :

$$(3.33) \quad rad^3(c) + h = rad^3(a) - l + b \implies rad^3(a) - rad^3(c) = h + l - b > 0$$

as $rad(a) > rad^{\frac{1.63}{1.37}}(c)$. We obtain the equation:

$$(3.34) \quad rad^3(a) - rad^3(c) = h + l - b = m > 0$$

Let $X = rad(a) - rad(c)$, then X is an integer root of the polynomial $H(X)$ defined as:

$$(3.35) \quad H(X) = X^3 + 3rad(ac)X - m = 0$$

To resolve the above equation, we denote $X = u + v$, It follows that u^3, v^3 are the roots of the polynomial $G(t)$ given by:

$$(3.36) \quad G(t) = t^2 - mt - rad^3(ac) = 0$$

The discriminant of $G(t)$ is $\Delta = m^2 + 4rad^3(ac) = \alpha^2$, $\alpha > 0$. The two real roots of (3.36) are:

$$(3.37) \quad t_1 = u^3 = \frac{m + \alpha}{2}, \quad t_2 = v^3 = \frac{m - \alpha}{2}$$

As $m = rad^3(a) - rad^3(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^3(c) > 0$, then from the expression of the discriminant Δ , it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$(3.38) \quad x^2 - y^2 = N$$

with $N = 4rad^3(ac) > 0$. From the expression of Δ above, we remark that α and m verify the following equations:

$$(3.39) \quad x + y = 2u^3 = 2rad^3(a)$$

$$(3.40) \quad x - y = -2v^3 = 2rad^3(c)$$

$$(3.41) \quad \text{then } x^2 - y^2 = N = 4rad^3(a).rad^3(c)$$

As (α, m) is a couple of solutions of the Diophantine equation (3.38) and $\alpha > m$, then $\exists d, d'$ positive integers with $d > d'$ and $N = d.d'$ so that :

$$(3.42) \quad d + d' = 2\alpha$$

$$(3.43) \quad d - d' = 2m$$

III-3-2-1-3-2-2-1- Now, we consider for example, the case $d = 4rad^3(a)$ and $d' = rad^3(c) \implies d > d'$. We rewrite the equations (3.42-3.43):

$$(3.44) \quad rad^3(a) + rad^3(c) = 2(rad^3(a) + rad^3(c)) \implies 2rad^3(a) = rad^3(c)$$

$$(3.45) \quad rad^3(a) - rad^3(c) = 2(rad^3(a) - rad^3(c)) \implies 2rad^3(a) = -rad^3(c)$$

Then the contradiction.

III-3-2-1-3-2-2-2- we consider the case $d = 4rad^3(c)rad^3(a)$ and $d' = 1 \implies d > d'$. We rewrite the equations (3.42-3.43):

$$(3.46) \quad 4rad^3(c)rad^3(a) + 1 = 2(rad^3(c) + rad^3(a)) \implies 2rad^3(c) = 1$$

$$(3.47) \quad 4rad^3(c)rad^3(a) - 1 = 2(rad^3(c) - rad^3(a)) \implies 2rad^3(c) = -1$$

Then the contradiction.

III-3-2-1-3-2-2-3- Let c_1 be the first factor of $rad(c)$. we consider the case $d = 4c_1rad^3(a)$ and $d' = \frac{rad^3(c)}{c_1} \implies d > d'$. We rewrite the equation (3.42):

$$(3.48) \quad 4c_1rad^3(a) + \frac{rad^3(c)}{c_1} = 2(rad^3(a) + rad^3(c)) \implies$$

$$(3.49) \quad d^3(a)(2c_1 - 1) = \frac{rad^3(c)}{c_1}(2c_1 - 1) \implies 2rad^3(a) = rad^2(c) \cdot \frac{rad(c)}{c_1}$$

$c_1 = 2$ or not, there is a contradiction.

The others cases of the expressions of d and d' not coprime so that $N = d.d'$ give also contradictions.

Let $Q(N)$ be the number of the solutions of (3.38), as $N \equiv 0 \pmod{4}$, then $Q(N) = [\tau(N/4)/2]$. From the study of some cases above, we obtain that $Q(N) < [(\tau(N)/4)/2]$. It follows the contradiction.

Then the cases $\mu_a \leq rad^2(a)$ and $c > rad^{3.26}(c)$ are impossible.

III-3-2-2 We suppose that $rad^{1.63}(c) < \mu_c \leq rad^2(c)$ and $\mu_a > rad^{1.63}(a)$:

III-3-2-2-1- Case $rad(c) < rad(a)$: As $c \leq rad^3(c) = rad^{1.63}(c).rad^{1.37}(c) \implies c < rad^{1.63}(c).rad^{1.37}(a) < rad^{1.63}(ac) < rad^{1.63}(abc) \implies \boxed{c < R^{1.63}}$.

III-3-2-2-2- Case $rad(a) < rad(c) < rad^{\frac{1.63}{1.37}}(a)$:
As $c \leq rad^3(c) \leq rad^{1.63}(c).rad^{1.37}(c) \implies c < rad^{1.63}(c).rad^{1.63}(a) < rad^{1.63}(abc) \implies \boxed{c < R^{1.63}}$.

III-3-2-2-3- Case $rad^{\frac{1.63}{1.37}}(a) < rad(c)$:

III-3-2-2-3-1- We suppose $rad^{1.63}(a) < \mu_a \leq rad^{2.26}(a) \implies a \leq rad^{1.63}(a).rad^{1.63}(a) \implies a < rad^{1.63}(a).rad^{1.37}(c) \implies c = a + b < 2a < 2rad^{1.63}(a).rad^{1.63}(c) < rad^{1.63}(abc) \implies c < R^{1.63} \implies \boxed{c < R^{1.63}}$.

III-3-2-2-3-2- We suppose $\mu_a > rad^{2.26}(a)$ and $\mu_c \leq rad^2(c)$. Using the same method as it was explicated in the paragraphs **III-3-2-1-3-2-** (permuting a, c), we arrive at a contradiction (see the appendix). It follows that the case $\mu_c = rad^2(c)$ and $\mu_a > rad^{2.26}(a)$ is impossible.

III-3-2-2-3-2-2- We suppose $a > rad^{3.26}(a)$ and large and $\mu_c < rad^2(c)$. Then $a = rad^3(a) + h, h > rad^3(a)$, h a positive integer and we can write $c + l = rad^3(c)$, $l > 0$. Then we obtain :

$$(3.50) \quad rad^3(c) - rad^3(a) = h + l + b > 0$$

as $rad(c) > rad^{\frac{1.63}{1.37}}(a)$. Let $X = rad(c) - rad(a)$, then X is an integer root of the polynomial $H(X)$ defined as:

$$(3.51) \quad H(X) = X^3 + 3rad(ac)X - m = 0$$

To resolve the above equation, we denote $X = u + v$. It follows that u^3, v^3 are the roots of the polynomial $G(t)$ given by:

$$(3.52) \quad G(t) = t^2 - mt - rad^3(ac) = 0$$

The discriminant of $G(t)$ is $\Delta = m^2 + 4rad^3(ac) = \alpha^2$, $\alpha > 0$. The two real roots of (3.52) are:

$$(3.53) \quad t_1 = u^3 = \frac{m + \alpha}{2}, \quad t_2 = v^3 = \frac{m - \alpha}{2}$$

As $m = rad^3(c) - rad^3(a) > 0$, we obtain that $\alpha = rad^3(a) + rad^3(c) > 0$, then from the expression of the discriminant Δ , it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$(3.54) \quad x^2 - y^2 = N$$

with $N = 4rad^3(ac) > 0$. It is the same case (permuting a and c) as the case above **III-3-2-1-3-2-2-** and we obtain contradictions.

Then the cases $\mu_c \leq rad^2(c)$ and $a > rad^{3.26}(a)$ are impossible.

III-3-3- Case $\mu_a > rad^{1.63}(a)$ and $\mu_c > rad^{1.63}(c)$: Taking into account the cases studied above, it remains to see the following two cases:

- $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a)$,
- $\mu_a > rad^2(a)$ and $\mu_c > rad^{1.63}(c)$.

III-3-3-1- We suppose $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a) \implies c > rad^3(c)$ and $a > rad^{2.63}(a)$. We can write $c = rad^3(c) + h$ and $a = rad^3(a) + l$ with h a positive integer and $l \in \mathbb{Z}$.

III-3-3-1-1- We suppose $rad(c) < rad(a)$. We obtain the equation:

$$(3.55) \quad rad^3(a) - rad^3(c) = h - l - b = m > 0$$

Let $X = rad(a) - rad(c)$, from the above equation, X is a real root of the polynomial:

$$(3.56) \quad H(X) = X^3 + 3rad(ac)X - m = 0$$

As above, to resolve (3.56), we denote $X = u + v$, It follows that u^3, v^3 are the roots of the polynomial $G(t)$ given by :

$$(3.57) \quad G(t) = t^3 - mt - rad^3(ac) = 0$$

The discriminant of $G(t)$ is:

$$(3.58) \quad \Delta = m^2 + 4rad^3(ac) = \alpha^2, \quad \alpha > 0$$

The two real roots of (3.57) are:

$$(3.59) \quad t_1 = u^3 = \frac{m + \alpha}{2}, \quad t_2 = v^3 = \frac{m - \alpha}{2}$$

As $m = rad^3(a) - rad^3(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^3(c) > 0$, then from the equation (3.58), it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$(3.60) \quad x^2 - y^2 = N$$

with $N = 4rad^3(ac) > 0$. From the equations (3.59), we remark that α and m verify the following equations:

$$(3.61) \quad x + y = 2u^3 = 2rad^3(a)$$

$$(3.62) \quad x - y = -2v^3 = 2rad^3(c)$$

$$(3.63) \quad \text{then } x^2 - y^2 = N = 4rad^3(a).rad^3(c)$$

Let $Q(N)$ be the number of the solutions of (3.60) and $\tau(N)$ is the number of suitable factorization of N , and using the same method as in the paragraph **III-3-2-2-3-2-2-** above, we obtain a contradiction.

III-3-3-1-2- We suppose $rad(a) < rad(c)$. We obtain the equation:

$$(3.64) \quad rad^3(c) - rad^3(a) = b + l - h = m > 0$$

Let X be the variable $X = rad(c) - rad(a)$, we use the similar calculations as in the paragraph above **III-3-3-1-1-**, we find a contradiction.

It follows that the case $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a)$ is impossible.

III-3-3-2- We suppose $\mu_a > rad^2(a)$ and $\mu_c > rad^{1.63}(c)$, we obtain $a > rad^3(a)$ and $c > rad^{2.63}(c)$. We can write $a = rad^3(a) + h$ and

$c = rad^3(c) + l$ with h a positive integer and $l \in \mathbb{Z}$.

The calculations are similar to those in the case **III-3-3-1-**. We obtain a contradiction.

It follows that the case $\mu_c > rad^{1.63}(c)$ and $\mu_a > rad^2(a)$ is impossible. \square

We can state the following important theorem:

Theorem 3.4. — *Let a, b, c positive integers relatively prime with $c = a + b$, then $c < rad^{1.63}(abc)$.*

From the theorem above, we can announce also:

Corollary 3.5. — *Let a, b, c positive integers relatively prime with $c = a + b$, then the conjecture $c < rad^2(abc)$ is true.*

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Appendix

III-3-2-2-3-2- We suppose $\mu_a > rad^{2.26}(a)$ and $\mu_c \leq rad^2(c)$

III-3-2-2-3-2-1- We consider the case $\mu_c = rad^2(c) \implies c = rad^3(c)$. Then, we obtain that $Y = rad(c)$ is a solution in positive integers of the equation:

$$(3.65) \quad Y^3 + 1 = a + b + 1 = c'$$

But it is the case $c' = 1 + c$.

III-3-2-2-3-2-1-1- We suppose that $c' = rad^n(c')$ with $n \geq 4$, we obtain the equation:

$$(3.66) \quad rad^n(c') - rad^3(c) = 1$$

But the solutions of the equation (3.66) are [5] : $(rad(c') = 3, n = 2, rad(c) = +2)$, it follows the contradiction with $n \geq 4$ and the case $c' = rad^n(c'), n \geq 4$ is to reject.

III-3-2-2-3-2-1-2-In the following, we will study the cases $\mu_{c'} = A.rad^n(c')$ with $rad(c') \nmid A, n \geq 0$. The above equation (3.65) can be written as :

$$(3.67) \quad (Y + 1)(Y^2 - Y + 1) = c'$$

Let δ any divisor of c' , then:

$$(3.68) \quad Y + 1 = \delta$$

$$(3.69) \quad Y^2 - Y + 1 = \frac{c'}{\delta} = c'' = \delta^2 - 3Y$$

We recall that $rad(c) > rad^{\frac{1.63}{1.37}}(a)$.

III-3-2-2-3-2-1-2-1- We suppose $\delta = l.rad(c')$. We have $\delta = l.rad(c') < c' = \mu'_c.rad(c') \implies l < \mu'_c$. As δ is a divisor of c' , then l is a divisor of μ'_c , we write $\mu'_c = l.m$. From $\mu'_c = l(\delta^2 - 3Y)$, we obtain:

$$m = l^2.rad^2(c') - 3rad(c) \implies 3rad(c) = l^2.rad^2(c') - m$$

A- Case $3|m \implies m = 3m', m' > 1$: As $\mu'_c = ml = 3m'l \implies 3|rad(c')$ and $(rad(c'), m')$ not coprime. We obtain:

$$rad(c) = l^2.rad(c').\frac{rad(c')}{3} - m'$$

It follows that c, c' are not coprime, then the contradiction.

B - Case $m = 3 \implies \mu'_c = 3l \implies c' = 3l.rad(c') = 3\delta = \delta(\delta^2 - 3Y) \implies \delta^2 = 3(1 + Y) = 3\delta \implies \delta = l.rad(c') = 3 \implies c' = 3\delta = 9 = c + 1 \implies c = 8$, then it is a trivial case.

III-3-2-2-3-2-1-2-2- We suppose $\delta = l.rad^2(c'), l \geq 2$. If $n = 0$ then $\mu_{c'} = A$ and from the equation above (3.69):

$$c'' = \frac{c'}{\delta} = \frac{\mu_{c'}.rad(c')}{l.rad^2(c')} = \frac{A.rad(c')}{l.rad^2(c')} = \frac{A}{l.rad(c')} \implies rad(c')|A$$

It follows the contradiction with the hypothesis above $rad(c') \nmid A$.

III-3-2-2-3-2-1-2-3- In the following, we suppose that $n > 0$.

If $l.rad(c') \nmid \mu_{c'}$ then the case is to reject. We suppose $l.rad(c')|\mu_{c'} \implies \mu_{c'} = m.l.rad(c')$, then $\frac{c'}{\delta} = m = \delta^2 - 3rad(c)$.

C' - Case $m = 1 = c'/\delta \implies \delta^2 - 3rad(c) = 1 \implies (\delta - 1)(\delta + 1) = 3rad(c) = rad(c)(\delta + 1) \implies \delta = 2 = l.rad^2(c')$, then the contradiction.

D' - Case $m = 3$, we obtain $3(1 + rad(c)) = \delta^2 = 3\delta \implies \delta = 3 = lrad^2(c')$. Then the contradiction.

E' - Case $m \neq 1, 3$, we obtain: $3rad(c) = l^2rad^4(c') - m \implies rad(c)$ and $rad(c')$ are not coprime. Then the contradiction.

III-3-2-2-3-2-1-2-4- We suppose $\delta = lrad^n(c')$, $l \geq 2$ with $n \geq 3$. From $c' = \mu_{c'}rad(c') = lrad^n(c')(\delta^2 - 3rad(c))$, we denote $m = \delta^2 - 3rad(c) = \delta^2 - 3Y$.

F' - As seen above (paragraphs C',D'), the cases $m = 1$ and $m = 3$ give contradictions, it follows the reject of these cases.

G' - Case $m \neq 1, 3$. Let q be a prime that divides m , it follows $q|\mu_{c'} \implies q = c'_{j'_0} \implies c'_{j'_0}|\delta^2 \implies c'_{j'_0}|3rad(c)$. Then $rad(c)$ and $rad(c')$ are not coprime. It follows the contradiction.

III-3-2-2-3-2-1-2-5- We suppose $\delta = \prod_{j \in J_1} c_j^{\beta_j}$, $\beta_j \geq 1$ with at least one $j_0 \in J_1$ with $\beta_{j_0} \geq 2$, $rad(c') \nmid \delta$. We can write:

$$(3.70) \quad \delta = \mu_\delta rad(\delta), \quad rad(c') = m rad(\delta), \quad m > 1, \quad (m, \mu_\delta) = 1$$

Then, we obtain:

$$(3.71) \quad \begin{aligned} c' = \mu_{c'} rad(c') &= \mu_{c'} m rad(\delta) = \delta(\delta^2 - 3Y) = \mu_\delta rad(\delta)(\delta^2 - 3Y) \implies \\ m \cdot \mu_{c'} &= \mu_\delta(\delta^2 - 3Y) \end{aligned}$$

- We suppose $\mu_{c'} = \mu_\delta \implies m = \delta^2 - 3Y = (\mu_{c'} rad(\delta))^2 - 3Y$. As $\delta < \delta^2 - 3Y \implies m > \delta \implies rad(c') > m > \mu_{c'} rad(\delta) > rad^3(c')$ because $\mu_{c'} > rad^{2.26}(c')$, it follows $rad(c') > rad^2(c')$. Then the contradiction.

- We suppose $\mu_{c'} < \mu_\delta$. As $rad(c) = \mu_\delta rad(\delta) - 1$, we obtain:

$$(3.72) \quad \begin{aligned} rad(c) > \mu_{c'} rad(\delta) - 1 > 0 &\implies rad(cc') > c' rad(\delta) - rad(c') > 0 \implies \\ c' > rad(cc') > c' rad(\delta) - rad(c') > 0 &\implies 1 > rad(\delta) - \frac{rad(c')}{c'} > 0, \quad rad(\delta) \geq 2 \\ &\implies \text{The contradiction} \end{aligned}$$

- We suppose $\mu_{c'} > \mu_\delta$. In this case, from the equation (3.71) and as $(m, \mu_\delta) = 1$, it follows we can write:

$$(3.73) \quad \mu_{c'} = \mu_1 \mu_2, \quad \mu_1, \mu_2 > 1$$

$$(3.74) \quad c' = \mu_{c'} rad(c') = \mu_1 \mu_2 rad(\delta) m = \delta(\delta^2 - 3Y)$$

$$(3.75) \quad \text{so that } m \mu_1 = \delta^2 - 3Y, \quad \mu_2 = \mu_\delta \implies \delta = \mu_2 rad(\delta)$$

**1- We suppose $(\mu_1, \mu_2) \neq 1$, then $\exists c'_{j_0}$ so that $c'_{j_0} | \mu_1$ and $c'_{j_0} | \mu_2$. But $\mu_\delta = \mu_2 \Rightarrow c'^2_{j_0} | \delta$. From $3Y = \delta^2 - m\mu_1 \Rightarrow c'_{j_0} | 3Y \Rightarrow c'_{j_0} | Y$ or $c'_{j_0} = 3$.

- If $c'_{j_0} | Y$, it follows the contradiction with $(c', c) = 1$.

- If $c'_{j_0} = 3$. We have $m\mu_1 = \delta^2 - 3Y = \delta^2 - 3(\delta - 1) \Rightarrow \delta^2 - 3\delta + 3 - m.\mu_1 = 0$.

As $3 | \mu_1 \Rightarrow \mu_1 = 3^k \mu'_1, 3 \nmid \mu'_1, k \geq 1$, we obtain:

$$(3.76) \quad \delta^2 - 3\delta + 3(1 - 3^{k-1}m\mu'_1) = 0$$

**1-1- We consider the case $k > 1 \Rightarrow 3 \nmid (1 - 3^{k-1}m\mu'_1)$. We apply Eisenstein criterion [6] to the polynomial $R(Z)$ given by:

$$(3.77) \quad R(Z) = Z^2 - 3Z + 3(1 - 3^{k-1}m\mu'_1)$$

then:

- $3 \nmid 1$, - $3 | (-3)$, - $3 | 3(1 - 3^{k-1}m\mu'_1)$, and - $3^2 \nmid 3(1 - 3^{k-1}m\mu'_1)$.

It follows that the polynomial $R(Z)$ is irreducible in \mathbb{Q} , then, the contradiction with $R(\delta) = 0$.

**1-2- We consider the case $k = 1$, then $\mu_1 = 3\mu'_1$ and $(\mu'_1, 3) = 1$, we obtain:

$$(3.78) \quad \delta^2 - 3\delta + 3(1 - m\mu'_1) = 0$$

* If $3 \nmid (1 - m.\mu'_1)$, we apply the same Eisenstein criterion to the polynomial $R'(Z)$ given by:

$$R'(Z) = Z^2 - 3Z + 3(1 - m\mu'_1)$$

and we find a contradiction with $R'(\delta) = 0$.

**1-2-2- We consider that $3 | (1 - m.\mu'_1) \Rightarrow m\mu'_1 - 1 = 3^i.h, i \geq 1, 3 \nmid h, h \in \mathbb{N}^*$. δ is an integer root of the polynomial $R'(Z)$:

$$(3.79) \quad \begin{aligned} R'(Z) = Z^2 - 3Z + 3(1 - m\mu'_1) = 0 &\Rightarrow \text{the discriminant of } R'(Z) \text{ is :} \\ \Delta = 3^2 + 3^{i+1} \times 4.h \end{aligned}$$

As the root δ is an integer, it follows that $\Delta = l^2 > 0$ with l a positive integer. We obtain:

$$(3.80) \quad \Delta = 3^2(1 + 3^{i-1} \times 4h) = l^2$$

$$(3.81) \quad \Rightarrow 1 + 3^{i-1} \times 4h = q^2 > 1, q \in \mathbb{N}^*$$

We can write the equation (3.78) as :

$$(3.82) \quad \delta(\delta - 3) = 3^{i+1}.h \Rightarrow 3^3 \mu'_1 \frac{\text{rad}(\delta)}{3}. (\mu'_1 \text{rad}(\delta) - 1) = 3^{i+1}.h \Rightarrow$$

$$(3.83) \quad \mu'_1 \frac{\text{rad}(\delta)}{3}. (\mu'_1 \text{rad}(\delta) - 1) = h$$

We obtain $i = 2$ and $q^2 = 1 + 12h = 1 + 4\mu'_1 \text{rad}(\delta)(\mu'_1 \text{rad}(\delta) - 1)$. Then, q satisfies :

$$(3.84) \quad q^2 - 1 = 12h \Rightarrow \frac{(q-1)}{2} \cdot \frac{(q+1)}{2} = 3h = (\mu'_1 \text{rad}(\delta) - 1) \cdot \mu'_1 \text{rad}(\delta) \Rightarrow$$

$$(3.85) \quad q - 1 = 2\mu'_1 \text{rad}(\delta) - 2$$

$$(3.86) \quad q + 1 = 2\mu'_1 \text{rad}(\delta)$$

It follows that $(q = x, 1 = y)$ is a solution of the Diophantine equation:

$$(3.87) \quad x^2 - y^2 = N$$

with $N = 12h > 0$. Let $Q(N)$ be the number of the solutions of (3.87) and $\tau(N)$ is the number of suitable factorization of N , then we announce the following result concerning the solutions of the Diophantine equation (3.87) (see theorem 27.3 in [7]):

- If $N \equiv 2 \pmod{4}$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3 \pmod{4}$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0 \pmod{4}$, then $Q(N) = [\tau(N/4)/2]$.

Let (α', m') , $\alpha', m' \in \mathbb{N}^*$ be another pair, solution of the equation (3.87), then $\alpha'^2 - m'^2 = x^2 - y^2 = N = 12h$, but $q = x$ and $1 = y$ satisfy the equation (3.86) given by $x + y = 2\mu'_1 \text{rad}(\delta)$, it follows α', m' verify also $\alpha' + m' = 2\mu'_1 \text{rad}(\delta)$, that gives $\alpha' - m' = 2(\mu'_1 \text{rad}(\delta) - 1)$, then $\alpha' = x = q = 2\mu'_1 \text{rad}(\delta)$ and $m' = y = 1$. So, we have given the proof of the uniqueness of the solutions of the equation (3.87) with the condition $x + y = 2\mu'_1 \text{rad}(\delta)$. As $N = 12h \equiv 0 \pmod{4} \implies Q(N) = [\tau(N/4)/2] = [\tau(3h)/2]$, the expression of $3h = \mu'_1 \cdot \text{rad}(\delta) \cdot (\mu'_1 \text{rad}(\delta) - 1)$, then $Q(N) = [\tau(3h)/2] > 1$. But $Q(N) = 1$, then the contradiction and the case $3|(1 - m \cdot \mu'_1)$ is to reject.

** We suppose that $(\mu_1, \mu_2) = 1$.

From the equation $m\mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1)$, we obtain that δ is a root of the following polynomial :

$$(3.88) \quad R(Z) = Z^2 - 3Z + 3 - m \cdot \mu_1 = 0$$

The discriminant of $R(Z)$ is:

$$(3.89) \quad \Delta = 9 - 4(3 - m \cdot \mu_1) = 4m \cdot \mu_1 - 3 = q^2 \quad \text{with } q \in \mathbb{N}^* \quad \text{as } \delta \in \mathbb{N}^*$$

- We suppose that $2|m\mu_1 \implies c'$ is even. Then $q^2 \equiv 5 \pmod{8}$, it gives a contradiction because a square is $\equiv 0, 1$ or $4 \pmod{8}$.

- We suppose c' an odd integer, then c is even. It follows $c = \text{rad}^3(c) \equiv 0 \pmod{8} \implies c' \equiv 1 \pmod{8}$. As $c' = \delta^2 - 3Y \cdot \delta$, we obtain $\delta^2 - 3Y \cdot \delta \equiv 1 \pmod{8}$. If $\delta^2 \equiv 1 \pmod{8} \implies -3Y \cdot \delta \equiv 0 \pmod{8} \implies 8|Y \cdot \delta \implies 4|\delta \implies c'$

is even. Then, the contradiction. If $\delta^2 \equiv 4 \pmod{8} \implies \delta \equiv 2 \pmod{8}$ or $\delta \equiv 6 \pmod{8}$. In the two cases, we obtain $2|\delta$. Then, the contradiction with c' an odd integer.

It follows that the case $\mu_a > rad^{2.26}(a)$ and $\mu_c = rad^2(c)$ is impossible.

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CHAPTER 4

IS THE *abc* CONJECTURE TRUE?

Abstract. — In this paper, we consider the *abc* conjecture. As the conjecture $c < rad^2(abc)$ is true, then we give the proof of the *abc* conjecture for $\epsilon \geq 1$ and for the case $\epsilon \in]0, 1[$, we consider that the *abc* conjecture is false, from the proof, we arrive in a contradiction.

Résumé. — Dans cet article, nous considérons la conjecture *abc*. Comme la conjecture $c < rad^2(abc)$ est vraie, nous donnons la preuve que la conjecture *abc* est vraie pour $\epsilon \geq 1$ et pour les cas $\epsilon \in]0, 1[$, supposant que la conjecture est fausse nous arrivons à une contradiction.

4.1. Introduction and notations

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$(4.1) \quad a = \prod_i a_i^{\alpha_i} = rad(a) \cdot \prod_i a_i^{\alpha_i - 1}$$

We note:

$$(4.2) \quad \mu_a = \prod_i a_i^{\alpha_i - 1} \implies a = \mu_a \cdot rad(a)$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Esterlé of Pierre et Marie Curie University (Paris 6) [4]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 4.1. — (*abc Conjecture*): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :

$$(4.3) \quad c < K(\epsilon) \cdot \text{rad}^{1+\epsilon}(abc)$$

where K is a constant depending only of ϵ .

The idea to try to write a paper about this conjecture was born after the publication in September 2018, of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$. So, I will give a simple proof that can be understood by undergraduate students.

We know that numerically, $\frac{\text{Log}c}{\text{Log}(\text{rad}(abc))} \leq 1.629912$ [4]. A conjecture was proposed that $c < \text{rad}^2(abc)$ [3]. It is the key to resolve the *abc* conjecture. In my paper, as the conjecture $c < \text{rad}^2(abc)$ holds (chapter 3), I propose an elementary proof of the *abc* conjecture.

4.2. The Proof of the *abc* conjecture

Proof. — We note $R = \text{rad}(abc)$ in the case $c = a + b$ or $R = \text{rad}(ac)$ in the case $c = a + 1$.

4.2.1. Case : $\epsilon \geq 1$. — As $c < R^2$ is true, we have $\forall \epsilon \geq 1$:

$$(4.4) \quad c < R^2 \leq R^{1+\epsilon} < K(\epsilon) \cdot R^{1+\epsilon}, \quad \text{with } K(\epsilon) = e, \quad \epsilon \geq 1$$

Then the *abc* conjecture is true.

4.2.2. Case: $\epsilon < 1$. — From the statement of the *abc* conjecture 4.1, we want to give a proof that $c < K(\epsilon)R^{1+\epsilon} \implies \text{Log}K(\epsilon) + (1+\epsilon)\text{Log}R - \text{Log}c > 0$.

For our proof, we proceed by contradiction of the *abc* conjecture. We suppose that the *abc* conjecture is false:

$$(4.5) \quad \exists \epsilon_0 \in]0, 1[, \forall K(\epsilon) > 0, \quad \exists c_0 = a_0 + b_0; \quad a_0, b_0, c_0 \text{ coprime so that} \\ c_0 > K(\epsilon_0)R_0^{1+\epsilon_0} \text{ and } \forall \epsilon \in]0, 1[, \quad c_0 > K(\epsilon)R_0^{1+\epsilon}$$

We choose the constant $K(\epsilon) = \frac{1}{e\epsilon^2}$. Let :

$$(4.6) \quad Y_{c_0}(\epsilon) = \frac{1}{\epsilon^2} + (1 + \epsilon)\text{Log}R_0 - \text{Log}c_0, \epsilon \in]0, 1[$$

From the above explications, if we will obtain $\forall \epsilon \in]0, 1[, Y_{c_0}(\epsilon) > 0 \implies c_0 < K(\epsilon)R_0^{1+\epsilon} \implies c_0 < K(\epsilon_0)R_0^{1+\epsilon_0}$, then the contradiction with (4.5).

About the function Y_{c_0} , we have:

$$\begin{aligned} \lim_{\epsilon \rightarrow 1} Y_{c_0}(\epsilon) &= 1 + \text{Log}(R_0^2/c_0) = \lambda > 0 \\ \lim_{\epsilon \rightarrow 0} Y_{c_0}(\epsilon) &= +\infty \end{aligned}$$

The function $Y_{c_0}(\epsilon)$ has a derivative for $\forall \epsilon \in]0, 1[$, we obtain:

$$(4.7) \quad Y'_{c_0}(\epsilon) = -\frac{2}{\epsilon^3} + \text{Log}R_0 = \frac{\epsilon^3 \text{Log}R_0 - 2}{\epsilon^3}$$

$$Y'_{c_0}(\epsilon) = 0 \implies \epsilon = \epsilon' = \sqrt[3]{\frac{2}{\text{Log}R_0}} \in]0, 1[\text{ for } R_0 \geq 8.$$

ϵ	0	ϵ'	1
$Y'(\epsilon)$	-	0	+
$Y(\epsilon)$	$+\infty$	$Y(\epsilon')$	$\lambda > 0$

FIGURE 1. Table of variations

Discussion from the table (Fig.: 1):

- If $Y_{c_0}(\epsilon') \geq 0$, it follows that $\forall \epsilon \in]0, 1[, Y_{c_0}(\epsilon) \geq 0$, then the contradiction with $Y_{c_0}(\epsilon_0) < 0 \implies c_0 > K(\epsilon_0)R_0^{1+\epsilon_0}$ and the supposition that the *abc* conjecture is false can not hold. Hence the *abc* conjecture is true for $\epsilon \in]0, 1[$.

- If $Y_{c_0}(\epsilon') < 0 \implies \exists 0 < \epsilon_1 < \epsilon' < \epsilon_2 < 1$, so that $Y_{c_0}(\epsilon_1) = Y_{c_0}(\epsilon_2) = 0$. Then we obtain $c_0 = K(\epsilon_1)R_0^{1+\epsilon_1} = K(\epsilon_2)R_0^{1+\epsilon_2}$. We recall the following definition:

Definition 4.2. — The number ξ is called algebraic number if there is at least one polynomial:

$$(4.8) \quad l(x) = l_0 + l_1x + \cdots + a_mx^m, \quad a_m \neq 0$$

with integral coefficients such that $l(\xi) = 0$, and it is called transcendental if no such polynomial exists.

We consider the equality :

$$(4.9) \quad c_0 = K(\epsilon_1)R_0^{1+\epsilon_1} \implies \frac{c_0}{R_0} = \frac{\mu_{c_0}}{\text{rad}(a_0b_0)} = e^{\frac{1}{\epsilon_1^2}} R_0^{\epsilon_1}$$

i) - We suppose that $\epsilon_1 = \beta_1$ is an algebraic number then $\beta_0 = 1/\epsilon_1^2$ and $\alpha_1 = R_0$ are also algebraic numbers. We obtain:

$$(4.10) \quad \frac{c_0}{R_0} = \frac{\mu_{c_0}}{\text{rad}(a_0b_0)} = e^{\frac{1}{\epsilon_1^2}} R_0^{\epsilon_1} = e^{\beta_0} \cdot \alpha_1^{\beta_1}$$

From the theorem (see theorem 3, page 196 in [1]):

Theorem 4.3. — $e^{\beta_0} \alpha_1^{\beta_1} \dots \alpha_n^{\beta_n}$ is transcendental for any nonzero algebraic numbers $\alpha_1, \dots, \alpha_n, \beta_0, \dots, \beta_n$.

we deduce that the right member $e^{\beta_0} \alpha_1^{\beta_1}$ of (4.10) is transcendental, but the term $\frac{\mu_{c_0}}{\text{rad}(a_0b_0)}$ is an algebraic number, then the contradiction and the case $Y_{c_0}(\epsilon') < 0$ is impossible. It follows $Y_{c_0}(\epsilon') \geq 0$ then the *abc* conjecture is true.

ii) - We suppose that ϵ_1 is transcendental, then $1/(\epsilon_1^2), e^{1/(\epsilon_1^2)}$ and $R_0^{\epsilon_1} = e^{\epsilon_1 \text{Log} R_0}$ are also transcendental, we obtain that c_0/R_0 is transcendental, then the contradiction with c_0/R_0 an algebraic number. It follows that $Y_{c_0}(\epsilon') \geq 0$ and the *abc* conjecture is true.

Then the proof of the *abc* conjecture is finished. As $c < R^2$ is true, we obtain that $\forall \epsilon > 0, \exists K(\epsilon) > 0$, if $c = a + b$ with a, b, c positive integers relatively coprime, then :

$$(4.11) \quad c < K(\epsilon) \cdot \text{rad}^{1+\epsilon}(abc)$$

and the constant $K(\epsilon)$ depends only of ϵ .

Q.E.D

Ouf, end of the mystery!

□

4.3. Conclusion

As $c < R^2$ is true, we have given an elementary proof of the *abc* conjecture. We can announce the important theorem:

Theorem 4.4. — *The abc conjecture is true:
For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then:*

$$(4.12) \quad c < K(\epsilon) \cdot \text{rad}^{1+\epsilon}(abc)$$

where K is a constant depending of ϵ .

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