Tutorial: Electromagnetic Violations of Newtonian Precepts in Basic Experiments

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Abstract Since acceleration is invariant under constant-velocity Galilean transformations, a system moving at constant velocity cannot, in Newtonian physics, exert new forces it doesn't already exert when it is at rest. But a bar magnet moving at nonzero constant velocity exerts a force on electric charges that it doesn't exert when it is at rest (Faraday's Law), and a charge moving at nonzero constant velocity exerts a torque on the needle of a magnetic compass that it doesn't exert when it is at rest (Biot-Savart Law). Thus basic electromagnetic experiments which are feasible in undergraduate or secondary-school physics labs illustrate the need to replace the Galilean transformations. That seems pedagogically much more compelling than the standard practice of merely discussing experiments which use extremely high-precision equipment such as Michelson interferometers. What should replace the Galilean transformations? The key qualification obviously is compatibility with the electromagnetic Laws. Those Laws can be presented as wave equations with source terms, and wherever the source terms are zero, the free waves travel exclusively at the fixed constant speed c. Thus electromagnetic-Law compatible coordinate transformations must preserve speed-c wave travel, which is the central property of the Lorentz transformations, replacing the time-coordinate preservation that is central to the Galilean transformations.

1. Basic electromagnetic experiments versus Galilean-transformation force invariance

The Galilean transformation of time and space coordinates due to travel at constant velocity $\mathbf{v}$ is given by,

$$t' = t \text{ and } \mathbf{r}' = \mathbf{r} - \mathbf{v}t. \tag{1.1a}$$

Therefore the constant-velocity-$\mathbf{v}$ Galilean transformation of velocity $d\mathbf{r}/dt$ simply subtracts $\mathbf{v}$ from $d\mathbf{r}/dt$,

$$d\mathbf{r}'/dt' = d(\mathbf{r} - \mathbf{v}t)/dt = d\mathbf{r}/dt - \mathbf{v}, \tag{1.1b}$$

and this Galilean transformation *leaves acceleration* $d^2\mathbf{r}/dt^2$ *invariant,*

$$d^2\mathbf{r}'/d(t')^2 = d(d\mathbf{r}'/dt')/dt' = d(d\mathbf{r}/dt - \mathbf{v})/dt = d^2\mathbf{r}/dt^2. \tag{1.1c}$$

Therefore, because forces *produce accelerations* in Newtonian physics, a constant-velocity-$\mathbf{v}$ Galilean transformation is *incapable* of introducing *new* forces which were *absent* before that transformation was made.

This Newtonian/Galilean precept notwithstanding, it was observed hundreds of years ago that the magnetic-dipole needle of a compass which is lying sufficiently close to a metal wire is deflected away from its equilibrium position of pointing toward magnetic north upon that wire being connected to a battery. It is surmised that the battery sets the invisible microscopic free electrons in the metal wire into motion with, at least on average, a nonzero constant speed that causes them to produce a magnetic field which is *absent* when the battery *isn't connected* and those free electrons are, at least on average, *at rest.*

Of course surmises about the state of motion of the completely invisible microscopic free electrons in a metal wire are hardly immediately persuasive. Such an experiment would be more compelling if the wire and its invisible microscopic free electrons were replaced by a macroscopic object which has been statically charged. Issues concerning such an approach include getting enough charge on such an object and/or getting its speed high enough to produce a strong enough magnetic field to visibly deflect a magnetic compass needle. More subtle concerns include too-rapid dissipation of the object's charge into the air around it, which might be ameliorated by artificial cooling and dehumidification of that air. The object's charge may also need to be shielded from air currents associated with its speed, or which occur spontaneously in its surroundings.

In 1831 Michael Faraday showed that thrusting a bar magnet lengthwise through the center of a metal wire coil produces a transient current in the coil that is detected by a galvanometer connected to the coil. Thus a bar magnet moving at a nonzero constant velocity in the direction of its magnetic moment apparently produces *a moving azimuthal electric field* that is *absent* when the magnet *is at rest,* but transiently drives the free electrons in the wire coil *around that coil* when the magnet *is moving.* Here one moving object, the bar magnet, is indeed *macroscopic,* so its motion, or the lack thereof, *is plain to see.* The moving azimuthal electric field that its motion supposedly produces is inferred, however, from supposed transient azimuthal motion of invisible microscopic free electrons in the metal wire coil.

It would be more compelling to *instead* verify that transient electric field by *the visible deflection* of a macroscopic entity which has been statically charged. A low-mass charged object which hangs downward

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by a thread directly above the bar magnet's horizontal-line trajectory should be deflected horizontally perpendicular to that trajectory (i.e., azimuthally), first toward one side and then toward the opposite side, as the magnet passes beneath it at constant velocity. As might be expected, this concept comes with its list of caveats and pitfalls. The object's charge must be great enough and its mass low enough to produce a visible deflection. (Of course the stronger the moving bar magnet's electric field is, the greater is the deflection of the charged object; that electric field strength increases with the magnet's velocity and the strength of its dipole moment.) Too-rapid dissipation of the object's charge into the air around it needs to be ameliorated, possibly by cooling and dehumidifying that air. The low-mass hanging charged object would be ultra-sensitive to deflection by stray air currents, and so would need to hang inside an airtight transparent case.

The deflection of the needle of a magnetic compass by nearby *moving* charges (but *not* by such charges *at rest)* became in due course the essence of the Biot-Savart Law of electromagnetic theory, and James Clerk Maxwell distilled Faraday's demonstration that a *moving* magnetic field (but *not* a *stationary* magnetic field) produces a moving electric field into Faraday's Law of electromagnetic theory.

These two Laws of electromagnetism obviously *flatly contradict* the Newtonian/Galilean precept pointed out below Eq. (1.1c) that a constant-velocity-$\mathbf{v}$ Galilean transformation is *incapable* of introducing *new* forces which were *absent* before that transformation was made.

The *existence* of this blatant contradiction of a basic Newtonian precept by basic Laws of electromagnetism that are distilled from experiment somehow *utterly failed to penetrate the consciousness of the physics community* until well after *the null result* of the Michelson-Morley experiment hoisted *another,* this time better appreciated, *red flag over a basic Newtonian precept,* namely the *additivity* of the constant velocity $\mathbf{v}$ of the Galilean transformation to general velocities which is displayed in Eq. (1.1b).

The algebraic progression from the Galilean coordinate transformation of Eq. (1.1a) to the astonishingly simple Galilean velocity and acceleration transformations of Eqs. (1.1b) and (1.1c) respectively *strongly depends on the invariance of time,* $t' = t$*,* which is *postulated* in Eq. (1.1a) of the Galilean transformation.

But if we ponder the Laws of electromagnetism, which are largely distilled from experiment, we don't encounter any clear motivation to postulate the invariance of time, $t' = t$, under constant-velocity transformations of the space and time coordinates. The electromagnetic Laws, however, can be recast into *wave equations with source terms,* where the constant c is *universally* the speed of *the free waves* that can exist anywhere in space and time that the source terms are zero. Since c is the *only* electromagnetic free wave speed which the electromagnetic Laws *permit,* coordinate transformations *must preserve that free wave speed to be compatible with the electromagnetic Laws.* The contention that the free wave speed must always be c is very strongly reinforced by the Michelson-Morley null result, which found no variation in the speed of light signals from sources traveling at varying velocities. We now turn our attention to the details of constructing the Lorentz transformations, which preserve the speed c of free light waves.

2. The Lorentz constant-velocity coordinate transformation which preserves the speed of light

Except for its salient property of preserving the speed of light, the Lorentz transformation should resemble the Galilean transformation as closely as feasible. In order to reduce algebraic complexity during the development of the Lorentz transformation, we initially work in the particular coordinate system whose x-axis points in the direction of the Lorentz transformation's constant velocity $\mathbf{v}$,

$$\mathbf{v} = (|\mathbf{v}|, 0, 0). \tag{2.1a}$$

The Galilean transformation is of course given by Eq. (1.1a),

$$t' = t \text{ and } \mathbf{r}' = \mathbf{r} - \mathbf{v}t.$$

and since,

$$\mathbf{r}' = (x', y', z') \text{ and } \mathbf{r} = (x, y, z), \tag{2.1b}$$

in this particular coordinate system the Galilean transformation is given by,

$$t' = t \text{ and } (x', y', z') = (x, y, z) - (|\mathbf{v}|, 0, 0)t, \tag{2.1c}$$

namely,

$$t' = t, \quad x' = x - |\mathbf{v}|t, \quad y' = y, \quad z' = z. \tag{2.1d}$$

We now pattern the Lorentz transformation in the particular coordinate system where $\mathbf{v} = (|\mathbf{v}|, 0, 0)$ on the Eq. (2.1d) Galilean transformation in that particular coordinate system as follows,

$$t' = \lambda(t - \sigma(x/|\mathbf{v}|)), \quad x' = \gamma(x - \kappa|\mathbf{v}|t), \quad y' = y, \quad z' = z, \tag{2.2a}$$

which we expect to reduce to the Eq. (2.1d) Galilean transformation as $|\mathbf{v}| \to 0$, so,

$$\lambda \to 1,\ \sigma \to 0,\ \gamma \to 1 \text{ and } \kappa \to 1 \text{ as } |\mathbf{v}| \to 0. \tag{2.2b}$$

Furthermore, in order for Eq. (2.2a) to make sense as a velocity-$\mathbf{v}$ coordinate transformation, we expect the *origin* $\mathbf{r}' = \mathbf{0}$ of the *primed* system to correspond to the point $\mathbf{r} = \mathbf{v}t$ of the unprimed system. Therefore if we insert $x' = 0$, $y' = 0$, and $z' = 0$ into Eq. (2.2a), we expect the consequences to be $x = |\mathbf{v}|t$, $y = 0$, and $z = 0$. We see from Eq. (2.2a) that this requirement implies that $\kappa = 1$, which isn't inconsistent with Eq. (2.2b). We therefore now *reiterate* Eqs. (2.2a) *and* (2.2b) *with* $\kappa = 1$,

$$t' = \lambda(t - \sigma(x/|\mathbf{v}|)), \quad x' = \gamma(x - |\mathbf{v}|t), \quad y' = y, \quad z' = z, \tag{2.3a}$$

which we expect to reduce to the Eq. (2.1d) Galilean transformation as $|\mathbf{v}| \to 0$, so,

$$\lambda \to 1,\ \sigma \to 0, \text{ and } \gamma \to 1 \text{ as } |\mathbf{v}| \to 0. \tag{2.3b}$$

Since the Eq. (2.3a) Lorentz transformation transforms *space and time coordinates,* the requirement that it preserve the speed of light is, *in terms of space and time coordinates,* that it preserve *the expanding spherical light surface,* i.e., that,

$$|\mathbf{r}'|^2 - (ct')^2 = |\mathbf{r}|^2 - (ct)^2 \text{ which implies that } (x')^2 + (y')^2 + (z')^2 - (ct')^2 = x^2 + y^2 + z^2 - (ct)^2. \tag{2.3c}$$

Inserting Eq. (2.3a) into the second equality in Eq. (2.3c) produces,

$$\gamma^2(x - |\mathbf{v}|t)^2 + y^2 + z^2 - \lambda^2(ct - \sigma(x/(|\mathbf{v}|/c)))^2 = x^2 + y^2 + z^2 - (ct)^2, \tag{2.3d}$$

which we regroup to read,

$$x^2(\gamma^2 - (\lambda^2\sigma^2/((|\mathbf{v}|/c)^2)) - 1) + 2x(ct)(\lambda^2(\sigma/(|\mathbf{v}|/c)) - \gamma^2(|\mathbf{v}|/c)) + (ct)^2(\gamma^2(|\mathbf{v}|/c)^2 + 1 - \lambda^2) = 0. \tag{2.3e}$$

The three entities x^2, $2x(ct)$ and $(ct)^2$ *are linearly independent,* so *their three coefficients in* Eq. (2.3e) *must individually vanish,* which produces *three equations for the three entities* σ, γ^2 *and* λ^2. The second of these three equations yields the interim result,

$$\sigma = (\gamma^2/\lambda^2)(|\mathbf{v}|/c))^2. \tag{2.3f}$$

Inserting Eq. (2.3f) into the first of the three equations implied by Eq. (2.3e) yields another interim result,

$$\gamma^2 - (\gamma^4/\lambda^2)(|\mathbf{v}|/c))^2 - 1 = 0, \tag{2.3g}$$

into which,

$$\lambda^2 = \gamma^2(|\mathbf{v}|/c)^2 + 1, \tag{2.3h}$$

from the third of the three equations implied by Eq. (2.3e) is inserted to produce,

$$\gamma^2 - ((\gamma^4(|\mathbf{v}|/c))^2)/(\gamma^2(|\mathbf{v}|/c)^2 + 1)) - 1 = 0, \tag{2.3i}$$

which simplifies to $\gamma^2 - \gamma^2(|\mathbf{v}|/c)^2 - 1 = 0$, yielding the result,

$$\gamma^2 = (1/(1 - (|\mathbf{v}|/c)^2)). \tag{2.3j}$$

Insertion of the Eq. (2.3j) result into Eq. (2.3h) yields,

$$\lambda^2 = (1/(1 - (|\mathbf{v}|/c)^2)) = \gamma^2. \tag{2.3k}$$

Insertion of Eq. (2.3k) into Eq. (2.3f) yields,

$$\sigma = (|\mathbf{v}|/c))^2. \tag{2.3l}$$

So taking into account the limits prescribed by Eq. (2.3b), our results are,

$$\lambda = \gamma = \big(1/\sqrt{1 - (|\mathbf{v}|/c)^2}\,\big) \text{ and } \sigma\ = (|\mathbf{v}|/c))^2. \tag{2.3m}$$

Therefore the Eq. (2.3a) Lorentz transformation in the particular coordinate system where $\mathbf{v} = (|\mathbf{v}|, 0, 0)$ is,

$$t' = \gamma(t - (|\mathbf{v}|/c^2)x), \quad x' = \gamma(x - |\mathbf{v}|t), \quad y' = y, \quad z' = z; \quad \gamma \stackrel{\text{def}}{=} \big(1/\sqrt{1 - (|\mathbf{v}|/c)^2}\,\big). \tag{2.3n}$$

We next *transcribe* the Eq. (2.3n) Lorentz transformation *to vector form,* which *automatically rescinds* its specialization to the *particular* coordinate system where $\mathbf{v} = (|\mathbf{v}|, 0, 0)$. To carry out *the transcription of* Eq. (2.3n) *to vector form,* we make *repeated use* of the three facts that $\mathbf{r}' = (x', y', z')$, $\mathbf{r} = (x, y, z)$ and $\mathbf{v} = (|\mathbf{v}|, 0, 0)$ in the *particular* coordinate system where the Lorentz transformation is given by Eq. (2.3n). To *begin* the vector-form transcription of Eq. (2.3n), we note that,

$$(\mathbf{v} \cdot \mathbf{r}) = |\mathbf{v}|x, \tag{2.4a}$$

so the time transformation part $t' = \gamma(t - (|\mathbf{v}|/c^2)x)$ of the Eq. (2.3n) Lorentz transformation is given *in vector form* by,

$$t' = \gamma(t - ((\mathbf{v} \cdot \mathbf{r})/c^2)). \tag{2.4b}$$

We next combine the the Eq. (2.4a) result that $(\mathbf{v} \cdot \mathbf{r}) = |\mathbf{v}|x$ with $\mathbf{v} = (|\mathbf{v}|, 0, 0)$ to obtain,

$$(x, 0, 0) = ((|\mathbf{v}|, 0, 0)(|\mathbf{v}|x)/|\mathbf{v}|^2) = (\mathbf{v}(\mathbf{v} \cdot \mathbf{r})/|\mathbf{v}|^2), \tag{2.4c}$$

where $(\mathbf{v}(\mathbf{v} \cdot \mathbf{r})/|\mathbf{v}|^2)$ is, of course, *in vector form,* and likewise that,

$$(x', 0, 0) = ((|\mathbf{v}|, 0, 0)(|\mathbf{v}|x')/|\mathbf{v}|^2) = (\mathbf{v}(\mathbf{v} \cdot \mathbf{r}')/|\mathbf{v}|^2). \tag{2.4d}$$

We further note that,

$$(0, y, z) = (x, y, z) - (x, 0, 0) = \mathbf{r} - (\mathbf{v}(\mathbf{v} \cdot \mathbf{r})/|\mathbf{v}|^2), \tag{2.4e}$$

and likewise that,

$$(0, y', z') = (x', y', z') - (x', 0, 0) = \mathbf{r}' - (\mathbf{v}(\mathbf{v} \cdot \mathbf{r}')/|\mathbf{v}|^2), \tag{2.4f}$$

Since Eq. (2.3n) implies that $(0, y', z') = (0, y, z)$, it is consequently true that,

$$\mathbf{r}' - (\mathbf{v}(\mathbf{v} \cdot \mathbf{r}')/|\mathbf{v}|^2) = \mathbf{r} - (\mathbf{v}(\mathbf{v} \cdot \mathbf{r})/|\mathbf{v}|^2). \tag{2.4g}$$

Eq. (2.3n) implies that $(x', 0, 0) = (\gamma(x - |\mathbf{v}|t), 0, 0) = \gamma(x, 0, 0) - \gamma(|\mathbf{v}|, 0, 0)t = \gamma(\mathbf{v}(\mathbf{v} \cdot \mathbf{r})/|\mathbf{v}|^2) - \gamma\mathbf{v}t$, where we have applied Eq. (2.4c) and $(|\mathbf{v}|, 0, 0) = \mathbf{v}$. Since in addition, $(x', 0, 0) = (\mathbf{v}(\mathbf{v} \cdot \mathbf{r}')/|\mathbf{v}|^2)$ from Eq. (2.4d), we have obtained from Eq. (2.3n) that,

$$(\mathbf{v}(\mathbf{v} \cdot \mathbf{r}')/|\mathbf{v}|^2) = \gamma(\mathbf{v}(\mathbf{v} \cdot \mathbf{r})/|\mathbf{v}|^2) - \gamma\mathbf{v}t, \tag{2.4h}$$

Now we add Eq. (2.4g) to Eq. (2.4h) to obtain,

$$\mathbf{r}' = \mathbf{r} + (\gamma - 1)(\mathbf{v}(\mathbf{v} \cdot \mathbf{r})/|\mathbf{v}|^2) - \gamma\mathbf{v}t, \tag{2.4i}$$

the space transformation part of the Lorentz transformation *in vector form.* Combining the Eq. (2.4b) time and the Eq. (2.4i) space transformation parts of the Lorentz transformation *in vector form* yields,

$$t' = \gamma(t - ((\mathbf{v} \cdot \mathbf{r})/c^2)) \text{ and } \mathbf{r}' = \mathbf{r} + (\gamma - 1)(\mathbf{v}(\mathbf{v} \cdot \mathbf{r})/|\mathbf{v}|^2) - \gamma\mathbf{v}t; \ \gamma \stackrel{\text{def}}{=} \big(1/\sqrt{1 - (|\mathbf{v}|/c)^2}\,\big). \tag{2.4j}$$

In Eq. (2.4j) the time and space parts of the Lorentz transformation don't have the same dimensions, which would hinder the introduction of the widely applicable (e.g., to electromagnetic physics) dimensionless matrix/tensor form of the Lorentz transformation. To modify Eq. (2.4j) to have dimensional homogeneity, we switch from using the time t to using the entity $x^0 \stackrel{\text{def}}{=} (ct)$, which has *spatial* dimension. For *a more compact notation,* we *also* switch to the dimensionless scaled velocity vector $\mathbf{b} \stackrel{\text{def}}{=} (\mathbf{v}/c)$,

$$(x^0)' = \gamma(x^0 - (\mathbf{b} \cdot \mathbf{r})) \text{ and } \mathbf{r}' = \mathbf{r} + (\gamma - 1)(\mathbf{b}(\mathbf{b} \cdot \mathbf{r})/|\mathbf{b}|^2) - \gamma\mathbf{b}x^0; \ \mathbf{b} \stackrel{\text{def}}{=} (\mathbf{v}/c), \gamma \stackrel{\text{def}}{=} \big(1 - |\mathbf{b}|^2\big)^{-\frac{1}{2}}. \tag{2.4k}$$

We next briefly return to the Galilean transformation to show that its *inverse* has a special property, a property which the inverse of the Lorentz transformation shares. The Galilean transformation is,

$$t' = t \text{ and } \mathbf{r}' = \mathbf{r} - \mathbf{v}t, \tag{2.5a}$$

and its *inverse* is easily calculated,

$$t = t' \text{ and } \mathbf{r} = \mathbf{r}' + \mathbf{v}t'. \tag{2.5b}$$

Therefore the inverse of the Galilean transformation has almost the same form as the Galilean transformation itself *except* that $\mathbf{v} \to -\mathbf{v}$. This property of the Galilean transformation, sometimes called *relativistic reciprocity,* is a logic-driven relationship between the two coordinate systems: *they are equivalent except for the velocity* $-\mathbf{v}$ which an observer at rest in the "moving" system attributes to the "stationary" system. Since relativistic reciprocity is a logic-driven relationship between the two coordinate systems, *it should also hold for the Lorentz transformation.* We next undertake the *lengthy* process of *inverting* the Lorentz transformation displayed in Eq. (2.6a) below *to check whether relativistic reciprocity holds,*

$$(x^0)' = \gamma(x^0 - (\mathbf{b}\cdot\mathbf{r})) \text{ and } \mathbf{r}' = \mathbf{r} + (\gamma-1)(\mathbf{b}(\mathbf{b}\cdot\mathbf{r})/|\mathbf{b}|^2) - \gamma\mathbf{b}x^0;\ \mathbf{b} \stackrel{\text{def}}{=} (\mathbf{v}/c),\ \gamma \stackrel{\text{def}}{=} \left(1-|\mathbf{b}|^2\right)^{-\frac{1}{2}}. \tag{2.6a}$$

We *start* inverting by solving the first equation given in Eq. (2.6a) for x^0, and solving the second one for $\mathbf{r}$,

$$x^0 = ((x^0)'/\gamma) + (\mathbf{b}\cdot\mathbf{r}) \text{ and } \mathbf{r} = \mathbf{r}' - (\gamma-1)(\mathbf{b}(\mathbf{b}\cdot\mathbf{r})/|\mathbf{b}|^2) + \gamma\mathbf{b}x^0;\ \mathbf{b} \stackrel{\text{def}}{=} (\mathbf{v}/c),\ \gamma \stackrel{\text{def}}{=} \left(1-|\mathbf{b}|^2\right)^{-\frac{1}{2}}. \tag{2.6b}$$

Our *next* step will be to take the dot product of the second equation given in Eq. (2.6b) with $\mathbf{b}$, followed by solving the result for $(\mathbf{b}\cdot\mathbf{r})$ in terms of $(\mathbf{b}\cdot\mathbf{r}')$ and x^0. Since the first equation given in Eq. (2.6b) gives x^0 in terms in terms of $(x^0)'$ and $(\mathbf{b}\cdot\mathbf{r})$, our *subsequent* step will be to solve that newly available *pair of equations* for x^0 in terms of $(x^0)'$ and $(\mathbf{b}\cdot\mathbf{r}')$, and also for $(\mathbf{b}\cdot\mathbf{r})$ in terms of $(x^0)'$ and $(\mathbf{b}\cdot\mathbf{r}')$. Insertion of those two results into the second equation given in Eq. (2.6b) then yields $\mathbf{r}$ in terms of $\mathbf{r}'$ and $(x^0)'$, completing the inversion. Our next step of taking the dot product of the second equation given in Eq. (2.6b) with $\mathbf{b}$, followed by solving the result for $(\mathbf{b}\cdot\mathbf{r})$ in terms of $(\mathbf{b}\cdot\mathbf{r}')$ and x^0, produces the second equation given in Eq. (2.6c), which is displayed below,

$$x^0 = ((x^0)'/\gamma) + (\mathbf{b}\cdot\mathbf{r}) \text{ and } (\mathbf{b}\cdot\mathbf{r}) = ((\mathbf{b}\cdot\mathbf{r}')/\gamma) + |\mathbf{b}|^2x^0;\ \mathbf{b} \stackrel{\text{def}}{=} (\mathbf{v}/c),\ \gamma \stackrel{\text{def}}{=} \left(1-|\mathbf{b}|^2\right)^{-\frac{1}{2}}. \tag{2.6c}$$

Solving the pair of equations given in Eq. (2.6c) for x^0 in terms of $(x^0)'$ and $(\mathbf{b}\cdot\mathbf{r}')$, and also for $(\mathbf{b}\cdot\mathbf{r})$ in terms of $(x^0)'$ and $(\mathbf{b}\cdot\mathbf{r}')$ yields,

$$x^0 = \gamma((x^0)' + (\mathbf{b}\cdot\mathbf{r}')) \text{ and } (\mathbf{b}\cdot\mathbf{r}) = \gamma((\mathbf{b}\cdot\mathbf{r}') + |\mathbf{b}|^2(x^0)');\ \mathbf{b} \stackrel{\text{def}}{=} (\mathbf{v}/c),\ \gamma \stackrel{\text{def}}{=} \left(1-|\mathbf{b}|^2\right)^{-\frac{1}{2}}. \tag{2.6d}$$

Comparison of the first equation given in Eq. (2.6d) to the first equation given in Eq. (2.6a) *shows adherence to relativistic reciprocity.* We next insert *both* of the equations given in Eq. (2.6d) into the second equation given in Eq. (2.6b), and then gather terms. The result of doing so is the second equation given in Eq. (2.6e), which is displayed below. The comparison of the second equation given in Eq. (2.6e) below to the second equation given in Eq. (2.6a) above *shows adherence to relativistic reciprocity.* (The first equation given in Eq. (2.6e) below is simply a repetition of the first equation given in Eq. (2.6d) above.) Eq. (2.6e) given below *is the full inverse of* Eq. (2.6a) given above, and *it shows complete adherence to relativistic reciprocity,*

$$x^0 = \gamma((x^0)' + (\mathbf{b}\cdot\mathbf{r}')) \text{ and } \mathbf{r} = \mathbf{r}' + (\gamma-1)(\mathbf{b}(\mathbf{b}\cdot\mathbf{r}')/|\mathbf{b}|^2) + \gamma\mathbf{b}(x^0)';\ \mathbf{b} \stackrel{\text{def}}{=} (\mathbf{v}/c),\ \gamma \stackrel{\text{def}}{=} \left(1-|\mathbf{b}|^2\right)^{-\frac{1}{2}}. \tag{2.6e}$$

We next verify Lorentz-transformation invariance of the expanding spherical light surface, i.e.,

$$|\mathbf{r}'|^2 - ((x^0)')^2 = |\mathbf{r}|^2 - (x^0)^2, \tag{2.7a}$$

where $\mathbf{r}'$ and $(x^0)'$ are given in terms of $\mathbf{r}$, x^0 and $\mathbf{b}$ by the Eq. (2.6a) Lorentz transformation, namely by,

$$\mathbf{r}' = \mathbf{r} + (\gamma-1)(\mathbf{b}(\mathbf{b}\cdot\mathbf{r})/|\mathbf{b}|^2) - \gamma\mathbf{b}x^0 \text{ and } (x^0)' = \gamma(x^0 - (\mathbf{b}\cdot\mathbf{r})). \tag{2.7b}$$

Since *nine terms* initially enter into the expression $|\mathbf{r}'|^2 - ((x^0)')^2$, that expression must be pored over at considerable length to verify that it actually simplifies to $|\mathbf{r}|^2 - (x^0)^2$,

$$\begin{gathered} |\mathbf{r}'|^2 - ((x^0)')^2 = \\ |\mathbf{r} + (\gamma-1)(\mathbf{b}(\mathbf{b}\cdot\mathbf{r})/|\mathbf{b}|^2) - \gamma\mathbf{b}x^0|^2 - (\gamma(x^0 - (\mathbf{b}\cdot\mathbf{r})))^2 = \\ |\mathbf{r}|^2 - (x^0)^2\left[\gamma^2(1-|\mathbf{b}|^2)\right] + 2(x^0)(\mathbf{b}\cdot\mathbf{r})\left[\gamma^2 - \gamma(\gamma-1) - \gamma\right] + ((\mathbf{b}\cdot\mathbf{r})/|\mathbf{b}|)^2\left[(\gamma-1)^2 + 2(\gamma-1) - \gamma^2|\mathbf{b}|^2\right] = \\ |\mathbf{r}|^2 - (x^0)^2, \end{gathered} \tag{2.7c}$$

because $|\mathbf{b}|^2 = 1 - (1/\gamma^2)$, so the expanding spherical light surface is Lorentz-transformation invariant.

The Eq. (2.7b) *vector form* of the Lorentz transformation is far less widely applicable than is a *dimensionless* 4×4 *symmetric matrix* (or second-rank tensor) *which resides within it.* That dimensionless Lorentz-transformation matrix allows momentum and energy, for example, as well as space and time coordinates, to be Lorentz-transformed. In this tutorial we wish to set time-independent electromagnetic potentials and fields, such as those of stationary charges and magnetic dipoles, into motion at constant velocity, for which the dimensionless Lorentz-transformation matrix *is indispensable.*

To carry out the *extraction* of the Lorentz-transformation's *dimensionless matrix elements* from the Lorentz transformation's Eq. (2.7b) vector form, we write $\mathbf{r}' = ((x^1)', (x^2)', (x^3)')$, $\mathbf{r} = (x^1, x^2, x^3)$ and $\mathbf{b} = (b^1, b^2, b^3)$. The Lorentz transformation's dimensionless matrix elements then *are the coefficients of* x^0, x^1, x^2 and x^3 in that transformation's Eq. (2.7b) vector form. For example, the homogeneous linear relation $(x^0)' = \gamma(x^0 - (\mathbf{b}\cdot\mathbf{r}))$ from the Eq. (2.7b) vector form of the Lorentz transformation is parsed as follows in order to obtain a subset of the Lorentz transformation's dimensionless matrix elements,

$$(x^0)' = \gamma(x^0 - (\mathbf{b}\cdot\mathbf{r})) = (\gamma)x^0 + \textstyle\sum_{i=1}^3 (-\gamma b^i)x^i = \Lambda^{00}(\mathbf{b})x^0 + \sum_{i=1}^3 \Lambda^{0i}(\mathbf{b})x^i, \tag{2.8a}$$

from which we read off the following *subset* of the sixteen dimensionless matrix elements of the Lorentz-transformation matrix (or second-rank tensor) $\Lambda^{\mu\nu}(\mathbf{b})$, $\mu = 0,1,2,3$ and $\nu = 0,1,2,3$,

$$\Lambda^{00}(\mathbf{b}) = \gamma; \quad \Lambda^{0i}(\mathbf{b}) = -\gamma b^i,\ i = 1,2,3. \tag{2.8b}$$

Passing now to the *remaining* homogeneous linear relation $\mathbf{r}' = \mathbf{r} + (\gamma - 1)(\mathbf{b}(\mathbf{b}\cdot\mathbf{r})/|\mathbf{b}|^2) - \gamma\mathbf{b}x^0$ of the Eq. (2.7b) vector form of the Lorentz transformation, the *three components* of this *vector relation* are,

$$(x^i)' = x^i + (\gamma - 1)(b^i(\mathbf{b}\cdot\mathbf{r})/|\mathbf{b}|^2) - \gamma b^i x^0,\ i = 1,2,3, \tag{2.8c}$$

from which we want to extract the dimensionless coefficients of x^0, x^1, x^2 and x^3. To do that we parse Eq. (2.8c) as follows,

$$(x^i)' = \textstyle\sum_{j=1}^3 \big(\delta^{ij} + (\gamma - 1)(b^i b^j/|\mathbf{b}|^2)\big)x^j + (-\gamma b^i)x^0 = \sum_{j=1}^3 \Lambda^{ij}(\mathbf{b})x^j + \Lambda^{i0}(\mathbf{b})x^0,\ i = 1,2,3, \tag{2.8d}$$

from which we read off the following *subset* of the sixteen dimensionless matrix elements of the Lorentz-transformation matrix (or second-rank tensor) $\Lambda^{\mu\nu}(\mathbf{b})$, $\mu = 0,1,2,3$ and $\nu = 0,1,2,3$,

$$\Lambda^{ij}(\mathbf{b}) = \big(\delta^{ij} + (\gamma - 1)(b^i b^j/|\mathbf{b}|^2)\big),\ i = 1,2,3 \text{ and } j = 1,2,3; \quad \Lambda^{i0}(\mathbf{b}) = -\gamma b^i,\ i = 1,2,3. \tag{2.8e}$$

Combining the results of Eq. (2.8b) with those of Eq. (2.8e) provides all sixteen dimensionless matrix elements of the Lorentz-transformation matrix (or second-rank tensor) $\Lambda^{\mu\nu}(\mathbf{b})$, $\mu = 0,1,2,3$ and $\nu = 0,1,2,3$,

$$\Lambda^{00}(\mathbf{b}) = \gamma; \quad \Lambda^{0i}(\mathbf{b}) = \Lambda^{i0}(\mathbf{b}) = -\gamma b^i,\ i = 1,2,3;$$
$$\Lambda^{ij}(\mathbf{b}) = \big(\delta^{ij} + (\gamma - 1)(b^i b^j/|\mathbf{b}|^2)\big),\ i = 1,2,3 \text{ and } j = 1,2,3. \tag{2.9a}$$

It is apparent by inspection of Eq. (2.9a) that the Lorentz-transformation matrix is *symmetric,*

$$\Lambda^{\mu\nu}(\mathbf{b}) = \Lambda^{\nu\mu}(\mathbf{b}),\ \mu = 0,1,2,3 \text{ and } \nu = 0,1,2,3, \tag{2.9b}$$

and the parity properties of its matrix elements under reversal of the sign of $\mathbf{b}$ are,

$$\Lambda^{00}(-\mathbf{b}) = \Lambda^{00}(\mathbf{b}); \quad \Lambda^{0i}(-\mathbf{b}) = -\Lambda^{0i}(\mathbf{b}),\ i = 1,2,3; \quad \Lambda^{ij}(-\mathbf{b}) = \Lambda^{ij}(\mathbf{b}),\ i = 1,2,3 \text{ and } j = 1,2,3. \tag{2.9c}$$

The the Eq. (2.9c) *parity properties* of the Lorentz-transformation matrix elements will, further on, help us to verify relativistic reciprocity, i.e., that the matrix $\Lambda(-\mathbf{b})$ is the *inverse* of the matrix $\Lambda(\mathbf{b})$. But we first must *abstract* from preservation of the expanding spherical light surface a property of the $\Lambda^{\mu\nu}(\mathbf{b})$ *themselves,* in which the coordinates *aren't present.* With the coordinates *present,* that preservation of course is,

$$((x^0)')^2 - \textstyle\sum_{k=1}^3((x^k)')^2 = (x^0)^2 - \sum_{k=1}^3 (x^k)^2, \text{ where } (x^\mu)' = \sum_{\alpha=0}^3 \Lambda^{\mu\alpha}(\mathbf{b})x^\alpha = \Lambda^{\mu\alpha}(\mathbf{b})x^\alpha. \tag{2.9d}$$

The final equality of Eq. (2.9d) communicates that *repeated Greek indices are to be understood as being summed over.* Eq. (2.9d) therefore can be written as,

$$\big(\Lambda^{0\alpha}(\mathbf{b})\Lambda^{0\beta}(\mathbf{b}) - \textstyle\sum_{k=1}^3 \Lambda^{k\alpha}(\mathbf{b})\Lambda^{k\beta}(\mathbf{b})\big)x^\alpha x^\beta = (x^0)^2 - \sum_{k=1}^3 (x^k)^2. \tag{2.9e}$$

At this point it is tremendously useful to write,

$$(x^0)^2 - \sum_{k=1}^{3}(x^k)^2 = \big(\eta^{\alpha\beta}\big)x^\alpha x^\beta, \text{ where } \eta^{00} = 1,\ \eta^{kk} = -1 \text{ for } k = 1,2,3 \text{ and } \eta^{\alpha\beta} = 0 \text{ for } \alpha \neq \beta. \quad (2.9f)$$

Since $\eta^{\alpha\beta}$ and $\big(\Lambda^{0\alpha}(\mathbf{b})\Lambda^{0\beta}(\mathbf{b}) - \sum_{k=1}^{3}\Lambda^{k\alpha}(\mathbf{b})\Lambda^{k\beta}(\mathbf{b})\big)$ above are *symmetric* under $\alpha \rightleftharpoons \beta$ *exchange,* the linear independence of ten of the sixteen coordinate products $x^\alpha x^\beta$ ensures the validity of the equations,

$$\big(\Lambda^{\alpha 0}(\mathbf{b})\Lambda^{0\beta}(\mathbf{b}) - \sum_{k=1}^{3}\Lambda^{\alpha k}(\mathbf{b})\Lambda^{k\beta}(\mathbf{b})\big) = \eta^{\alpha\beta}. \quad (2.9g)$$

In Eq. (2.9g) the $\Lambda^{\kappa\alpha}(\mathbf{b})$ of Eq. (2.9e) are replaced by $\Lambda^{\alpha\kappa}(\mathbf{b})$, which is permitted by the symmetry of the $\Lambda^{\mu\nu}(\mathbf{b})$ indices (see Eq. (2.9b)). As required, Eq. (2.9g) abstracts a property of the $\Lambda^{\mu\nu}(\mathbf{b})$ *themselves* from invariance of the expanding spherical light surface; the coordinates *aren't present.* The $\Lambda^{\mu\nu}(\mathbf{b})$ given by Eq. (2.9a) are readily (if a bit tediously) *verified to actually satisfy* Eq. (2.9g). In light of its form, the left side of Eq. (2.9g) can be more *tidily* presented as $\Lambda^{\alpha\kappa}(\mathbf{b})\eta^{\kappa\lambda}\Lambda^{\lambda\beta}(\mathbf{b})$. The consequent *matrix* equation,

$$\Lambda(\mathbf{b})\eta\Lambda(\mathbf{b}) = \eta, \quad (2.9h)$$

is in fact *the standard way* Eq. (2.9g) is presented—together with pointing out that therefore Lorentz transformations are homogeneous linear 4×4 mappings *which preserve* η. The dryly abstract "preservation of η" extends the reach of Lorentz-transformation invariance *far beyond* the expanding spherical light surface of the space and time coordinates.

The more detailed Eq. (2.9g) version of the extremely tidy Eq. (2.9h) is better suited to carrying out calculations, however. We next use Eq. (2.9g), together with the simple parity properties of the transformation matrix elements $\Lambda^{\mu\nu}(\mathbf{b})$ noted in Eq. (2.9c), to show that $\Lambda(-\mathbf{b})$ is the *inverse* of $\Lambda(\mathbf{b})$, i.e., that $\Lambda(\mathbf{b})$ *adheres to relativistic reciprocity.*

We begin by using those parity properties, and also the properties of $\eta^{\alpha\beta}$, to show that for specific subsets of the four possible values which its particular index α can assume, Eq. (2.9g) is equivalent to,

$$\big(\Lambda^{\alpha 0}(-\mathbf{b})\Lambda^{0\beta}(\mathbf{b}) + \sum_{k=1}^{3}\Lambda^{\alpha k}(-\mathbf{b})\Lambda^{k\beta}(\mathbf{b})\big) = \delta^{\alpha\beta}. \quad (2.9i)$$

We now show specifically that Eq. (2.9g) is equivalent to Eq. (2.9i) when α has the one of the values i, where $i = 1,2,3$. In those cases, Eq. (2.9g) reads,

$$\big(\Lambda^{i0}(\mathbf{b})\Lambda^{0\beta}(\mathbf{b}) - \sum_{k=1}^{3}\Lambda^{ik}(\mathbf{b})\Lambda^{k\beta}(\mathbf{b})\big) = \eta^{i\beta}.$$

It is easily verified that $\eta^{i\beta} = -\delta^{i\beta}$. Also, according to the Eq. (2.9c) parity rules, $\Lambda^{i0}(\mathbf{b}) = -\Lambda^{i0}(-\mathbf{b})$, whereas $\Lambda^{ik}(\mathbf{b}) = \Lambda^{ik}(-\mathbf{b})$. Therefore, in the cases that $\alpha = i$, where $i = 1,2,3$, Eq. (2.9g) is equivalent to,

$$-\big(\Lambda^{i0}(-\mathbf{b})\Lambda^{0\beta}(\mathbf{b}) + \sum_{k=1}^{3}\Lambda^{ik}(-\mathbf{b})\Lambda^{k\beta}(\mathbf{b})\big) = -\delta^{i\beta},$$

which is equivalent to,

$$\big(\Lambda^{i0}(-\mathbf{b})\Lambda^{0\beta}(\mathbf{b}) + \sum_{k=1}^{3}\Lambda^{ik}(-\mathbf{b})\Lambda^{k\beta}(\mathbf{b})\big) = \delta^{i\beta},$$

which is precisely Eq. (2.9i) when $\alpha = i = 1,2,3$.

Passing next to the case that $\alpha = 0$, Eq. (2.9g) reads,

$$\big(\Lambda^{00}(\mathbf{b})\Lambda^{0\beta}(\mathbf{b}) - \sum_{k=1}^{3}\Lambda^{0k}(\mathbf{b})\Lambda^{k\beta}(\mathbf{b})\big) = \eta^{0\beta}.$$

Here we note that $\eta^{0\beta} = \delta^{0\beta}$, $\Lambda^{00}(\mathbf{b}) = \Lambda^{00}(-\mathbf{b})$ and $\Lambda^{0k}(\mathbf{b}) = -\Lambda^{0k}(-\mathbf{b})$. Therefore, when $\alpha = 0$, Eq. (2.9g) is equivalent to,

$$\big(\Lambda^{00}(-\mathbf{b})\Lambda^{0\beta}(\mathbf{b}) + \sum_{k=1}^{3}\Lambda^{0k}(-\mathbf{b})\Lambda^{k\beta}(\mathbf{b})\big) = \delta^{0\beta}.$$

which is precisely Eq. (2.9i) when $\alpha = 0$.

Thus Eq. (2.9g) is equivalent to Eq. (2.9i) *for all four possible values of* α*,* i.e.,

$$\big(\Lambda^{\alpha 0}(-\mathbf{b})\Lambda^{0\beta}(\mathbf{b}) + \sum_{k=1}^{3}\Lambda^{\alpha k}(-\mathbf{b})\Lambda^{k\beta}(\mathbf{b})\big) = \delta^{\alpha\beta},$$

which implies that,

$$\big(\Lambda^{\alpha\kappa}(-\mathbf{b})\Lambda^{\kappa\beta}(\mathbf{b})\big) = \delta^{\alpha\beta},$$

which in turn implies that,

$$\big(\Lambda(-\mathbf{b})\Lambda(\mathbf{b})\big)^{\alpha\beta} = \mathrm{I}^{\alpha\beta},$$

so,

$$\Lambda(-\mathbf{b})\Lambda(\mathbf{b}) = \mathrm{I},$$

and therefore,

$$\Lambda(-\mathbf{b}) = (\Lambda(\mathbf{b}))^{-1}. \tag{2.9j}$$

Eq. (2.9j) shows that the Lorentz transformation matrix given by Eq. (2.9a) adheres to relativistic reciprocity.

Since our interest in Lorentz transformation is this tutorial is its application to electromagnetic fields, and since those fields are governed by differential equations, we now work out the Lorentz transformations of *the two differential operators* that enter into the differential equations which govern the scalar and vector electromagnetic potentials. The *fundamental differential operator* of the electromagnetic differential equations is the space-time gradient,

$$(\partial/\partial x^{\mu}) = ((\partial/\partial x^{0}), \nabla_{\mathbf{r}}), \tag{2.10a}$$

which is *a critical ingredient* of "the equation of continuity" that enforces local charge conservation, and which *underlies* the d'Alembertian differential operator,

$$(\partial/\partial x^{\mu})\eta^{\mu\nu}(\partial/\partial x^{\nu}) = \left(\partial^{2}/\partial(x^{0})^{2} - \nabla_{\mathbf{r}}^{2}\right), \tag{2.10b}$$

that is the essence *of the crucially-important* electromagnetic wave equations.

The Lorentz transformation of the space-time gradient $(\partial/\partial x^{\mu})$ is of course $(\partial/\partial(x^{\mu})')$, where $(x^{\mu})' = \Lambda^{\mu\beta}(\mathbf{b})x^{\beta}$. We next wish to work out the *coefficients* of this Lorentz transformation's *representation as a linear combination of its untransformed components* $(\partial/\partial x^{\alpha})$. Since the space-time gradient is a simple first-order differential operator, we can *formally* obtain *that particular linear representation of its Lorentz transformation* $(\partial/\partial(x^{\mu})')$ *by an appropriate application of the calculus chain rule,* specifically,

$$(\partial/\partial(x^{\mu})') = (\partial x^{\alpha}/\partial(x^{\mu})')(\partial/\partial x^{\alpha}). \tag{2.11a}$$

However, the *coefficients* $(\partial x^{\alpha}/\partial(x^{\mu})')$ obtained in Eq. (2.11a) can *only* be evaluated after working out the *untransformed* coordinates x^{α} *in terms of the transformed ones* $(x^{\sigma})'$, $\sigma = 0, 1, 2, 3$, which *compels* application of the *inverse* $(\Lambda(\mathbf{b}))^{-1}$ of the Lorentz-transformation matrix $\Lambda(\mathbf{b})$. In detail, the x^{α} are worked out as linear combinations of the $(x^{\sigma})'$, $\sigma = 0, 1, 2, 3$, by applying the twin facts that,

$$(x^{\sigma})' = \Lambda^{\sigma\beta}(\mathbf{b})x^{\beta} \text{ and } \left((\Lambda(\mathbf{b}))^{-1}\right)^{\alpha\sigma}\Lambda^{\sigma\beta}(\mathbf{b})x^{\beta} = \delta^{\alpha\beta}x^{\beta} = x^{\alpha}, \tag{2.11b}$$

which together imply that,

$$x^{\alpha} = \left((\Lambda(\mathbf{b}))^{-1}\right)^{\alpha\sigma}(x^{\sigma})'. \tag{2.11c}$$

Inserting the Eq. (2.11c) result into the first factor on the right side of Eq. (2.11a) then yields,

$$(\partial/\partial(x^{\mu})') = \left(\partial\left[\left((\Lambda(\mathbf{b}))^{-1}\right)^{\alpha\sigma}(x^{\sigma})'\right]/\partial(x^{\mu})'\right)(\partial/\partial x^{\alpha}) = \left((\Lambda(\mathbf{b}))^{-1}\right)^{\alpha\mu}(\partial/\partial x^{\alpha}), \tag{2.11d}$$

Since from the Eq. (2.9j) principle of relativistic reciprocity we know that $(\Lambda(\mathbf{b}))^{-1} = \Lambda(-\mathbf{b})$, and since the matrix elements of $\Lambda(\mathbf{b})$ are *symmetric* (see Eq. (2.9b)), the *final form* of the Eq. (2.11d) Lorentz transformation of the space-time *gradient* in terms of the untransformed components of that gradient is,

$$(\partial/\partial(x^{\mu})') = \Lambda^{\mu\alpha}(-\mathbf{b})(\partial/\partial x^{\alpha}), \tag{2.11e}$$

whose *form* only differs from the *form* of the Lorentz transformation of the space-time *coordinates,* which is $(x^{\mu})' = \Lambda^{\mu\alpha}(\mathbf{b})x^{\alpha}$, *by the reversal of the sign of* $\mathbf{b}$.

We now turn to the Lorentz transformation of the Eq. (2.10b) d'Alembertian differential operator $(\partial/\partial x^{\mu})\eta^{\mu\nu}(\partial/\partial x^{\nu})$, which is transparently underlain by the Eq. (2.10a) space-time gradient $(\partial/\partial x^{\mu})$. We are in a position to readily evaluate its Lorentz transformation $(\partial/\partial(x^{\mu})')\eta^{\mu\nu}(\partial/\partial(x^{\nu})')$, where $(x^{\mu})' = \Lambda^{\mu\alpha}(\mathbf{b})x^{\alpha}$ and $(x^{\nu})' = \Lambda^{\nu\alpha}(\mathbf{b})x^{\alpha}$, by *first* applying the space-time gradient Lorentz-transformation result of Eq. (2.11e), *followed* by applying the Lorentz-transformation invariance of $\eta^{\mu\nu}$, i.e., that $\Lambda(\mathbf{b})\eta\Lambda(\mathbf{b}) = \eta$ which is set out in Eq. (2.9h). Thus,

$$(\partial/\partial(x^{\mu})')\eta^{\mu\nu}(\partial/\partial(x^{\nu})') = \Lambda^{\mu\alpha}(-\mathbf{b})(\partial/\partial x^{\alpha})\eta^{\mu\nu}\Lambda^{\nu\beta}(-\mathbf{b})(\partial/\partial x^{\beta}) =$$
$$(\partial/\partial x^{\alpha})\Lambda^{\alpha\mu}(-\mathbf{b})\eta^{\mu\nu}\Lambda^{\nu\beta}(-\mathbf{b})(\partial/\partial x^{\beta}), \tag{2.12a}$$

where we have applied Eq. (2.11e) to the two Lorentz-transformed space-time gradient parts of the Lorentz-transformed d'Alembertian differential operator, rearranged the order of some of the multiplicative factors, and utilized the index symmetry of $\Lambda^{\mu\alpha}(-\mathbf{b})$. We next apply Eq. (2.9h) to collapse the group $\Lambda^{\alpha\mu}(-\mathbf{b})\eta^{\mu\nu}\Lambda^{\nu\beta}(-\mathbf{b})$ in the center of the final expression of Eq. (2.12a) to $\eta^{\alpha\beta}$, which implies that,

$$(\partial/\partial(x^\mu)')\eta^{\mu\nu}(\partial/\partial(x^\nu)') = (\partial/\partial x^\alpha)\eta^{\alpha\beta}(\partial/\partial x^\beta), \tag{2.12b}$$

so the Lorentz-transformed d'Alembertian differential operator is *itself,* i.e., the d'Alembertian differential operator is *invariant* under Lorentz transformations, just as is the expanding spherical light surface.

3. Setting time-independent electromagnetic fields into motion at constant velocity

In this tutorial we are interested in conceivable undergraduate or high school physics lab electromagnetic experiments which violate Newtonian precepts. Galilean constant-velocity transformations leave accelerations *invariant,* so *additional* forces *cannot* be generated by constant-velocity transformations in Newtonian physics. That consequence of Newtonian physics is resoundingly *refuted* by *electromagnetic* experiments: a moving, but *not a stationary,* charge is accompanied by a moving azimuthal *magnetic* field; a moving, but *not a stationary,* dipole magnet is accompanied by a moving *electric* field.

It has been known for hundreds of years that a metal wire will deflect the needle of a magnetic compass which is sufficiently close to the wire when that wire is connected to a battery, but it would be more compelling if a moving macroscopic object which had been statically charged was substituted for the invisible microscopic free electrons in the metal wire which the battery caused to move. In 1831 Michael Faraday showed that a bar magnet moving in the direction of its dipole moment is accompanied by a moving azimuthal electric field which can transiently propel the free electrons in a metal wire coil around that coil. The presence of the moving electric field when the magnet is moving (and its absence when the magnet is at rest) would be more compelling if that electric field transiently deflected a low-mass macroscopic object which had been statically charged and was hanging downward by a thread immediately above the magnet's horizontal trajectory, instead of the moving magnet transiently deflecting the invisible microscopic free electrons in a metal wire coil.

The physically incorrect invariance of acceleration in constant-velocity Galilean transformations is very strongly related to the Galilean/Newtonian postulate that constant-velocity transformations *leave time invariant.* Time invariance and acceleration invariance are *both* notably *absent* from Lorentz constant-velocity transformations, which are guided by the equations of electromagnetism, wherein free waves which *always* travel *at the completely fixed speed* c are utterly ubiquitous.

In this section we work out and discuss the magnetic field of a point charge that moves at an "everyday speed" ($|\mathbf{v}| \ll c$), and the electric field of a similarly moving point magnetic dipole. But on the way to those results we work out the equations of electromagnetism in four-potential form, and develop the Lorentz transformation of that four-potential.

We begin with a quick review of the Laws which govern the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$,

$$\text{Coulomb's Law: } \nabla_\mathbf{r}\cdot\mathbf{E} = 4\pi d^0, \quad \text{Faraday's Law: } \nabla_\mathbf{r}\times\mathbf{E} + \partial\mathbf{B}/\partial x^0 = 0,$$

$$\text{Gauss' Law: } \nabla_\mathbf{r}\cdot\mathbf{B} = 0, \quad \text{Biot-Savart/Maxwell Law: } \nabla_\mathbf{r}\times\mathbf{B} - \partial\mathbf{E}/\partial x^0 = 4\pi\mathbf{d}, \tag{3.1}$$

where $d^0 \stackrel{\text{def}}{=} \rho$, the charge density, $\mathbf{d} \stackrel{\text{def}}{=} (\mathbf{j}/c)$, the current density divided by c, and, of course, $x^0 \stackrel{\text{def}}{=} ct$.

When the magnetic field $\mathbf{B}$ and electric field $\mathbf{E}$ are attributed as follows to a four-potential $A^\mu = (A^0, \mathbf{A})$,

$$\mathbf{B} = \nabla_\mathbf{r}\times\mathbf{A} \quad\text{and}\quad \mathbf{E} = -\nabla_\mathbf{r}A^0 - \partial\mathbf{A}/\partial x^0, \tag{3.2}$$

then Gauss' Law and Faraday's Law are satisfied. Coulomb's Law and the Biot-Savart/Maxwell Law become,

$$-\nabla_\mathbf{r}^2 A^0 - \partial(\nabla_\mathbf{r}\cdot\mathbf{A})/\partial x^0 = 4\pi d^0 \quad\text{and}\quad \nabla_\mathbf{r}(\nabla_\mathbf{r}\cdot\mathbf{A}) - \nabla_\mathbf{r}^2\mathbf{A} + \nabla_\mathbf{r}(\partial A^0/\partial x^0) + \partial^2\mathbf{A}/\partial(x^0)^2 = 4\pi\mathbf{d}. \tag{3.3a}$$

At this point it is important to take note of the fact that Eq. (3.2) *doesn't pin down* $A^\mu = (A^0, \mathbf{A})$ *uniquely.* It is readily verified that, given *an arbitrary scalar function* $X(x^0, \mathbf{r})$ (which has the appropriate dimension), then if $(A^0, \mathbf{A})$ satisfies Eq. (3.2), *so does* $(A^0 - \partial X/\partial x^0,\ \mathbf{A} + \nabla_\mathbf{r}X)$*,* the "gauge indeterminism" of $(A^0, \mathbf{A})$.

We therefore now take advantage of this scalar-function degree of "gauge freedom" in $(A^0, \mathbf{A})$ *to require that* $(A^0, \mathbf{A})$ *satisfies the following scalar equation,*

$$\nabla_{\mathbf{r}} \cdot \mathbf{A} = -\partial A^0/\partial x^0, \tag{3.3b}$$

which is known as the "Lorentz condition". The consequence of the Lorentz condition for the equation pair of Eq. (3.3a) is that they *simplify* to now read,

$$\partial^2 A^0/\partial(x^0)^2 - \nabla_{\mathbf{r}}^2 A^0 = 4\pi d^0 \quad \text{and} \quad \partial^2 \mathbf{A}/\partial(x^0)^2 - \nabla_{\mathbf{r}}^2 \mathbf{A} = 4\pi \mathbf{d}. \tag{3.3c}$$

We see that in Eq. (3.3c) the d'Alembertian differential operator $\left(\partial^2/\partial(x^0)^2 - \nabla_{\mathbf{r}}^2\right)$ acts on both parts of the electromagnetic four-potential $A^\mu = (A^0, \mathbf{A})$, so it will be convenient to denote the charge density and current density divided by c on the right sides of the two equations in Eq. (3.3c) as $d^\mu = (d^0, \mathbf{d})$. The Lorentz condition of Eq. (3.3b) can be written in terms of $A^\mu = (A^0, \mathbf{A})$ and the space-time gradient differential operator $(\partial/\partial x^\mu) = ((\partial/\partial x^0), \nabla_{\mathbf{r}})$, which was introduced in Eq. (2.10a), as $(\partial/\partial x^\mu)A^\mu(x^\sigma) = 0$. Thus the Laws of electromagnetism are expressed in four-potential form as follows,

$$\left((\partial/\partial x^\alpha)\eta^{\alpha\beta}(\partial/\partial x^\beta)\right)A^\mu(x^\sigma) = 4\pi d^\mu(x^\sigma) \quad \text{and} \quad (\partial/\partial x^\mu)A^\mu(x^\sigma) = 0. \tag{3.4}$$

If Eq. (3.4) has been solved for $A^\mu = (A^0, \mathbf{A})$, the $\mathbf{E}$ and $\mathbf{B}$ fields can then be obtained by using Eq. (3.2).

Applying the operator $(\partial/\partial x^\mu)$ to both sides of the first equation in Eq. (3.4) and summing over the index μ produces zero on the left side *because of the second equation in* Eq. (3.4), i.e., because of the Lorentz condition. Therefore this procedure *must produce zero on the right side as well,* i.e.,

$$(\partial/\partial x^\mu)d^\mu(x^\sigma) = 0. \tag{3.5a}$$

Eq. (3.5a) is called the charge density/current "equation of continuity"; *it enforces local charge conservation.*

It is self-evident that any constant-velocity Lorentz transformation of a charge density/current entity $d^\mu(x^\sigma)$ is *itself* a charge density/current entity, and *as such* it is *obliged* to adhere to the equation of continuity *in order to enforce local charge conservation.* Therefore we would expect the Lorentz transformation $(d^\mu)'(x^\sigma)$ of the charge density/current entity $d^\mu(x^\sigma)$ to satisfy,

$$(\partial/\partial x^\alpha)\left((d^\alpha)'(x^\sigma)\right) = 0. \tag{3.5b}$$

Another property that we would expect of the Lorentz transformation $(d^\alpha)'(x^\sigma)$ of the charge density/current entity $d^\alpha(x^\sigma)$ is that $(d^\alpha)'(x^\sigma)$ would be a homogeneous linear transformation of the four components of $d^\alpha(x^\sigma)$, but *with each of those components evaluated at the Lorentz-transformed coordinates,* i.e.,

$$(d^\alpha)'(x^\sigma) = \Omega^{\alpha\mu} d^\mu((x^\sigma)'), \text{ where } (x^\sigma)' = \Lambda^{\sigma\beta}(\mathbf{b})x^\beta. \tag{3.5c}$$

Putting Eq. (3.5c) into Eq. (3.5b) produces,

$$(\partial/\partial x^\alpha)(\Omega^{\alpha\mu} d^\mu((x^\sigma)')) = 0, \text{ where } (x^\sigma)' = \Lambda^{\sigma\beta}(\mathbf{b})x^\beta. \tag{3.5d}$$

The question now is, *what matrix* $\Omega^{\alpha\mu}$ *ensures that* Eq. (3.5d) *holds,* given that the charge density/current entity $d^\mu(x^\sigma)$ is such that Eq. (3.5a) holds? A radical shortcut to answering that question turns out to exist, namely the systematic replacement of all occurrences of the independent variable x^σ in Eq. (3.5a) by its Lorentz-transformed counterpart $(x^\sigma)' = \Lambda^{\sigma\beta}(\mathbf{b})x^\beta$, which changes Eq. (3.5a) to,

$$(\partial/\partial(x^\mu)')d^\mu((x^\sigma)') = 0, \text{ where } (x^\sigma)' = \Lambda^{\sigma\beta}(\mathbf{b})x^\beta. \tag{3.5e}$$

Using Eq. (2.11e), we replace the differential operator $(\partial/\partial(x^\mu)')$ in Eq. (3.5e) by $\Lambda^{\mu\alpha}(-\mathbf{b})(\partial/\partial x^\alpha)$ and then rearrange the order of factors to obtain,

$$(\partial/\partial x^\alpha)(\Lambda^{\mu\alpha}(-\mathbf{b})\, d^\mu((x^\sigma)')) = 0, \text{ where } (x^\sigma)' = \Lambda^{\sigma\beta}(\mathbf{b})x^\beta. \tag{3.5f}$$

Comparison of Eq. (3.5d) to the result obtained in Eq. (3.5f) shows that,

$$\Omega^{\alpha\mu} = \Lambda^{\mu\alpha}(-\mathbf{b}) = \Lambda^{\alpha\mu}(-\mathbf{b}), \tag{3.5g}$$

where $\Lambda^{\alpha\mu}(-\mathbf{b})$ is, of course, *symmetric* in its index pair $\alpha\mu$.

We now insert the result given above by Eq. (3.5g) into Eqs. (3.5c) to obtain the Lorentz transformation $(d^\mu)'(x^\sigma)$ of the charge density/current entity $d^\mu(x^\sigma)$,

$$(d^\mu)'(x^\sigma) = \Lambda^{\mu\nu}(-\mathbf{b})\, d^\nu((x^\sigma)'), \text{ where } (x^\sigma)' = \Lambda^{\sigma\beta}(\mathbf{b})x^\beta. \tag{3.6}$$

In Eq. (3.4) the electromagnetic four-potential $A^\mu(x^\sigma)$ is linked to its charge density/current source $d^\mu(x^\sigma)$ *by only the d'Alembertian differential operator* $((\partial/\partial x^\alpha)\eta^{\alpha\beta}(\partial/\partial x^\beta))$, which is Lorentz-transformation *invariant* (see Eq. (2.12b)). Therefore, the Lorentz-transformation *character* of $A^\mu(x^\sigma)$ *must be identical to that of* $d^\mu(x^\sigma)$*,* i.e.,

$$(A^\mu)'(x^\sigma) = \Lambda^{\mu\nu}(-\mathbf{b})A^\nu((x^\sigma)'), \text{ where } (x^\sigma)' = \Lambda^{\sigma\beta}(\mathbf{b})x^\beta. \tag{3.7}$$

A consequence of the Eq. (3.7) Lorentz-transformation result for $A^\mu(x^\sigma)$ is that, just as all Lorentz transformations of the charge density/current $d^\mu(x^\sigma)$ *satisfy the equation of continuity,* all Lorentz transformations of the electromagnetic four-potential $A^\mu(x^\sigma)$ *satisfy the Lorentz condition.*

We next present the Eq. (3.7) Lorentz transformation of $A^\mu(x^\sigma) = (A^0(x^0,\, \mathbf{r}),\; \mathbf{A}(x^0,\, \mathbf{r}))$ *in vector form,* analogous to the Eq. (2.4k) presentation of the *space-time* Lorentz transformation *in vector form,*

$$(A^0)'(x^0,\mathbf{r}) = \gamma(A^0((x^0)',\mathbf{r}') + (\mathbf{b}\cdot\mathbf{A}((x^0)',\mathbf{r}'))) \text{ and}$$
$$\mathbf{A}'(x^0,\mathbf{r}) = \mathbf{A}((x^0)',\mathbf{r}') + (\gamma-1)(\mathbf{b}(\mathbf{b}\cdot\mathbf{A}((x^0)',\mathbf{r}'))/|\mathbf{b}|^2) + \gamma\mathbf{b}A^0((x^0)',\mathbf{r}'), \text{ where}$$
$$(x^0)' = \gamma(x^0 - (\mathbf{b}\cdot\mathbf{r})) \text{ and } \mathbf{r}' = \mathbf{r} + (\gamma-1)(\mathbf{b}(\mathbf{b}\cdot\mathbf{r})/|\mathbf{b}|^2) - \gamma\mathbf{b}x^0. \tag{3.8}$$

In this tutorial we will only Lorentz-transform the time-independent electromagnetic potentials of stationary point sources. Thus there will be *no dependence* on the variable $(x^0)'$, which we therefore systematically drop. We will also only Lorentz-transform these time-independent stationary systems to "everyday speeds", so $|\mathbf{b}| \ll 1$. Therefore we systematically drop all effects of order $|\mathbf{b}|^2$ or higher. Since $\gamma = \big(1/\sqrt{1-|\mathbf{b}|^2}\big) = 1 + \mathrm{O}(|\mathbf{b}|^2)$, we set γ to unity. Under these specialized conditions, Eq. (3.8) becomes,

$$(A^0)'(x^0,\mathbf{r}) = A^0(\mathbf{r}') + (\mathbf{b}\cdot\mathbf{A}(\mathbf{r}')) + \mathrm{O}(|\mathbf{b}|^2) \text{ and } \mathbf{A}'(x^0,\mathbf{r}) = \mathbf{A}(\mathbf{r}') + \mathbf{b}A^0(\mathbf{r}') + \mathrm{O}(|\mathbf{b}|^2), \text{ where}$$
$$\mathbf{r}' = \mathbf{r} - \mathbf{b}x^0 + \mathrm{O}(|\mathbf{b}|^2) = \mathbf{r} - \mathbf{v}t + \mathrm{O}(|\mathbf{b}|^2). \tag{3.9}$$

Upon its return to "everyday notation" which utilizes $t = (x^0/c)$ and $\mathbf{v} = \mathbf{b}c$, Eq. (3.9) becomes,

$$(A^0)'(\mathbf{r},t) = A^0(\mathbf{r}-\mathbf{v}t) + ((\mathbf{v}/c)\cdot\mathbf{A}(\mathbf{r}-\mathbf{v}t)) + \mathrm{O}(|\mathbf{v}/c|^2) \text{ and}$$
$$\mathbf{A}'(\mathbf{r},t) = \mathbf{A}(\mathbf{r}-\mathbf{v}t) + (\mathbf{v}/c)A^0(\mathbf{r}-\mathbf{v}t) + \mathrm{O}(|\mathbf{v}/c|^2). \tag{3.10}$$

These $|\mathbf{v}/c| \ll 1$ Lorentz transformations of initially time-independent electromagnetic potentials *also yield the corresponding Lorentz-transformed electric and magnetic fields.* We now work out the corresponding Lorentz-transformed magnetic field $\mathbf{B}'(\mathbf{r},t)$,

$$\mathbf{B}'(\mathbf{r},t) = \nabla_{\mathbf{r}}\times\mathbf{A}'(\mathbf{r},t) = \nabla_{\mathbf{r}}\times(\mathbf{A}(\mathbf{r}-\mathbf{v}t) + (\mathbf{v}/c)A^0(\mathbf{r}-\mathbf{v}t)) + \mathrm{O}(|\mathbf{v}/c|^2) =$$
$$\mathbf{B}(\mathbf{r}-\mathbf{v}t) - (\mathbf{v}/c)\times(\nabla_{\mathbf{r}}A^0(\mathbf{r}-\mathbf{v}t)) + \mathrm{O}(|\mathbf{v}/c|^2) =$$
$$\mathbf{B}(\mathbf{r}-\mathbf{v}t) + (\mathbf{v}/c)\times\mathbf{E}(\mathbf{r}-\mathbf{v}t)) + \mathrm{O}(|\mathbf{v}/c|^2). \tag{3.11a}$$

We next work out the Lorentz-transformed electric field $\mathbf{E}'(\mathbf{r},t)$ which corresponds to the above conditions,

$$\mathbf{E}'(\mathbf{r},t) = -\nabla_{\mathbf{r}}(A^0)'(\mathbf{r},t) - (1/c)(\partial/\partial t)\mathbf{A}'(\mathbf{r},t) + \mathrm{O}(|\mathbf{v}/c|^2) =$$
$$-\nabla_{\mathbf{r}}(A^0(\mathbf{r}-\mathbf{v}t) + ((\mathbf{v}/c)\cdot\mathbf{A}(\mathbf{r}-\mathbf{v}t))) - (1/c)(\partial/\partial t)(\mathbf{A}(\mathbf{r}-\mathbf{v}t) + (\mathbf{v}/c)A^0(\mathbf{r}-\mathbf{v}t)) + \mathrm{O}(|\mathbf{v}/c|^2) =$$
$$\mathbf{E}(\mathbf{r}-\mathbf{v}t) - \nabla_{\mathbf{r}}((\mathbf{v}/c)\cdot\mathbf{A}(\mathbf{r}-\mathbf{v}t)) + ((\mathbf{v}/c)\cdot\nabla_{\mathbf{r}})(\mathbf{A}(\mathbf{r}-\mathbf{v}t) + (\mathbf{v}/c)A^0(\mathbf{r}-\mathbf{v}t)) + \mathrm{O}(|\mathbf{v}/c|^2) =$$
$$\mathbf{E}(\mathbf{r}-\mathbf{v}t) - (\mathbf{v}/c)\times(\nabla_{\mathbf{r}}\times\mathbf{A}(\mathbf{r}-\mathbf{v}t)) + \mathrm{O}(|\mathbf{v}/c|^2) =$$
$$\mathbf{E}(\mathbf{r}-\mathbf{v}t) - (\mathbf{v}/c)\times\mathbf{B}(\mathbf{r}-\mathbf{v}t) + \mathrm{O}(|\mathbf{v}/c|^2), \tag{3.11b}$$

where the Eq. (3.11b) term $\big((\mathbf{v}/c)\big((\mathbf{v}/c)\cdot\nabla_{\mathbf{r}}\big)A^0(\mathbf{r}-\mathbf{v}t)\big)$ was dropped because it is of order $|\mathbf{v}/c|^2$.

Displayed properly as a pair, the foregoing $|\mathbf{v}/c| \ll 1$ Lorentz transformations of initially time-independent electric and magnetic fields are,

$$\mathbf{E}'(\mathbf{r},t) = \mathbf{E}(\mathbf{r}-\mathbf{v}t) - (\mathbf{v}/c)\times\mathbf{B}(\mathbf{r}-\mathbf{v}t) + \mathrm{O}(|\mathbf{v}/c|^2),$$
$$\mathbf{B}'(\mathbf{r},t) = \mathbf{B}(\mathbf{r}-\mathbf{v}t) + (\mathbf{v}/c)\times\mathbf{E}(\mathbf{r}-\mathbf{v}t) + \mathrm{O}(|\mathbf{v}/c|^2). \tag{3.12}$$

For the stationary point charge, $\mathbf{E}(\mathbf{r}) = q(\mathbf{r}/|\mathbf{r}|^3)$ and $\mathbf{B}(\mathbf{r}) = \mathbf{0}$. Therefore when the point charge has constant velocity $\mathbf{v}$ we read off from Eq. (3.12) that,

$$\mathbf{B}'(\mathbf{r},t) = q(((\mathbf{v}/c)\times\mathbf{r})/|\mathbf{r}-\mathbf{v}t|^3) + \mathrm{O}(|\mathbf{v}/c|^2),$$
$$\mathbf{E}'(\mathbf{r},t) = q((\mathbf{r}-\mathbf{v}t)/|\mathbf{r}-\mathbf{v}t|^3) + \mathrm{O}(|\mathbf{v}/c|^2). \tag{3.13}$$

The Eq. (3.13) magnetic field, which vanishes when $\mathbf{v} = \mathbf{0}$ (in contradiction to Newtonian precepts), is azimuthal. Arranging the trajectory of a charged object to run along a magnetic north-south line should enhance the deflection of the needle of a magnetic compass when the charged object passes immediately beneath the compass.

For the stationary point magnetic dipole, $\mathbf{E}(\mathbf{r}) = \mathbf{0}$ and $\mathbf{B}(\mathbf{r}) = ((3\mathbf{r}(\mathbf{r}\cdot\mathbf{m}) - \mathbf{m}|\mathbf{r}|^2)/|\mathbf{r}|^5)$. Therefore when the point magnetic dipole has constant velocity $\mathbf{v}$ we read off from Eq. (3.12) that,

$$\mathbf{E}'(\mathbf{r},t) = ((-3((\mathbf{v}/c)\times\mathbf{r})((\mathbf{r}-\mathbf{v}t)\cdot\mathbf{m}) + ((\mathbf{v}/c)\times\mathbf{m})|\mathbf{r}-\mathbf{v}t|^2)/|\mathbf{r}-\mathbf{v}t|^5) + \mathrm{O}(|\mathbf{v}/c|^2),$$
$$\mathbf{B}'(\mathbf{r},t) = ((3(\mathbf{r}-\mathbf{v}t)((\mathbf{r}-\mathbf{v}t)\cdot\mathbf{m}) - \mathbf{m}|\mathbf{r}-\mathbf{v}t|^2)/|\mathbf{r}-\mathbf{v}t|^5) + \mathrm{O}(|\mathbf{v}/c|^2). \quad (3.14)$$

The Eq. (3.14) electric field, which vanishes when $\mathbf{v} = \mathbf{0}$ (in contradiction to Newtonian precepts), is azimuthal when the velocity $\mathbf{v}$ is *parallel* to the magnetic dipole's magnetic moment $\mathbf{m}$. That azimuthal electric field can transiently propel the invisible microscopic free electrons in a metal wire coil through whose center the dipole passes (Faraday), or it can transiently deflect a low-mass charged macroscopic object which is hanging downward by a thread immediately above the dipole's horizontal trajectory.