

Alternate methods for judging the significance of Hub Test results

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Abstract

Observations of astronomical objects include vectors that are transverse, perpendicular to the direction to the objects themselves. If a set of transverse vectors are correlated, this information may reflect on the objects themselves or of the intervening medium through which the observations are made or otherwise. One measure of correlation made by the Hub Test determines how well transverse vectors on the sky align with the other points on the sky. Any such result must be accompanied by an estimate of its significance, *i.e.* how likely is it that randomly directed transverse vectors would be as well aligned. Generating random data and calculating the significance is a mathematical exercise that can be done in many ways with varying degrees of reliability. Given a sample with observed transverse vectors, one can repeatedly replace the observed vectors with randomly directed vectors to build probability distributions of random data results. This method of finding significance is time consuming and computationally intensive. Two alternatives are presented here. A standard reference Library has been created by treating many simulated samples, organized by the number of sources and the size of the sample region. Assuming an observed sample has these properties in range of the Library's resources, the significances of its results can be estimated by interpolating the Library data. Another method, easier but less reliable than the interpolations, fits the Library data with functions, so finding significance reduces to substituting the observed sample's number of sources and region dimensions into formulas. An Appendix supplies computer software that can be used to apply the alternative methods to observed samples. Links to a ready-to-run, self-contained Mathematica notebook is provided in the references.

Keywords: Alignment; Hub Test; Transverse Vectors; Polarization; Jets

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0. Preface

The pdf version of this notebook is available online from the viXra archive.

To find the ready-to-run notebook follow the link in Ref. 1. The notebooks in this series were created using Wolfram Mathematica, Version Number: 12.1, Ref. 2.

Note(s):

(1) Some numerical quantities in the pdf version may differ from the live ready-to-run version in Ref. 1 because the ready-to-run version may have been run after the pdf was produced. The ready-to-run version and the pdf version may be updated out-of-synch.

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1. Introduction

Given a set of transverse vectors on the sky, one may ask if their directions are correlated. Applications include the polarization direction of electromagnetic radiation and the direction of asymmetries such as jets. One test of alignment, the Hub Test, extends the transverse directions, making Great Circle geodesics on the Celestial Sphere. The transverse directions are perfectly aligned if they intersect at some point H on the sphere. The directions are well-aligned when they converge in a small area near some point H_{\min} . The Hub Test can find correlations for samples with hubs H_{\min} that are near the sources as well as the distant Hubs of other alignment tests.

The basic quantity that measures the convergence of the Great Circles is a function of position on the Celestial Sphere, denoted $\bar{\eta}(H)$. The function $\bar{\eta}(H)$ is an acute angle defined at all points of the Celestial Sphere, except at the sources. The smallest alignment angle $\bar{\eta}_{\min}^{\text{obs}}$ and the largest avoidance angle $\bar{\eta}_{\max}^{\text{obs}}$ provide two independent measures of the correlation of the set of transverse vectors. This article concentrates on alignment, $\bar{\eta}_{\min}^{\text{obs}}$. The Appendix puts both measures of correlation on an equal footing. The Hub Test is described briefly in Sec. 2 and references made there.

The problem is to determine the significance of the $\bar{\eta}_{\min}^{\text{obs}}$ calculated for a given sample. By the significance of the smallest alignment angle $\bar{\eta}_{\min}^{\text{obs}}$ we mean the likelihood that random transverse directions would deliver as good a result as is found with the observed data. For the Hub Test measure of transverse vector alignment, this means finding out how likely random directions would yield a value $\bar{\eta}_{\min}^{\text{random}}$ less than or equal to $\bar{\eta}_{\min}^{\text{obs}}$. The significance is sometimes called the “ p -value”.

By its definition, the most direct process we consider is what we call the ‘direct’ method, “Direct Method A”. Make many random runs, *i.e.* duplicates of the calculations with randomly directed transverse vectors, and find a formula that fits the probability distribution of the smallest alignment angles $\bar{\eta}_{\min}^{\text{random}}$ from the random runs of data. Assuming that the fitting formula is accurate along the ‘tail’ of the distribution, we can estimate the significance by integrating the probability distribution to find the likelihood that random results $\bar{\eta}_{\min}^{\text{random}}$ have values below the smallest observed alignment angle $\bar{\eta}_{\min}^{\text{obs}}$. Direct Method A is the most reliable

of the three methods considered in this article. It is discussed in Sec. 2 along with the brief description of the Hub Test.

To avoid the time-consuming Direct Method A, we present two alternatives. We have constructed simulated samples and made many random runs for each sample. The probability distributions for random run results $\bar{\eta}_{\min}^{\text{random}}$ have been calculated by applying Direct Method A, but here for random transverse vector samples. The process, discussed in Sec. 3. gives us a Library of probability distributions. The fitting functions are two-parameter fits, like Gaussians, with a probability distribution determined by the peak parameter η_0^{\min} and the half-width parameter σ^{\min} . In this way we only need to save two numbers to determine the probability distribution for each simulated sample. Oh, wait, each parameter has a standard error found when the distribution is fit, so four numbers are recorded for each simulation sample.

The Library consists of the relevant properties of the simulated samples and the four parameters needed to replicate the probability distributions of the two measures of correlation $\bar{\eta}_{\min}^{\text{random}}$ and $\bar{\eta}_{\max}^{\text{random}}$. The Library also includes the standard errors for these parameters, so that estimates of the accuracy of the process can be made.

Once the Library is constructed, an interpolation determines the probability distribution required for an observed sample. This method of estimating significance is termed “Interpolation Method B”. Method B is presented in Sec. 3 with the Library construction. Interpolation Method B is limited by the range of the collection of samples with random runs. For example, observed samples with more sources than any sample in the Library would require an extrapolation. Since extrapolation is less accurate than an interpolation between data points, best practice constrains the application of Method B to the range of samples in the Library.

Function Method C finds two functions that fit the Library data, the peak η_0^{\min} and half-width σ^{\min} . Similar remarks apply to the other measure of correlation, $\bar{\eta}_{\max}^{\text{obs}}$. The two functions η_0^{\min} and σ^{\min} depend on a sample’s number of sources and its root-mean-square radius. Once values of the two parameters are determined, the probability distribution can be constructed and integrated to find the chance that random runs have better, smaller values of $\bar{\eta}_{\min}^{\text{random}}$ than the observed sample’s $\bar{\eta}_{\min}^{\text{obs}}$. The formulas needed for Function Method C are included in this article, in Sec. 4.

We have previously studied three samples. The significances in those studies were estimated with Direct Method A. The papers are accessible, already appearing in the on-line non-refereed literature. Two samples have 27 and 13 QSO radio sources and one sample has 99 stars from our Galaxy with polarized visible starlight; see Refs. 3,4,5.

In Sec. 5 of this article, the significances of the three samples are determined with Interpolation Method B and Formula Method C. In future work, the plan is to apply Interpolation Method B and/or Function Method C to catalogs, mapping the significances of Hub Test’s assessment of the correlations of cataloged data.

Concluding remarks appear in Sec. 6.

2. The Hub Test and Direct Method A

The Hub Test, Ref. 6, is based on the alignment of transverse directions with directions toward other points on the Celestial Sphere. In Fig. 1, the “alignment angle” η is the acute angle η between two great circles at S , $0^\circ \leq \eta \leq 90^\circ$. The alignment angle η measures how well the polarization direction \hat{v}_ψ matches the direction \hat{v}_H toward the point H . Perfect alignment occurs when $\eta = 0^\circ$ and the two great circles overlap. Perpendicular great circles, $\eta = 90^\circ$, indicates maximum “avoidance” of the polarization direction \hat{v}_ψ with the point H on the sphere. The halfway value, $\eta = 45^\circ$, favors neither alignment nor avoidance.

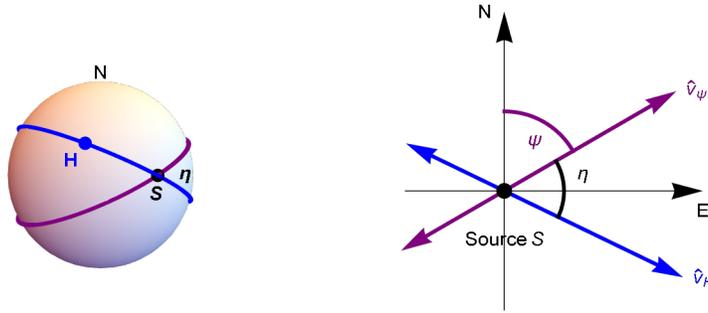


Figure 1: The Celestial sphere is pictured on the left and on the right is the plane tangent to the sphere at the source S . The linear polarization direction \hat{v}_ψ lies in the tangent plane and determines the purple great circle on the sphere. A point H on the sphere together with the point S determine a second great circle, the blue circle drawn on the sphere. Clearly, H and S must be distinct in order to determine a great circle. The angle η measures the alignment of the polarization direction ψ with the point H .

With N sources $S_i, i = 1, \dots, N$, there are N alignment angles η_{iH} at each point H . One can calculate an average alignment angle $\bar{\eta}$ at H ,

$$\bar{\eta}(H) = \frac{1}{N} \sum_{i=1}^N \eta_{iH}, \tag{1}$$

where

$$\cos(\eta_{iH}) = | \hat{v}_\psi \cdot \hat{v}_H |. \tag{2}$$

Each angle η_{iH} is taken to be the acute angle solving (2). Then the average alignment angle $\bar{\eta}(H)$ at the point H must also be acute.

The alignment angle $\bar{\eta}(H)$ is a function of position H on the sphere. See Figs. 2 and 3. It is symmetric across diameters, $\bar{\eta}(H) = \bar{\eta}(-H)$, because great circles are symmetric across diameters.

For random polarization directions, the average $\bar{\eta}(H)$ should be near 45° , since each alignment angle η_{iH} is acute, $0^\circ \leq \eta_{iH} \leq 90^\circ$, and random polarization directions should not favor large values or small values of η_{iH} , and, therefore, average to about 45° . Points H where the average alignment angle $\bar{\eta}(H)$ is smaller than 45° , the great circles tend to converge and where the angle $\bar{\eta}(H)$ is larger than 45° , the great circles can be said to diverge. The extremes of the function $\bar{\eta}(H)$ measure extreme convergence and extreme divergence of the great circles determined by the polarization directions.

In this article and notebook, we often use “min” to label the smallest alignment angle $\bar{\eta}_{\min}$, the minimum value of the function $\bar{\eta}(H)$, Eq. (1). The associated points on the Celestial Sphere are the “hubs” H_{\min} and $-H_{\min}$. Thus “min” is associated with convergence of the polarization directions. For divergence, the hubs H_{\max} and $-H_{\max}$ locate places where the polarization directions most avoid, as indicated by the largest alignment angle $\bar{\eta}_{\max}$, the maximum value of the function $\bar{\eta}(H)$. Thus, we very often label an avoidance related quantity with “max”.

random-based results for $\bar{\eta}_{\min}$ is then approximated by a suitable fitting function. Aside from a scale factor that normalizes the distribution, the fitting function of the histogram is the probability distribution of the random results $\bar{\eta}_{\min}$. Having found a function that approximates the probability distribution, one estimates the likelihood that random runs return better results than the observed $\bar{\eta}_{\min}^{\text{obs}}$ and that is the significance of $\bar{\eta}_{\min}^{\text{obs}}$.

One of the samples we discuss here has 27 QSOs and has been treated elsewhere, Ref. 3. The significances of $\bar{\eta}_{\min}$ and $\bar{\eta}_{\max}$ for the 27 QSO sample were found by Direct Method A. The histograms are displayed in Fig. 4. The histogram for the largest avoidance angle $\bar{\eta}_{\max}$ is on the right in Fig. 4. The symmetry between max and min extremes of the function $\bar{\eta}(H)$ both describe valid correlations of the polarization directions.

There is much in common between convergence and divergence, both are extremes of the alignment angle function $\bar{\eta}(H)$. For brevity and because mutual alignment is of central interest in the literature, only convergence is discussed in Part I the Article. Both are treated equally in Part II the Appendix.

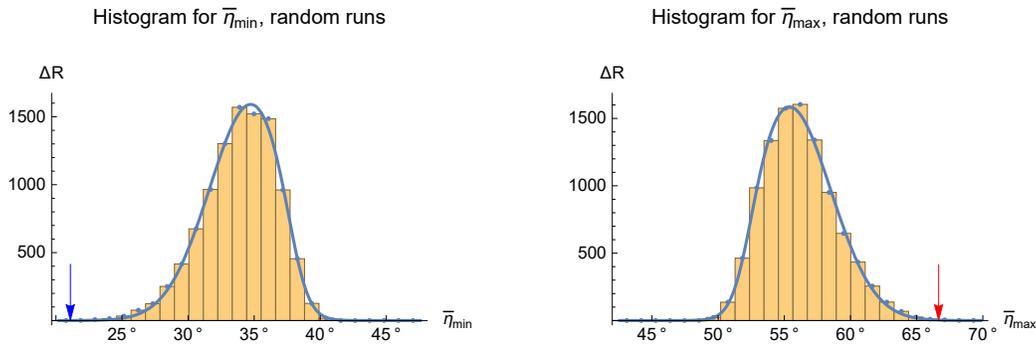


Figure 4: Direct Method A applied to a sample of 27 QSOs, Ref. 3. Histograms for both the smallest alignment angle $\bar{\eta}_{\min}$ and the largest avoidance angle $\bar{\eta}_{\max}$ are displayed. Adding the heights of the bars in a histogram yields the total number of random runs, here $\Sigma \Delta R = R = 10,000$ runs. The height ΔR of each bar is the number of runs with results in the indicated bin, so the fraction $\Delta R/R$ is the probability that values land in the bin. The observed polarization directions give the values indicated by the arrows.

3. Creating a Library of Random Run Results, Interpolation Method B

One way to avoid generating and analyzing thousands of randomly directed samples is to generate and analyze thousands of artificially created simulated samples beforehand and collect parameters sufficient to estimate the probability distribution for a wide range of samples by interpolation. The collection of parameters can be called a “Library”. The process is called “Interpolation Method B”.

To begin with, we must create a set of samples to analyze. Let us choose to arrange the sources in square arrays, so that they are spread out uniformly over the region. This forces the number of sources to be perfect squares. Currently, the Library has nine values of N ranging from $N = 9$ to $N = 225$, in detail: $N = 9, 16, 25, 36, 49, 64, 81, 121, 225$.

It is convenient to pretend that all samples can be approximated as circular regions with a “radius”. Thus, to this end, we build the square arrays using a nominal radius, “ ρ_{Nominal} ”. The size of each square is determined by averaging two squares, a large square with the nominal circle inscribed and a smaller square that has vertices on the nominal circle. The side of the square array for the simulated sample is the average of the inscribed and superscribed squares. Thus we choose values for the radii ρ_{Nominal} and construct square arrays of samples. As of this writing, $\rho_{\text{Nominal}} = 0.25^\circ, 0.33^\circ, 0.50^\circ, 1.0^\circ, 2.0^\circ, 3.0^\circ, 4.0^\circ, 8.0^\circ, 42^\circ$, ten values in

total.

For calculations and comparing the extent of different samples, the best “radius” measure may be the root-mean-square radius ρ_{RMS} . The RMS radius is a well-recognized measure of the size of the region containing the sources.

We calculate the RMS radius of the square arrays built for the Library and use the RMS radii ρ_{RMS} in calculations, while we organize the samples by the number of sources N and the nominal radius ρ_{Nominal} . In total, there are nine values for N and ten values of ρ_{Nominal} , both sets are listed above. Each combination $(N, \rho_{\text{Nominal}})$ has its own value for ρ_{RMS} , so there are 90 ρ_{RMS} values, one for each combination $(N, \rho_{\text{Nominal}})$. Fig. 5 displays one sample constructed for the Library.

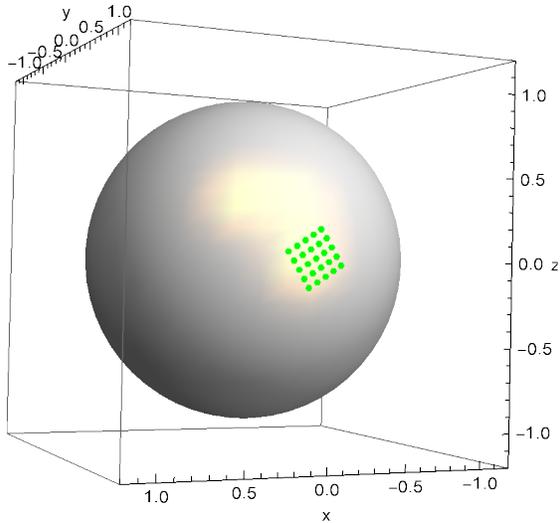


Figure 5: A sample with $N = 25$ sources and a nominal radius $\rho_{\text{Nominal}} = 8^\circ$. The sample is rotated to avoid coincidences with coordinate axes. The length of the side of the square is set so that the square approximates a circle with the nominal radius. For calculations and comparison with any observed samples, the root-mean-square radius is found. Here, the root-mean-square radius is $\rho_{\text{RMS}} = 6.84^\circ$.

Once the samples are created, the sources are given randomly directed transverse vectors. The function $\bar{\eta}(H)$, Eq. (1), is calculated and the smallest alignment angle $\bar{\eta}_{\text{min}}$ is determined. At the time of writing, there are 90 samples and each is processed with 10,000 random runs. The histograms for the square array of $N = 25$ sources are displayed in Fig. 6 and 7.

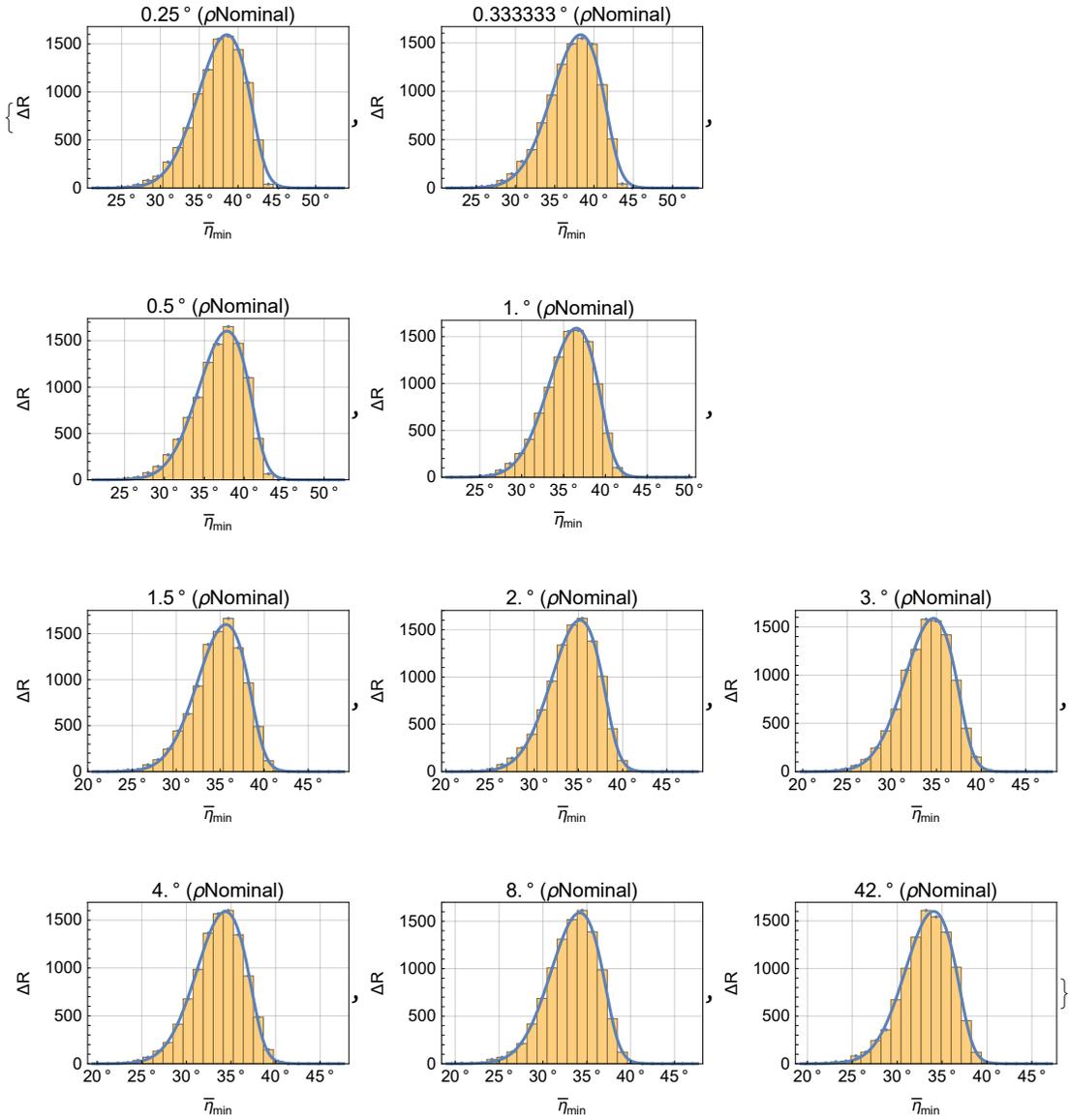


Figure 6: Thumbnails of the ten histograms for the values of $\bar{\eta}_{\min}$ from random runs with $N = 25$. The plots are labelled with the nominal radius ρ_{Nominal} . Function that fit the distributions are plotted as blue lines. Note that all the distributions tilt and are distorted from a symmetric Gaussian shape.

A glance at Fig. 6 above reveals that the distributions for $\bar{\eta}_{\min}$ are steeper on the high side toward $\eta \rightarrow 45^\circ$. This behavior is repeated for all such distributions and appears to be an inherent property of the statistics.

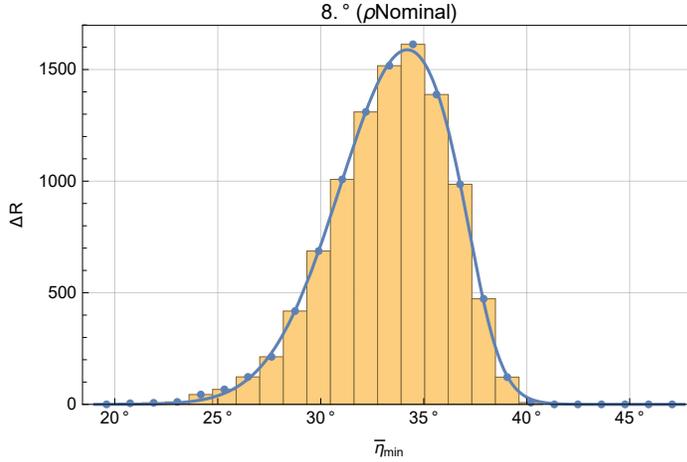


Figure 7: Enlarged view of the histogram and fit for $N = 25$ and $\rho_{\text{Nominal}} = 8^\circ$. The distribution of the $R = 10,000$ values of the smallest alignment angle $\bar{\eta}_{\min}$ from the random runs. As noted above with Fig. 4, the histogram approximates the shape of the probability distribution, aside from a normalizing scale factor. The distribution is fit by a Gaussian multiplied by a step function. The step function vanishes to the right of the peak and is unity on the more important left side where the smallest alignment angles $\bar{\eta}_{\min}$ for well-aligned samples are found.

A histogram of the smallest alignment angles $\bar{\eta}_{\min}$ from $R = 10,000$ runs is enlarged from Fig. 6 and displayed in Fig. 7. The height of a bar in Fig. 7 is the number ΔR of the random runs that gave a value of $\bar{\eta}_{\min}$ in the interval $\delta\eta$, the width of a bin. Thus, the quantity $\Delta R/R$ is the likelihood that random runs give a value of $\bar{\eta}_{\min}$ in the bin. The histogram has the shape of the probability distribution $P_{\min}(\eta)$ for $\bar{\eta}_{\min}$. The probability distribution is normalized, $1 = \int P_{\min} d\eta$. Since the sum $\sum \Delta R = R$, it follows that $1 = \sum \frac{\Delta R}{R\delta\eta} \delta\eta$ and that $P_{\min}(\eta) \approx \frac{\Delta R}{R\delta\eta}$. Therefore, by fitting the histogram, we obtain the probability distribution $P_{\min}(\eta)$, within a numerical factor.

There is a complication. Look closely at the distribution in Fig. 7. The right side, the side toward $\bar{\eta}_{\min} \rightarrow \pi/4 \sim 0.79$, has a steeper slope than the left side, the side toward $\bar{\eta}_{\min} \rightarrow 0$. Thus, the low $\bar{\eta}_{\min}$ side is favored and probability is pushed from the right side to the left side. A simple, symmetrical Gaussian would not fit the data well. The fitting curve shown in blue in Fig. 7 combines a Gaussian with a unit step-function, that is unity to the left of the peak, and zero to the right. We have

$$P_{\min}(\eta) = \left(\frac{\text{norm}}{\sigma (2\pi)^{1/2}} \right) \left(\mathbf{1} + e^{4 \frac{(\eta - \eta_0 - \sigma)}{\sigma}} \right)^{-1} e^{-\frac{1}{2} \left(\frac{\eta - \eta_0}{\sigma} \right)^2}, \quad (3)$$

where ‘‘norm’’ is a scale factor that normalizes the probability density. One finds that $\text{norm} \approx 1.22$. There are two free variables that can be chosen to fit a random run histogram like the one shown in Fig. 7. We call η_0 the ‘peak’ value and σ is the ‘half-width’. The presence of the step function $\left(\mathbf{1} + e^{4 \frac{(\eta - \eta_0 - \sigma)}{\sigma}} \right)^{-1}$ moves the peak and half-width a little from their Gaussian values, *i.e.* their values without the step function.

It is remarkable that the fitting function has two parameters just like a Gaussian, namely the peak η_0 and the half-width σ . The step function is introduced without an associated parameter. Given the form in Eq. (3), the problem of determining the significance of an observed $\bar{\eta}_{\min}$ reduces to estimating the values of two parameters, η_0 and σ , that determine the probability distribution $P_{\min}(\eta)$.

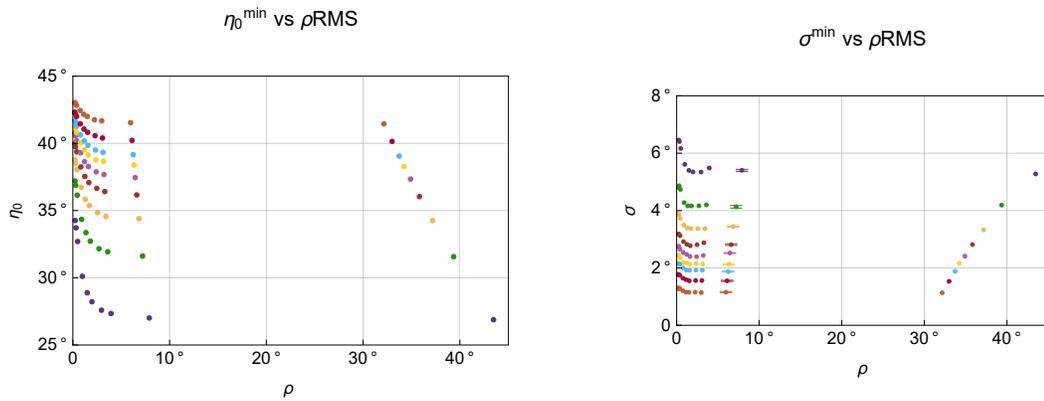
For small N one confronts an issue with distribution formulas. Gaussians and probability distributions like that in Eq. (3) return non-zero probabilities for unphysical values of the variables. The acute angle $\bar{\eta}_{\min}$ can not be negative even though the probability distribution in Eq. (3) assigns a probability to negative values of η . The problem is unimportant when there are a

sufficient number of sources because low values of $\bar{\eta}_{\min}$ are unlikely. With more sources, the likelihood of alignment decreases, as is evident from the left plot in Fig. 8. Thus, valid statistics require a minimum number of sources; we take 7. The number of sources N must be at least 7, $N \geq 7$ for the significance of negative $\bar{\eta}_{\min}$ to be sufficiently small.

To find the significance, or p -value, of the smallest alignment angle $\bar{\eta}_{\min}$, one finds the likelihood of smaller random run values by integrating $P_{\min}(\eta)$ from below,

$$p(\bar{\eta}_{\min}) = \int_{-\infty}^{\bar{\eta}_{\min}} P_{\min}(\eta) d\eta \quad (4)$$

By Eqs. (3) and (4) the significance of the correlated behavior indicated by the smallest alignment angle $\bar{\eta}_{\min}$ rests on the two probability distribution parameters η_0^{\min} and σ^{\min} .



$\{N, \text{color}\} =$
 $\{\{9, \text{purple}\}, \{16, \text{green}\}, \{25, \text{orange}\}, \{36, \text{brown}\}, \{49, \text{purple}\}, \{64, \text{yellow}\}, \{81, \text{blue}\}, \{121, \text{red}\}, \{225, \text{brown}\}\}$

Figure 8: The Library data, 90 values each of η_0 and σ in Eq. (3) for the random run distributions, as in Fig. 7, of the smallest alignment angle $\bar{\eta}_{\min}$. *Left.* The parameter η_0^{\min} locates the peak of a random run distribution. For a given number of sources N , the parameter η_0^{\min} curves upward for ρRMS less than about $\rho\text{RMS} \lesssim 4^\circ$. The standard errors of the η_0^{\min} are smaller than the dots and are not displayed. *Right.* Similar behavior is found for the half-width σ^{\min} , with constant values at large ρRMS and curving upward below about 4° . The standard errors for σ^{\min} are large enough to see. Here the standard errors are plotted for the column of dots near $\rho\text{RMS} \approx 6^\circ$ to 8° . For a given N , the other standard errors for the σ^{\min} have similar magnitudes as those displayed.

As of this writing, the Library has 90 sets of η_0 and σ for 90 sets of source number N and root-mean-square radius ρRMS . See Fig. 8. Look at the data in the left plot in Fig. 8. Most of the change occurs in the first 5° of ρRMS , while the radii ρRMS go out all the way to 42° . It would be better, at least for appearances and possibly for the interpolation itself, to spread the data out more evenly. To do that, we replace the variables N and ρRMS with functions of N and ρRMS and use the functions of N and ρRMS as variables rather than N and ρRMS themselves.

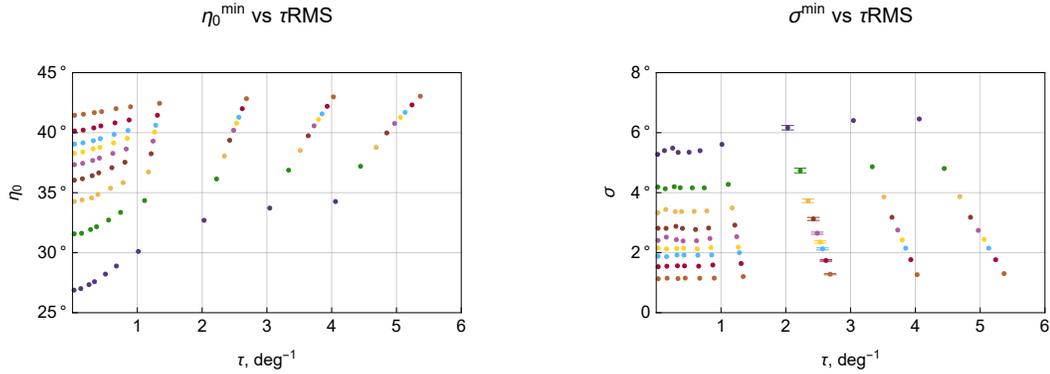
One motivation for changing variables is visible in Fig. 5: the N sources occupy a square array. The quantity N is associated with a square patch of the Celestial Sphere, an area. By taking the square root, *i.e.* $N^{1/2}$, one obtains a ‘linear’ quantity. Then, both $N^{1/2}$ and ρRMS are associated with angular quantities.

For a given number of sources, a glance at Fig. 8 shows that large radius samples, $\rho\text{RMS} \gg 1^\circ$, show little variation in the values

of η_0^{\min} or σ^{\min} . Most of the variation occurs for small angles. By the operation of multiplicative inversion, one can define a new variable $\tau\text{RMS} \equiv 1/\rho\text{RMS}$ to replace ρRMS . Treating $N^{1/2}$ similarly, we define $w \equiv 1/N^{1/2}$ and use w and τRMS as the variables for the fit. Fig. 9 plots the Library data in terms of the variables τRMS and w . The new, preferred variables are

$$w \equiv N^{-1/2} \text{ and } \tau\text{RMS} \equiv \rho\text{RMS}^{-1}. \quad (5)$$

Clearly one can write any formula in terms of w and τRMS as a formula in terms of N and ρRMS . However, the Library data spreads out better with w and τRMS than with N and ρRMS . See Fig. 9.



$$\{N, \text{color-code}, w = N^{-1/2}\} =$$

$$\left\{ \{9., \blacksquare, 0.333333\}, \{16., \blacksquare, 0.25\}, \{25., \blacksquare, 0.2\}, \{36., \blacksquare, 0.166667\}, \{49., \blacksquare, 0.142857\}, \right.$$

$$\left. \{64., \blacksquare, 0.125\}, \{81., \blacksquare, 0.111111\}, \{121., \blacksquare, 0.0909091\}, \{225., \blacksquare, 0.0666667\} \right\}$$

Figure 9: A change in the preferred variables for the fits to probability distribution parameters η_0^{\min} and σ^{\min} . Left: The Library data in Fig. 8 displayed for the peak alignment angle η_0^{\min} as a function of $\tau\text{RMS} \equiv 1/\rho\text{RMS}$, the inverse sample radius. Note that the data is spaced nicely when compared with Fig. 8. The big gap from 8° to 42° in Fig. 6 is now a small gap from 0.125 deg^{-1} to 0.024 deg^{-1} . And the curved structure below $\rho = 5^\circ$ for the η_0^{\min} in Fig. 8 is spread out in Fig. 9 above $\tau = 0.2 \text{ deg}^{-1}$. We prefer $w = N^{-1/2}$ and $\tau\text{RMS} = \rho\text{RMS}^{-1}$ as the variables for the fits to the parameters η_0^{\min} and σ^{\min} .

For Interpolation Method B, interpolation is the way to get values of the peak η_0 and the half-width σ in the probability distribution Eq. (3) that are appropriate for a given experimentally observed sample. For an observed sample whose number of sources N^{obs} and RMS radius $\rho\text{RMS}^{\text{obs}}$ are in the range of the Library's resources, a quick conversion gives $w = N^{-1/2}$ and $\tau\text{RMS} = \rho\text{RMS}^{-1}$. Then interpolation of the data in Fig. 9 produces values of η_0 and σ . With Eq. (3), these values determine the probability distribution $P_{\min}(\eta)$ for the observed sample. By integrating the distribution $P_{\min}(\eta)$ over all η less than the observed sample's $\bar{\eta}_{\min}^{\text{obs}}$, one finds the significance or p -value, $p(\bar{\eta}_{\min})$, of the smallest alignment angle $\bar{\eta}_{\min}^{\text{obs}}$ calculated for the observed sample, as in Eq. (4).

Method B may not give reliable parameters η_0 and σ when the number of sources N^{obs} and/or the RMS radius $\rho\text{RMS}^{\text{obs}}$ is/are outside the range of Library data. In such cases, the process is extrapolation. While extrapolation is much less reliable than interpolation, it does yield values of η_0 and σ allowing one to determine a probability distribution $P_{\min}(\eta)$. However, it might be better to consider Formula Method C whenever the observed sample's N and ρRMS leads to an extrapolation of the Library data to find η_0 and σ .

4. Fitting the Library and Formula Method C

“Function Method C”, like Interpolation Method B, is based on the Library data in Figs. 8 and 9. Both Methods B and C avoid the time-consuming random-runs needed with Direct Method A. Function Method C finds functions that fit the two parameters η_0 and σ listed in Library data table. Then, with these two functions, one substitutes the N and ρ RMS of the sample into the formulas and that gives values for the peak η_0 and half-width σ . The two parameters are all that is needed by Eq. (3) to construct the probability function P_{\min} and then find the significance of an observed smallest alignment angle $\bar{\eta}_{\min}^{\text{obs}}$.

Therefore, we need to find functions $\eta_0^{\min}(w, \tau\text{RMS})$ and $\sigma^{\min}(w, \tau\text{RMS})$ that fit the Library data. Recall that we prefer the variables $\tau\text{RMS} \equiv 1/\rho\text{RMS}$ and $w \equiv 1/N^{1/2}$ to the number of sources N and the root-mean-square radius ρRMS . The Library data is plotted in Fig. 9. Looking closely at the left graph in Fig. 9, one sees that η_0^{\min} is relatively flat for small τ and η_0^{\min} is flat again for large τ . There is a ‘squiggle’ for middle values of τ from $\tau = 0.2 \text{ deg}^{-1}$ to $\tau = 1 \text{ deg}^{-1}$. This behavior is reminiscent of the hyperbolic tangent, $\tanh(u) = \sinh(u) / \cosh(u)$, displayed in Fig. 10.

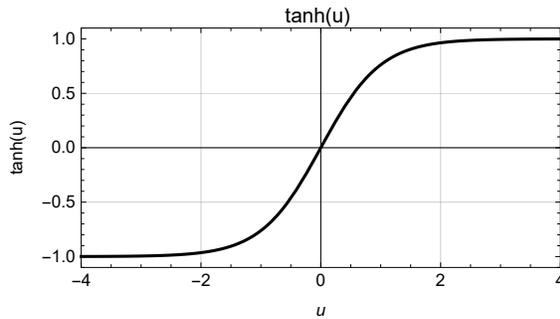


Figure 10: The hyperbolic function $\tanh(u)$. The squiggle in the η_0^{\min} data in Fig. 9 is most apparent with the $N = 9$ values. By rescaling and translating the $\tanh(u)$ function displayed here, one can fit the Library data for η_0^{\min} and σ^{\min} in Fig. 9 quite well.

By contorting the $\tanh(u)$ with an application of polynomials and varying their coefficients to get a least squares fit, we get the following functions: Ref. 7.

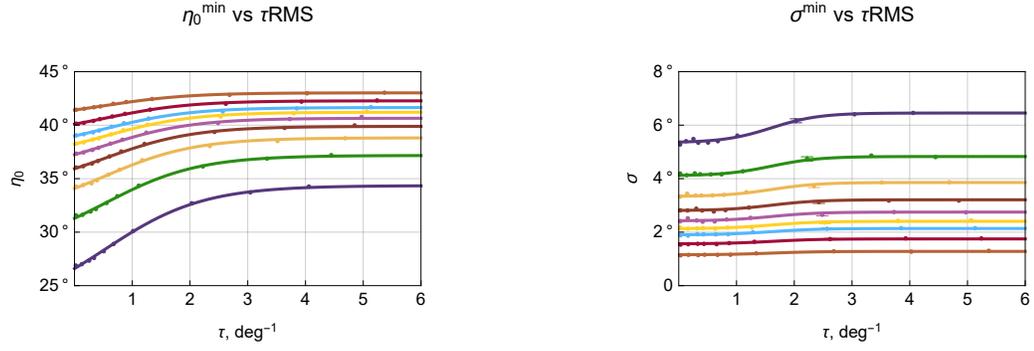
$$\eta_0^{\min}(w, \tau) = 45.027 - w \{47.39 + 7.32 w - 17.79 \tanh[(0.710 - 0.349 w) (-0.535 + \tau)] \} \quad (6)$$

$$\sigma^{\min}(w, \tau) = 0.25 w \{73.57 - 8.29 w + (3.09 + 10.66 w) \tanh[1.22 (-1.607 + \tau)] \} \quad (7)$$

where $w = N^{-1/2}$ and $\tau = \tau\text{RMS} = 1/\rho\text{RMS}$. The functions and the Library data are displayed in Fig. 11.

By eye, the fit to the Library values of η_0^{\min} in the Left plot of Fig. 11 is better than the fit to the σ^{\min} on the Right. In Part II the Appendix, it is found that the percent difference of the fit of $\eta_0^{\min}(w, \tau)$ in Eq. (6) to the Library data averages 0.16%, where by ‘average’ we mean an arithmetic mean, and the % diff. is always better than 0.78% For $\sigma^{\min}(w, \tau)$, one finds that the functions fit the Library to 3.5% or better, with an average of 0.91%. Thus, the 7-parameter fit in Eq. (6) fits 90 Library data points of η_0^{\min} to a few tenths of a percent, while the 6-parameter function in Eq. (7) fits the 90 values of σ^{\min} to a percent or so. See Figs. A6 and A7 in Part II the Appendix.

Graphing the functions in Eqs. (6) and (7) along with the Library data that the functions fit, we have



$$\{N, \text{color-code}, w = N^{-1/2}\} =$$

$$\left\{ \{9., \text{purple}, 0.333333\}, \{16., \text{green}, 0.25\}, \{25., \text{yellow}, 0.2\}, \{36., \text{brown}, 0.166667\}, \{49., \text{purple}, 0.142857\}, \right.$$

$$\left. \{64., \text{yellow}, 0.125\}, \{81., \text{blue}, 0.111111\}, \{121., \text{red}, 0.0909091\}, \{225., \text{brown}, 0.0666667\} \right\}$$

Figure 11: Fitting functions for the two alignment angle probability distribution parameters η_0^{\min} (Left) and σ^{\min} (Right). Compare the purple and light brown curves, *i.e.* bottom and top for η_0^{\min} and top and bottom for σ^{\min} . The curves flatten out for large N , *i.e.* small w . The dependence on the region radius decreases as N increases.

It is strange, but true, that the polynomials in Eqs. (6) and (7) are, in a sense, self-truncating. As shown in detail in Ref. 7, the polynomials blur and become fuzzy by the addition of a term with the next higher power. For example, consider the linear expression $47.39 + 7.32 w$ in Eq. 6. The standard error for the coefficient 7.32 of w is about 18%. If we try instead to fit a quadratic, $a + b w + c w^2$, then the new b and c have such huge standard errors that they are equivalent to zero, $b \approx 0$ and $c \approx 0$. We see that the linear fit gives a definite result while a quadratic fit does not give a definite result. One can say that adding $c w^2$ makes the fit blurred and fuzzy, so the power series is self-truncating.

With Formula Method C, we now have three methods to determine the parameters η_0^{\min} and σ^{\min} for the probability distribution Eq. (3) needed to determine significance or p -value in Eq. (4). Let us see how the three methods work in practice with three samples.

5. Applying the Three Methods to Three Sets of Experimental Data

In previous work, Refs. 3,4,5, the Direct Method A was used to find the significance of the alignment of polarization directions. Two of the samples involve radio QSOs and one sample has polarized light from Milky Way Galaxy stars. The data analyzed in the QSO studies are from a catalog JVAS1450, Ref. 8. The JVAS1450 catalog adds object-specific data to a subset of an earlier catalog JVAS/CLASS 8.4-GHz, Ref. 9. The observations of the 99 Stars sample are contained in two catalogs, Heiles 2000 and Berdyugin 2014, Refs. 10,11.

In the previous work, all three samples had their significance determined by Direct Method A. For each sample, 10,000 random runs were generated with random polarization directions at the sites of the observed sources. The histogram distributions for the 10,000 smallest alignment angle $\bar{\eta}_{\min}$ were fit with functions of the form in Eq. (3). This gave values for the two adjustable parameters for the fitting function, the value of $\bar{\eta}_{\min}$ at the peak, η_0^{\min} , and the half-width σ^{\min} where the distribution is down by a factor of $e^{-1/2}$ from the peak value. The terminology is for Gaussian distributions; the presence of a step-function in Eq. (3) modifies the meaning slightly.

Sample	N	$\rho\text{RMS, deg}$	$\bar{\eta}_{\min, \text{deg}}$
27 QSOs	27	6.82492	21.094
13 QSOs	13	4.72812	10.865
99 Stars	99	6.83803	7.007

Sample	$\eta_0^{\min, \text{deg}}$	$\sigma^{\min, \text{deg}}$
27 QSOs	34.923 ± 0.029	3.272 ± 0.034
13 QSOs	30.26 ± 0.17	4.64 ± 0.20
99 Stars	39.947 ± 0.018	1.737 ± 0.021

Table 1: Three previously studied samples of observed polarizations, Refs. 3,4,5. *Left:* The number of sources N and the root-mean-square radius of the sample ρ_{RMS} are the two properties of observed samples that Methods B and C need as input. These and the calculated smallest alignment angle $\bar{\eta}_{\text{min}}$ depend on observations. *Right:* This table contains results that do not depend on the observed polarizations. By calculations with random run simulations with Direct Method A, one obtains a random run distribution for each sample that is fit by a function in the form of Eq. (3). Each of the three fitting functions has two parameters, η_0^{min} and σ^{min} , which are listed in the table on the right. Recall that the alignment angle at the peak is η_0^{min} and the half-width is σ^{min} .

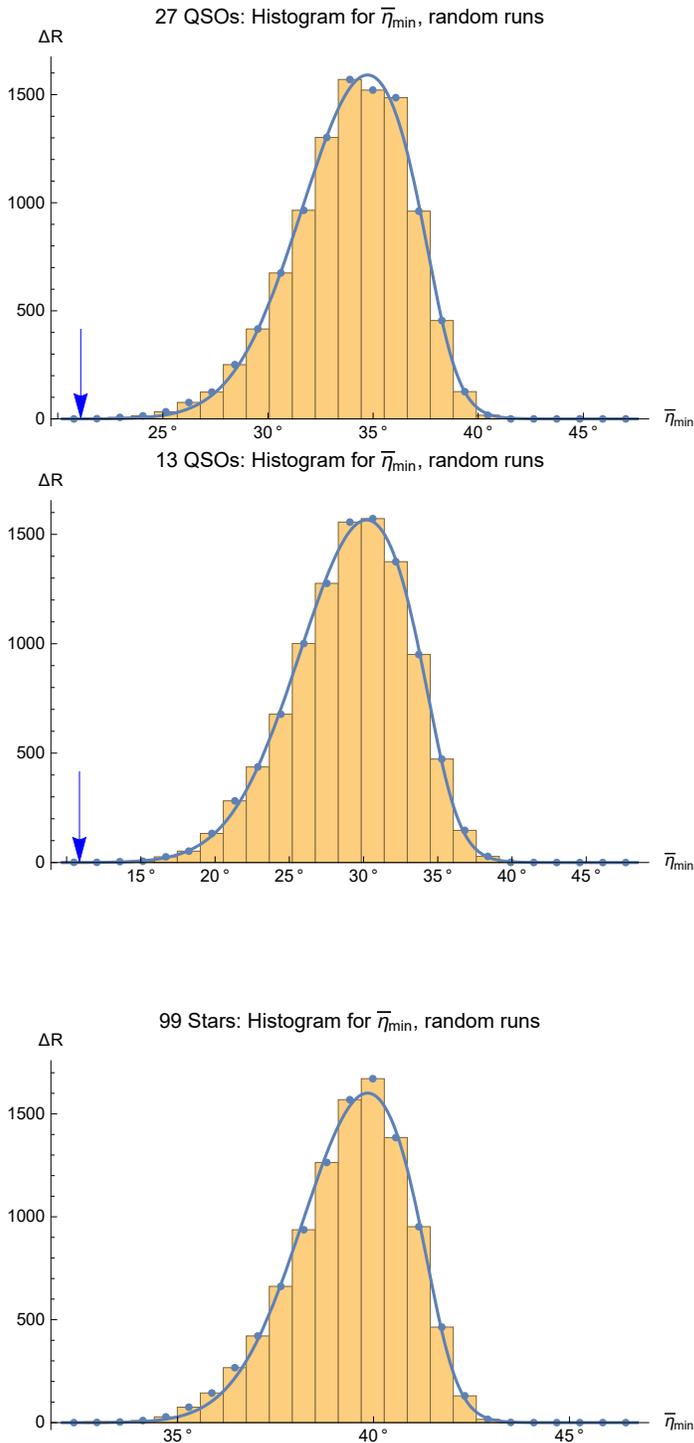


Figure 12: Histograms and fits for random runs found with Direct Method A for all three samples, 27 QSOs, 13 QSOs, and 99 Stars. The Blue arrows mark the observed smallest alignment angles $\bar{\eta}_{\min}$. The value of $\bar{\eta}_{\min}$ for the 99 Stars is off-scale at $\bar{\eta}_{\min} = 7.01^\circ$. The fraction of the random run results to the left of one of the Blue arrows indicates the significance of the observed smallest alignment angle $\bar{\eta}_{\min}$. The significance of the alignments for the 27 QSOs and the 13 QSOs is apparent, while $\bar{\eta}_{\min} = 7.01^\circ$ for the 99 Stars is so far down the $\bar{\eta}_{\min}$ -axis that the significance is exceptional. For the 99 Stars, the likelihood that random data would be better aligned is infinitesimal.

As discussed earlier, since Direct Method A uses the correct location of the sources and has the correct radius ρ_{RMS} with the correct number N of sources, Method A should be more reliable than Interpolation Method B or Formula Method C. Therefore, we interpret the following as comparing the inferior Methods B and C with the superior Method A.

All three methods rely on the same probability distribution formula, Eq. (3). The methods differ in how the parameters for the location of the peak, η_0^{\min} , and the half-width, σ^{\min} , are calculated. Therefore, let us compare the values of η_0^{\min} and σ^{\min} from B and C with the parameters from A. See Table 2.

Sample	A (degrees)	B (degrees)	C (degrees)
27 QSOs	34.923 ± 0.029	34.796 ± 0.016	34.80 ± 0.19
13 QSOs	30.26 ± 0.17	30.293 ± 0.033	30.36 ± 0.27
99 Stars	39.947 ± 0.018	39.703 ± 0.015	39.73 ± 0.10

(a) peak η_0^{\min} :

Sample	A (degrees)	B (degrees)	C (degrees)
27 QSOs	3.272 ± 0.034	3.299 ± 0.020	3.229 ± 0.022
13 QSOs	4.64 ± 0.20	4.59 ± 0.04	4.55 ± 0.04
99 Stars	1.737 ± 0.021	1.700 ± 0.018	1.729 ± 0.009

(b) half-width σ^{\min} :

Table 2: Distribution parameters η_0^{\min} and σ^{\min} in the probability distribution formula Eq. (3). The \pm values are the standard errors reported in fitting the data with the Mathematica command NonlinearModelFit. (a) Compared with Method A, the values of η_0^{\min} from Methods B and C overlap with A except for the B-value of η_0^{\min} for the 27 QSOs and the 99 Stars. (b) The half-widths σ^{\min} , in degrees, for the three methods. The B and C values for σ^{\min} are within the listed standard error of the Method A values. Also see the displays in Figs. 13 and 14, where these comments may be more apparent.

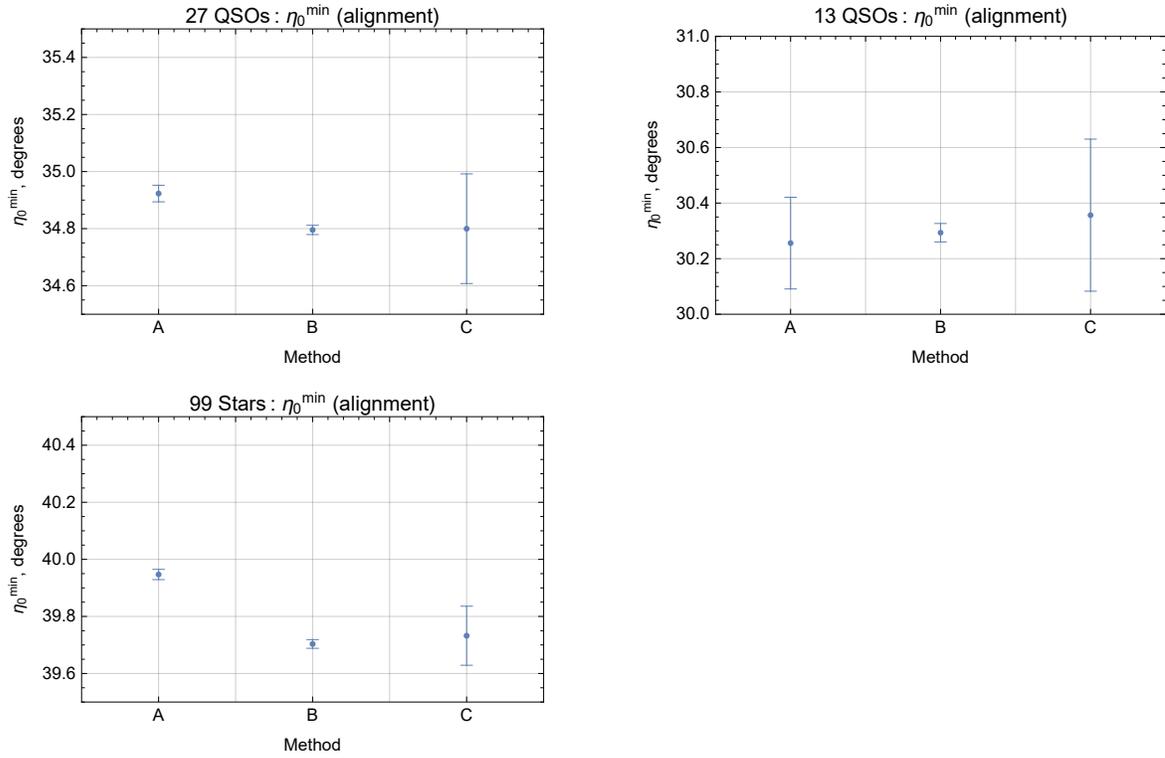


Figure 13: Compare parameter η_0^{\min} for the 3 methods. Method B has small standard errors, so it is precise, but it is not accurate for the 27 QSOs and 99 Stars when compared with A.

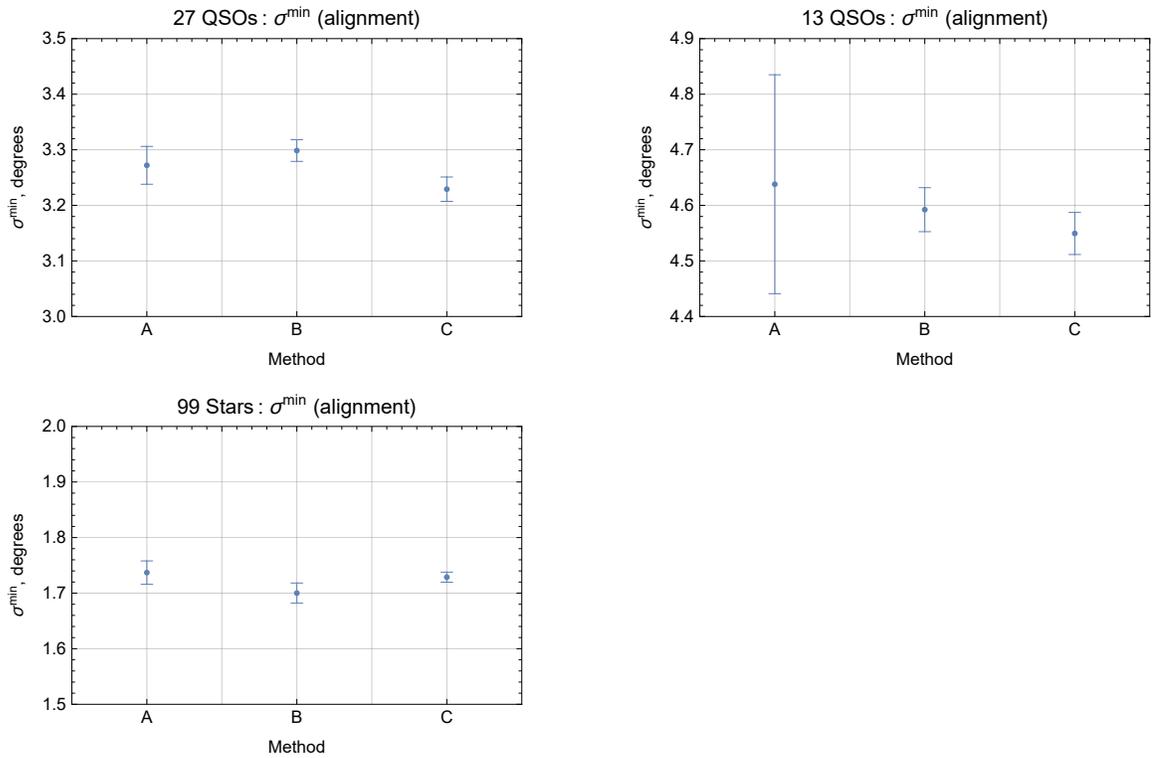


Figure 14: Compare parameter σ^{\min} for the 3 methods with 3 samples. Methods B and C give values that overlap with A. The range of σ^{\min} is the same, $\Delta\sigma = 0.5^\circ$, in all three graphs, so the three samples can be compared with each other.

Finally, we display the significances of the alignments of the three samples with B and C and compare those with the significances found by A. The calculations can be found in Part II the Appendix. Even though the distribution parameters η_0 do not always agree within the error bars, the significances, p -values, for all three methods A,B,C, do agree within the tolerances. One effect to notice in Fig. 15 is the asymmetry of the plus and minus uncertainties and that is most evident for the 99 Stars.

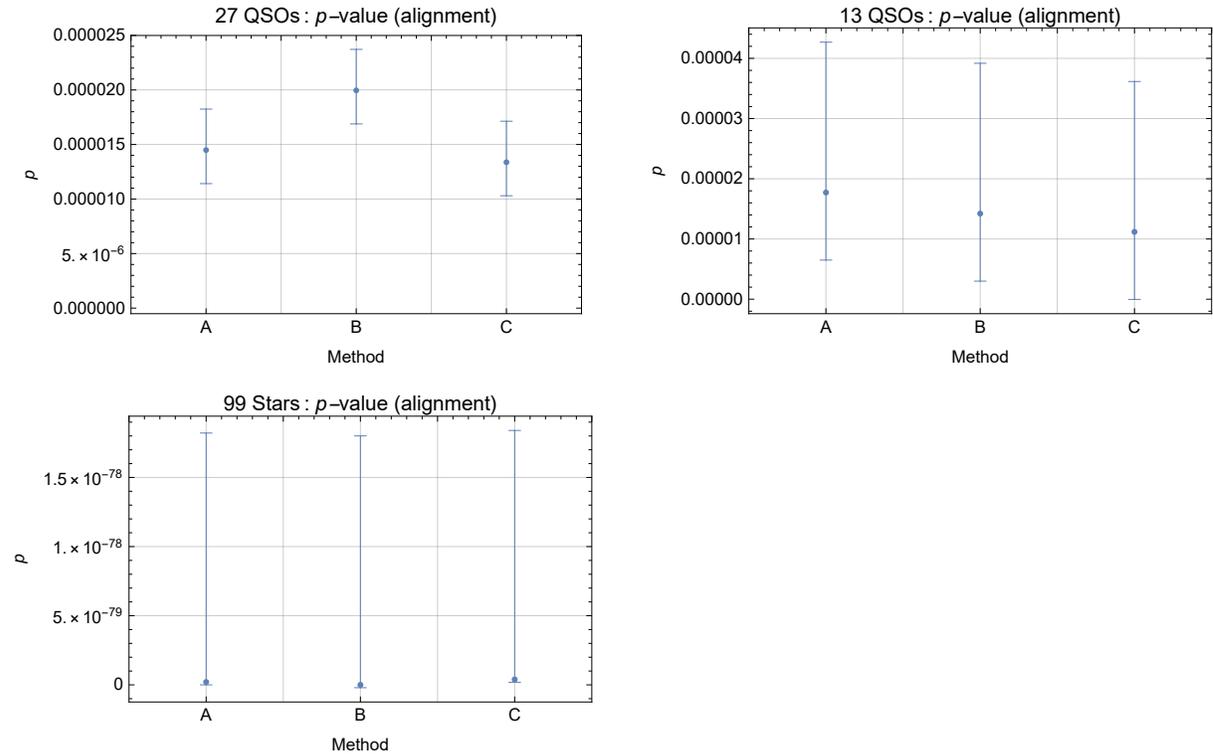


Figure 15: The significance of the smallest alignment angle $\bar{\eta}_{\min}$ for three samples as determined by Methods A, B, and C. The error bars are due to the standard errors of the parameters η_0^{\min} and σ^{\min} . For all three samples, the significances determined by the Interpolation Method B and the Formula Method C agree with the Direct Method A, within the error bars drawn here.

6. Concluding Remarks

The article presents two methods, Interpolation Method B and Formula Method C, to calculate the significance of the smallest alignment angle $\bar{\eta}_{\min}$ in the Hub Test. Both methods, B and C, provide quick alternatives to the laborious process of analyzing thousands of random runs required with Direct Method A.

The altered-Gaussian distribution form of the probability distributions of thousands of random runs appears to be universal. Adjusting the two parameters, the peak η_0 and the half-width σ , provides excellent fits to the 90 Library samples and the 3 experimentally observed samples considered in this article. Methods B and C give significances that are the same as the significance from Method A within standard errors of the distribution parameters for the three samples.

One application of Interpolation Method B and Formula Method C might be mapping the significance of catalogued data, perhaps catalogs of polarized sources or jet directions. A possible strategy is to use Methods B or C to locate regions of interest. Once a sample is found to be significantly aligned by Methods B or C, one might apply Direct Method A to be more certain of the result. Or not.

Whatever the project, having Methods B and C should make the Hub Test easier to apply.

References

1. R. Shurtleff, the ready-to-run Mathematica version of this notebook is available at the following URL:
<https://www.dropbox.com/s/iu4q2dtan9lgj90/20211112InterpolateAndFormula2a.nb?dl=0>
<https://www.wolframcloud.com/obj/shurtleffr/Published/20211112InterpolateAndFormula2a.nb>
2. Wolfram Research, Inc., Mathematica, Version 12.1, Champaign, IL (2020).
3. Shurtleff, R., “Evaluating the Alignment of the Polarized Radio Waves from 27 QSOs in a Region near the NGP”, <https://vixra.org/abs/2105.0091> (2021).
4. Shurtleff, R., “Evaluating the Alignment of the Polarized Radio Waves from 13 QSOs in Ursa Major”, <https://vixra.org/abs/2110.0077> (2021).
5. Shurtleff, R., “Evaluating the Alignment of the Polarized Starlight from 99 Stars in a Region off the Disk of the Milky Way”, <https://vixra.org/abs/2104.0074> (2021).
6. Shurtleff, R., “Indirect polarization alignment with points on the sky, the Hub Test”, <https://vixra.org/abs/2011.0026> (2020).
7. Shurtleff, R., the ready-to-run Mathematica version of the notebook “20211116AlternateRandomRunStatsDegrees.nb” is available at the following URLs:
<https://www.dropbox.com/s/3yotly80wzn3kpc/20211116AlternateRandomRunStatsDegrees.nb?dl=0>
<https://www.wolframcloud.com/obj/shurtleffr/Published/20211116AlternateRandomRunStatsDegrees.nb>
8. See Table 1 QSOs and Sec. 2 in Pelgrims, V. and Hutsemékers, D., Polarization alignments of quasars from the JVAS/CLASS 8.4-GHz surveys, MNRAS, 450, 4161–4173, doi: 10.1093/mnras/stv917, arXiv:astro-ph:1503.03482 (2015). The JVAS1450 catalog was kindly emailed to me by V. Pelgrims .
9. Jackson, N., Battye, R. A., Browne, I. W. A., Joshi, S., Muxlow, T. W. B., and Wilkinson, P. N., A survey of polarization in the JVAS/CLASS flat-spectrum radio source surveys - I. The data and catalogue production, MNRAS, 376, 371–377, doi: 10.1111/j.1365-2966.2007.11442.x , arXiv:astro-ph/0703273 (2007).
10. Heiles, C., An agglomeration of stellar polarization catalogs, Astron. J. 119, 923 (2000).
11. Berdyugin A., Piirola V., Teerikorpi P., Astron. Astrophys. 561, A24 DOI: 10.1051/0004-6361/201322604 (2014).

Part II the Appendix

00. Preface

The problem of “significance” is to determine the likelihood that random polarizations directions would have better alignment or avoidance than the observed polarization directions. We make polarization directions the topic discussed. But, more generally, one can analyze observations of any transverse directions on the sky.

Note that Part I the Article discusses the measure of alignment, the smallest alignment angle $\bar{\eta}_{\min}$ exclusively. However, the Hub Test also determines correlations of the transverse directions with points on the Celestial Sphere that the great circles avoid. In that case, the relevant quantity is the largest avoidance angle $\bar{\eta}_{\max}$ which is the maximum of the alignment function $\bar{\eta}(H)$. Both alignment and avoidance are treated equally in this part, Part II the Appendix.

A1. Introduction and Probability Distributions, Significance Formulas

Given a sample whose sources have measured transverse directions, say electromagnetic polarization or physical features like jets, then the alignment angle function $\bar{\eta}(H)$, Eq. (1) and Figs. 2,3, can be produced defined at all point H on the sphere except at the sources themselves. The directions are best aligned with the points H_{\min} and $-H_{\min}$ where the smallest alignment angle occurs, $\bar{\eta}_{\min}$. Thus $\bar{\eta}_{\min}$ gauges alignment. The largest alignment angle $\bar{\eta}_{\max}$ is found at hubs H_{\max} and $-H_{\max}$ and gauges a different correlation, avoidance.

The probability distributions for alignment and avoidance of samples with randomly oriented polarization directions involve two parameters η_0 and σ , one set for alignment, η_0^{\min} and σ^{\min} , and one set for avoidance, η_0^{\max} and σ^{\max} . We choose

$$P_{\min}(\eta) = \left(\frac{\text{norm}}{\sigma (2\pi)^{1/2}} \right) \left(1 + e^{4 \frac{(\eta - \eta_0 - \sigma)}{\sigma}} \right)^{-1} e^{-\frac{1}{2} \left(\frac{\eta - \eta_0}{\sigma} \right)^2}, \quad (\text{A1})$$

and

$$P_{\max}(\eta) = \left(\frac{\text{norm}}{\sigma (2\pi)^{1/2}} \right) \left(1 + e^{-4 \frac{(\eta - \eta_0 + \sigma)}{\sigma}} \right)^{-1} e^{-\frac{1}{2} \left(\frac{\eta - \eta_0}{\sigma} \right)^2}, \quad (\text{A2})$$

where norm is the normalization constant, $\text{norm} = 1.22029$, so that the definite integral over all η of the probability distributions is equal to one.

To find the significance, or p -value, of the smallest alignment angle $\bar{\eta}_{\min}$, one finds the likelihood of smaller random run values by integrating $P_{\min}(\eta)$ from below,

$$p(\bar{\eta}_{\min}) = \int_{-\infty}^{\bar{\eta}_{\min}} P_{\min}(\eta) \, d\eta \quad (\text{A3})$$

For avoidance, the largest avoidance angle $\bar{\eta}_{\max}$ is likely not as large as random runs with a significance of

$$p(\bar{\eta}_{\max}) = \int_{\bar{\eta}_{\max}}^{\infty} P_{\max}(\eta) \, d\eta. \quad (\text{A4})$$

By Eqs. (A1 - A4) the calculation of significances for a sample of correlated behavior indicated by the smallest alignment angle $\bar{\eta}_{\min}$ and the largest avoidance angle $\bar{\eta}_{\max}$ rests on obtaining values of the four probability distribution parameters η_0^{\min} , σ^{\min} , η_0^{\max} , and σ^{\max} .

The Direct Method A finds the four needed parameters η_0^{\min} , σ^{\min} , η_0^{\max} , and σ^{\max} by generating many, many random runs and analyzing the results. This notebook takes the four parameters from a wide selection of samples that have been analyzed elsewhere in other notebooks and estimates the four parameters for any given sample based on this ‘‘Library’’ of parameters from similar samples. Interpolation Method B makes an interpolation of the Library data, while Formula Method C employs a set of four functions that fit the Library data for the four needed parameters η_0^{\min} , σ^{\min} , η_0^{\max} , and σ^{\max} .

Sec. A2 describes the Library. Interpolation Method B is the topic of Sec. A3 and Sec. A4 has Formula Method C. The three previously studied samples are compared in Sec. A5.

A2 The Library

A2a. Preliminary

Definitions:

homeDirectory location of this notebook and many associated files. Other directories are used also.

mean the arithmetic average of a set of numbers, $\frac{1}{N} \sum_{i=1}^N n_i$

stanDev the standard deviation. Given a set of N numbers n_i with mean value m , the standard deviation is $\left(\frac{1}{N} \sum_{i=1}^N (n_i - m)^2\right)^{1/2}$, the square root of the average of the squares of the differences of the numbers with the mean. Note that we divide by N to get the average of the deviations squared.

norm a constant used to normalize the distribution so the integral of probability is 1.

probMIN0, probMAX0 probability distributions for η , dependent on parameters η_0, σ . [alignment (MIN) and avoidance (MAX)]

signiMIN0($\bar{\eta}_{\min}, \eta_0^{\min}, \sigma^{\min}$) significance of $\bar{\eta}_{\min}$, dependent on probability distribution parameters $\eta_0^{\min}, \sigma^{\min}$. (alignment)

signiMAX0($\bar{\eta}_{\max}, \eta_0^{\max}, \sigma^{\max}$) significance of $\bar{\eta}_{\max}$, dependent on probability distribution parameters $\eta_0^{\max}, \sigma^{\max}$. (alignment)

```
In[1]:= Print["The date and time that this statement was evaluated: ", Now]
Print["The computer time expended so far is ", TimeUsed[], " seconds."]
```

The date and time that this statement was evaluated: Mon 20 Dec 2021 13:04:48 GMT-5.

The computer time expended so far is 0.5 seconds.

```
In[3]:= homeDirectory =
"C:\\Users\\shurt\\Dropbox\\HOME_DESKTOP-0MRE50J\\SendXXX_CJP_CEJPetc\\SendViXra\\
20200715AlignmentMethod\\20210505AlignmentMethodv4\\20210515Clump1QSOsNearNGP";
```

```
In[4]:= mean[data_] := (1/Length[data]) Sum[data[[i4]], {i4, Length[data]}];
(* arithmetic average *)
stanDev[data_] :=
((1/Length[data]) Sum[(data[[i5]] - mean[data])^2, {i5, Length[data]}])^1/2
(*standard deviation*)
```

```
In[6]:= (* y = ((η - η0)/σ); dy = dη/σ *)
(* The normalization factor "norm" is needed to make
the integral of the probability distribution equal to unity. *)
norm = (1/(2 π)^1/2 NIntegrate[(1 + e^(4 (y-1)))^-1 e^(-y^2/2), {y, -∞, ∞}])^-1;
Print["The normalization scale factor is norm = ",
norm, " for both alignment probMIN0 and avoidance probMAX0."];
```

The normalization scale factor is norm = 1.22029 for both alignment probMIN0 and avoidance probMAX0.

```
In[8]:= probMIN0[η_, η0_, σ_] := (norm/(σ (2 π)^1/2)) (1 + e^(4 (η-η0)/σ))^-1 e^(-1/2 (η-η0)/σ)^2
signiMIN0[η_, η0_, σ_] := NIntegrate[probMIN0[η1, η0, σ], {η1, -∞, η}]
probMAX0[η_, η0_, σ_] := (norm/(σ (2 π)^1/2)) (1 + e^(-4 (η-η0)/σ))^-1 e^(-1/2 (η-η0)/σ)^2
signiMAX0[η_, η0_, σ_] := NIntegrate[probMAX0[η1, η0, σ], {η1, η, ∞}]
```

The significance signiMIN0[η, η_0, σ] is the Integral of probMIN0, i.e. signiMIN0 = $\int_{-\infty}^{\eta} P_{\text{MIN}}(\eta) d\eta$.

The significance $\text{signiMAX0}[\eta, \eta_0, \sigma]$ is the Integral of probMAX0 , i.e. $\text{signiMAX0} = \int_{\eta}^{\infty} P_{\text{MAX}}(\eta) d\eta$.

`In[12]:= (* {{nSrc, ρRgnRadius}, {x0min, dx0min}, {bmin, dbmin},
 {amin, damin}, {x0max, dx0max}, {bmax, dbmax}, {amax, damax},
 {σθrHminToCenter, θrHminToCenter}, {σθrHmaxToCenter, θrHmaxToCenter}} *)`

A2b. The Library data

Definitions:

`fitData0` This table is output from another notebook. Not all of the data is relevant here.

`powr[i,j]`, `fitDataReduced1` The number of digits are adjusted so that the standard error has two significant figures, 2 sig figs

`fitData` Parameters of the alignment (min) and avoidance (max) random run distributions. Originally in radians, converted to degrees below.

1a. `nSrc[i]` Number of sources 1b. `ρNomi[i]` Nominal radius, deg. 1c. `ρRMSi[i]` RMS radius, deg.

2a. `η0mini[i]` peak alignment distribution 2b. `dη0mini[i]` standard error

3a. `σmini[i]` half-width alignment distr. 3b. `dσmini[i]` standard error

4a. `η0maxi[i]` peak alignment distribution 4b. `dη0maxi[i]` standard error

5a. `σmaxi[i]` half-width alignment distr. 5b. `dσmaxi[i]` standard error

`wi[i]` inverse square root of the number of sources, $w = 1/N^{1/2}$

`τRMSi[i]` inverse RMS radius, in deg.^{-1}

`nSrcList` list of the number of sources in the Library samples

`wList` list of the inverse square root of the number of sources for Library samples

`ρNomList` list of the nominal radii, deg., of the Library samples

`τNomList` list of the inverse nominal radii

`colomnSrcList` assigned colors for the numbers of sources `nSrc` and the corresponding `w`

`colorρList` assigned colors for the nominal radii `ρNominal` and the corresponding inverses `τNominal`

`nSrcColorKeyLegend` for the number of sources `nSrc`

`wNColorKey` Legend for the number of sources `nSrc` and $w = nSrc^{-1/2}$

`ρNomColorKey` Legend for the nominal radii `ρNominal`

`ρτNomColorKey` Legend for the nominal radii `ρNominal` and their inverses, and

`τNominal`

`idsFORnSrc[iN]` ID #s in `fitData`, one set for each value of `nSrc`

`idsFORρNom[iρ]` ID #s in `fitData`, one set for each value of `ρNominal`

`lppη0min[iN]` list plot of Library data, alignment peak parameter η_0^{min} versus RMS radius ρ_{RMS}

`eta0VSradiusDataMin` Combined plot of the `lppη0min` for all Library data

`lppσmamin, sigmaMINvsRadiusData` Library data plots, except alignment half-width parameter σ^{min} replaces η_0^{min}

`lppη0max, eta0VSradiusDataMax` Library data plots, except avoidance peak η_0^{max} replaces η_0^{min}

`lppσmamax, sigmaMAXvsRadiusData` Plots of Library data

`lppτη0minPlusMinus[iN]` list plot of Library data, alignment peak parameter η_0^{min} versus RMS radius ρ_{RMS} with error bars

for one of the ρ Nominal values

Similar list plots: $\text{Ipr}\sigma_{\text{maminPlusMinus}}$, $\sigma_{\text{MINvsTauData}}$, $\text{Ipr}\eta_{0\text{maxPlusMinus}[iN]}$, $\eta_{0\text{maxVs}\tau\text{Data}}$, $\text{Ipr}\sigma_{\text{maxPlusMi-}}$
 $\text{nus}[iN]$, $\sigma_{\text{MAXvsTauData}}$.

```
In[15]:= (*In the last two hidden cells,
a table fitData0 is imported that has all the needed data for the Library,
but also extra data that is not needed. The following table,
fitData, is the result of removing the irrelevant quantities. And
it has been altered so the sig. figs. match the standard errors. *)
fitData1 = {{ {9., 0.004363, 0.0043, 10000.}, {0.598, 0.0013}, {0.1127, 0.0016},
  {0.977, 0.001700}, {0.1128, 0.002}}, {{ {9., 0.005818, 0.005734, 10000.},
  {0.5885, 0.0016}, {0.1118, 0.0019}, {0.9868, 0.001}, {0.1107, 0.001200}},
  {{ {9., 0.008727, 0.008601, 10000.}, {0.5707, 0.0011}, {0.1076, 0.0013},
  {1.00503, 0.000860}, {0.1069, 0.001}}, {{ {9., 0.017453, 0.017202, 10000.},
  {0.52541, 0.00068}, {0.09791, 0.000810}, {1.05165, 0.00079}, {0.09636, 0.000950}},
  {{ {9., 0.02618, 0.025803, 10000.}, {0.50422, 0.000610}, {0.09425, 0.000730},
  {1.07266, 0.000740}, {0.09407, 0.000890}}, {{ {9., 0.034907, 0.034406, 10000.},
  {0.49249, 0.00084}, {0.09335, 0.001}, {1.08406, 0.000770}, {0.09356, 0.000910}},
  {{ {9., 0.05236, 0.051615, 10000.}, {0.48150, 0.00076}, {0.09325, 0.000910},
  {1.09181, 0.00072}, {0.09353, 0.000860}}, {{ {9., 0.069813, 0.068831, 10000.},
  {0.47716, 0.000600}, {0.09572, 0.00072}, {1.0949, 0.0005}, {0.09324, 0.000600}},
  {{ {9., 0.139626, 0.137827, 10000.}, {0.47140, 0.00051}, {0.0943, 0.000610},
  {1.10081, 0.000610}, {0.09566, 0.000730}}, {{ {9., 0.733038, 0.758894, 10000.},
  {0.46917, 0.00062}, {0.09209, 0.00075}, {1.10139, 0.00043000}, {0.09214, 0.000520}},
  {{ {16., 0.004363, 0.003926, 10000.}, {0.6493, 0.0011}, {0.0839, 0.0014},
  {0.92532, 0.000950}, {0.083, 0.0011}}, {{ {16., 0.005818, 0.005234, 10000.},
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```

```

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```

```

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 {0.72336, 0.00016}, {0.019770, 0.00019}, {0.84781, 0.0001400}, {0.02023, 0.00017}}};

```

```
In[16]:= fitData = fitData1;
```

```
In[17]:= (*Identify the items in the fitData table with functions having recognizable names.*)
nSrci[i_] := fitData[[i, 1, 1]]
(*Convert fitData radians to DEGREES:*)
ρNomi[i_] := fitData[[i, 1, 2]]  $\left(\frac{360.}{2. \pi}\right)$  (*Use the nominal radius in degrees*)
ρRMSi[i_] := fitData[[i, 1, 3]]  $\left(\frac{360.}{2. \pi}\right)$  (*Use the RMS radius *)
ηθmini[i_] := fitData[[i, 2, 1]]  $\left(\frac{360.}{2. \pi}\right)$ 
dηθmini[i_] := fitData[[i, 2, 2]]  $\left(\frac{360.}{2. \pi}\right)$ 
σmini[i_] := fitData[[i, 3, 1]]  $\left(\frac{360.}{2. \pi}\right)$ 
dσmini[i_] := fitData[[i, 3, 2]]  $\left(\frac{360.}{2. \pi}\right)$ 
ηθmaxi[i_] := fitData[[i, 4, 1]]  $\left(\frac{360.}{2. \pi}\right)$ 
dηθmaxi[i_] := fitData[[i, 4, 2]]  $\left(\frac{360.}{2. \pi}\right)$ 
σmaxi[i_] := fitData[[i, 5, 1]]  $\left(\frac{360.}{2. \pi}\right)$ 
dσmaxi[i_] := fitData[[i, 5, 2]]  $\left(\frac{360.}{2. \pi}\right)$ 
```

```
In[28]:= wi[i_] :=  $\frac{1}{nSrci[i]^{1/2}}$  (*w = 1/N1/2*)
τRMSi[i_] :=  $\frac{1}{\rhoRMSi[i]}$  (* inverse RMS radius in inverse degrees*)
```

FYI: Much of the following is copied from "20211116AlternateRandomRunStatsDegrees.nb"

```
In[30]:= nSrcList = Union[Table[nSrci[i], {i, Length[fitData]}]];
wList = Table[ $\frac{1}{nSrcList[[i]]^{1/2.}}$ , {i, Length[nSrcList]}];
Print["The number of sources for samples in the Library are nSrc = ", nSrcList, ". "]
Print["Their inverse square roots are w = ", wList, ". "]
The number of sources for samples in the Library are nSrc =
{9., 16., 25., 36., 49., 64., 81., 121., 225.}.
Their inverse square roots are w =
{0.333333, 0.25, 0.2, 0.166667, 0.142857, 0.125, 0.111111, 0.0909091, 0.0666667} .
```

```

In[34]=  $\rho$ NomList = Union[Table[ $\rho$ Nomi[i], {i, Length[fitData]}]]; (*degrees*)
 $\tau$ NomList = Table[ $\frac{1}{\rho$ NomList[[i]]}, {i, Length[ $\rho$ NomList]}]; (*degrees-1*)
Print["The list of nominal radii is  $\rho =$ ",  $\rho$ NomList, " , in degrees. "]
Print["Their inverses are  $\tau$ Nom = ",  $\tau$ NomList, " , in deg-1."]

The list of nominal radii is  $\rho = \{0.249981, 0.333347, 0.50002,$ 
  0.999983, 1.5, 2.00002, 3.00001, 3.99999, 7.99998, 42.} , in degrees.
Their inverses are  $\tau$ Nom = {4.0003, 2.99988, 1.99992, 1.00002,
  0.666665, 0.499994, 0.333333, 0.250001, 0.125, 0.0238095} , in deg-1.

In[38]= (*Set up the color codes.*)
colorSrcList =
  Table[ColorData["Atoms", "ColorList"][[i]], {i, {55, 56, 57, 70, 74, 79, 89, 118, 65}}];
color $\rho$ List = Table[ColorData["Atoms", "ColorList"][[i]],
  {i, {52, 53, 54, 67, 71, 76, 86, 115, 62, 77}}];
nSrcColorKey = Table[{nSrcList[[i]], colornSrcList[[i]]}, {i, Length[nSrcList]}];
wNColorKey = {{nSrcList[[1]], colornSrcList[[1]], "1/3"},
  {nSrcList[[2]], colornSrcList[[2]], "1/4"}, {nSrcList[[3]],
  colornSrcList[[3]], "1/5"}, {nSrcList[[4]], colornSrcList[[4]], "1/6"},
  {nSrcList[[5]], colornSrcList[[5]], "1/7"}, {nSrcList[[6]], colornSrcList[[6]],
  "1/8"}, {nSrcList[[7]], colornSrcList[[7]], "1/9"}, {nSrcList[[8]],
  colornSrcList[[8]], "1/11"}, {nSrcList[[9]], colornSrcList[[9]], "1/15"}];
 $\rho$ NomColorKey = Table[ $\rho$ Nomi[i], color $\rho$ List[[i]]], {i, Length[ $\rho$ NomList]}];
 $\rho$  $\tau$ NomColorKey =
  Table[ $\rho$ Nomi[i] "o", color $\rho$ List[[i]],  $\rho$ Nomi[i]-1], {i, Length[ $\rho$ NomList]}];

In[43]= Print[" ",
  Text@Grid[Prepend[Table[{colornSrcList[[i]], nSrcList[[i]]}, {i, Length[nSrcList]}],
    {"Display code", "N"}], Spacings → {Automatic, .6},
    Dividers → {{Darker[Gray, .6], {Lighter[Gray, .5]}, Darker[Gray, .6]},
      {Darker[Gray, .6], Darker[Gray, .6], {False}, Darker[Gray, .6]}]],
  " ", Text@Grid[Prepend[Table[{color $\rho$ List[[i]],
    {"1/4", "1/3", "1/2", "1", "3/2", "2", "3", "4", "8", "42"}[[i]] "o"},
    {i, Length[ $\rho$ NomList]}], {"Display code", " $\rho$  (Nominal)"}], Spacings → {Automatic, .6},
    Dividers → {{Darker[Gray, .6], {Lighter[Gray, .5]}, Darker[Gray, .6]},
      {Darker[Gray, .6], Darker[Gray, .6], {False}, Darker[Gray, .6]}]]]
Print["Table A1: Color codes for variables  $N$  and  $\rho$ Nominal, used to label the Library
  data. Left: Display codes for the number of sources  $N$ . Right: Display codes for
  the nominal sample radius  $\rho$ Nominal. While we prefer the root-mean-square radius
   $\rho$ RMS for calculations, the RMS radius depends on  $N$  and does not work as a label. "]

```

Display code	N	Display code	ρ (Nominal)
	9.		$1/4^\circ$
	16.		$1/3^\circ$
	25.		$1/2^\circ$
	36.		1°
	49.		$3/2^\circ$
	64.		2°
	81.		3°
	121.		4°
	225.		8°
			42°

Table A1: Color codes for variables N and ρ_{Nominal} , used to label the Library data. Left: Display codes for the number of sources N . Right: Display codes for the nominal sample radius ρ_{Nominal} . While we prefer the root-mean-square radius ρ_{RMS} for calculations, the RMS radius depends on N and does not work as a label.

```
In[45]:= Table[idsFORnSrc[iN] = {}, {iN, Length[nSrcList]}];
(* ids give fitData table entries with nSrc sources.*)
Table[For[i = 1, i ≤ Length[fitData], i++, If[(nSrci[i] - nSrcList[[iN]])2 < 1,
AppendTo[idsFORnSrc[iN], i]], {iN, Length[nSrcList]}];
iN = 3;
Print["For example, idsFORnSrc[" , iN,
"] gives the ids for the records in the fitData table for iN = " , iN,
" and nSrc = " , nSrcList[[iN]], " They are ids = " , idsFORnSrc[iN], "."]
Clear[
iN]

For example, idsFORnSrc[3] gives the ids for the records in the fitData table for iN =
3 and nSrc = 25. They are ids = {21, 22, 23, 24, 25, 26, 27, 28, 29, 30}.

In[50]:= Table[idsFORρNom[iρ] = {}, {iρ, Length[ρNomList]}];
Table[For[i = 1, i ≤ Length[fitData], i++, If[Abs[ $\frac{(\rho_{\text{Nomi}}[i] - \rho_{\text{NomList}}[[i\rho]])}{\text{Min}[\rho_{\text{Nomi}}[i], \rho_{\text{NomList}}[[i\rho]]]}$ ] < 10-2,
AppendTo[idsFORρNom[iρ], i]], {iρ, Length[ρNomList]}];
iρ = 3;
Print["For example, idsFORρNom[" , iρ,
"] gives the ids for the records in the fitData table for iρ = " ,
iρ, " with nominal radius ρ = " , ρNomList[[iρ]],
" degrees; record numbers: " , idsFORρNom[iρ], "."]
Clear[iρ]

For example, idsFORρNom[3] gives the ids for the records in the fitData table for iρ = 3
with nominal radius ρ = 0.50002 degrees; record numbers: {3, 13, 23, 33, 43, 53, 63, 73, 83}.

In[55]:= (*Do not DROP!! Needed as legends for some figures in Part I the Article.*)
(*Print["{N, color-code, w = N-1/2} = " , wNColorKey]
Print["{ρNom, color-code, τNom = ρNom-1} = " , ρτNomColorKey ]
*)
```

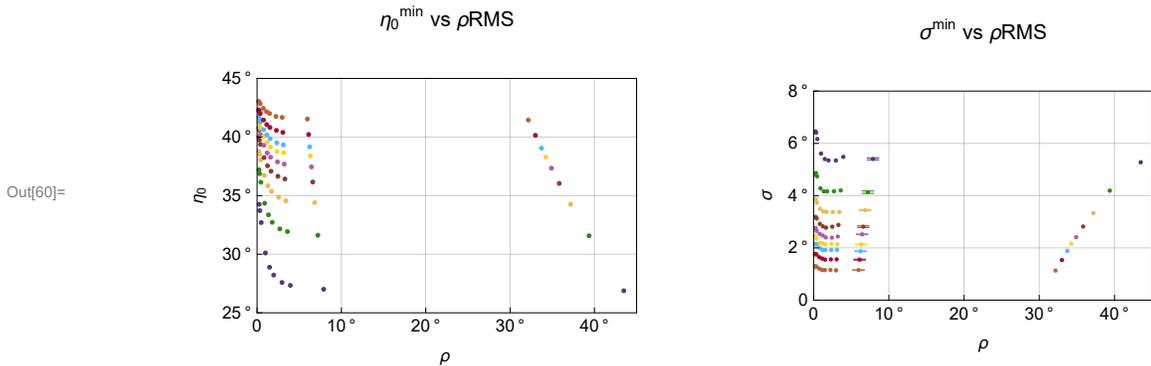
```

In[56]:= lpprhoηmin[iN_] := ListPlot[Table[{ρRMSi[i], η0mini[i]}, {i, idsFORnSrc[iN]}],
  PlotRange → {{0, 45}, {25, 45}}, FrameLabel → {"ρ", "η0"},
  FrameTicks → { {{25, 25 °}, {30, 30 °}, {35, 35 °}, {40, 40 °}, {45, 45 °}}, None },
  {{{0, 0 °}, {10, 10 °}, {20, 20 °}, {30, 30 °}, {40, 40 °}}, None }},
  PlotLabel → "η0min vs ρRMS", PlotStyle → colornSrcList[[iN]],
  GridLines → Automatic, Frame → True, ImageSize → 72 × 6];
eta0VSradiusDataMin = Show[Table[lpprhoηmin[iN], {iN, Length[wList]}]];
(*Print[" {N, color} = ", nSrcColorKey]*)

In[58]:= lpprhoσmin[iN_] := ListPlot[Table[{ρRMSi[i],
  If[8.5 < Mod[i, Length[ρNomList]] < 9.5, Around[σmini[i], dσmini[i]], σmini[i]]],
  {i, idsFORnSrc[iN]}], PlotRange → {{0.0, 45}, {0.0, 8.}}, FrameLabel → {"ρ", "σ"},
  FrameTicks → { {{0, 0 °}, {2, 2 °}, {4, 4 °}, {6, 6 °}, {8, 8 °}}, None },
  {{{0, 0 °}, {10, 10 °}, {20, 20 °}, {30, 30 °}, {40, 40 °}}, None }},
  PlotLabel → "σmin vs ρRMS", PlotStyle → colornSrcList[[iN]], GridLines → Automatic,
  Frame → True, IntervalMarkersStyle → colornSrcList[[iN]], ImageSize → 72 × 4];
sigmaMINvsRadiusData = Show[Table[lpprhoσmin[iN], {iN, Length[wList]}]];

In[60]:= GraphicsRow[{eta0VSradiusDataMin, sigmaMINvsRadiusData}, ImageSize → 72 × 8]
Print[" {N, color} = ", nSrcColorKey]
Print["Figure A1: The Library data for probability distribution parameters η0min
  and σmin plotted with respect to ρRMS. Fig. A1 is the same as Fig. 6."]

```



```

{N, color} =
{{9., ■}, {16., ■}, {25., ■}, {36., ■}, {49., ■}, {64., ■}, {81., ■}, {121., ■}, {225., ■}}

```

Figure A1: The Library data for probability distribution parameters η_0^{\min} and σ^{\min} plotted with respect to ρ RMS. Fig. A1 is the same as Fig. 6.

```

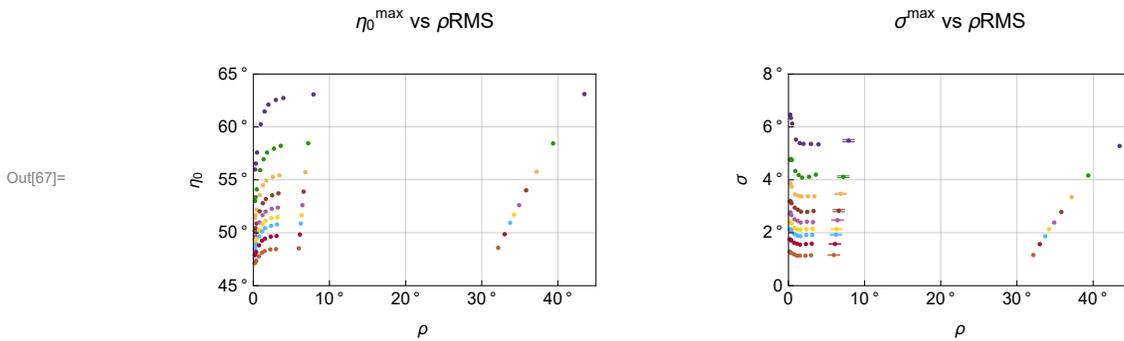
In[63]:= lpprhoηmax[iN_] := ListPlot[Table[{ρRMSi[i], η0maxi[i]}, {i, idsFORnSrc[iN]}],
  PlotRange → {{0, 45}, {45, 65}}, FrameLabel → {"ρ", "η0"},
  FrameTicks → { {{45, 45 °}, {50, 50 °}, {55, 55 °}, {60, 60 °}, {65, 65 °}}, None },
  {{{0, 0 °}, {10, 10 °}, {20, 20 °}, {30, 30 °}, {40, 40 °}}, None }},
  PlotLabel → "η0max vs ρRMS", PlotStyle → colornSrcList[[iN]],
  GridLines → Automatic, Frame → True, ImageSize → 72 × 5]
eta0VSradiusDataMax = Show[Table[lpprhoηmax[iN], {iN, Length[wList]}]];
(*Print[" {N, color} = ", nSrcColorKey]*)

```

```
In[65]:= lppSigmaMax[iN_] := ListPlot[Table[{ρRMSi[i],
  If[8.5 < Mod[i, Length[ρNomList]] < 9.5, Around[σmaxi[i], dσmaxi[i], σmaxi[i]],
  {i, idsFORnSrc[iN]}], PlotRange → {{0.0, 45}, {0.0, 8.}}, FrameLabel → {"ρ", "σ"},
  FrameTicks → { {{0, 0°}, {2, 2°}, {4, 4°}, {6, 6°}, {8, 8°}}, None } ,
  {{0, 0°}, {10, 10°}, {20, 20°}, {30, 30°}, {40, 40°}}, None } },
  PlotLabel → "σmax vs ρRMS", PlotStyle → colornSrcList[[iN]], GridLines → Automatic,
  Frame → True, IntervalMarkersStyle → colornSrcList[[iN]], ImageSize → 72 × 4];
sigmaMAXvsRadiusData = Show[Table[lppSigmaMax[iN], {iN, Length[wList]}]];
```

```
In[67]:= GraphicsRow[{eta0VSRadiuDataMax, sigmaMAXvsRadiusData}, ImageSize → 72 × 8]
Print["{N, color} = ", nSrcColorKey]
Print["Figure A2: The Library data for distribution parameters
  η0max and σmax plotted with respect to ρRMS. Like the min data in Figs
  6 and A1, the max data for constant N has curves for ρ below 5°."]

```



```
{N, color} =
{{9., Purple}, {16., Green}, {25., Orange}, {36., Brown}, {49., Purple}, {64., Yellow}, {81., Blue}, {121., Red}, {225., Brown}}
```

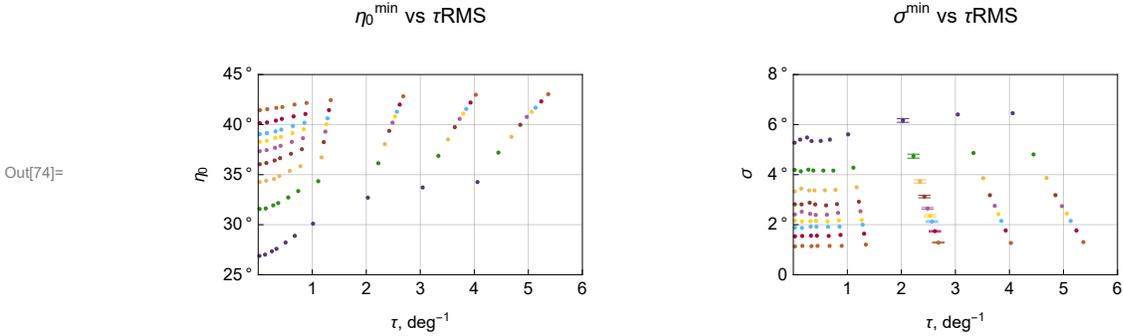
Figure A2: The Library data for distribution parameters

η_0^{\max} and σ^{\max} plotted with respect to ρ RMS. Like the min data in Figs 6 and A1, the max data for constant N has curves for ρ below 5° .

```
In[70]:= lprη0minPlusMinus[iN_] :=
ListPlot[Table[{τRMSi[i], If[2.5 < Mod[i, Length[ρNomList]] < 3.5,
  Around[η0mini[i], dη0mini[i], η0mini[i]], {i, idsFORnSrc[iN]}],
  PlotRange → {{0, 6}, {25, 45}}, FrameLabel → {"τ, deg-1", "η0"},
  FrameTicks → { {{25, 25°}, {30, 30°}, {35, 35°}, {40, 40°}, {45, 45°}}, None } ,
  {Table[i, {i, 6}], None } }, PlotLabel → "η0min vs τRMS",
  PlotStyle → colornSrcList[[iN]], GridLines → Automatic, Frame → True,
  IntervalMarkers → "Tubes", IntervalMarkersStyle → colornSrcList[[iN]], ImageSize → 72 × 5]
η0minVSτData = Show[Table[lprη0minPlusMinus[iN], {iN, Length[wList]}]];
```

```
In[72]:= lprSigmaMinPlusMinus[iN_] :=
ListPlot[Table[{τRMSi[i], If[2.5 < Mod[i, Length[ρNomList]] < 3.5,
  Around[σmini[i], dσmini[i], σmini[i]], {i, idsFORnSrc[iN]}],
  PlotRange → {{0, 6}, {0, 8}}, FrameLabel → {"τ, deg-1", "σ"}, FrameTicks →
  { {{0, 0°}, {2, 2°}, {4, 4°}, {6, 6°}, {8, 8°}}, None } , {Table[i, {i, 6}], None } },
  PlotLabel → "σmin vs τRMS", PlotStyle → colornSrcList[[iN]], GridLines → Automatic,
  Frame → True, IntervalMarkersStyle → colornSrcList[[iN]], ImageSize → 72 × 4]
sigmaMINvsTauData = Show[Table[lprSigmaMinPlusMinus[iN], {iN, Length[wList]}]];
```

```
In[74]:= GraphicsRow[{η0minVSτData, sigmaMINvsTauData}, ImageSize → 72 × 8]
Print[" {N, color, w = N-1/2} = ", wNColorKey]
Print["Figure A3: The Library data for probability distribution parameters η0min
and σmin plotted with respect to τRMS. (Fig. A3 is a copy of Fig. 7.)"]
```



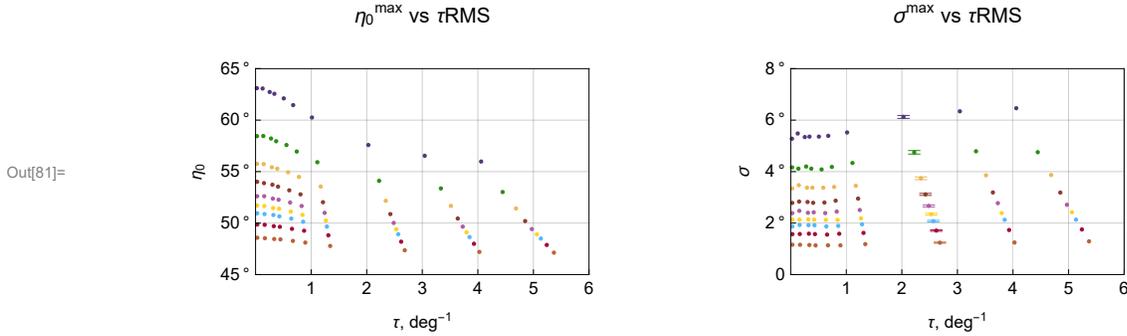
```
{N, color, w = N-1/2} = {{9., ■, 1/3}, {16., ■, 1/4}, {25., ■, 1/5}, {36., ■, 1/6},
{49., ■, 1/7}, {64., ■, 1/8}, {81., ■, 1/9}, {121., ■, 1/11}, {225., ■, 1/15}}
```

Figure A3: The Library data for probability distribution parameters η_0^{\min} and σ^{\min} plotted with respect to τ_{RMS} . (Fig. A3 is a copy of Fig. 7.)

```
In[77]:= lpτη0maxPlusMinus[iN_] :=
ListPlot[Table[{τRMSi[i], If[2.5 < Mod[i, Length[ρNomList]] < 3.5,
Around[η0maxi[i], dη0maxi[i], η0maxi[i]]], {i, idsFORnSrc[iN]}],
PlotRange → {{0, 6}, {45, 65}}, FrameLabel → {"τ, deg-1", "η0"},
FrameTicks → { {{45, 45 °}, {50, 50 °}, {55, 55 °}, {60, 60 °}, {65, 65 °}}, None } ,
{Table[i, {i, 6}], None }}, PlotLabel → "η0max vs τRMS",
PlotStyle → colornSrcList[[iN]], GridLines → Automatic, Frame → True,
IntervalMarkers → "Tubes", IntervalMarkersStyle → colornSrcList[[iN]], ImageSize → 72 × 5]
η0maxVSτData = Show[Table[lpτη0maxPlusMinus[iN], {iN, Length[wList]}]]];
```

```
In[79]:= lpτSigmaxPlusMinus[iN_] :=
ListPlot[Table[{τRMSi[i], If[2.5 < Mod[i, Length[ρNomList]] < 3.5,
Around[σmaxi[i], dσmaxi[i], σmaxi[i]]], {i, idsFORnSrc[iN]}],
PlotRange → {{0, 6}, {0, 8}}, FrameLabel → {"τ, deg-1", "σ"}, FrameTicks →
{ {{0, 0 °}, {2, 2 °}, {4, 4 °}, {6, 6 °}, {8, 8 °}}, None } , {Table[i, {i, 6}], None }},
PlotLabel → "σmax vs τRMS", PlotStyle → colornSrcList[[iN]], GridLines → Automatic,
Frame → True, IntervalMarkersStyle → colornSrcList[[iN]], ImageSize → 72 × 4]
sigmaMAXvsTauData = Show[Table[lpτSigmaxPlusMinus[iN], {iN, Length[wList]}]]];
```

```
In[81]:= GraphicsRow[{ $\eta_0^{\max}$ VS $\tau$ Data, sigmaMAXvsTauData}, ImageSize  $\rightarrow$  72  $\times$  8]
Print[" {N, color, w} = ", wNColorKey]
Print["Figure A4: The Library data for probability
distribution parameters  $\eta_0^{\max}$  and  $\sigma^{\max}$  plotted with respect to
 $\tau$ RMS. Compared with Fig. A2, the data is spread out nicely. "]
```



```
{N, color, w} = {{9.,  $\blacksquare$ , 1/3}, {16.,  $\blacksquare$ , 1/4}, {25.,  $\blacksquare$ , 1/5}, {36.,  $\blacksquare$ , 1/6},
{49.,  $\blacksquare$ , 1/7}, {64.,  $\blacksquare$ , 1/8}, {81.,  $\blacksquare$ , 1/9}, {121.,  $\blacksquare$ , 1/11}, {225.,  $\blacksquare$ , 1/15}}
```

Figure A4: The Library data for probability distribution parameters η_0^{\max} and σ^{\max} plotted with respect to τ RMS. Compared with Fig. A2, the data is spread out nicely here.

A3. Interpolation Method B

The Library constructed for Method B in Sec. A2 is essentially a table of the values of the four parameters, η_0^{\min} , σ^{\min} , η_0^{\max} , and σ^{\max} , needed to determine the probability distributions and significances in Eqs. (A1-A4).

Instead of the variables N and ρ RMS, the number of sources and the root-mean-square radius, we choose to consider the four parameters as functions of w and τ RMS, the inverse square root of N and the inverse of the radius ρ RMS,

$$w = N^{-1/2} \quad \text{and} \quad \tau\text{RMS} = \rho\text{RMS}^{-1}. \quad (\text{A5})$$

The Library data is graphed in Figs. 9 and A1 for the smallest alignment angle $\bar{\eta}_{\min}$. The data for avoidance, *i.e.* the largest avoidance angle $\bar{\eta}_{\max}$, is plotted in Figure A4 in terms of w and τ RMS.

Definitions:

Tables: $w\tau\eta_0^{\min}\text{Lib}$, $w\tau d\eta_0^{\min}\text{Lib}$, $w\tau\eta_0^{\max}\text{Lib}$, $w\tau d\eta_0^{\max}\text{Lib}$, $w\tau\sigma^{\min}\text{Lib}$, $w\tau d\sigma^{\min}\text{Lib}$, $w\tau\sigma^{\max}\text{Lib}$, $w\tau d\sigma^{\max}\text{Lib}$

The tables $w\tau\eta_0^{\min}\text{Lib}$... have Library data in the form $[(w, \tau\text{RMS}), \text{quantity}]$ where “quantity” is one of the parameters or their standard errors: η_0^{\min} , $d\eta_0^{\min}$, σ^{\min} , $d\sigma^{\min}$, η_0^{\max} , $d\eta_0^{\max}$, σ^{\max} , $d\sigma^{\max}$

The associated interpolation functions are $\eta_0^{\min}\text{Bint}$, $d\eta_0^{\min}\text{Bint}$, $\eta_0^{\max}\text{Bint}$, $d\eta_0^{\max}\text{Bint}$, $\sigma^{\min}\text{Bint}$, $d\sigma^{\min}\text{Bint}$, $\sigma^{\max}\text{Bint}$, $d\sigma^{\max}\text{Bint}$

nExample, ρ RMSexample Random values of the number of sources and the RMS radius are used to illustrate the distribution parameter outputs of Interpolation Method B

Setting up the interpolations takes two steps. First a tables of the data are constructed. Each table has the form $\{w, \tau\text{RMS}$,

parameter}. Second, the interpolation for each parameter is defined. There are four parameters η_0^{\min} , σ^{\min} , η_0^{\max} , and σ^{\max} and each one has a standard error $d\eta_0^{\min}$, $d\sigma^{\min}$, $d\eta_0^{\max}$, and $d\sigma^{\max}$ developed in the fitting process.

```
In[84]:= wτη0minLib = Table[{wi[i], τRMSi[i], η0mini[i]}, {i, Length[fitData]}];
wτdη0minLib = Table[{wi[i], τRMSi[i], dη0mini[i]}, {i, Length[fitData]}];
wτη0maxLib = Table[{wi[i], τRMSi[i], η0maxi[i]}, {i, Length[fitData]}];
wτdη0maxLib = Table[{wi[i], τRMSi[i], dη0maxi[i]}, {i, Length[fitData]}];
wτσminLib = Table[{wi[i], τRMSi[i], σmini[i]}, {i, Length[fitData]}];
wτdσminLib = Table[{wi[i], τRMSi[i], dσmini[i]}, {i, Length[fitData]}];
wτσmaxLib = Table[{wi[i], τRMSi[i], σmaxi[i]}, {i, Length[fitData]}];
wτdσmaxLib = Table[{wi[i], τRMSi[i], dσmaxi[i]}, {i, Length[fitData]}];

In[92]:= η0minBint = Interpolation[wτη0minLib]; (* int - interpolation function*)
dη0minBint = Interpolation[wτdη0minLib];
η0maxBint = Interpolation[wτη0maxLib];
dη0maxBint = Interpolation[wτdη0maxLib];
σminBint = Interpolation[wτσminLib];
dσminBint = Interpolation[wτdσminLib];
σmaxBint = Interpolation[wτσmaxLib];
dσmaxBint = Interpolation[wτdσmaxLib];
```

... **Interpolation:** Interpolation on unstructured grids is currently only supported for InterpolationOrder->1 or InterpolationOrder->All. Order will be reduced to 1.

By the rules of interpolations, when the variables w and τ are in the range of the Library data, then Mathematica finds an average value from the surrounding Library data points. In terms of the variables w and τ RMS, the ranges are

$$\frac{1}{15} \leq w \leq \frac{1}{3} \quad \text{and} \quad 0.024 \text{ deg}^{-1} \leq \tau\text{RMS} \leq 4 \text{ deg}^{-1} \quad (\text{Ranges of Interpolation Variables}) \quad (\text{A6})$$

The values for τ RMS are only approximate because the limits shown are values of $1/\rho$ Nominal and the nominal values only approximate the root-mean-square values.

```
In[100]:= (*An example of using the interpolating functions.*)
nExample = RandomInteger[{9, 225}]; ρRMSexample = RandomReal[{0.25, 12}];
Print["The most likely value of the two random
run distribution parameters η0^min and σ^min (alignment) for ",
nExample, " sources in a region with RMS radius ", ρRMSexample,
"° is found by interpolation of Library data: "]
Print["Given N = ", nExample, " and ρRMS = ", ρRMSexample,
"°, we find that w = N^-1/2 = ", N[nExample^-1/2], ", and τ = ρRMS^-1 = ", ρRMSexample^-1,
". The interpolations give η0^min = ", η0minBint[nExample^-1/2, ρRMSexample^-1], "°, σ^min = ",
σminBint[nExample^-1/2, ρRMSexample^-1], "°, η0^max = ", η0maxBint[nExample^-1/2, ρRMSexample^-1],
"°, σ^max = ", σmaxBint[nExample^-1/2, ρRMSexample^-1], "°." ]
Print["Compare these values with the Library data plotted in Figs. A3 and A4."]
```

The most likely value of the two random run distribution parameters η_{θ}^{\min} and σ^{\min} (alignment) for 25 sources in a region with RMS radius 3.08468° is found by interpolation of Library data: Given $N = 25$ and $\rho_{\text{RMS}} = 3.08468^{\circ}$, we find that $w = N^{-1/2} = 0.2$, and $\tau = \rho_{\text{RMS}}^{-1} = 0.324183$. The interpolations give $\eta_{\theta}^{\min} = 34.6589^{\circ}$, $\sigma^{\min} = 3.3701^{\circ}$, $\eta_{\theta}^{\max} = 55.3769^{\circ}$, $\sigma^{\max} = 3.3757^{\circ}$. Compare these values with the Library data plotted in Figs. A3 and A4.

A4. Fit the Library Data with Functions, Function Method C

Applying Interpolation Method B when one or both sample's variables are outside the Library data set, results in Mathematica extrapolating, not interpolating. Instead of interpolating Library data points that surround the sample's variables, Mathematica guesses what lies beyond the Library's boundaries. In such a case or for other situations that arise, one can apply alternative Formula Method C to find the distribution parameters η_0^{\min} , σ^{\min} , η_0^{\max} , and σ^{\max} and, with Eqs. (A4, A5), the significances desired.

Formula Method C finds formulas to fit the four distribution parameters $\eta_0^{\min}(w, \tau_{\text{RMS}})$, $\sigma^{\min}(w, \tau_{\text{RMS}})$, $\eta_0^{\max}(w, \tau_{\text{RMS}})$, and $\sigma^{\max}(w, \tau_{\text{RMS}})$.

Definitions:

Alignment:

Library data fitting functions:

`eta0minFit[w,τ]`, `eta0minFitbig[w,τ]`, `eta0minFitsmall[w,τ]`, `deta0minFit[w,τ]`
`sigmaminFit[w,τ]`, `sigminFitBig[w,τ]`, `sigminFitSmall[w,τ]`, `dsigmaminFit[w,τ]`

Plots of the Library data fitting function for the iN^{th} value of w :

`plotTauEtamin[iN]`, `plotTauEtaminbig[iN]`, `plotTauEtaminsmall[iN]`,
`plotTausigmamin[iN]`, ...

Display of the fitting functions for all values of w :

`eta0MinVSTauFit` (Big, Best, Small) and `eta0MinVSTauFit0` (Best only)

`sortPercentDiffEta0minfit` percent differences between the Library data and the relevant fitting function, here for η_0^{\min} .

`sortPercentDiffSigmaminfit` Same, but for σ^{\min}

Avoidance:

REPEAT ALL OF THE ABOVE AGAIN, BUT THIS TIME WITH "MAX", NOT "MIN".

`eta0maxFit[w,τ]`, ..., `sortPercentDiffSigmamaxfit`

The fitting functions are derived in another notebook, "20211116AlternateRandomRunStatsDegrees.nb", Ref. 7. Mathematica supplies a lot of extra digits when the formulas are copied. Please forgive the appearance. I am rounding the coefficients to two more places than the change in value from best to big. For example, the best value of the first coefficient in `eta0minFit(w,τ)` is `45.02686206815433`` with a value in `eta0minFitbig(w,τ)` of `45.04343933099011``. These values are replaced by 45.0269 and 45.0434, respectively.

```

In[104]:= (*Equation (A7)*)
eta0minFit[w_, τ_] :=
  45.0269 - w (47.386 + 7.32 w - 17.789 Tanh[(0.7096 - 0.3488 w) (-0.5348 + τ)])

In[105]:= (*Equation (A8)*) eta0minFitbig[w_, τ_] :=
  45.0434 - w (47.031 + 6.83 w + (-17.789 + 0.302 Sign[(-0.7096 + 0.3488 w) (-0.5348 + τ)])
    Tanh[(-0.5348 + τ + 0.0254 Sign[0.7096 - 0.3488 w])
      (0.7096 + w (-0.3488 + 0.0321 Sign[-0.5348 + τ]) + 0.0137 Sign[-0.5348 + τ])])

In[106]:= (*Equation (A9)*) eta0minFitsmall[w_, τ_] :=
  45.0103 - w (47.741 + 7.81 w + (-17.789 - 0.302 Sign[(-0.7096 + 0.3488 w) (-0.5348 + τ)])
    Tanh[(-0.5348 + τ - 0.0254 Sign[0.7096 - 0.3488 w])
      (0.7096 + w (-0.3488 - 0.0321 Sign[-0.5348 + τ]) - 0.0137 Sign[-0.5348 + τ])])

In[107]:= deta0minFit[w_, τ_] := eta0minFitbig[w, τ] - eta0minFit[w, τ]

In[108]:=
plotTauEtamin[iN_] :=
  Plot[eta0minFit[wList[[iN]], τ], {τ, 0, 6}, PlotStyle → colornSrcList[[iN]]];
plotTauEtaminbig[iN_] := Plot[eta0minFitbig[wList[[iN]], τ],
  {τ, 0, 6}, PlotStyle → colornSrcList[[iN]]];
plotTauEtaminsmall[iN_] := Plot[eta0minFitsmall[wList[[iN]], τ],
  {τ, 0, 6}, PlotStyle → colornSrcList[[iN]]];

In[111]:= eta0MinVSTauFit =
  Show[Table[Show[{lpτη0minPlusMinus[iN], plotTauEtamin[iN], plotTauEtaminbig[iN],
    plotTauEtaminsmall[iN]}], {iN, Length[nSrcList]}], ImageSize → 72 × 5];
eta0MinVSTauFit0 = Show[Table[Show[{lpτη0minPlusMinus[iN], plotTauEtamin[iN]}],
  {iN, Length[nSrcList]}], ImageSize → 72 × 5];

In[113]:= (*Equation (A10)*)
sigmaminFit[w_, τ_] := 0.25 w (73.570 - 8.29 w + (3.093 + 10.658 w) Tanh[1.2161 (-1.6072 + τ)])

In[114]:= (*Equation (A11)*)
sigminFitBig[w_, τ_] := 0.25 w
  (73.679 - 7.86 w + (3.093 + w (10.658 + 0.508 Sign[-1.6072 + τ]) + 0.126 Sign[-1.6072 + τ])
    Tanh[(-1.6072 + τ + 0.0202 Sign[3.093 + 10.658 w]) (1.2161 + 0.0441 Sign[-1.6072 + τ])])

In[115]:= (*Equation (A12)*)
sigminFitSmall[w_, τ_] := 0.25 w
  (73.460 - 8.73 w + (3.093 + w (10.658 - 0.508 Sign[-1.6072 + τ]) - 0.126 Sign[-1.6072 + τ])
    Tanh[(-1.6072 + τ - 0.0202 Sign[3.093 + 10.658 w]) (1.2161 - 0.0441 Sign[-1.6072 + τ])])

In[116]:= dsigmaminFit[w_, τ_] := sigminFitBig[w, τ] - sigmaminFit[w, τ]

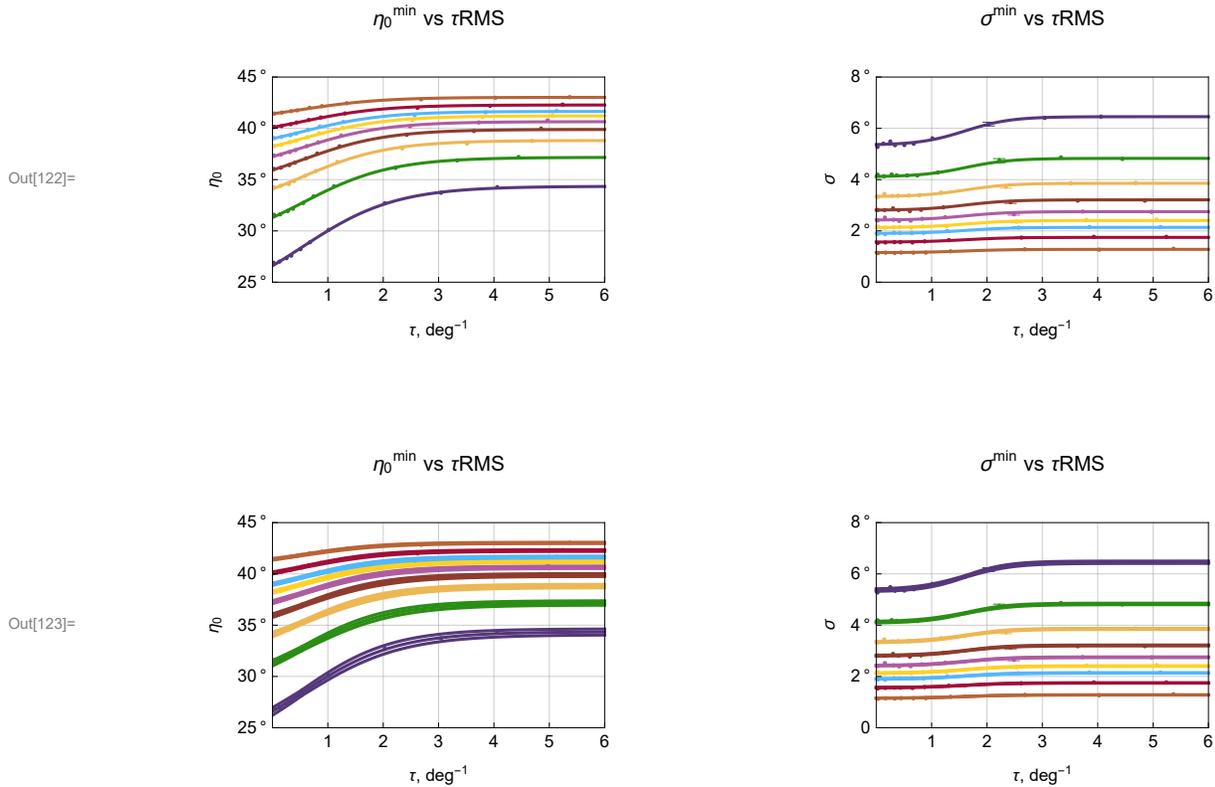
In[117]:=
plotTausigmamin[iN_] :=
  Plot[sigmaminFit[wList[[iN]], τ], {τ, 0, 6}, PlotStyle → colornSrcList[[iN]]];
plotTausigmaminbig[iN_] := Plot[sigminFitBig[wList[[iN]], τ],
  {τ, 0, 6}, PlotStyle → colornSrcList[[iN]]];
plotTausigmaminsmall[iN_] := Plot[sigminFitSmall[wList[[iN]], τ],
  {τ, 0, 6}, PlotStyle → colornSrcList[[iN]]];

```

```

In[120]:= sigmaMinVSTauFit =
  Show[Table[Show[{lpτSigmaminPlusMinus[iN], plotTausigmamin[iN], plotTausigmaminbig[iN],
    plotTausigmaminsmall[iN]}], {iN, Length[nSrcList]}], ImageSize → 72 × 5];
sigmaMinVSTauFitθ = Show[Table[Show[{lpτSigmaminPlusMinus[iN], plotTausigmamin[iN]}],
  {iN, Length[nSrcList]}], ImageSize → 72 × 5];
In[122]:= GraphicsRow[{etaθMinVSTauFitθ, sigmaMinVSTauFitθ}, ImageSize → 72 × 9]

```



```

{N, color, w} = {{9., #1, 1/3}, {16., #2, 1/4}, {25., #3, 1/5}, {36., #4, 1/6},
  {49., #5, 1/7}, {64., #6, 1/8}, {81., #7, 1/9}, {121., #8, 1/11}, {225., #9, 1/15}}

```

Figure A5: The fitting functions and Library data for alignment parameters η_0^{\min} , σ^{\min} . *Top*: The best fit to the data. *Bottom*: One sigma accuracy of the fit. For the η_0^{\min} fit, this means that each of the 7 parameters in Eq. (6) is replaced by a value that differs from its best value by its standard error divided by $\sqrt{7}$, with the expectation that this will move the curve by one σ . We plot Big, Small, and Best curves.

```

In[126]:= sortPercentDiffEta0minfit =
  Sort[Flatten[Table[100. Abs[( $\eta_{\theta}^{\min}[i] - \text{eta0minFit}[wi[i], \tau\text{RMSi}[i]]$ ) /  $\eta_{\theta}^{\min}[i]$ ],
    {i, Length[fitData]}]]];
ListPlot[sortPercentDiffEta0minfit, PlotRange → All, FrameLabel → {"", "% diff  $\eta_{\theta}^{\min}$  "},
  PlotLabel → " $\eta_{\theta}^{\min}$  fit", GridLines → Automatic, Frame → True]
Print["Figure A6: The percent difference of the fit to the
  data for  $\eta_{\theta}^{\min}$ . The average, i.e. arithmetic mean, %diff is ",
  mean[sortPercentDiffEta0minfit], "%, with the largest %diff = ",
  sortPercentDiffEta0minfit[[-1]], "%." ]

```

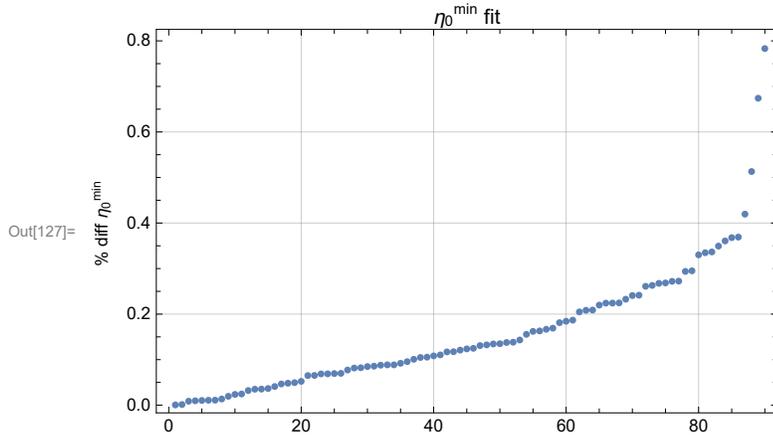


Figure A6: The percent difference of the fit to the data for η_{θ}^{\min} . The average, i.e. arithmetic mean, %diff is 0.161082%, with the largest %diff = 0.783077%.

```

In[129]:= sortPercentDiffSigmaminfit =
  Sort[Flatten[Table[100. Abs[( $\sigma_{\min}[i]$  - sigmaminFit[wi[i],  $\tau_{\text{RMSi}}[i]$ ]) /  $\sigma_{\min}[i]$ ],
    {i, Length[fitData]}]]];
ListPlot[sortPercentDiffSigmaminfit, PlotRange → All, FrameLabel → {"", "% diff  $\sigma_{\min}$  "},
  PlotLabel → " $\sigma^{\min}$  fit", GridLines → Automatic, Frame → True]
Print["Figure A7: The percent difference of
  the fit to the data for  $\sigma^{\min}$ . The mean %diff is ",
  mean[sortPercentDiffSigmaminfit], "%, with the largest %diff = ",
  sortPercentDiffSigmaminfit[[-1]], "%." ]

```

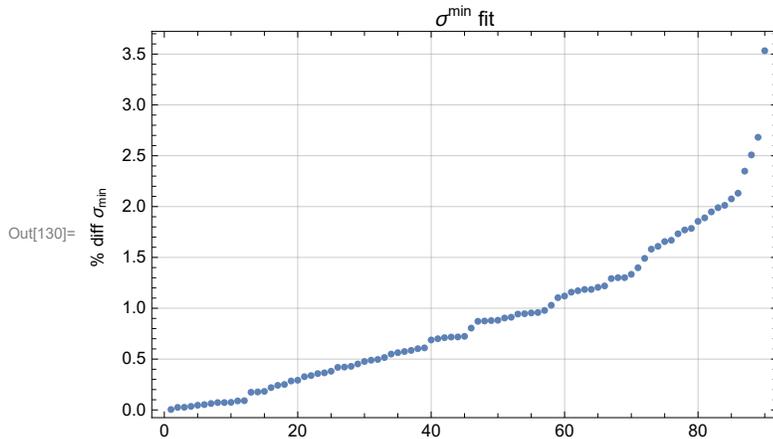


Figure A7: The percent difference of the fit to the data for σ^{\min} . The mean %diff is 0.909478%, with the largest %diff = 3.53309%.

```

In[132]:= (*Equation (A13)*)
eta0maxFit[w_,  $\tau$ _] := 45.1455 + w (44.230 + 8.35 w - 14.632 Tanh[0.6808 (-0.8608 +  $\tau$ )])

In[133]:= (*Equation (A14)*)
eta0maxFitbig[w_,  $\tau$ _] := 45.1632 + w (44.483 + 8.85 w + (-14.632 + 0.179 Sign[-0.8608 +  $\tau$ ])
  Tanh[(-0.8777 +  $\tau$ ) (0.6808 + 0.0106 Sign[0.8608 -  $\tau$ ])])

In[134]:= (*Equation (A15)*)
eta0maxFitsmall[w_,  $\tau$ _] := 45.1279 + w (43.977 + 7.85 w + (-14.632 - 0.179 Sign[-0.8608 +  $\tau$ ])
  Tanh[(-0.8439 +  $\tau$ ) (0.6808 - 0.0106 Sign[0.8608 -  $\tau$ ])])

In[135]:= deta0maxFit[w_,  $\tau$ _] := eta0maxFitbig[w,  $\tau$ ] - eta0maxFit[w,  $\tau$ ]

In[136]:=
plotTauEtamax[iN_] :=
  Plot[eta0maxFit[wList[[iN]],  $\tau$ ], { $\tau$ , 0, 6}, PlotStyle → colornSrcList[[iN]]];
plotTauEtamaxbig[iN_] := Plot[eta0maxFitbig[wList[[iN]],  $\tau$ ],
  { $\tau$ , 0, 6}, PlotStyle → colornSrcList[[iN]]];
plotTauEtamaxsmall[iN_] := Plot[eta0maxFitsmall[wList[[iN]],  $\tau$ ],
  { $\tau$ , 0, 6}, PlotStyle → colornSrcList[[iN]]];

```

```

In[139]:= eta0MaxVSTauFit =
  Show[Table[Show[{lp $\tau\eta_0^{\max}$ PlusMinus[iN], plotTauEtamax[iN], plotTauEtamaxbig[iN],
    plotTauEtamaxsmall[iN]}], {iN, Length[nSrcList]}], ImageSize  $\rightarrow$  72  $\times$  5];
eta0MaxVSTauFit0 = Show[Table[Show[{lp $\tau\eta_0^{\max}$ PlusMinus[iN], plotTauEtamax[iN]}],
  {iN, Length[nSrcList]}], ImageSize  $\rightarrow$  72  $\times$  5];

In[141]:= (*Equation (A16)*)
sigmamaxFit[w_,  $\tau$ _] := 0.25 w (73.287 - 8.11 w + (2.773 + 11.126 w) Tanh[1.2850 (-1.6242 +  $\tau$ )])

In[142]:= (*Equation (A17)*)
sigmaxFitBig[w_,  $\tau$ _] := 0.25 w
(73.400 - 7.66 w + (2.773 + w (11.126 + 0.521 Sign[-1.6242 +  $\tau$ ]) + 0.129 Sign[-1.6242 +  $\tau$ ])
  Tanh[(-1.6242 +  $\tau$  + 0.0210 Sign[2.773 + 11.126 w]) (1.2850 + 0.0494 Sign[-1.6242 +  $\tau$ )])

In[143]:= (*Equation (A18)*)
sigmaxFitSmall[w_,  $\tau$ _] := 0.25 w
(73.174 - 8.567 w + (2.773 + w (11.126 - 0.521 Sign[-1.6242 +  $\tau$ ]) - 0.129 Sign[-1.6242 +  $\tau$ ])
  Tanh[(-1.6242 +  $\tau$  - 0.0210 Sign[2.773 + 11.126 w]) (1.2850 - 0.0494 Sign[-1.6242 +  $\tau$ )])

In[144]:= dsigmamaxFit[w_,  $\tau$ _] := sigmaxFitBig[w,  $\tau$ ] - sigmamaxFit[w,  $\tau$ ]

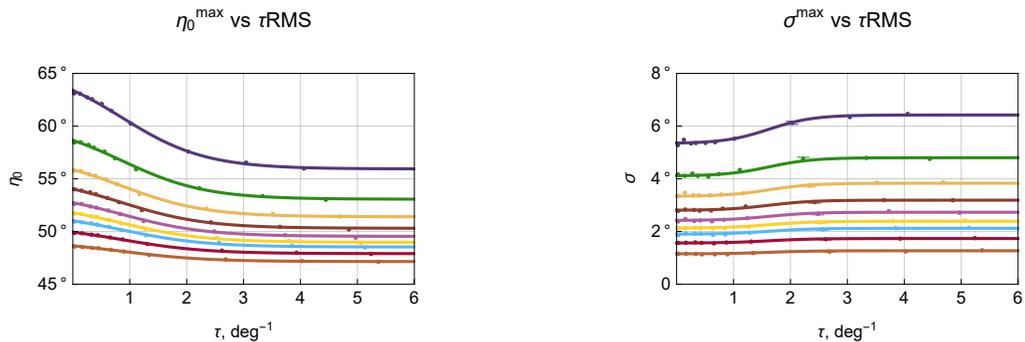
In[145]:=
plotTausigmamax[iN_] :=
  Plot[sigmamaxFit[wList[[iN]],  $\tau$ ], { $\tau$ , 0, 6}, PlotStyle  $\rightarrow$  colornSrcList[[iN]]];
plotTausigmamaxbig[iN_] := Plot[sigmaxFitBig[wList[[iN]],  $\tau$ ],
  { $\tau$ , 0, 6}, PlotStyle  $\rightarrow$  colornSrcList[[iN]]];
plotTausigmamaxsmall[iN_] := Plot[sigmaxFitSmall[wList[[iN]],  $\tau$ ],
  { $\tau$ , 0, 6}, PlotStyle  $\rightarrow$  colornSrcList[[iN]]];

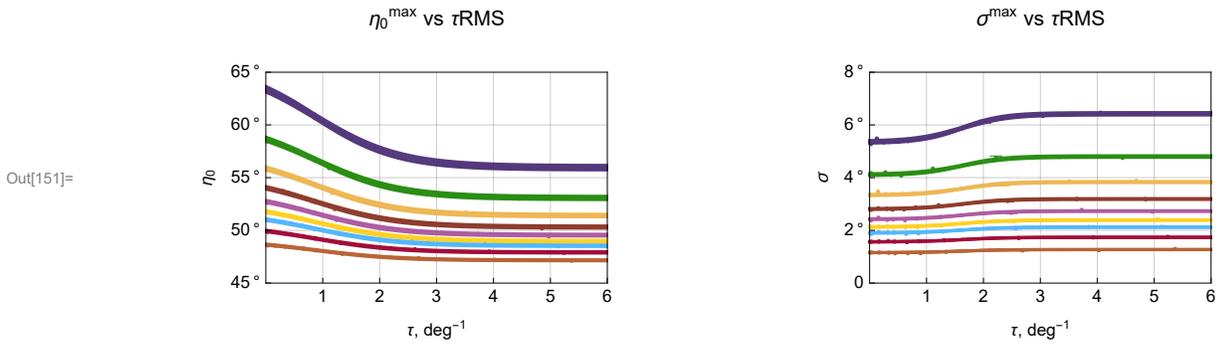
In[148]:= sigmaMaxVSTauFit =
  Show[Table[Show[{lp $\tau$ SigmamaxPlusMinus[iN], plotTausigmamax[iN], plotTausigmamaxbig[iN],
    plotTausigmamaxsmall[iN]}], {iN, Length[nSrcList]}], ImageSize  $\rightarrow$  72  $\times$  5];
sigmaMaxVSTauFit0 = Show[Table[Show[{lp $\tau$ SigmamaxPlusMinus[iN], plotTausigmamax[iN]}],
  {iN, Length[nSrcList]}], ImageSize  $\rightarrow$  72  $\times$  5];

In[150]:= GraphicsRow[{eta0MaxVSTauFit0, sigmaMaxVSTauFit0}, ImageSize  $\rightarrow$  72  $\times$  9]

```

Out[150]=





```
{N, color, w} = {{9., #1, 1/3}, {16., #2, 1/4}, {25., #3, 1/5}, {36., #4, 1/6},
{49., #5, 1/7}, {64., #6, 1/8}, {81., #7, 1/9}, {121., #8, 1/11}, {225., #9, 1/15}}
```

Figure A8: The fitting functions and Library data for avoidance parameters η_θ^{\max} , σ^{\max} . *Top*: The best fit to the data. *Bottom*: One sigma accuracy of the fit. For what this means, see Fig. A5.

```
In[154]:= sortPercentDiffEta0maxfit =
Sort[Flatten[Table[100. Abs[(eta0maxi[i] - eta0maxFit[wi[i], tauRMSi[i]]) / eta0maxi[i]],
{i, Length[fitData]}]]];
ListPlot[sortPercentDiffEta0maxfit, PlotRange -> All, FrameLabel -> {"", "% diff eta_0^max"},
PlotLabel -> "eta_0^max fit", GridLines -> Automatic, Frame -> True]
Print["Figure A9: The percent difference of fit to data
for eta_0^max. The average, i.e. arithmetic mean, %diff is ",
mean[sortPercentDiffEta0maxfit], "%, with the largest %diff = ",
sortPercentDiffEta0maxfit[[-1]], "%." ]
```

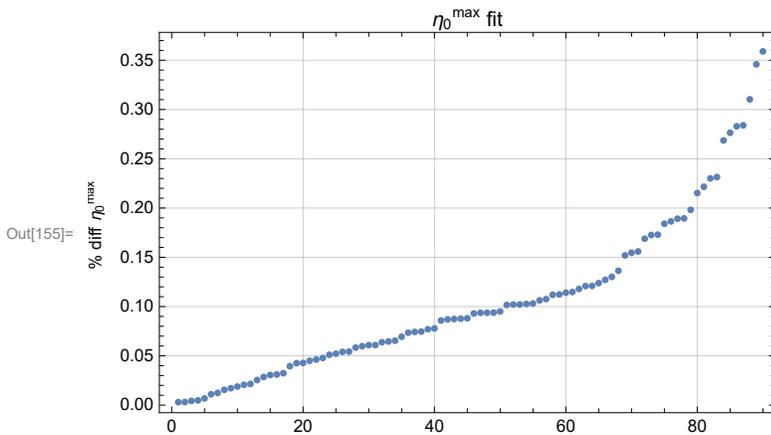


Figure A9: The percent difference of fit to data for η_θ^{\max} . The average, i.e. arithmetic mean, %diff is 0.106133%, with the largest %diff = 0.358982%.

```
In[157]:= sortPercentDiffSigmamaxfit =
  Sort[Flatten[Table[100. Abs[(σmaxi[i] - sigmamaxFit[wi[i], τRMSi[i]]) / σmaxi[i]],
    {i, Length[fitData]}]]];
ListPlot[sortPercentDiffSigmamaxfit, PlotRange → All, FrameLabel → {"", "% diff σmax"},
  PlotLabel → "σmax fit", GridLines → Automatic, Frame → True]
Print["Figure A10: The percent diff of fit to data for σmax. The average %diff is ",
  mean[sortPercentDiffSigmamaxfit], "%, with the largest %diff = ",
  sortPercentDiffSigmamaxfit[[-1]], "%." ]
```

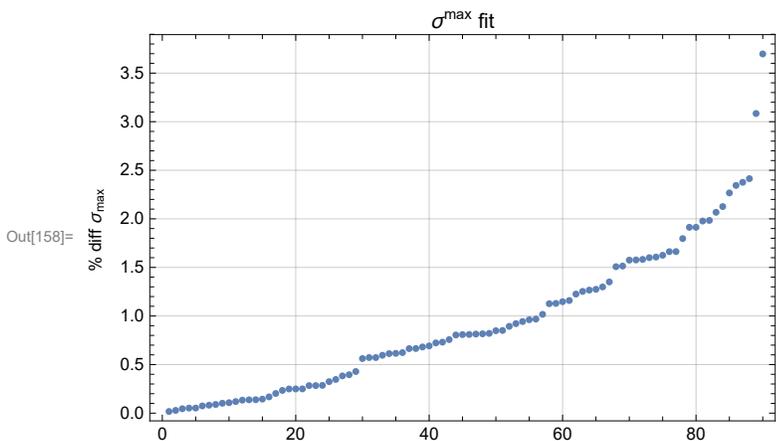


Figure A10: The percent diff of fit to data for σ^{\max} . The average %diff is 0.942796%, with the largest %diff = 3.69747%.

A5. Compare the three methods A,B,C for three previously studied samples 27 QSOs, 13 QSOs, 99 Stars

Definitions:

qsosStarsColumns Labels for the columns of the table of data from the three samples
 qsosStars3Samples data for the three samples

qsosStarsPropertiesmin Labels as above, but restricted to alignment for the article in Part I
 qsosStarsDistParametersmin data restricted to alignment for display in the article in Part I

From previous work with the three samples, the distribution parameters $\eta_0^{\min}, d\eta_0^{\min}, \sigma^{\min}, d\sigma^{\min}, \eta_0^{\max}, d\eta_0^{\max}, \sigma^{\max}, d\sigma^{\max}$.
 These were all found with Direct Method A, hence the names:

$$\eta_0^{\min A}, d\eta_0^{\min A}, \sigma^{\min A}, d\sigma^{\min A}, \eta_0^{\max A}, d\eta_0^{\max A}, \sigma^{\max A}, d\sigma^{\max A},$$

The same quantities for the three samples, but found with Interpolation Method B and Formula Method C. Same names except 'A' replaced with 'B' or 'C'.

For B: $\eta_0^{\min B}, \dots, d\sigma^{\max B}$

For C: $\eta_0^{\min C}, \dots, d\sigma^{\max C}$

compare η_0^{\min} values of η_0^{\min} for the three samples by the three methods; the value includes error bars

compare{ σ_{\min} , $\eta_{0\max}$, σ_{\max} } Ditto for the other parameters

lowy{ $\eta_{0\min}$, ..., σ_{\max} } PlotRange for graphics

lp{ $\eta_{0\min}$, $\eta_{0\max}$, σ_{\min} , σ_{\max} }Compare[i_] Compare the parameter { $\eta_{0\min}$, $\eta_{0\max}$, σ_{\min} , σ_{\max} } (pick one) for the i^{th} sample

sigminA, B, C significances of alignment angle η_0^{\min} for the three samples by Methods A, B, C

Standard errors of the significances for the three samples:

dsigminAPLUS[i_], dsigminAMINUS[i_], dsigminBPLUS[i_], dsigminBMINUS[i_], dsigminCPLUS[i_], dsigminCMI-
NUS[i_]

lpSigminCompare[i_] Compare the significances found by the three methods for the i^{th} sample

Repeat sigminA, B, C ----- lpSigminCompare[i_] for avoidance "max"

Previous studies of three observed samples, 27 QSOs, 13 QSOs, 99 Stars, produced two types of data. First there is the number of sources, the root-mean-square radius and the extremes, $\bar{\eta}_{\min}$ and $\bar{\eta}_{\max}$, of the alignment function $\bar{\eta}(H)$. The second type of data comes from running the process with random polarization directions replacing the observed directions. The second type of data reduces to the location of the peak, η_0^{\min} and η_0^{\max} , and the half-widths, σ^{\min} and σ^{\max} , of the random run distributions of $\bar{\eta}_{\min}$ and $\bar{\eta}_{\max}$. We put the two types of data in separate tables.

threeSamplesObservedData:

1. Sample 2. N , number of sources 3. ρ_{RMS} , region's root-mean-square radius 4. smallest alignment angle $\bar{\eta}_{\min}$ in degrees 5. largest avoidance angle $\bar{\eta}_{\max}$ in degrees

threeSamplesRandomRunParameters:

1. Sample 2. η_0^{\min} , most likely in random runs 3. $d\eta_0^{\min}$, standard error 4. σ^{\min} , half-width of random run distribution 5. $d\sigma^{\min}$, standard error 6. η_0^{\max} , most likely in random runs 7. $d\eta_0^{\max}$, standard error 8. σ^{\max} , half-width of random run distribution 9. $d\sigma^{\max}$, standard error

```
In[160]:= threeSamplesObservedColumns = {"Sample", "N", " $\rho_{\text{RMS}}$ ", " $\bar{\eta}_{\min}$ ", " $\bar{\eta}_{\max}$ "};
          "27 QSOs" 27 6.82492 21.094 66.660
threeSamplesObserved = "13 QSOs" 13 4.72812 10.865 62.665;
          "99 Stars" 99 6.83803 7.007 83.122
```

```
In[162]:= threeSamplesRandomRunColumns =
          {"Sample", " $\eta_0^{\min}$ ", " $d\eta_0^{\min}$ ", " $\sigma^{\min}$ ", " $d\sigma^{\min}$ ", " $\eta_0^{\max}$ ", " $d\eta_0^{\max}$ ", " $\sigma^{\max}$ ", " $d\sigma^{\max}$ "};
threeSamplesRandomRunParameters =
          "27 QSOs" 34.923 0.029 3.272 0.034 55.200 0.031 3.279 0.037
          "13 QSOs" 30.256 0.165 4.638 0.197 60.168 0.171 4.811 0.204;
          "99 Stars" 39.947 0.018 1.737 0.021 50.074 0.011 1.765 0.013
```

```
In[164]= Print["", Text@Grid[Prepend[threeSamplesObserved,
  {"Sample", "N", "ρRMS, deg", "η̄min, deg", "η̄max, deg"}], Spacings → {Automatic, .6},
  Dividers → {{Darker[Gray, .6], {Lighter[Gray, .5]}, Darker[Gray, .6]},
  {Darker[Gray, .6], Darker[Gray, .6], {False}, Darker[Gray, .6]}}] ]
Print["Table A2: Number of sources, root-mean-square radius, and
  extremes of the alignment angle function η̄(H) for the three
  samples from previously 'published' work. See Refs. 3,4,5. "]
```

Sample	N	ρRMS, deg	η̄ _{min} , deg	η̄ _{max} , deg
27 QSOs	27	6.82492	21.094	66.66
13 QSOs	13	4.72812	10.865	62.665
99 Stars	99	6.83803	7.007	83.122

Table A2: Number of sources, root-mean-square radius, and extremes of the alignment angle function $\bar{\eta}(H)$ for the three samples from previously 'published' work. See Refs. 3,4,5.

```
In[166]= Print["",
  Text@Grid[Prepend[threeSamplesRandomRunParameters, {"Sample", "η0min", "dη0min",
  "σmin", "dσmin", "η0max", "dη0max", "σmax", "dσmax"}], Spacings → {Automatic, .6},
  Dividers → {{Darker[Gray, .6], {Lighter[Gray, .5]}, Darker[Gray, .6]},
  {Darker[Gray, .6], Darker[Gray, .6], {False}, Darker[Gray, .6]}}] ]
Print["Table A3: Parameters and their standard errors for the fits to
  the random run distributions Eqs. (A1) and (A2), obtained with
  Direct Method A. All of the parameter values are in degrees. "]
```

Sample	η ₀ ^{min}	dη ₀ ^{min}	σ ^{min}	dσ ^{min}	η ₀ ^{max}	dη ₀ ^{max}	σ ^{max}	dσ ^{max}
27 QSOs	34.923	0.029	3.272	0.034	55.2	0.031	3.279	0.037
13 QSOs	30.256	0.165	4.638	0.197	60.168	0.171	4.811	0.204
99 Stars	39.947	0.018	1.737	0.021	50.074	0.011	1.765	0.013

Table A3: Parameters and their standard errors for the fits to the random run distributions Eqs. (A1) and (A2), obtained with Direct Method A. All of the parameter values are in degrees.

The following section creates table for the article, Part I the Article. The article in Part I does not include the topic of avoidance, so “max” quantities are excluded.

```
In[168]= (*This table is created for Part I the Article*)threeSamplesPropertiesmin =
  Partition[Flatten[Table[threeSamplesObserved[[i, j]], {i, 3}, {j, 4}]], 4]
Out[168]= {{27 QSOs, 27, 6.82492, 21.094},
  {13 QSOs, 13, 4.72812, 10.865}, {99 Stars, 99, 6.83803, 7.007}}

In[169]= (*This table is created for Part I the Article*)threeSamplesDistParametersmin =
  Partition[Flatten[Table[{threeSamplesRandomRunParameters[[i, 1]], Around[
  threeSamplesRandomRunParameters[[i, 2]], threeSamplesRandomRunParameters[[i, 3]]],
  Around[threeSamplesRandomRunParameters[[i, 4]],
  threeSamplesRandomRunParameters[[i, 5]]}], {i, 3}], 3]
Out[169]= {{27 QSOs, 34.923 ± 0.029, 3.272 ± 0.034},
  {13 QSOs, 30.26 ± 0.17, 4.64 ± 0.20}, {99 Stars, 39.947 ± 0.018, 1.737 ± 0.021}}
```

```

In[170]:= (*This table is created for Part I the Article*)
Print["",
  Text@Grid[Prepend[threeSamplesPropertiesmin, {"Sample", "N", " $\rho$ RMS, deg", " $\bar{\eta}_{\min}$ , deg"}],
    Spacings → {Automatic, .6},
    Dividers → {{Darker[Gray, .6], {Lighter[Gray, .5]}, Darker[Gray, .6]},
      {Darker[Gray, .6], Darker[Gray, .6], {False}, Darker[Gray, .6]}}, " ",
  Text@Grid[Prepend[threeSamplesDistParametersmin, {"Sample", " $\eta_0^{\min}$ , deg", " $\sigma^{\min}$ , deg"}],
    Spacings → {Automatic, .6},
    Dividers → {{Darker[Gray, .6], {Lighter[Gray, .5]}, Darker[Gray, .6]},
      {Darker[Gray, .6], Darker[Gray, .6], {False}, Darker[Gray, .6]}}, " "]
Print["See Part I the Article, this is Table 1. The Article
  focusses on alignment (min), not avoidance (max). "]

```

Sample	N	ρ RMS, deg	$\bar{\eta}_{\min}$, deg
27 QSOs	27	6.82492	21.094
13 QSOs	13	4.72812	10.865
99 Stars	99	6.83803	7.007

Sample	η_0^{\min} , deg	σ^{\min} , deg
27 QSOs	34.923 ± 0.029	3.272 ± 0.034
13 QSOs	30.26 ± 0.17	4.64 ± 0.20
99 Stars	39.947 ± 0.018	1.737 ± 0.021

See Part I the Article, this is Table 1.

The Article focusses on alignment (min), not avoidance (max).

```

In[172]:=  $\eta_0^{\min}A$  = Table[threeSamplesRandomRunParameters[[i, 2]],
  {i, Length[threeSamplesRandomRunParameters]}]; (*degrees*)
 $d\eta_0^{\min}A$  = Table[threeSamplesRandomRunParameters[[i, 3]],
  {i, Length[threeSamplesRandomRunParameters]}]; (*degrees*)
 $\sigma^{\min}A$  = Table[threeSamplesRandomRunParameters[[i, 4]],
  {i, Length[threeSamplesRandomRunParameters]}]; (*degrees*)
 $\delta\sigma^{\min}A$  = Table[threeSamplesRandomRunParameters[[i, 5]],
  {i, Length[threeSamplesRandomRunParameters]}]; (*degrees*)

```

```

In[176]:=  $\eta_0^{\max}A$  = Table[threeSamplesRandomRunParameters[[i, 6]],
  {i, Length[threeSamplesRandomRunParameters]}]; (*degrees*)
 $d\eta_0^{\max}A$  = Table[threeSamplesRandomRunParameters[[i, 7]],
  {i, Length[threeSamplesRandomRunParameters]}]; (*degrees*)
 $\sigma^{\max}A$  = Table[threeSamplesRandomRunParameters[[i, 8]],
  {i, Length[threeSamplesRandomRunParameters]}]; (*degrees*)
 $\delta\sigma^{\max}A$  = Table[threeSamplesRandomRunParameters[[i, 9]],
  {i, Length[threeSamplesRandomRunParameters]}]; (*degrees*)

```

```

In[180]:=  $\eta_0^{\min}B$  = Table[ $\eta_0^{\min}B_{\text{int}}$ [threeSamplesObserved[[i, 2]]-1/2, threeSamplesObserved[[i, 3]]-1],
  {i, Length[threeSamplesObserved]}]; (*degrees*)
 $d\eta_0^{\min}B$  = Table[ $d\eta_0^{\min}B_{\text{int}}$ [threeSamplesObserved[[i, 2]]-1/2,
  threeSamplesObserved[[i, 3]]-1], {i, Length[threeSamplesObserved]}]; (*degrees*)
 $\sigma^{\min}B$  = Table[ $\sigma^{\min}B_{\text{int}}$ [threeSamplesObserved[[i, 2]]-1/2, threeSamplesObserved[[i, 3]]-1],
  {i, Length[threeSamplesObserved]}]; (*degrees*)
 $\delta\sigma^{\min}B$  = Table[ $\delta\sigma^{\min}B_{\text{int}}$ [threeSamplesObserved[[i, 2]]-1/2, threeSamplesObserved[[i, 3]]-1],
  {i, Length[threeSamplesObserved]}]; (*degrees*)

```

In[184]:=

```

 $\eta\theta_{\max B}$  = Table[ $\eta\theta_{\max \text{BinT}}$ [threeSamplesObserved[[i, 2]]-1/2, threeSamplesObserved[[i, 3]]-1],
  {i, Length[threeSamplesObserved]}]; (*degrees*)
 $d\eta\theta_{\max B}$  = Table[ $d\eta\theta_{\max \text{BinT}}$ [threeSamplesObserved[[i, 2]]-1/2,
  threeSamplesObserved[[i, 3]]-1], {i, Length[threeSamplesObserved]}]; (*degrees*)
 $\sigma_{\max B}$  = Table[ $\sigma_{\max \text{BinT}}$ [threeSamplesObserved[[i, 2]]-1/2, threeSamplesObserved[[i, 3]]-1],
  {i, Length[threeSamplesObserved]}]; (*degrees*)
 $d\sigma_{\max B}$  = Table[ $d\sigma_{\max \text{BinT}}$ [threeSamplesObserved[[i, 2]]-1/2, threeSamplesObserved[[i, 3]]-1],
  {i, Length[threeSamplesObserved]}]; (*degrees*)

```

In[188]:= $\eta\theta_{\min C}$ =

```

Table[ $\eta\theta_{\min \text{Fit}}$ [threeSamplesObserved[[i, 2]]-1/2, threeSamplesObserved[[i, 3]]-1],
  {i, Length[threeSamplesObserved]}]; (*degrees*)
 $d\eta\theta_{\min C}$  = Table[ $d\eta\theta_{\min \text{Fit}}$ [threeSamplesObserved[[i, 2]]-1/2,
  threeSamplesObserved[[i, 3]]-1], {i, Length[threeSamplesObserved]}]; (*degrees*)
 $\sigma_{\min C}$  = Table[ $\sigma_{\min \text{Fit}}$ [threeSamplesObserved[[i, 2]]-1/2,
  threeSamplesObserved[[i, 3]]-1], {i, Length[threeSamplesObserved]}]; (*degrees*)
 $d\sigma_{\min C}$  = Table[ $d\sigma_{\min \text{Fit}}$ [threeSamplesObserved[[i, 2]]-1/2,
  threeSamplesObserved[[i, 3]]-1], {i, Length[threeSamplesObserved]}]; (*degrees*)

```

In[192]:=

```

 $\eta\theta_{\max C}$  =
  Table[ $\eta\theta_{\max \text{Fit}}$ [threeSamplesObserved[[i, 2]]-1/2, threeSamplesObserved[[i, 3]]-1],
    {i, Length[threeSamplesObserved]}]; (*degrees*)
 $d\eta\theta_{\max C}$  = Table[ $d\eta\theta_{\max \text{Fit}}$ [threeSamplesObserved[[i, 2]]-1/2,
  threeSamplesObserved[[i, 3]]-1], {i, Length[threeSamplesObserved]}]; (*degrees*)
 $\sigma_{\max C}$  = Table[ $\sigma_{\max \text{Fit}}$ [threeSamplesObserved[[i, 2]]-1/2,
  threeSamplesObserved[[i, 3]]-1], {i, Length[threeSamplesObserved]}]; (*degrees*)
 $d\sigma_{\max C}$  = Table[ $d\sigma_{\max \text{Fit}}$ [threeSamplesObserved[[i, 2]]-1/2,
  threeSamplesObserved[[i, 3]]-1], {i, Length[threeSamplesObserved]}]; (*degrees*)

```

```

In[196]:= compare $\eta\theta_{\min}$  = Table[{threeSamplesObserved[[i, 1]], Around[ $\eta\theta_{\min A}$ [[i]],  $d\eta\theta_{\min A}$ [[i]] ],
  Around[ $\eta\theta_{\min B}$ [[i]],  $d\eta\theta_{\min B}$ [[i]] ], Around[ $\eta\theta_{\min C}$ [[i]],  $d\eta\theta_{\min C}$ [[i]] ]}, {i, 3}];
compare $\sigma_{\min}$  = Table[{threeSamplesObserved[[i, 1]], Around[ $\sigma_{\min A}$ [[i]],  $d\sigma_{\min A}$ [[i]] ],
  Around[ $\sigma_{\min B}$ [[i]],  $d\sigma_{\min B}$ [[i]] ], Around[ $\sigma_{\min C}$ [[i]],  $d\sigma_{\min C}$ [[i]] ]}, {i, 3}];

```

Probability distribution parameters for $\bar{\eta}_{\min}$ with the three Methods:

(a) peak η_{θ}^{\min} :

Sample	A (degrees)	B (degrees)	C (degrees)
27 QSOs	34.923 \pm 0.029	34.796 \pm 0.016	34.80 \pm 0.19
13 QSOs	30.26 \pm 0.17	30.293 \pm 0.033	30.36 \pm 0.27
99 Stars	39.947 \pm 0.018	39.703 \pm 0.015	39.73 \pm 0.10

(b) half-width σ^{\min} :

Sample	A (degrees)	B (degrees)	C (degrees)
27 QSOs	3.272 \pm 0.034	3.299 \pm 0.020	3.229 \pm 0.022
13 QSOs	4.64 \pm 0.20	4.59 \pm 0.04	4.55 \pm 0.04
99 Stars	1.737 \pm 0.021	1.700 \pm 0.018	1.729 \pm 0.009

Table A4: Probability distribution parameters for $\bar{\eta}_{\min}$ with the three Methods, Direct Method A, Interpolation Method B, Function Method C. This is Table 2 in Part I the Article.

[Caption Table 2: Distribution parameters η_{θ}^{\min} and σ^{\min} in the probability distribution formula Eq. (3). (a) The values of the parameters for the location of the peak η_{θ}^{\min} for the three methods. Compared with A, the B and C values overlap with A except for the B-value of η_{θ}^{\min} for the 27 QSOs and the 99 Stars. (b) The half-widths σ^{\min} , in degrees, for the three methods. The B and C values for σ^{\min} are within the listed standard error of the Method A values. Also see the displays in Figs. 11 and 12.]

In[201]= `compare η_{θ}^{\max} = Table[{threeSamplesObserved[[i, 1]], Around[η_{θ}^{\max} A[[i]], d η_{θ}^{\max} A[[i]]], Around[η_{θ}^{\max} B[[i]], d η_{θ}^{\max} B[[i]]], Around[η_{θ}^{\max} C[[i]], d η_{θ}^{\max} C[[i]]]}, {i, 3}];`
`compare σ^{\max} = Table[{threeSamplesObserved[[i, 1]], Around[σ^{\max} A[[i]], d σ^{\max} A[[i]]], Around[σ^{\max} B[[i]], d σ^{\max} B[[i]]], Around[σ^{\max} C[[i]], d σ^{\max} C[[i]]]}, {i, 3}];`

(a) peak η_{θ}^{\max}

Sample	A	B	C
27 QSOs	55.200 \pm 0.031	55.315 \pm 0.023	55.24 \pm 0.14
13 QSOs	60.17 \pm 0.17	59.799 \pm 0.029	59.74 \pm 0.21
99 Stars	50.074 \pm 0.011	50.332 \pm 0.016	50.34 \pm 0.08

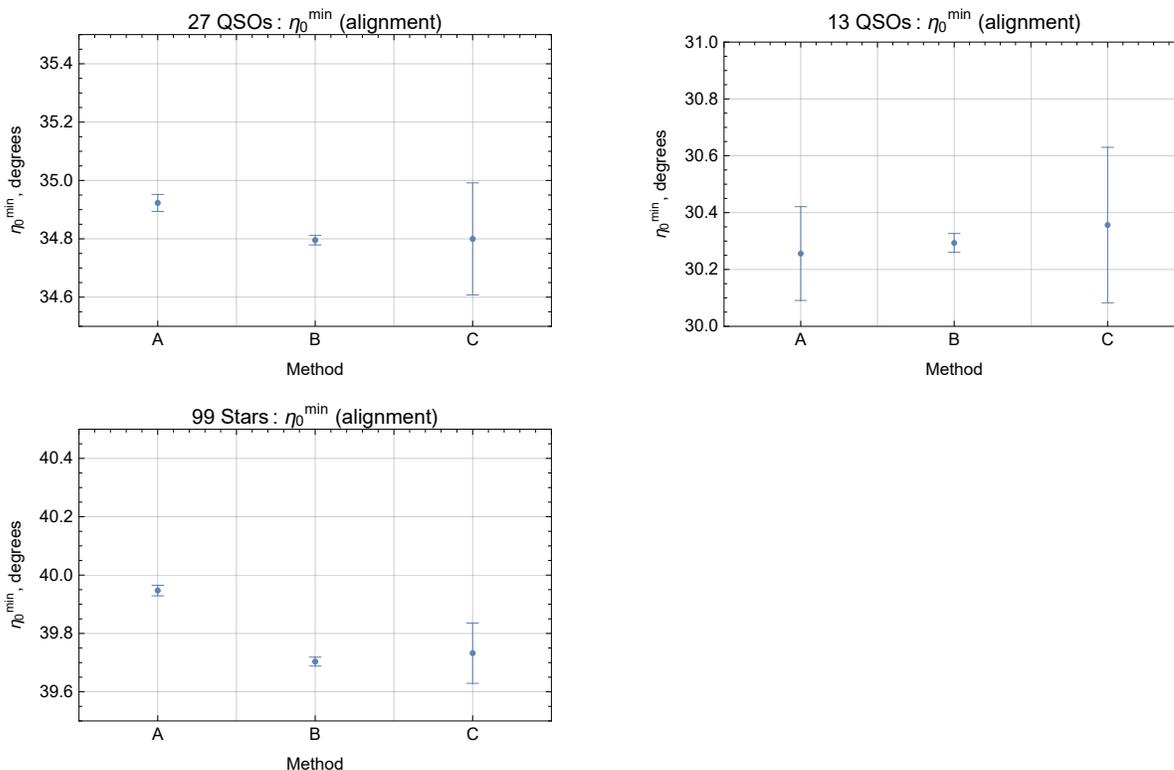
(b) half-width σ^{\max} :

Sample	A	B	C
27 QSOs	3.28 \pm 0.04	3.328 \pm 0.028	3.225 \pm 0.022
13 QSOs	4.81 \pm 0.20	4.539 \pm 0.034	4.54 \pm 0.04
99 Stars	1.765 \pm 0.013	1.743 \pm 0.019	1.727 \pm 0.009

Table A5: Probability distribution parameters for $\bar{\eta}_{\max}$ with the three Methods, Direct Method A, Interpolation Method B, Function Method C. Here are the distribution parameters η_{θ}^{\max} and σ^{\max} in the probability distribution formula Eq. (A2). (a) The values of the parameters for the location of the peak η_{θ}^{\max} for the three methods. Compared with A, the B and C values do not overlap with A except for the C-value of η_{θ}^{\max} for the 27 QSOs. (b) The half-widths σ^{\max} , in degrees, for the three methods. Again most of the B and C values for σ^{\max} are not within the listed standard error of the Method A values. Also see the displays in Figs. 11 and 12.

```
In[205]:= lowyηθmin = {34.5, 30.0, 39.5};
lpηθminCompare[i_] := ListPlot[{Around[ηθminA[[i]], dηθminA[[i]]],
    Around[ηθminB[[i]], dηθminB[[i]]], Around[ηθminC[[i]], dηθminC[[i]]]},
    PlotRange → {{0.5, 3.5}, {lowyηθmin[[i]], lowyηθmin[[i]] + 1.}},
    GridLines → Automatic, Frame → True, FrameLabel → {"Method", "ηθmin, degrees"},
    FrameTicks → {Automatic, {{1., "A"}, {2., "B"}, {3., "C"}}, Automatic}},
    PlotLabel → threeSamplesObserved[[i, 1]] ": ηθmin (alignment)", ImageSize → 72 × 5]
lpηθminCompare[2];
```

```
In[208]:= GraphicsGrid[
    {{lpηθminCompare[1], lpηθminCompare[2]}, {lpηθminCompare[3]}}, ImageSize → 72 × 9]
Print["Figure A11: This is Fig. 11 in Part I the Article. [Fig. 11 Caption: Compare
    parameter ηθmin for the 3 methods with 3 samples. Method B has small standard
    errors, so it is precise, but it is not always accurate when compared with A.]"]
```

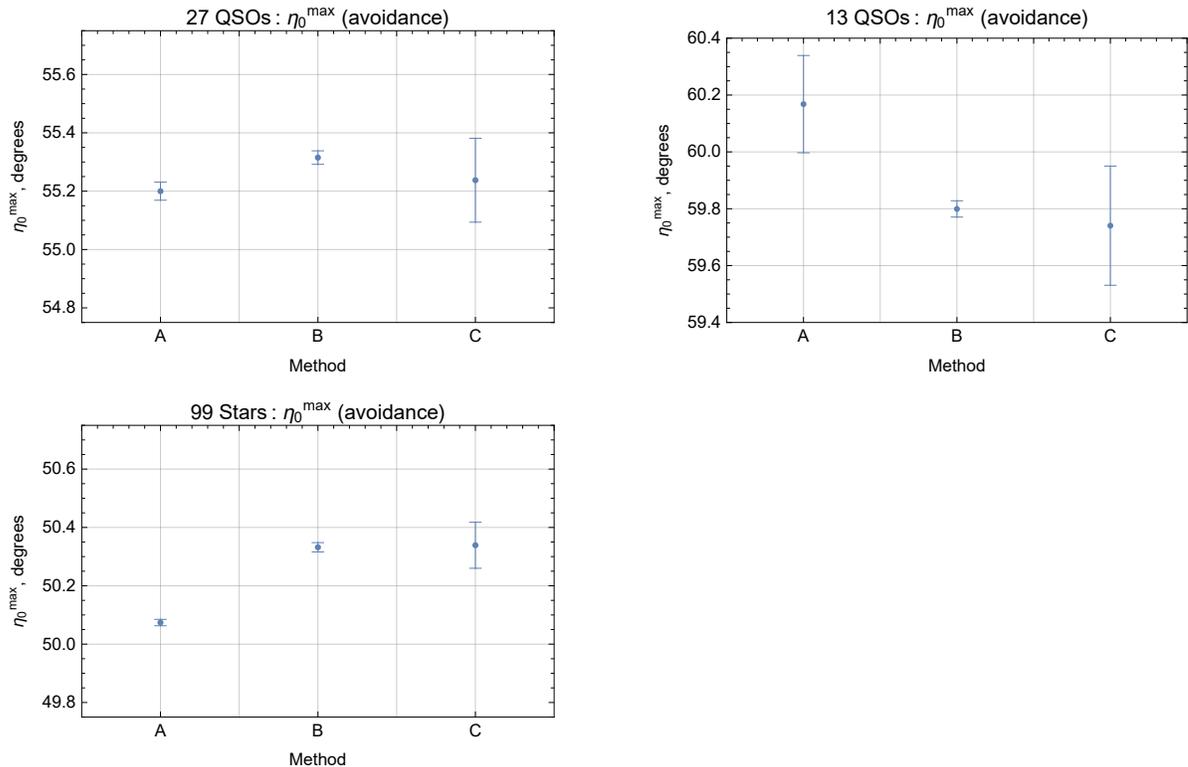


Out[208]=

Figure A11: This is Fig. 11 in Part I the Article. [Fig. 11 Caption: Compare parameter η_{θ}^{\min} for the 3 methods with 3 samples. Method B has small standard errors, so it is precise, but it is not always accurate when compared with A.]

```
In[210]:= lowyηθmax = {54.75, 59.4, 49.75};
lpηθmaxCompare[i_] := ListPlot[{Around[ηθmaxA[[i]], dηθmaxA[[i]]],
    Around[ηθmaxB[[i]], dηθmaxB[[i]]], Around[ηθmaxC[[i]], dηθmaxC[[i]]]},
    PlotRange → {{0.5, 3.5}, {lowyηθmax[[i]], lowyηθmax[[i]] + 1.}},
    GridLines → Automatic, Frame → True, FrameLabel → {"Method", "ηθmax, degrees"},
    FrameTicks → {Automatic, {{1., "A"}, {2., "B"}, {3., "C"}}, Automatic}},
    PlotLabel → threeSamplesObserved[[i, 1]] ": ηθmax (avoidance)", ImageSize → 72 × 5]
lpηθmaxCompare[1];
```

```
In[213]:= GraphicsGrid[
  {{lpη0maxCompare[1], lpη0maxCompare[2]}, {lpη0maxCompare[3]}}, ImageSize → 72 × 9]
Print["Figure A12: Compare parameter  $\eta_0^{\max}$  for the 3 methods with 3 samples. As with
 $\eta_0^{\min}$  in Fig. A11, Method B tends to be precise, but not always accurate,
when compared with A. Method C has larger error bars than B, but only the 27
QSO sample value agrees with Method A, within the standard errors shown." ]
```

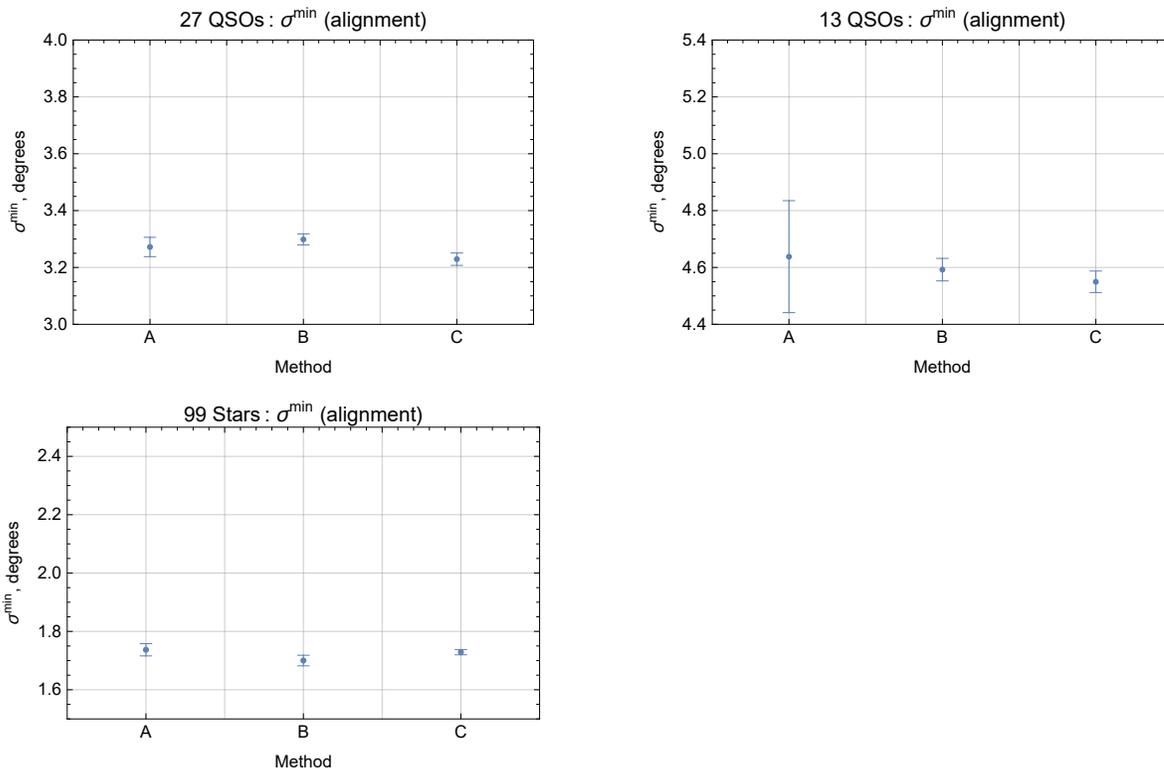


```
Out[213]=
```

Figure A12: Compare parameter η_0^{\max} for the 3 methods with 3 samples. As with η_0^{\min} in Fig. A11, Method B tends to be precise, but not always accurate, when compared with A. Method C has larger error bars than B, but only the 27 QSO sample value agrees with Method A, within the standard errors shown.

```
In[215]:= lowyomin = {3.0, 4.4, 1.5};
lpominCompare[i_] := ListPlot[{Around[ominA[[i]], domina[[i]], Around[ominB[[i]],
  domineB[[i]], Around[ominC[[i]], domineC[[i]]}], PlotRange → {{0.5, 3.5},
  {lowyomin[[i]], lowyomin[[i]] + 1.}}, GridLines → Automatic, Frame → True, FrameLabel →
  {"Method", " $\sigma^{\min}$ , degrees"}, FrameTicks → {Automatic, {{1., "A"}, {2., "B"}, {3., "C"}},
  Automatic}}, FrameTicks → {Automatic, {{1., "A"}, {2., "B"}, {3., "C"}}, Automatic}},
  PlotLabel → threeSamplesObserved[[i, 1]] ":  $\sigma^{\min}$  (alignment)", ImageSize → 72 × 5]
lpominCompare[2];
```

```
In[217]:= GraphicsGrid[
  {{lpominCompare[1], lpominCompare[2]}, {lpominCompare[3]}}, ImageSize -> 72 x 9]
Print["Figure A13: This is Fig. 12 in Part I the Article.
  [Fig. 12 Caption: Compare parameter  $\sigma^{\min}$  for the 3 methods with
  3 samples. Methods B and C give values that overlap with A.]"]
```

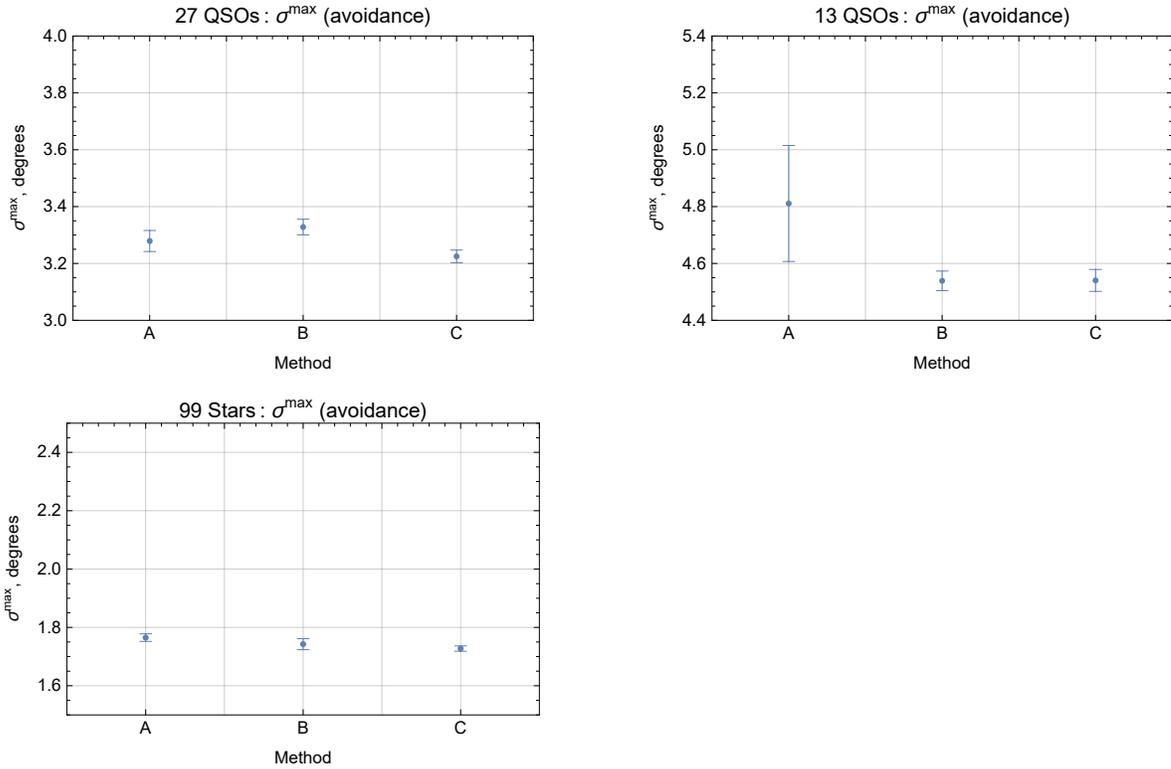


```
Out[217]=
```

Figure A13: This is Fig. 12 in Part I the Article. [Fig. 12 Caption: Compare parameter σ^{\min} for the 3 methods with 3 samples. Methods B and C give values that overlap with A.]

```
In[219]:= lowyomax = {3.0, 4.4, 1.5};
lpomaxCompare[i_] := ListPlot[{Around[omaxA[[i]], domaxA[[i]]], Around[omaxB[[i]],
  domaxB[[i]]], Around[omaxC[[i]], domaxC[[i]]]}, PlotRange -> {{0.5, 3.5},
  {lowyomax[[i]], lowyomax[[i]] + 1.}}, GridLines -> Automatic, Frame -> True, FrameLabel ->
  {"Method", " $\sigma^{\max}$ , degrees"}, FrameTicks -> {Automatic, {{1., "A"}, {2., "B"}, {3., "C"}},
  Automatic}}, FrameTicks -> {Automatic, {{1., "A"}, {2., "B"}, {3., "C"}}, Automatic}},
  PlotLabel -> threeSamplesObserved[[i, 1]] " :  $\sigma^{\max}$  (avoidance)", ImageSize -> 72 x 5]
lpomaxCompare[2];
```

```
In[221]:= GraphicsGrid[
  {{lpomaxCompare[1], lpomaxCompare[2]}, {lpomaxCompare[3]}}, ImageSize -> 72 x 9]
Print["Figure A14: Compare parameter  $\sigma^{\max}$  for the 3 methods with 3 samples. Notice
the similarity between this figure and Fig. A13. The half-widths  $\sigma^{\max}$  for
avoidance and the half-widths  $\sigma^{\min}$  for alignment are alike in their values. "]
```



```
Out[221]=
```

Figure A14: Compare parameter σ^{\max} for the 3 methods with 3 samples.
Notice the similarity between this figure and Fig. A13. The half-widths σ^{\max} for avoidance and the half-widths σ^{\min} for alignment are alike in their values.

```
In[223]:= sigmaA = Table[signiMIN0[threeSamplesObserved[[i, 4]], eta0minA[[i]], sigmaA[[i]]],
  {i, Length[threeSamplesObserved]}];
sigmaB = Table[signiMIN0[threeSamplesObserved[[i, 4]], eta0minB[[i]], sigmaB[[i]]],
  {i, Length[threeSamplesObserved]}];
sigmaC = Table[signiMIN0[threeSamplesObserved[[i, 4]], eta0minC[[i]], sigmaC[[i]]],
  {i, Length[threeSamplesObserved]}];
```

```

In[226]:= dsigminAPLUS[i_] := signiMIN0[threeSamplesObserved[[i, 4]],
  η0minA[[i]] - dη0minA[[i]], ominA[[i]] + dminA[[i]] ] - sigminA[[i]];
dsigminAMINUS[i_] := sigminA[[i]] - signiMIN0[threeSamplesObserved[[i, 4]],
  η0minA[[i]] + dη0minA[[i]], ominA[[i]] - dminA[[i]] ];
dsigminBPLUS[i_] := signiMIN0[threeSamplesObserved[[i, 4]],
  η0minB[[i]] - dη0minB[[i]], ominB[[i]] + dminB[[i]] ] - sigminB[[i]];
dsigminBMINUS[i_] := sigminB[[i]] - signiMIN0[threeSamplesObserved[[i, 4]],
  η0minB[[i]] + dη0minB[[i]], ominB[[i]] - dminB[[i]] ];
dsigminCPLUS[i_] := signiMIN0[threeSamplesObserved[[i, 4]],
  η0minC[[i]] - dη0minC[[i]], ominC[[i]] + dminC[[i]] ] - sigminC[[i]];
dsigminCMINUS[i_] := sigminC[[i]] - signiMIN0[threeSamplesObserved[[i, 4]],
  η0minC[[i]] + dη0minC[[i]], ominC[[i]] - dminC[[i]] ];

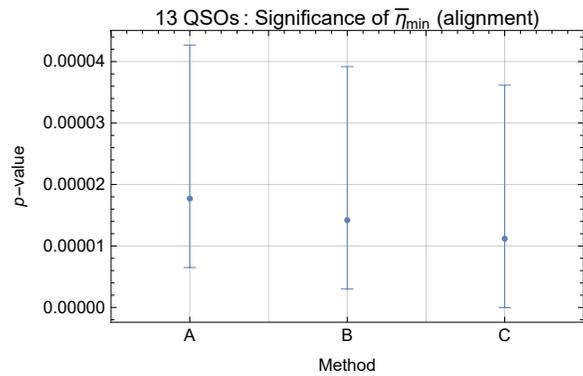
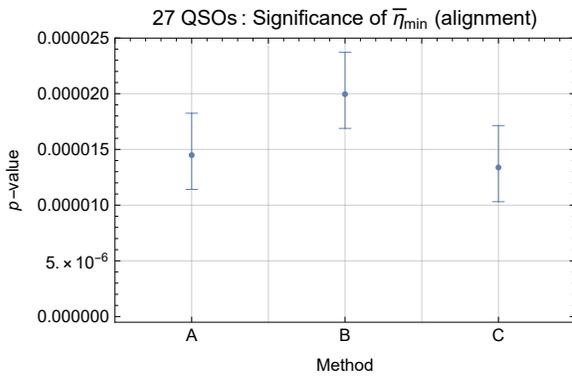
In[232]:= lpSigminCompare[i_] :=
  ListPlot[ {Around[sigminA[[i]], {dsigminAMINUS[i], dsigminAPLUS[i]}],
    Around[sigminB[[i]], {dsigminAMINUS[i], dsigminAPLUS[i]}],
    Around[sigminC[[i]], {dsigminAMINUS[i], dsigminAPLUS[i]}] },
  PlotRange → {{0.5, 3.5}, All}, GridLines → Automatic,
  Frame → True, FrameLabel → {"Method", "p-value"},
  FrameTicks → {Automatic, {{1., "A"}, {2., "B"}, {3., "C"}}, Automatic}, PlotLabel →
    threeSamplesObserved[[i, 1]] ": Significance of  $\bar{\eta}_{\min}$  (alignment)", ImageSize → 72 × 5]
lpSigminCompare[1];

```

```

In[234]:= GraphicsGrid[
  {{lpSigminCompare[1], lpSigminCompare[2]}, {lpSigminCompare[3]}}, ImageSize → 72 × 9]

```



Out[234]=

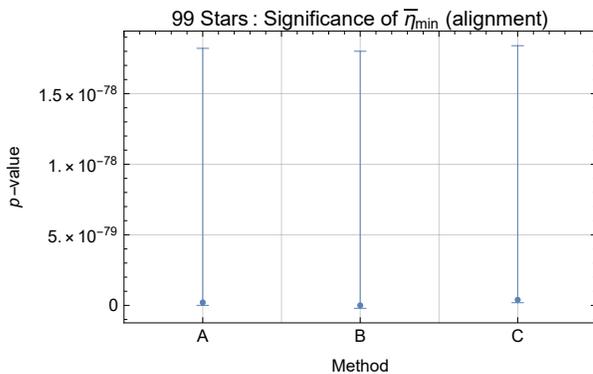


Figure A15: This is Fig. 13 in Part I the Article. [Fig. 13 Caption: The significances of the smallest alignment angle $\bar{\eta}_{\min}$ for three samples as determined by Methods A, B, and C. The error bars are due to the standard errors of the parameters η_{θ}^{\min} and σ^{\min} . For all three samples, the significances determined by the Interpolation Method B and the Formula Method C agree with the Direct Method A.]

```

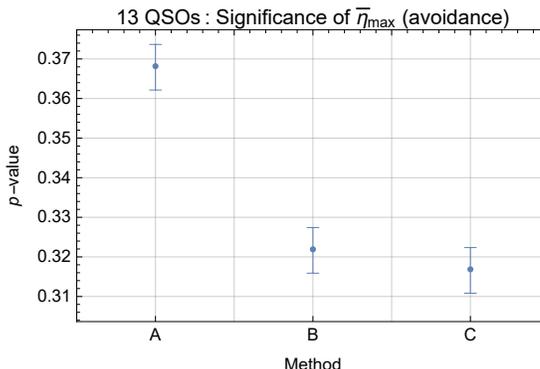
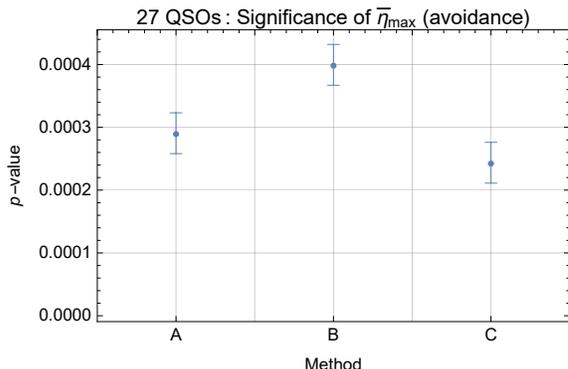
In[236]:= sigmaxA = Table[signiMAX0[threeSamplesObserved[[i, 5]],  $\eta_{\theta\max A}[[i]]$ ,  $\sigma_{\max A}[[i]]$ ],
  {i, Length[threeSamplesObserved]};
sigmaxB = Table[signiMAX0[threeSamplesObserved[[i, 5]],  $\eta_{\theta\max B}[[i]]$ ,  $\sigma_{\max B}[[i]]$ ],
  {i, Length[threeSamplesObserved]};
sigmaxC = Table[signiMAX0[threeSamplesObserved[[i, 5]],  $\eta_{\theta\max C}[[i]]$ ,  $\sigma_{\max C}[[i]]$ ],
  {i, Length[threeSamplesObserved]};

In[239]:= dsigmaxAPLUS[i_] := signiMAX0[threeSamplesObserved[[i, 5]],
   $\eta_{\theta\max A}[[i]] - d\eta_{\theta\max A}[[i]]$ ,  $\sigma_{\max A}[[i]] + d\sigma_{\max A}[[i]]$ ] - sigmaxA[[i]];
dsigmaxAMAXUS[i_] := sigmaxA[[i]] - signiMAX0[threeSamplesObserved[[i, 5]],
   $\eta_{\theta\max A}[[i]] + d\eta_{\theta\max A}[[i]]$ ,  $\sigma_{\max A}[[i]] - d\sigma_{\max A}[[i]]$ ];
dsigmaxBPLUS[i_] := signiMAX0[threeSamplesObserved[[i, 5]],
   $\eta_{\theta\max B}[[i]] - d\eta_{\theta\max B}[[i]]$ ,  $\sigma_{\max B}[[i]] + d\sigma_{\max B}[[i]]$ ] - sigmaxB[[i]];
dsigmaxBMAXUS[i_] := sigmaxB[[i]] - signiMAX0[threeSamplesObserved[[i, 5]],
   $\eta_{\theta\max B}[[i]] + d\eta_{\theta\max B}[[i]]$ ,  $\sigma_{\max B}[[i]] - d\sigma_{\max B}[[i]]$ ];
dsigmaxCPLUS[i_] := signiMAX0[threeSamplesObserved[[i, 5]],
   $\eta_{\theta\max C}[[i]] - d\eta_{\theta\max C}[[i]]$ ,  $\sigma_{\max C}[[i]] + d\sigma_{\max C}[[i]]$ ] - sigmaxC[[i]];
dsigmaxCMAXUS[i_] := sigmaxC[[i]] - signiMAX0[threeSamplesObserved[[i, 5]],
   $\eta_{\theta\max C}[[i]] + d\eta_{\theta\max C}[[i]]$ ,  $\sigma_{\max C}[[i]] - d\sigma_{\max C}[[i]]$ ];

In[245]:= lpSigmaxCompare[i_] :=
  ListPlot[{Around[sigmaxA[[i]], {dsigmaxAMAXUS[i], dsigmaxAPLUS[i]}],
    Around[sigmaxB[[i]], {dsigmaxAMAXUS[i], dsigmaxAPLUS[i]}],
    Around[sigmaxC[[i]], {dsigmaxAMAXUS[i], dsigmaxAPLUS[i]}]},
  PlotRange → {{0.5, 3.5}, All}, GridLines → Automatic,
  Frame → True, FrameLabel → {"Method", "p-value"},
  FrameTicks → {Automatic, {{1., "A"}, {2., "B"}, {3., "C"}}, Automatic}}, PlotLabel →
  threeSamplesObserved[[i, 1]] ": Significance of  $\bar{\eta}_{\max}$  (avoidance)", ImageSize → 72 × 5]
lpSigmaxCompare[1];

```

```
In[247]:= GraphicsGrid[
  {{lpSigmaxCompare[1], lpSigmaxCompare[2]}, {lpSigmaxCompare[3]}}, ImageSize -> 72 x 9]
```



```
Out[247]=
```

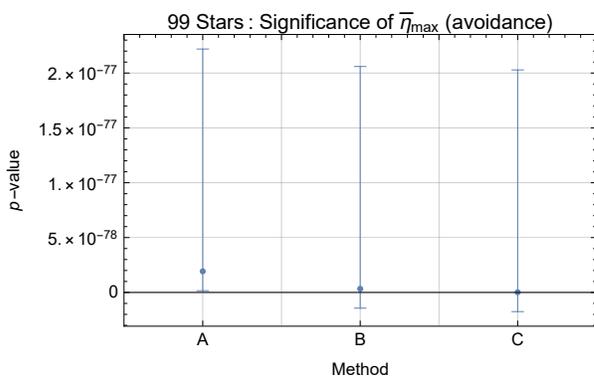


Figure A16: The significance of the largest avoidance angle $\bar{\eta}_{\max}$ for three samples as determined by Methods A, B, and C. The error bars are due to the standard errors of the parameters η_{θ}^{\max} and σ^{\max} . Three of six B and C significances agree with those of A; three do not. Note that the 13 QSO sample's $\bar{\eta}_{\max}$ is not significant for avoidance. For the 13 QSO sample, 30% or 40% (i.e. p -value = 0.30 to 0.40) of the $\bar{\eta}_{\max}$ for like-samples with randomly directed polarization directions are better, larger than the observed polarization directions' result $\bar{\eta}_{\max}$. Better random run results are too likely for the 13 QSO sample to qualify as significant. One needs 5% for 'significant' and 1% qualifies as 'very significant'. This shows that significant alignment does not guarantee significant avoidance.

```
In[249]:= Print["The date and time that this statement was evaluated : ", Now]
```

The date and time that this statement was evaluated : Mon 20 Dec 2021 13:04:58 GMT-5.