Using the Partial Sums of the Alternating Harmonic Series to prove the Harmonic Series is divergent

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Abstract: Many proofs of the divergence of the harmonic series have been given since the first proof by Nicole Oresme (1323-1382). In this article we shall give a simple proof using the partial sums of the alternating harmonic series. A simple consequence of this is an approximation that follows as a corollary. We then show that every harmonic number is the sum of partial sums of the alternating harmonic series. Finally as a corollary we show that the sequence of subseries of the harmonic series is converging to $ln2$.

Keywords: Harmonic Series, Alternating Harmonic Series, Proof

The harmonic series is:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots$$

The first known person to show that this infinite series diverges was Nicole Oresme (1323-1382). His idea was to compare the harmonic series with another divergent series.

Proof:

$$1+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots$$

$$\geq 1+\left(\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots$$

$$=1+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\cdots$$

Therefore, since $1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$ diverges so must the harmonic series.

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For our purposes we shall merely state that a convergent series related to the harmonic series is the alternating harmonic series:

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=ln2=0.693147\ldots$$

Now, the partial sums of the harmonic series are the harmonic numbers:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad for\ n = 1, 2, 3, \dots .$$

The partial sums of the alternating harmonic series are related to the harmonic numbers with the only difference being in the positive/negative signs. So we establish a connection between both partial sums.

Lemma:

$$\sum_{k=1}^{2^n} \frac{1}{k} = \sum_{k=1}^{2^n} \frac{(-1)^{k-1}}{k} + \sum_{k=1}^{2^{n-1}} \frac{1}{k}, \quad for\ n = 1, 2, 3, \dots .$$

Proof:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n - 1} + \frac{1}{2^n} - \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2^n - 1} - \frac{1}{2^n}\right)$$

$$= (1 - 1) + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{2^n - 1} - \frac{1}{2^n - 1}\right) + \left(\frac{1}{2^n} + \frac{1}{2^n}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^{n-1}}$$

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Theorem 1: The sequence $\{H_n\}$ for $n = 1, 2, 3, \dots$ is divergent.

Proof: As the identity holds for $n = 1, 2, 3, \dots$ we construct the subsequence $\{H_{2^n}\}$ as follows:

$$H_2 = 1 + \frac{1}{2} = \left(1 - \frac{1}{2}\right) + 1$$

$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right) + \left(1 + \frac{1}{2}\right)$$

$$= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right) + H_2$$

$$= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right) + \left(1 - \frac{1}{2}\right) + 1$$

$$H_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8}\right) + \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$$

$$= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8}\right) + H_4$$

$$= \left(1 - \frac{1}{2} + \cdots + \frac{1}{7} - \frac{1}{8}\right) + \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right) + \left(1 - \frac{1}{2}\right) + 1$$

And the pattern continues for $H_{16}, H_{32}, H_{64}, \dots, H_{2^n}, \dots$. Therefore, as each consecutive harmonic number has an additional partial sum on the r.h.s. the subsequence $\{H_{2^n}\}$ is unbounded. Hence, the sequence $\{H_n\}$ is divergent.

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Corollary: Theorem 1 gives the following approximation:

$$H_{2^n} \approx (n+1)ln2$$

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For the next theorem we need to the following lemmas.

Lemma A:

$$\sum_{k=1}^{2n} \frac{1}{k} = \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} + \sum_{k=1}^{n} \frac{1}{k}, \quad for\, n = 1, 2, 3, \dots . \qquad (A)$$

Proof:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} - \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n}\right)$$

$$= (1-1) + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n-1}\right) + \left(\frac{1}{2n} + \frac{1}{2n}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

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Lemma B:

$$\sum_{k=1}^{2n+1} \frac{1}{k} = \sum_{k=1}^{2n+1} \frac{(-1)^{k-1}}{k} + \sum_{k=1}^{n} \frac{1}{k}, \quad for\, n = 1, 2, 3, \dots . \qquad (B)$$

Proof:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} - \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{2n} + \frac{1}{2n+1}\right)$$

$$= (1-1) + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{2n} + \frac{1}{2n}\right) + \left(\frac{1}{2n+1} - \frac{1}{2n+1}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

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Theorem 2: Every harmonic number is the sum of partial sums of the alternating harmonic series.

Proof: Similar to the previous proof using A and B allows us to systematically construct the harmonic numbers as follows:

$$H_1 = 1 = \mathbf{1}$$

$(A): n = 1$ $$H_2 = 1 + \frac{1}{2} = \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}}\right) + \mathbf{1}$$

$(B): n = 1$ $$H_3 = 1 + \frac{1}{2} + \frac{1}{3} = \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}}\right) + \mathbf{1}$$

$(A): n = 2$ $$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}}\right) + \left(1 + \frac{1}{2}\right)$$

$$= \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}}\right) + \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}}\right) + \mathbf{1}$$

$(B): n = 2$ $$H_5 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}}\right) + \left(1 + \frac{1}{2}\right)$$

$$= \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}}\right) + \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}}\right) + \mathbf{1}$$

$$H_6 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}} - \frac{\mathbf{1}}{\mathbf{6}}\right) + \left(1 + \frac{1}{2} + \frac{1}{3}\right)$$

$$= \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}} - \frac{\mathbf{1}}{\mathbf{6}}\right) + \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}}\right) + \mathbf{1}$$

$$H_7 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}} - \frac{\mathbf{1}}{\mathbf{6}} + \frac{\mathbf{1}}{\mathbf{7}}\right) + \left(1 + \frac{1}{2} + \frac{1}{3}\right)$$

$$= \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}} - \frac{\mathbf{1}}{\mathbf{6}} + \frac{\mathbf{1}}{\mathbf{7}}\right) + \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}}\right) + \mathbf{1}$$

$$H_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}} + \frac{\mathbf{1}}{\mathbf{5}} - \frac{\mathbf{1}}{\mathbf{6}} + \frac{\mathbf{1}}{\mathbf{7}} - \frac{\mathbf{1}}{\mathbf{8}}\right) + \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$$

$$= \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \cdots + \frac{\mathbf{1}}{\mathbf{7}} - \frac{\mathbf{1}}{\mathbf{8}}\right) + \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{3}} - \frac{\mathbf{1}}{\mathbf{4}}\right) + \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}}\right) + \mathbf{1}$$

$$\vdots$$

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Corollary: The sequence of finite subseries of the infinite harmonic series is converging to $ln2$.

Proof: Using Lemmas A and B we have:

$$\frac{1}{2} = \left(1 - \frac{1}{2}\right)$$

$$\frac{1}{2} + \frac{1}{3} = \left(1 - \frac{1}{2} + \frac{1}{3}\right)$$

$$\frac{1}{3} + \frac{1}{4} = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\right)$$

$$\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\right)$$

$$\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}\right)$$

$$\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}=\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}\right)$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}=\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}\right)$$

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