A Proof of the Erdös-Straus Conjecture

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Abstract

First, divide all integers ≥2 into 8 kinds, and that formulate each of 7 kinds therein into a sum of 3 unit fractions.

For the unsolved kind, again divide it into 3 genera, and that formulate each of 2 genera therein into a sum of 3 unit fractions. For the unsolved genus, further divide it into 5 sorts, and that formulate each of 3 sorts therein into a sum of 3 unit fractions. For two unsolved sorts i.e. 4/(49+120c) and 4/(121+120c) where c≥0, let us depend on logical deduction to prove them separately.

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1. Introduction

The Erdös-Straus conjecture relates to Egyptian fractions. In 1948, Paul Erdös conjectured that for any integer n≥2, there are invariably 4/n=1/x+1/y+1/z, where x, y and z are positive integers; [1].

Later, Ernst G. Straus further conjectured that x, y and z satisfy $x\neq y$, $y\neq z$ and $z\neq x$, because there are convertible $1/2r+1/2r=1/(r+1)+1/r(r+1)$ and $1/(2r+1)+1/(2r+1)=1/(r+1)+1/(r+1)(2r+1)$ where $r\geq 1$; [2].

Thus, the Erdös conjecture and the Straus conjecture are equivalent from each other, and they are called the Erdös-Straus conjecture collectively.

As a general rule, the Erdös-Straus conjecture states that for every integer $n\geq 2$, there are positive integers x, y and z, such that $4/n=1/x+1/y+1/z$. Yet, it is still one both unproved and un-negated conjecture hitherto; [3].

2. Divide integers≥2 into 8 kinds and that formulate 7 kinds therein

First, divide integers≥2 into 8 kinds, i.e. 8k+1, 8k+2, 8k+3, 8k+4, 8k+5, 8k+6, 8k+7 and 8k+8, where $k\geq 0$, and arrange them as follows orderly:

K\n=	8k+1,	8k+2,	8k+3,	8k+4,	8k+5,	8k+6,	8k+7,	8k+8
0,	①,	2,	3,	4,	5,	6,	7,	8,
1,	9,	10,	11,	12,	13,	14,	15,	16,
2,	17,	18,	19,	20,	21,	22,	23,	24,
3,	25,	26,	27,	28,	29,	30,	31,	32,
…,	…,	…,	…,	…,	…,	…,	…,	…,

Excepting n=8k+1, formulate each of other 7 kinds into $1/x+1/y+1/z$:

(1) For n=8k+2, there are $4/(8k+2)=1/(4k+1)+1/(4k+2)+1/(4k+1)(4k+2)$;

(2) For n=8k+3, there are $4/(8k+3)=1/(2k+2)+1/(2k+1)(2k+2)+1/(2k+1)(2k+3)$;

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(3) For n=8k+4, there are 4/(8k+4)=1/(2k+3)+1/(2k+2)(2k+3)+1/(2k+1)(2k+2);

(4) For n=8k+5, there are 4/(8k+5)=1/(2k+2)+1/(8k+5)(2k+2)+1/(8k+5)(k+1);

(5) For n=8k+6, there are 4/(8k+6)=1/(4k+3)+1/(4k+4)+1/(4k+3)(4k+4);

(6) For n=8k+7, there are 4/(8k+7)=1/(2k+3)+1/(2k+2)(2k+3)+1/(2k+2)(8k+7);

(7) For n=8k+8, there are 4/(8k+8)=1/(2k+4)+1/(2k+2)(2k+3)+1/(2k+3)(2k+4).

By this token, n as above 7 kinds of integers be suitable to the conjecture.

3. Divide the unsolved kind into 3 genera and that formulate 2 genera therein

For the unsolved kind n=8k+1 with k≥1, divide it by the modulus 3 into 3 genera to (1) the remainder 0, (2) the remainder 1 and (3) the remainder 2. Excepting the genus (2), formulate each of other 2 genera as listed below:

(8) For n=8k+1 by the modulus 3 to the remainder 0, i.e. let k=3t+1 with t≥0, then there are 4/(8k+1)=1/(8k+1)/3+1/(8k+2)+1/(8k+1)(8K+2) with k≥1, of course, (8k+1)/3 at here is an integer.

(9) For n=8k+1 by the modulus 3 to the remainder 2, i.e. let k=3t+2 with t≥0, then there are 4/(8k+1)=1/(8k+2)/3+1/(8k+1)+1/(8k+1)(8k+2)/3 with k≥2, of course, (8k+2)/3 at here is an integer.

4. Divide the unsolved genus into 5 sorts and that formulate 3 sorts therein

For the unsolved genus n= 8k+1 by the modulus 3 to the remainder 1, i.e.

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let k=3t with t≥1, further divide it into 5 sorts, i.e. 25+120c, 49+120c, 73+120c, 97+120c and 121+120c, where c≥0, as listed below.

C,	25+120c,	49+120c,	73+120c,	97+120c,	121+120c,
0,	25,	49,	73,	97,	121,
1,	145,	169,	193,	217,	241,
2,	205,	289,	313,	337,	361,
…,	…,	…,	…,	…,	…,

Excepting n=49+120c and 121+120c, formulate other 3 sorts as as follows:

(10) For n=25+120c, there are 4/(25+120c)=1/(25+120c)+1/(50+240c)+1/(10+48c);

(11) For n=73+120c, there are 4/(73+120c)=1/(73+120c)(10+15c)+1/(20+30c)+ 1/(73+120c)(4+6c) ;

(12) For n=97+120c, there are 4/(97+120c)=1/(25+30c)+1/(97+120c)(50+60c)+ 1/(97+120c)(10+12c).

For each of listed above 12 equations that express 4/n=1/x+1/y+1/z, please each reader self to make a check respectively.

5. Proving the sort 4/(49+120c)=1/x+1/y+1/z

For a proof of the sort 4/49+120c, it means that when c is equal to each of positive integers plus 0, there are 4/(49+120c)=1/x+1/y+1/z .

After c is endowed with any value, 4/(49+120c) can be substituted by infinitely more a sum of 2 fractions, and that these fractions are different from one another:

$4/(49+120c)$

$= 1/(13+30c) + 3/(13+30c)(49+120c)$

$= 1/(14+30c) + 7/(14+30c)(49+120c)$

$= 1/(15+30c) + 11/(15+30c)(49+120c)$

$= 1/(16+30c) + 15/(16+30c)(49+120c)$

…

$= 1/(13+\alpha+30c) + (3+4\alpha)/(13+\alpha+30c)(49+120c)$, where $\alpha \geq 0$ and $c \geq 0$

…

As listed above, it is observed that we can first let $1/(13+\alpha+30c)=1/x$, after that, prove $(3+4\alpha)/(13+\alpha+30c)(49+120c)=1/y +1/z$.

Proof· When c=0, such as α=1, then the fraction $4/(49+120c)$ got is exactly 4/49, and that there is $4/49=1/14 + 1/99 +1/(98\times99)$;

When c=1, such as α=9, then the fraction $4/(49+120c)$ got is exactly 4/169, and that there is $4/169=1/52 + 1/(2\times169) + 1/(2^2\times169)$.

This manifests that when c=0 and 1, $4/(49+120c)$ has been expressed into a sum of 3 unit fractions respectively.

Excepting $1/(13+\alpha+30c)=1/x$, in following paragraphs, let us analyze and prove $(3+4\alpha)/(13+\alpha+30c)(49+120c) = 1/y+1/z$ by c=2k and c=2k+1 concurrently, where $k \geq 1$.

For the numerator 3+4α, excepting itself as an integer, also can express it into the sum of two integers, i.e. 1+(2+4α), 2+(1+4α), 3+(4α), (3+3α)+α,

$(1+\alpha)+(2+3\alpha)$, $(2+\alpha)+(1+3\alpha)$, $(3+\alpha)+3\alpha$, $(3+2\alpha)+2\alpha$ and $(2+2\alpha)+(1+2\alpha)$.

For the denominator $(13+\alpha+30c)(49+120c)$, in reality merely need us to convert $13+\alpha+30c$, and that can continue to have $49+120c$.

For $13+\alpha+30c$, after α is endowed with values≥ 0, because begin with each constant i.e. 13, 14, 15…p…, there is $\alpha\geq 0$ in like wise, so $13+\alpha+30c$ can be converted to $p+\alpha+30c$ where $p\geq 13$, $\alpha\geq 0$, and $c\geq 0$.

Such being the case, so let $c=2k$, then $(3+4\alpha)/(p+\alpha+30c)$ is exactly $(3+4\alpha)/(p+\alpha+60k)$; again let $c=2k+1$, then $(3+4\alpha)/(p+\alpha+30c)$ is exactly $(3+4\alpha)/(p+\alpha+60k+30)$, where $k\geq 1$.

In fractions $(3+4\alpha)/(p+\alpha+60k)$ and $(3+4\alpha)/(p+\alpha+60k+30)$, the denominator $p+\alpha+60k$ can be every integer≥ 73, and the denominator $p+\alpha+60k+30$ can be every integer≥ 103. On the other, for the numerator $3+4\alpha$, either it is an integer or the sum of two integers as listed above.

In any case, not only each of numerators as listed above is smaller than a corresponding denominator $p+\alpha+60k$ or $p+\alpha+60k+30$, but also $p+\alpha+60k$ and $p+\alpha+60k+30$ contain respectively integers of whole multiples of $3+4\alpha$ and either of two integers which divide $3+4\alpha$ into.

Therefore, $(3+4\alpha)/(p+\alpha+60k)$ can be expressed into a sum of two unit fractions, and $(3+4\alpha)/(p+\alpha+60k+30)$ can be expressed into a sum of two unit fractions too, in which case $p\geq 13$, $\alpha\geq 0$ and $k\geq 1$, also $k\geq 1$ i.e. $c\geq 2$.

If $3+4\alpha$ serve as an integer, and from this get an unit fraction, then can multiply the denominator by 2 to make a sum of two identical unit fractions, afterwards again convert them into the sum of two each other's -distinct unit fractions by the formula 1/2r+1/2r=1/(r+1)+1/r(r+1).

Let a sum of two unit fractions which expresses (3+4α)/(p+α+60k) into be written as 1/μ+1/ν, again let a sum of two unit fractions which expresses (3+4α)/(p+α+60k+30) into be written as 1/φ+1/ψ, where μ, ν, φ and ψ express positive integers.

For 1/μ+1/ν and 1/φ+1/ψ, multiply every denominator by 49+120c reserved in the front, then get 1/μ(49+120c)+1/ν(49+120c)=1/y+1/z and 1/φ(49+120c)+1/ψ(49+120c)=1/y +1/z.

To sum up, we have proved 4/(49+120c)=1/(13+α+30c)+1/y+1/z, to wit 4/(49+120c)=1/x+1/y+1/z.

6. Proving the sort 4/(121+120c)=1/x+1/y+1/z

For a proof of the sort 4/(121+120c), it means that when c is equal to each of positive integers plus 0, there are 4/(121+120c)=1/x+1/y+1/z.

After c is endowed with any value, 4/(121+120c) can be substituted by infinitely more a sum of 2 fractions, and that these fractions are different from one another :

4/(121+120c)

= 1/(31+30c) + 3/(31+30c)(121+120c),

= 1/(32+30c) + 7/(32+30c)(121+120c),

$= 1/(33+30c) + 11/(33+30c)(121+120c)$,

$= 1/(34+30c) + 15/(34+30c)(121+120c)$,

…

$= 1/(31+\alpha+30c) + (3+4\alpha)/(31+\alpha+30c)(121+120c)$, where $\alpha\geq 0$ and $c\geq 0$.

…

As listed above, it is observed that we can first let $1/(31+\alpha+30c)=1/x$, after that, prove $(3+4\alpha)/(31+\alpha+30c)(121+120c)=1/y +1/z$.

***Proof*:** When c=0, such as $\alpha=2$, then the fraction $4/(121+120c)$ got is exactly 4/121, and that there is $4/121=1/33+1/(3\times 11^2+1)+ 1/(3\times 11^2)(3\times 11^2+1)$;

When c=1, such as $\alpha=2$, then the fraction $4/(121+120c)$ got is exactly 4/241, and that there is $4/241=1/63+1/(2\times 3\times 241)+1/(2\times 3^2\times 7\times 241)$.

This manifests that when c=0 and 1, $4/(121+120c)$ has been expressed into a sum of 3 unit fractions respectively.

Excepting $1/(31+\alpha+30c)=1/x$, in following paragraphs, let us analyze and prove $(3+4\alpha)/(31+\alpha+30c)(121+120c) = 1/y+1/z$ by c=2k and c=2k+1 concurrently, where $k\geq 1$.

For the numerator $3+4\alpha$, excepting itself as an integer, also can express it into the sum of two integers, i.e. $1+(2+4\alpha)$, $2+(1+4\alpha)$, $3+(4\alpha)$, $(3+3\alpha)+\alpha$, $(1+\alpha)+(2+3\alpha)$, $(2+\alpha)+(1+3\alpha)$, $(3+\alpha)+3\alpha$, $(3+2\alpha)+2\alpha$ and $(2+2\alpha)+(1+2\alpha)$.

For the denominator $(31+\alpha+30c)(121+120c)$, in reality merely need us to convert $31+\alpha+30c$, and that can continue to have $121+120c$.

For $31+\alpha+30c$ after α is endowed with values ≥ 0, because begin with each constant i.e. 31, 32, 33…q…, there is $\alpha\geq 0$ in like wise, so $31+\alpha+30c$ can be converted to $q+\alpha+30c$, where $q\geq 31$, $\alpha\geq 0$ and $c\geq 0$.

Such being the case, so let $c=2k$, then $(3+4\alpha)/(q+\alpha+30c)$ is exactly $(3+4\alpha)/(q+\alpha+60k)$; again let $c=2k+1$, then $(3+4\alpha)/(q+\alpha+30c)$ is exactly $(3+4\alpha)/(q+\alpha+60k+30)$, where $k\geq 1$.

In fractions $(3+4\alpha)/(q+\alpha+60k)$ and $(3+4\alpha)/(q+\alpha+60k+30)$, the denominator $q+\alpha+60k$ can be every integer≥ 91, and the denominator $q+\alpha+60k+30$ can be every integer≥ 121. On the other, for the numerator $3+4\alpha$, either it is an integer or the sum of two integers as listed above.

In any case, not only each of numerators as listed above is smaller than a corresponding denominator $q+\alpha+60k$ or $q+\alpha+60k+30$, but also $q+\alpha+60k$ and $q+\alpha+60k+30$ contain respectively integers of whole multiples of $3+4\alpha$ and either of two integers which divide $3+4\alpha$ into.

Therefore, $(3+4\alpha)/(q+\alpha+60k)$ can be expressed into a sum of two unit fractions, and $(3+4\alpha)/(q+\alpha+60k+30)$ can be expressed into a sum of two unit fractions too, in which case $q\geq 31$, $\alpha\geq 0$ and $k\geq 1$, also $k\geq 1$ i.e. $c \geq 2$.

If $3+4\alpha$ serve as an integer, and from this get an unit fraction, then can multiply the denominator by 2 to make a sum of two identical unit fractions, afterwards again convert them into the sum of two each other's -distinct unit fractions by the formula $1/2r +1/2r = 1/(r+1) +1/r(r+1)$.

Let a sum of two unit fractions which expresses $(3+4\alpha)/(q+\alpha+60k)$ into be written as $1/\beta+1/\xi$, again let a sum of two unit fractions which expresses $(3+4\alpha)/(q+\alpha+60k+30)$ into be written as $1/\eta+1/\delta$, where β, ξ, η and δ express positive integers.

For $1/\beta+1/\xi$ and $1/\eta+1/\delta$, multiply every denominator by $121+120c$ reserved in the front, then get $1/\beta(121+120c)+1/\xi(121+120c) =1/y+1/z$ and $1/\eta(121+120c)+1/\delta(121+120c) =1/y+1/z$.

To sum up, we have proved $4/(121+120c) =1/(31+\alpha+30c)+1/y+1/z$, to wit $4/(121+120c) = 1/x+1/y+1/z$.

The proof was thus brought to a close. As a consequence, the Erdös-Straus conjecture is tenable.

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