Evaluating the Alignment of 106 Linearly Polarized Optical QSOs

Richard Shurtleff *

Abstract

A sample of partially linearly polarized optical quasars (QSOs) is analyzed by the Hub Test. The data originates in an online catalog of 355 such QSOs, collected and published by others. Without their efforts this article would not be possible. The 106 QSOs populate a region with a radius of about 40° centered on a point in the sky near the South Galactic Pole. We find the polarization directions to be extremely well aligned. Besides supplying convincing evidence that the alignment is not due to chance, additional quantities are calculated that describe the collective behavior of the polarization directions. The alignment function mapped onto the Celestial Sphere provides a satisfying visual representation. This article is a Mathematica notebook.

Keywords: Polarization; Alignment; Computer Program; Uncertainties; Quasars; Hub Test; Large Scale Structure

*Department of Sciences, Wentworth Institute of Technology, 550 Huntington Avenue, Boston, MA, USA, 02115, orcid.org/0000-0001-5920-759X, e-mail addresses: shurtleffr@wit.edu, momentummatrix@yahoo.com

0. Preface

UPDATES: Notes plus errata and other changes to the online pdf version may appear here.

(1) The pdf version of this notebook differs from the ready-to-run notebook which is available via the link in Ref. 1.
(1a) Dealing with nR = 10,000 random runs and 10,000 uncertainty runs presents practical logistical problems.
(1b) The pdf version has nR = 10,000 random runs and 10,000 uncertainty runs. The ready-to-run version has nR = 2000 runs because 2000 runs takes much less computer time than 10,000 runs.
(1b) You can select nR to be any number of runs in Sec. 3 Settings. Setting nR = 200 runs gives results that can be compared to the 10,000 run values. And nR = 200 processes quickly.
(1c) The pdf version has the random run generating cell and the uncertainty run generating cell inactive, as comments. The needed data has been saved in .dat files. The random runs were generated separately from the uncertainty runs and saved. Separating and saving avoided overwhelming my computer.
(2) The filename for the associated notebook: “20210205IntermediateKitFor193BestOpticalQSORegions2.nb”. This file is available for download via the link in Ref. 1.
(3) The numerical values quoted in the Concluding Remarks in Sec. 8 are associated with the random runs and uncertainty runs with the pdf version. Other sets of random runs and uncertainty runs should alter those numerical values. They are unlikely to change much.

```
In[1]:= Print["The date and time that this statement was evaluated: ", Now]

The date and time that this statement was evaluated:  Wed 17 Feb 2021 12:45:26 GMT−5.
```

While this article is a Mathematica notebook, it is difficult, perhaps impossible, to run from the pdf version. A link to a ready-to-run version, Ref. 1, is provided for convenience.
The Hub Test is explained in some detail in Ref. 2, “Indirect polarization alignment with points on the sky, the Hub Test”.
A template for performing calculations similar to those in this notebook, but with other data, can be found online, Ref. 3. These notebooks were created using Wolfram Mathematica, Version Number: 12.1, Ref. 4.
The formulas for creating Aitoff plots were found on Wikipedia, Ref. 5.

CONTENTS

1. Introduction

Given a collection of astronomical sources with linearly polarized electromagnetic emissions, one can evaluate the mutual alignment of the polarization directions. In this paper, we apply the Hub Test to judge the alignment of a set of partially polarized quasars (QSOs). Since QSOs are distant extragalactic sources, any alignment of their polarizations is remarkable, possibly providing evidence of large-scale structures or perhaps reflecting the action of the intervening medium through which the QSOs are viewed. The alignment of the sample analyzed here is matched by only about one in hundreds of thousands of randomly polarized samples.

In Ref. 6, the region called “A3” contains 22 of the 106 objects in the sample analyzed in this notebook. The original article, Ref. 6, which includes the QSO catalog as an Appendix, provides far ranging discussions of alignment mechanisms and other effects.

However intriguing, interpretation of the results are beyond the scope of this notebook. For example contamination by interstellar polarization in our Galaxy is considered and dismissed as unlikely to be able to explain the alignments uncovered by their tests. In Ref. 8, alignments detected from the catalog data are shown to imply that “quasar spin axes are likely parallel to their host large-scale structures.” These and similar topics show how the problem of alignment is interesting and important in astronomy and astrophysics. However, they are not considered in this notebook. One hopes that the Hub Test can help with such investigations.

One motivation for constructing this notebook is to present an application of the Hub Test. The tests in Ref. 6 differ from the Hub Test because their tests compare polarization directions directly, while the Hub Test is indirect. The Hub Test infers alignment from determining the alignment of the polarization directions with points on the sky. Essentially the test looks for the convergence of a number of great circles.

1a. The Hub Test

The Hub Test, Ref. 2, answers the question of alignment indirectly by looking at the great circles on the Celestial sphere determined by the polarization directions. At each source one great circle has a tangent parallel to the polarization direction. The collection of these great circles appear over the sphere in regions of various densities, regions of convergence and divergence.

Start with a single source depicted in Fig. 1.

Out[ ]=

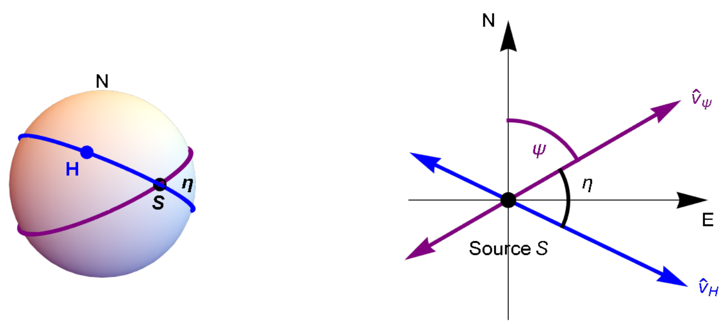


Figure 1: The Celestial sphere is pictured on the left and on the right is the plane tangent to the sphere at the source S. The linear polarization direction $\hat{v}_\psi$ lies in the tangent plane and determines the purple great circle on the sphere. A point H on the sphere and the location S of the source determine a second great circle, the blue circle drawn on the sphere at the left. Clearly, H and S must be distinct points on the sphere. The angle η, with $0° \leq \eta \leq 90°$, measures the alignment of the polarization direction with the point H. Perfect alignment occurs when $\eta = 0°$ and the two great circles overlap. Perpendicular great circles, $\eta = 90°$, indicates maximum avoidance of the polarization direction $\hat{v}_\psi$ with the point H on the sphere.

The basic concept includes “avoidance”, as well as alignment. Avoidance is high when the two directions $\hat{v}_\psi$ and $\hat{v}_H$ differ by a large angle, $\eta \rightarrow 90°$. Perpendicular great circles at S , $\eta = 90°$, would indicate the maximum avoidance of the polarization direction and the point on the sphere.

With many sources S_i, $i = 1, ..., N$, there are N alignment angles η_{iH} for the point H. To quantify the alignment of the N sources with the point H, calculate the arithmetic average alignment angle at H,

$$\overline{\eta}(\text{H}) = \frac{1}{N} \sum_{i=1}^{N} \eta_{\text{iH}} \;. \qquad (1)$$

The alignment angle $\overline{\eta}(\text{H})$ is a function of position H on the sphere. Since great circles that contain the point H also contain the diametrically opposite point -H, the function $\overline{\eta}(\text{H})$ is symmetric across diameters. The function $\overline{\eta}(\text{H})$ is also a measure of convergence and divergence of the great circles determined by the polarization directions. Where the alignment function $\overline{\eta}(\text{H})$ is small, the circles converge, where $\overline{\eta}(\text{H})$ is large, the circles diverge.

The polarization directions are best aligned with the point H_{min} where the alignment angle is a minimum $\overline{\eta}_{\text{min}}$. The polarization directions most avoid the point $H_{\text{max}}$where the function $\overline{\eta}(\text{H})$ takes its maximum value $\overline{\eta}_{\text{max}}$.

The location of their most extreme convergence is a “hub”, H_{min}, called the “alignment hub”. The most extreme divergence is another hub, H_{max}, the “avoidance hub”. Alignment and avoidance are symmetrical concepts with the Hub Test.

The Hub Test of alignment is based on the idea that the polarization directions are well-aligned with each other when they are well-aligned with some point of convergence, the hub H_{min}.

1b. Statistics

To judge the significance of the alignment or avoidance of the observed polarization directions for a given sample of N sources, one estimates how likely it would be for a sample with random polarization directions to yield the observed minimum alignment angle $\overline{\eta}_{min}$ and the maximum value $\overline{\eta}_{max}$.

By running the calculations with the same locations for the sources, but supplying those sources with random polarization directions, one can build up the statistics. The distributions of the results provide estimates of the likelihood that random data would produce certain results.

An alternative method is available. Generic formulas are presented in the Statistics Section of an earlier notebook, Ref. 3. Those formulas rely on the number of sources N and a rough size of their reach on the sphere. The results are less accurate than the formulas developed here that are obtained with random runs specialized to the sources analyzed in this notebook. The specialized statistics used here are more accurate than those obtained by applying the generic formulas in Ref. 3.

2. Preliminary

Consider a sphere in 3 dimensional Euclidean space. See Fig. 1 in the Introduction. The sphere is called the “Celestial sphere” or simply the “sphere” or sometimes “the sky”. The center of the sphere is the origin of a 3D Cartesian coordinate system with coordinates (x, y, z). The direction of the positive z-axis is associated with “North”. Right ascension, RA or α, and declination, dec or δ, are measured as usual with the direction of the positive x-axis along (RA,dec) = (0°, 0°). Declination δ = 90° indicates the North pole, the direction from the origin (0,0,0) to (0,0,1) .

From a point-of-view located outside the sphere, as in the left-hand sketch in Fig. 1, one pictures a source S plotted on the sphere and, in the 2D tangent plane at S, local North is upward and local East is to the right. See the right-hand sketch in Fig. 1. A “position angle” at the point S on the sphere, such as the angle ψ in Fig. 1, is measured in the 2D plane tangent to the sphere at S. The position angle ψ is measured clockwise from local North with East to the right.

The rest of this section contains some useful formulas that are helpful since we often mix spherical and Cartesian coordinates.

Definitions:

(α,δ) — Right Ascension RA and declination dec of a point on the sphere. Sometimes we use radians, sometimes degrees.
er(α,δ) — radial unit vector in a Cartesian coordinate system from the Origin to the point on the sphere with (RA,dec) = (α,δ), with α,δ in radians
eN(α,δ) — unit vector along local North at the point (α,δ) on the sphere, with α,δ in radians
eE(α,δ) — unit vector along local East at the point (α,δ) on the sphere, with α,δ in radians
αFROMr$(\hat{r})$ — RA for the point on the sphere determined by radial unit vector $\hat{r}$, result in radians
δFROMr$(\hat{r})$ — dec for the point on the sphere determined by radial unit vector $\hat{r}$, result in radians

```
In[2]:= (* For a Source at (RA,dec) = (α,δ): er, eN,
eE are unit vectors from Origin to Source, local North, local East, resp.  *)
er[α_, δ_] := er[α, δ] = {Cos[α] Cos[δ], Sin[α] Cos[δ], Sin[δ]}
eN[α_, δ_] := eN[α, δ] = {-Cos[α] Sin[δ], -Sin[α] Sin[δ], Cos[δ]}
eE[α_, δ_] := eE[α, δ] = {-Sin[α], Cos[α], 0}
Print["Check er.er = 1, er.eN = 0, er.eE = 0,
   eN.eN = 1, eN.eE = 0,eE.eE = 1, erXeE = eN, eEXeN = er, eNXer = eE: ",
 {0} == Union[Flatten[Simplify[{er[α, δ].er[α, δ] - 1, er[α, δ].eN[α, δ], er[α, δ].eE[α, δ],
      eN[α, δ].eN[α, δ] - 1, eN[α, δ].eE[α, δ], eE[α, δ].eE[α, δ] - 1, Cross[er[α, δ], eE[α, δ]] -
       eN[α, δ], Cross[eE[α, δ], eN[α, δ]] - er[α, δ], Cross[eN[α, δ], er[α, δ]] - eE[α, δ]}]]]]

Check er.er = 1, er.eN = 0, er.eE = 0, eN.eN
   = 1, eN.eE = 0,eE.eE = 1, erXeE = eN, eEXeN = er, eNXer = eE: True
```

Get (α,δ) in radians from radial vector r, with $-\pi < \alpha < +\pi$ and $\frac{-\pi}{2} < \delta < \frac{+\pi}{2}$

```
In[6]:= αFROMr[r_] := N[ArcTan[Abs[r[[2]] / r[[1]]]]] /; (r[[2]] ≥ 0 && r[[1]] > 0)
αFROMr[r_] := N[π - ArcTan[Abs[r[[2]] / r[[1]]]]] /; (r[[2]] ≥ 0 && r[[1]] < 0)
αFROMr[r_] := N[-π + ArcTan[Abs[r[[2]] / r[[1]]]]] /; (r[[2]] < 0 && r[[1]] < 0)
αFROMr[r_] := N[-ArcTan[Abs[r[[2]] / r[[1]]]]] /; (r[[2]] < 0 && r[[1]] > 0)
αFROMr[r_] := π / 2. /; (r[[2]] ≥ 0 && r[[1]] == 0)
αFROMr[r_] := -(π / 2.) /; (r[[2]] < 0 && r[[1]] == 0)

In[12]:= δFROMr[r_] := N[ArcTan[r[[3]] / (√(r[[1]]^2 + r[[2]]^2))]] /; (√(r[[1]]^2 + r[[2]]^2) > 0)
δFROMr[r_] := Sign[r[[3]]] (π / 2.) /; (√(r[[1]]^2 + r[[2]]^2) == 0)
```

3. Input and Settings

3a. Selection Process:

The selection of the 106 optical polarized QSOs proceeds as follows. The relevant calculations appear in a private notebook, “20210121HubTestOpticalQSOs24Deg.nb”.

The selection process employs the same grid developed below, a grid of 10,518 equally spaced points on the sphere. Each of the 10,518 grid points serves as the center of 10,518 regions each with a radius of 24°. The QSOs in the catalog make 10,518 samples with populations ranging from 0 to 73 QSOs. Seven sources are required for the generic statistics formulas to be sufficiently accurate, see Ref. 3. Of the 10,518 regions, just 4224 regions were populated with 7 or more QSOs.

Applying the generic statistics formulas in Ref. 3, just 418 of the 4224 regions showed very significant alignments, meaning that 1% or fewer randomly polarized 24° radius regions with the same number of sources would be better aligned. We chose to focus on the 200 most significant regions, an arbitrary choice. From Fig. 2 below it is clear that all but 7 of the 200 most significant regions are collected together, with a large gap separating the collection of 193 region centers from the group of 7. In this article, we make a sample of the QSOs in the 193 (= 200 - 7) regions in the cloud of regions depicted in Fig. 2. The 193 regions are populated with the 106 QSO sources analyzed in this notebook, completing the selection process.

Out[ ]=

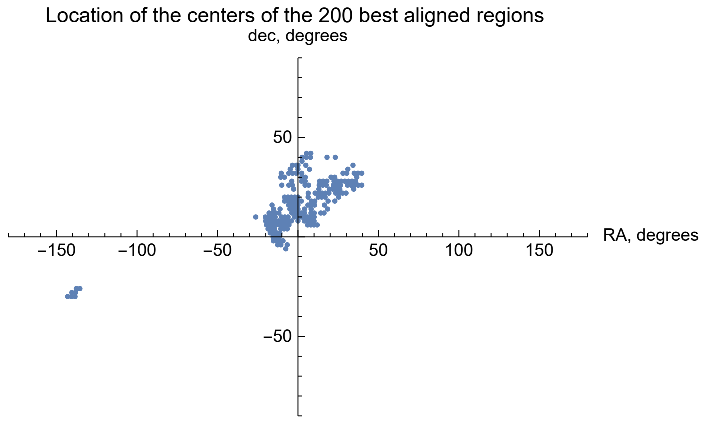


Figure 2: The centers of the 200 most significantly aligned regions. The dots indicate the centers of 24° radius regions; no sources are plotted. The significances of alignment range from 0.000016 to 0.0015 , i.e. 1 in 63,000 to 1 in 670 random runs. By significance, we mean the fraction of randomly directed polarized samples that were better aligned. The most significantly aligned region has 54 QSOs and is centered at (RA,dec) = (-12°, 8 °}. Note the island of 7 regions at (RA,dec) = (-140°,-30°). These 7 regions are thereby distinguished from the 193 others and dropped from consideration. (Figure and numerical values copied from "20210121HubTestOpticalQSOs24Deg.nb".)

Out[ ]=

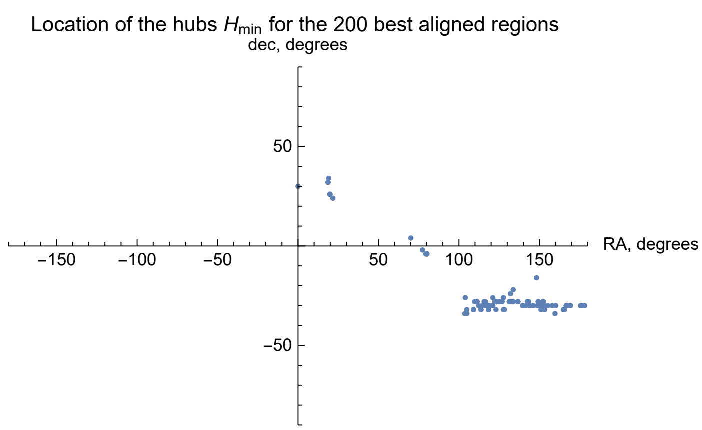


Figure 3: The points $H_{\min}$ where the alignment function $\overline{\eta}(\mathrm{H})$ has the minimum value $\overline{\eta}_{\min}$, for the 200 most significantly aligned regions. The most significant region has its hub $H_{\min}$ located at (RA,dec) = (123°, -28°) in the band of $H_{\min}$ hubs stretching from RA = 100° to RA = 180°. (Figure and numerical results copied from "20210121HubTestOpticalQSOs24Deg.nb".)

Definitions:

gridSpacing separation in degrees between grid points on a constant latitude circle and separation of constant latitude circles. There is no bunching at the poles.

dηContourPlot separation of successive contour lines on the map in Sec. 6b, in degrees

dataDirectory folder on the computer where data files and other relevant files are stored

nR number of runs generated from random polarization directions and also the number of runs generated with polarization directions consistent with the uncertainty in the observed directions

dataFromTheCatalog data for the QSOs in the sample, copied from the original catalog

nSrc number of sources in the sample

nameSrc Object coordinate name (B1950)

rSrc radial unit vectors in Cartesian coordinates from origin to sources

αSrc RA = α of the position of the sources, in radians, $-\pi <= \alpha <= +\pi$
δSrc dec = δ of the position of the sources, in radians, $-\pi/2 <= \delta <= +\pi/2$
eNSrc unit vectors along local North in the tangent planes of the sources
eESrc unit vectors along local East in the tangent planes of the sources
ψn Optical polarization position angle (PPA), in radians
$\sigma\psi$n Uncertainty of PPA, in radians

3b. Settings

In[14]:=
```
gridSpacing = 2.(*, in degrees. This is a setting.*);
Print["The grid points are separated by ",
 gridSpacing, "° arcs along latitude and longitude."]
```

```
The grid points are separated by 2.° arcs along latitude and longitude.
```

In[16]:=
```
dηContourPlot = 4;(*, in degrees. This is a setting.*)
```

In[17]:=
```
dataDirectory =
  "C:\\Users\\shurt\\Dropbox\\HOME_DESKTOP-0MRE5OJ\\SendXXX_CJP_CEJPetc\\SendViXra\\
    20200715AlignmentMethod\\20200715AlignmentMMAnotebooks\\StarterKit\\20210201
    Optical200MostSigRegionsCombined";(*This is a setting.*)
```

In[18]:=
```
nR = 10000;
(*number of runs with random ψ for statistics and with various ψ allowed
 by uncertainty for determining the uncertainty in the results. The number
 of runs can be changed by hand at the relevant "For" statements below. *)
```

3c. Inputs Note: The angles α, δ, ψ, $\sigma\psi$ are expected to be input in radians.

For the 106 QSOs in the 193 of the 200 most significant regions, the relevant data from the catalog, Refs. 6 and 7, is entered as input below in this section. For details on the data, consult the “ReadMe” and other files included with the catalog online; see Ref. 7.

The description of the entries for each object in the catalog:

1. Object coordinate name (B1950), 2. Redshift z, 3. % Optical polarization degree, 4. Uncertainty of % polarization degree, 5. Optical polarization position angle (PPA), in degrees, 6. Uncertainty of PPA, in degrees, 7. Reference code, detailed in file “refs.dat” in Ref. 7.

The catalog data is copied below as the quantity “dataFromTheCatalog”. The 106 objects analyzed in this notebook are the *i*th entries in the 355 object catalog, where *i* = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 28, 29, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 44, 45, 47, 50, 53, 56, 58, 62, 274, 277, 279, 280, 281, 282, 284, 286, 288, 291, 292, 293, 294, 297, 300, 301, 302, 303, 304, 305, 306, 308, 310, 311, 314, 315, 317, 318, 319, 320, 321, 322, 323, 324, 325, 326, 327, 328, 329, 330, 331, 334, 335, 336, 337, 338, 339, 340, 341, 342, 344, 345, 346, 347, 348, 349, 350, 351, 353, 354, 355}.

In[19]:=
```
(*Input*)
dataFromTheCatalog = {{"B*0003-066", 0.347`, 3.5`, 1.6`, 160, 12, 4}, {"B*0003+158",
    0.45`, 0.62`, 0.16`, 114, 7, 1}, {"B*0004+017", 1.711`, 1.29`, 0.28`, 122, 6, 8},
   {"B*0010-002", 2.145`, 1.7`, 0.77`, 116, 13, 8}, {"B*0013-004", 2.084`,
    1.03`, 0.33`, 115, 10, 0}, {"B*0017+154", 2.012`, 1.14`, 0.52`, 137, 13, 3},
```

```
{"B*0019+011", 2.124`, 0.76`, 0.19`, 26, 7, 8}, {"B*0021-022", 2.296`, 0.7`,
 0.32`, 170, 14, 0}, {"B*0024+224", 1.118`, 0.63`, 0.29`, 90, 14, 2},
{"B*0025-018", 2.076`, 1.16`, 0.52`, 109, 13, 8},
{"B*0029+002", 2.226`, 0.75`, 0.34`, 158, 14, 0}, {"B*0038+280", 0.194`,
 2.16`, 0.27`, 103, 3, 10}, {"B*0047+278", 0.277`, 2.28`, 0.75`, 49, 9, 10},
{"B*0048+292", 0.136`, 2.47`, 0.49`, 98, 5, 10}, {"B*0050+124", 0.061`,
 0.61`, 0.08`, 8, 3, 1}, {"B*0051+291", 1.828`, 0.8`, 0.38`, 119, 14, 3},
{"B*0055+157", 0.211`, 0.67`, 0.28`, 15, 13, 10}, {"B*0059+261", 0.194`,
 2.11`, 0.61`, 120, 8, 10}, {"B*0100+130", 2.66`, 0.84`, 0.29`, 112, 10, 2},
{"B*0103+257", 0.411`, 6.03`, 0.54`, 114, 2, 10}, {"B*0105+215", 0.285`,
 5.45`, 0.99`, 119, 5, 10}, {"B*0106+013", 2.107`, 1.87`, 0.84`, 143, 13, 3},
{"B*0109-014", 1.758`, 1.77`, 0.35`, 76, 6, 8}, {"B*0110+297", 0.363`,
 2.6`, 1.15`, 63, 13, 2}, {"B*0117+213", 1.493`, 0.61`, 0.2`, 102, 9, 1},
{"B*0117+197", 0.087`, 0.74`, 0.26`, 128, 11, 10}, {"B*0119+041", 0.637`,
 4.2`, 1.1`, 59, 6, 4}, {"B*0123+257", 2.358`, 1.63`, 0.81`, 140, 14, 3},
{"B*0130+242", 0.457`, 1.7`, 0.52`, 110, 9, 2}, {"B*0133+207", 0.425`,
 1.62`, 0.36`, 49, 6, 3}, {"B*0137-018", 2.232`, 1.12`, 0.29`, 61, 8, 0},
{"B*0137-010", 0.33`, 0.63`, 0.31`, 154, 14, 2}, {"B*0138-097", 0.733`,
 3.6`, 1.5`, 168, 11, 4}, {"B*0145+042", 2.029`, 2.7`, 0.32`, 131, 3, 0},
{"B*0146+017", 2.909`, 1.23`, 0.21`, 141, 5, 8}, {"B*0148+090", 0.299`,
 1.21`, 0.54`, 139, 13, 3}, {"B*0154+169", 0.213`, 1.44`, 0.47`, 66, 9, 10},
{"B*0204+292", 0.11`, 1.07`, 0.21`, 117, 6, 11}, {"B*0205+024", 0.155`,
 0.72`, 0.17`, 22, 7, 2}, {"B*0214+108", 0.408`, 1.13`, 0.22`, 121, 6, 2},
{"B*0231+244", 0.31`, 2.57`, 0.46`, 99, 5, 10}, {"B*0239+006", 2.071`, 1.47`,
 0.24`, 167, 5, 11}, {"B*0310+209", 0.094`, 1.53`, 0.43`, 147, 8, 10},
{"B*0322+176", 0.328`, 1.23`, 0.38`, 119, 8, 10}, {"B*0346+127", 0.21`,
 2.23`, 0.73`, 69, 9, 10}, {"B*2105-065", 0.644`, 1.12`, 0.22`, 147, 6, 11},
{"B*2121+050", 1.878`, 10.7`, 2.9`, 68, 6, 4}, {"B*2128-088", 1.983`, 0.61`,
 0.27`, 171, 14, 11}, {"B*2129-072", 2.048`, 1.78`, 0.32`, 44, 5, 11},
{"B*2131-021", 0.557`, 16.9`, 4.`, 93, 1, 4}, {"B*2132-011", 1.66`, 0.83`,
 0.25`, 113, 9, 11}, {"B*2139-085", 0.57`, 0.79`, 0.22`, 160, 8, 11},
{"B*2141+040", 0.463`, 0.84`, 0.25`, 111, 9, 11}, {"B*2145+067", 0.99`,
 0.6`, 0.2`, 138, 11, 4}, {"B*2155-152", 0.672`, 22.6`, 1.1`, 7, 2, 4},
{"B*2201-185", 1.814`, 1.43`, 0.51`, 7, 10, 8}, {"B*2203-188", 0.619`,
 1.26`, 0.29`, 31, 7, 11}, {"B*2203-215", 0.577`, 0.99`, 0.3`, 47, 9, 11},
{"B*2208-173", 1.21`, 1.`, 0.24`, 148, 7, 11}, {"B*2216-038", 0.901`,
 1.1`, 0.4`, 139, 11, 4}, {"B*2216-091", 0.75`, 0.72`, 0.31`, 1, 14, 11},
{"B*2219+196", 0.366`, 7.19`, 1.14`, 109, 4, 10}, {"B*2219+197", 0.211`,
 0.95`, 0.23`, 138, 7, 10}, {"B*2223-052", 1.404`, 13.6`, 0.4`, 133, 1, 4},
{"B*2223+197", 0.147`, 1.38`, 0.56`, 58, 13, 10}, {"B*2225-055", 1.981`,
 4.37`, 0.29`, 162, 2, 0}, {"B*2227-088", 1.562`, 9.2`, 0.87`, 173, 3, 6},
{"B*2230+025", 2.147`, 0.68`, 0.29`, 119, 14, 0}, {"B*2230+114", 1.037`,
 7.3`, 0.3`, 118, 1, 4}, {"B*2240-260", 0.774`, 14.78`, 0.21`, 131, 1, 11},
{"B*2243-123", 0.63`, 1.25`, 0.26`, 156, 6, 6}, {"B*2247+140", 0.237`,
 1.39`, 0.38`, 75, 8, 2}, {"B*2247+015", 1.128`, 1.11`, 0.25`, 82, 7, 11},
{"B*2251+113", 0.323`, 1.`, 0.15`, 49, 4, 2}, {"B*2251+158", 0.859`, 2.9`,
 0.3`, 144, 3, 4}, {"B*2251+244", 2.328`, 1.34`, 0.67`, 113, 14, 3},
{"B*2251+006", 1.15`, 0.89`, 0.26`, 129, 9, 11}, {"B*2253-115", 1.33`,
 0.81`, 0.23`, 130, 8, 11}, {"B*2254+024", 2.09`, 1.67`, 0.75`, 2, 13, 6},
{"B*2255-282", 0.926`, 2.`, 0.4`, 112, 6, 4}, {"B*2300+254", 0.331`, 4.38`,
 1.16`, 140, 7, 10}, {"B*2301+060", 1.268`, 3.69`, 0.26`, 163, 2, 11},
```

```
{"B*2302-279", 1.435`, 0.82`, 0.21`, 9, 7, 11}, {"B*2308+098", 0.432`,
 1.14`, 0.16`, 105, 4, 1}, {"B*2317-006", 1.889`, 1.85`, 0.3`, 164, 5, 11},
{"B*2320-035", 1.411`, 9.56`, 0.2`, 90, 1, 11}, {"B*2332-017", 1.184`,
 4.86`, 0.19`, 92, 1, 11}, {"B*2333-101", 1.76`, 0.99`, 0.34`, 160, 10, 11},
{"B*2335-027", 1.072`, 3.55`, 0.3`, 110, 2, 11}, {"B*2340-036", 0.896`,
 0.87`, 0.25`, 130, 8, 2}, {"B*2341-235", 2.82`, 0.64`, 0.2`, 122, 9, 9},
{"B*2342+120", 0.199`, 1.01`, 0.24`, 127, 6, 10}, {"B*2344+184", 0.138`,
 1.01`, 0.32`, 88, 10, 11}, {"B*2345-167", 0.576`, 4.9`, 1.5`, 70, 8, 4},
{"B*2345+002", 1.946`, 0.91`, 0.3`, 134, 10, 11}, {"B*2347-105", 1.31`,
 1.05`, 0.29`, 106, 8, 11}, {"B*2349-010", 0.174`, 0.91`, 0.21`, 143, 7, 2},
{"B*2350+008", 2.156`, 1.59`, 0.26`, 27, 5, 11}, {"B*2351-154", 2.665`,
 3.73`, 1.56`, 13, 12, 2}, {"B*2353+283", 0.731`, 1.43`, 0.54`, 76, 11, 3},
{"B*2353-008", 2.936`, 1.81`, 0.34`, 16, 5, 11}, {"B*2354-117", 0.949`,
 2.`, 0.4`, 105, 6, 4}, {"B*2354+002", 0.41`, 0.67`, 0.3`, 74, 14, 11},
{"B*2356-006", 1.757`, 1.46`, 0.33`, 158, 7, 11}, {"B*2357-129", 0.868`,
 4.12`, 0.2`, 151, 1, 11}, {"B*2358+022", 1.872`, 2.12`, 0.51`, 45, 7, 8}};
```

In[20]:=
```
nSrc = Length[dataFromTheCatalog];(*calculated from Input.*)
Print["There are ", nSrc, " sources."]
```

There are 106 sources.

Coordinates:

The coordinate name for each object contains the Right Ascension and Declination. For example “B*2320-035” has (RA,dec) = (α,δ) = (23hr 20min, -3.5°).

RA: The 23hr 20min is converted to radians: $\left(23+\frac{20}{60}\right)\left(\frac{2\pi}{24}\right) = 6.1087$ radians, which is close to 2π. We want RA $(=\alpha)$ to be between $-\pi$ and $+\pi$, i.e. $-\pi \le \alpha \le +\pi$, so we subtract 2π from the RA and get $6.1087 - 2\pi = -0.1745$, which can be found in the values for αSrc below.

Dec: The -3.5° is converted to radians: $(-3.5\,°)\left(\frac{2\pi}{360\,°}\right) = -0.0611$ radians, and that value can be found with δSrc below.

In[22]:=
```
(*names of objects from the catalog. Determined by Input.*)
nameSrc = Table[dataFromTheCatalog[[i1, 1]], {i1, nSrc}];
nameSrc[[86]](*The object discussed in regard to the above Coordinate calculation.*)
```

Out[23]= B*2320-035

In[24]:=
```
(*The Right Ascensions of the sources Si. Determined by Input. *)
αSrc = {0.0131`, 0.0131`, 0.0175`, 0.0436`, 0.0567`, 0.0742`, 0.0829`, 0.0916`,
   0.1047`, 0.1091`, 0.1265`, 0.1658`, 0.2051`, 0.2094`, 0.2182`, 0.2225`, 0.24`,
   0.2574`, 0.2618`, 0.2749`, 0.2836`, 0.288`, 0.3011`, 0.3054`, 0.336`, 0.336`,
   0.3447`, 0.3622`, 0.3927`, 0.4058`, 0.4232`, 0.4232`, 0.4276`, 0.4581`, 0.4625`,
   0.4712`, 0.4974`, 0.5411`, 0.5454`, 0.5847`, 0.6589`, 0.6938`, 0.829`, 0.8814`,
   0.9861`, -0.7636`, -0.6938`, -0.6632`, -0.6589`, -0.6501`, -0.6458`, -0.6152`,
   -0.6065`, -0.589`, -0.5454`, -0.5192`, -0.5105`, -0.5105`, -0.4887`, -0.4538`,
   -0.4538`, -0.4407`, -0.4407`, -0.4232`, -0.4232`, -0.4145`, -0.4058`, -0.3927`,
   -0.3927`, -0.3491`, -0.336`, -0.3185`, -0.3185`, -0.3011`, -0.3011`, -0.3011`,
   -0.3011`, -0.2923`, -0.288`, -0.2836`, -0.2618`, -0.2574`, -0.2531`, -0.2269`,
   -0.1876`, -0.1745`, -0.1222`, -0.1178`, -0.1091`, -0.0873`, -0.0829`,
   -0.0785`, -0.0698`, -0.0654`, -0.0654`, -0.0567`, -0.048`, -0.0436`, -0.0393`,
   -0.0305`, -0.0305`, -0.0262`, -0.0262`, -0.0175`, -0.0131`, -0.0087`};
```

In[25]:=
```
(*The Declinations of the sources Si. Determined by Input.*)
δSrc = {-0.1152`, 0.2758`, 0.0297`, -0.0035`, -0.007`, 0.2688`, 0.0192`, -0.0384`, 0.391`,
   -0.0314`, 0.0035`, 0.4887`, 0.4852`, 0.5096`, 0.2164`, 0.5079`, 0.274`, 0.4555`,
   0.2269`, 0.4485`, 0.3752`, 0.0227`, -0.0244`, 0.5184`, 0.3718`, 0.3438`, 0.0716`,
   0.4485`, 0.4224`, 0.3613`, -0.0314`, -0.0175`, -0.1693`, 0.0733`, 0.0297`, 0.1571`,
   0.295`, 0.5096`, 0.0419`, 0.1885`, 0.4259`, 0.0105`, 0.3648`, 0.3072`, 0.2217`,
   -0.1134`, 0.0873`, -0.1536`, -0.1257`, -0.0367`, -0.0192`, -0.1484`, 0.0698`,
   0.1169`, -0.2653`, -0.3229`, -0.3281`, -0.3752`, -0.3019`, -0.0663`, -0.1588`,
   0.3421`, 0.3438`, -0.0908`, 0.3438`, -0.096`, -0.1536`, 0.0436`, 0.199`, -0.4538`,
   -0.2147`, 0.2443`, 0.0262`, 0.1972`, 0.2758`, 0.4259`, 0.0105`, -0.2007`, 0.0419`,
   -0.4922`, 0.4433`, 0.1047`, -0.4869`, 0.171`, -0.0105`, -0.0611`, -0.0297`, -0.1763`,
   -0.0471`, -0.0628`, -0.4102`, 0.2094`, 0.3211`, -0.2915`, 0.0035`, -0.1833`, -0.0175`,
   0.014`, -0.2688`, 0.4939`, -0.014`, -0.2042`, 0.0035`, -0.0105`, -0.2251`, 0.0384`};
```

In[26]:=
```
rSrc = Table[er[ αSrc[[i]], δSrc[[i]] ],  {i, nSrc}];(*calculated from Input.*)
eNSrc = Table[eN[ αSrc[[i]], δSrc[[i]] ], {i, nSrc}];(*calculated from Input.*)
eESrc = Table[eE[ αSrc[[i]], δSrc[[i]]  ], {i, nSrc}];(*calculated from Input.*)
```

The polarization position angles (PPA) ψ and their uncertainties $\sigma\psi$ are given in degrees in the catalog. They are converted to radians below.

In[29]:=
```
(*The polarization position angles in radians for
 the EM radiation from the sources.  Determined by Input. *)
ψn = {2.7925`, 1.9897`, 2.1293`, 2.0246`, 2.0071`, 2.3911`, 0.4538`, 2.9671`,
   1.5708`, 1.9024`, 2.7576`, 1.7977`, 0.8552`, 1.7104`, 0.1396`, 2.0769`, 0.2618`,
   2.0944`, 1.9548`, 1.9897`, 2.0769`, 2.4958`, 1.3265`, 1.0996`, 1.7802`, 2.234`,
   1.0297`, 2.4435`, 1.9199`, 0.8552`, 1.0647`, 2.6878`, 2.9322`, 2.2864`, 2.4609`,
   2.426`, 1.1519`, 2.042`, 0.384`, 2.1118`, 1.7279`, 2.9147`, 2.5656`, 2.0769`,
   1.2043`, 2.5656`, 1.1868`, 2.9845`, 0.7679`, 1.6232`, 1.9722`, 2.7925`, 1.9373`,
   2.4086`, 0.1222`, 0.1222`, 0.5411`, 0.8203`, 2.5831`, 2.426`, 0.0175`, 1.9024`,
   2.4086`, 2.3213`, 1.0123`, 2.8274`, 3.0194`, 2.0769`, 2.0595`, 2.2864`, 2.7227`,
   1.309`, 1.4312`, 0.8552`, 2.5133`, 1.9722`, 2.2515`, 2.2689`, 0.0349`, 1.9548`,
   2.4435`, 2.8449`, 0.1571`, 1.8326`, 2.8623`, 1.5708`, 1.6057`, 2.7925`, 1.9199`,
   2.2689`, 2.1293`, 2.2166`, 1.5359`, 1.2217`, 2.3387`, 1.85`, 2.4958`, 0.4712`,
   0.2269`, 1.3265`, 0.2793`, 1.8326`, 1.2915`, 2.7576`, 2.6354`, 0.7854`};
```

In[30]:=
```
(*The uncertainties in the polarization
 position angles in radians.  Determined by Input. *)
σψn = {0.2094`, 0.1222`, 0.1047`, 0.2269`, 0.1745`, 0.2269`, 0.1222`, 0.2443`,
   0.2443`, 0.2269`, 0.2443`, 0.0524`, 0.1571`, 0.0873`, 0.0524`, 0.2443`, 0.2269`,
   0.1396`, 0.1745`, 0.0349`, 0.0873`, 0.2269`, 0.1047`, 0.2269`, 0.1571`, 0.192`,
   0.1047`, 0.2443`, 0.1571`, 0.1047`, 0.1396`, 0.2443`, 0.192`, 0.0524`, 0.0873`,
   0.2269`, 0.1571`, 0.1047`, 0.1222`, 0.1047`, 0.0873`, 0.0873`, 0.1396`, 0.1396`,
   0.1571`, 0.1047`, 0.1047`, 0.2443`, 0.0873`, 0.0175`, 0.1571`, 0.1396`, 0.1571`,
   0.192`, 0.0349`, 0.1745`, 0.1222`, 0.1571`, 0.1222`, 0.192`, 0.2443`, 0.0698`,
   0.1222`, 0.0175`, 0.2269`, 0.0349`, 0.0524`, 0.2443`, 0.0175`, 0.0175`, 0.1047`,
   0.1396`, 0.1222`, 0.0698`, 0.0524`, 0.2443`, 0.1571`, 0.1396`, 0.2269`, 0.1047`,
   0.1222`, 0.0349`, 0.1222`, 0.0698`, 0.0873`, 0.0175`, 0.0175`, 0.1745`, 0.0349`,
   0.1396`, 0.1571`, 0.1047`, 0.1745`, 0.1396`, 0.1745`, 0.1396`, 0.1222`, 0.0873`,
   0.2094`, 0.192`, 0.0873`, 0.1047`, 0.2443`, 0.1222`, 0.0175`, 0.1222`};
```

4. Grid

We avoid bunching at the poles by taking into account the diminishing radii of constant latitude circles as the latitude approaches the poles. Successive grid points along any latitude or along any longitude make an arc that subtends the same central angle dθ.
We grid one hemisphere at a time, then the hemispheres are combined.

Definitions:

gridSpacing	separation in degrees between grid points on a constant latitude circle and separation of constant latitude circles. Set by the user in Sec. 2.
dθ	grid spacing in radians
αpointH,δpointH	RA and dec of the grid points H_j
grid, gridN, gridS	tables of points on the sphere, "grid points", including associated information. See listing below for "grid" table entries
nGrid	number of grid points
rGrid	radial unit vectors from origin to grid points, in 3D Cartesian coordinates
αGrid	RAs for the grid points
δGrid	decs for the grid points

Tables:

grid, gridN **and** ***gridS***
1. sequential point # 2. RA index 3. dec index 4. RA (rad) 5. dec (rad) 6. Cartesian coordinates of the grid point

In[31]:=
```
(*When gridSpacing = 2°, we get a 2°x2° grid.*)
Print["The grid spacing is a setting that was chosen in Sec. 3 to be gridSpacing = ",
 gridSpacing, "°."]
dθ = ((2. π) / 360.) gridSpacing; (*Convert gridSpacing to radians*)
```

The grid spacing is a setting that was chosen in Sec. 3 to be gridSpacing = 2.°.

The grid spacing is a setting that was chosen in Sec. 3 to be gridSpacing = 2.°.

In[33]:=
```
(*The Northern Grid "gridN". *)
gridN = {}; idN = 1;

For[δj = 0., δj < π / (2. dθ), δj++, δpointH = δj dθ;
 For[ ai = 0., ai < Ceiling[ ((2. π) / dθ) (Cos[δpointH] + 0.01)],
  ai++, αpointH = ai dθ / (Cos[δpointH] + 0.01);
  AppendTo[gridN, {idN, ai, δj, αpointH, δpointH, er[αpointH, δpointH]}];
  idN = idN + 1
 ]]
```

In[35]:=
```
(*The Southern Grid "gridS". *)
gridS = {}; idS = 1;

For[δj = 1., δj < π / (2. dθ), δj++, δpointH = -δj dθ;
 (*Print["{δj,δpointH} = ",{δj,δpointH}];*)For[ ai = 0.,
  ai < Ceiling[ ((2. π) / dθ) (Cos[δpointH] + 0.01)], ai++, αpointH = ai dθ / (Cos[δpointH] + 0.01);
   (*Print["{ai,αpointH} = ",{ai,αpointH}];*)
  AppendTo[gridS, {idS, ai, δj, αpointH, δpointH, er[αpointH, δpointH]}];
  idS = idS + 1
 ]]
```

In[37]:=
```
grid = {}; j = 1;
For[jN = 1, jN ≤ Length[gridN], jN++, AppendTo[grid,
  {j, gridN[[jN, 2]], gridN[[jN, 3]], gridN[[jN, 4]], gridN[[jN, 5]], gridN[[jN, 6]]}];
 j = j + 1]
For[jS = 1, jS ≤ Length[gridS], jS++, AppendTo[grid,
  {j, gridS[[jS, 2]], gridS[[jS, 3]], gridS[[jS, 4]], gridS[[jS, 5]], gridS[[jS, 6]]}];
 j = j + 1]
nGrid = Length[grid];
```

In[41]:=
```
αGrid = Table[αFROMr[grid[[j, 6]] ], {j, Length[grid]}];
δGrid = Table[δFROMr[grid[[j, 6]] ], {j, Length[grid]}];
rGrid = Table[grid[[j, 6]] , {j, Length[grid]}];
```

In[44]:=
```
Print["There are ", nGrid, " points on the grid. "]
```

There are 10 518 points on the grid.

5. Significance

The problem of "significance" is to determine the likelihood that random polarizations directions would have better alignment or avoidance than the observed polarization directions.

The alignment of the 106 quasars is so remarkable that we take the time to give the statistics special treatment. Instead of relying on the estimates and formulas in Sec. 4 of the Intermediate level notebook in Ref. 3, we perform 10,000 runs with random ψ replacing the observed ψ of the 106 QSOs. Thus the probability distribution and significance formulas are tailored to the task at hand. The main effect is to reduce the uncertainty, plus/minus, values of the results.

To determine the probability distributions and related formulas, we made many runs with random data and fit the results. Each of the 10,000 runs involved the same 106 observed locations of the quasars, the only change to the process is the introduction of random polarization directions ψ, a different set of 106 random values ψ for each run. We fit the 10,000 sets of results with approximating functions and determined probability distributions for the best alignment angle $\overline{\eta}_{\min}$ and the best avoidance angle $\overline{\eta}_{\max}$. The significance of the observed $\overline{\eta}_{\min}$ is then the likelihood that the approximating functions that fit the random runs would give a smaller value of $\overline{\eta}_{\min}$. The statistical formulas found by this method are specific to these 106 quasars.

Many of the significances found for quantities in this notebook are a few parts in tens of millions. Since there were only 10,000 random runs, a comparatively small number compared to tens of millions, an essential assumption is that the exponential tails of the fitting function are suitable approximations. Thus, tiny values of significance may be less accurate than they appear. Yet even if the accuracy is suspect, the extremely small significances found for some quantities in this notebook should be interpreted as showing the likelihood of chance providing the same outcome is essentially nill.

5a. Random Run Generator

Definitions:

rSrcxrGrid unit vector $S_i \times H_j$ in the direction of the cross product of the radial vector to a source with the radial vector to a grid point, $S_i \times H_j$

runRandomData data generator in the random runs that is needed in the calculations that follow. See listing below for a description of the entries

ψRandomData the random PPA ψ values used in the runs

rSrcxψSrc unit vector, $S_i \times \psi_i$, cross product of the radial vector to the source with the vector in the direction of the polarization

jηBarToGrid average alignment angle the sources make with each grid point together with the grid point index j, *i.e.* $\{j, \overline{\eta}(H_j)\}$

jηBarMin, jηBarMax jηBarToGrid, $\{j, \overline{\eta}(H_j)\}$, for the smallest and largest $\overline{\eta}$

ηBarMinRandomData the second entry in jηBarMin, *i.e.* $\overline{\eta}_{\min}$

ηBarMaxRandomData the second entry in jηBarMax, *i.e.* $\overline{\eta}_{\max}$

```
In[45]:= rSrcxrGrid1 = Table[ Cross[ rSrc[[i]], rGrid[[j]] ] ,  {i, nSrc}, {j, nGrid}];
(*first step: raw cross product. These are not unit vectors*)
rSrcxrGrid =
  Table[ rSrcxrGrid1[[i, j]] / (rSrcxrGrid1[[i, j]].rSrcxrGrid1[[i, j]] + 0.000001)^1/2. ,
   {i, nSrc}, {j, nGrid}];(*unit vectors*)
Clear[rSrcxrGrid1];
(*rSrcxrGrid: table of the unit vectors perpendicular to the plane
  of the great circle containing the source Si and the grid point Hj*)
```

In[48]:=
```
(*
runRandomData={};ψRandomData={};nRunPrint=0;
For[nRun=1,nRun≤nR,nRun++,
 If[nRun>nRunPrint,Print["At the start of run ",nRun,", the time is ",
   TimeUsed[]," seconds and the memory in use is ",MemoryInUse[]," bytes."];
  nRunPrint=nRunPrint+500];
   ψSrc=Table[RandomReal[{0.00001,π-0.00001}],{i,nSrc}];
 (*table of PPA angles ψ for the sources in region j0, in radians*)
 rSrcxψSrc = Table[ Sin[ψSrc[[i]]]eNSrc[[i]]-Cos[ψSrc[[i]]] eESrc[[i]],    {i,nSrc}];
 (*table of the cross product of rSrc and vector in direction of ψSrc,
 a unit vector*)jηBarToGrid = Table[{j,(1/nSrc)Sum[ArcCos[
       Abs[  rSrcxψSrc[[i]].rSrcxrGrid[[i,j]]  ] - 0.000001  ],{i,nSrc}]},{j,nGrid}];
 (*
 {grid point #, value of the alignment angle ηnHj[j] averaged over all sources,
  in radians}*) sortjηBarToGrid=Sort[jηBarToGrid,#1[[2]]<#2[[2]]&];
 (*jηBarToGrid, {j,ηj}, but sorted with the smallest alignment angles first
 *)
 jηBarMin=sortjηBarToGrid[[1]]; (*  {j,ηj}, at the grid point Hj with minimum η̅*)
 jηBarMax=sortjηBarToGrid[[-1]]; (*  {j,ηj},
 at the grid point Hj with maximum η̅*)AppendTo[ψRandomData,{nRun,ψSrc}];
 AppendTo[runRandomData,{nRun,{ jηBarMin[[2]],{αGrid [ [ jηBarMin[[1]]  ]],
     δGrid [[  jηBarMin[[1]]  ]]}},{ jηBarMax[[2]],{αGrid [[  jηBarMax[[1]]  ]],
     δGrid [[  jηBarMax[[1]]  ]]}}} ](*collect data*)    ]
*)
```

Hint: You can save memory if you do not get the "ψRandomData" in the following cell. The values of ψ in ψRandomData are not needed in any of the following calculations.

In[49]:=
```
SetDirectory[dataDirectory]; (*Save memory space; ψRandomData is not used below.*)
(*Put[ψRandomData,"20210216psiRandomData.dat"  ] *) (*Save a new "ψRandomData"*)
(*ψRandomData=Get["20210203psiRandomData.dat"];*) (*Get an old "ψRandomData"*)
```

Hint: Saving "runRandomData" avoids the time it takes to complete the "For" statement. Do not forget to make the "For" statement into a remark so that it doesn't run.

In[50]:=
```
 SetDirectory[dataDirectory];
(*Put[runRandomData,"20210216runRandomData.dat"  ] *) (*Save a new "runRandomData".*)
runRandomData = Get["20210203runRandomData.dat"];
 (*Get an old "runRandomData".*)
```

In[52]:=
```
Print["The number of runs in the runRandomData table is ", Length[runRandomData], "."]
```

```
The number of runs in the runRandomData table is 10000.
```

In[53]:=
```
ηBarMinRandomData = Table[runRandomData[[i1, 2, 1]] , {i1, Length[runRandomData]}];
ηBarMaxRandomData = Table[runRandomData[[i1, 3, 1]] , {i1, Length[runRandomData]}];
```

5b. Analysis of the Random Run Results

For a single source, allowing its polarization direction to be randomly directed means the angle ψ in Fig. 1 can have any value from 0° to 180°, with no particular value favored. For a given grid point H, the alignment angle η is acute, *i.e.* $0° \leq \eta \leq 90°$. With random ψ, the alignment angle is also random, so no value of η is favored over any other. On average, one expects η to be 45°.

For the average alignment angle $\overline{\eta}(H)$, Eq. 1, with a sample of N sources with random polarization directions, one expects 45° to again be the most likely value for any H. However there will a minimum $\overline{\eta}_{\text{min}}$ at some hub H_{min} and a maximum $\overline{\eta}_{\text{max}}$ at some other hub H_{max}. In Ref. 2, we argue that the sum of N now-random numbers is much like a random walk which is well-known to increase with the number of steps N like $N^{1/2}$. Here the “number of steps” is the number of sources N. Since we get an average by dividing by N, the difference $(\frac{\pi}{4} - \overline{\eta})$ should be proportional to $N^{-1/2}$.

The point is that we expect many random runs to produce a most likely value of $\overline{\eta}_{\text{min}}$ and a most likely value of $\overline{\eta}_{\text{max}}$, with nearby values less likely. Thus, we expect some sort of Gaussian-like distributions.

The distributions for minimum alignment angle $\overline{\eta}_{\text{min}}$ and maximum avoidance angle $\overline{\eta}_{\text{max}}$ look like slanted Gaussians, each distribution slanted away from $\eta = \pi/4 = 45°$. To accommodate the behavior, the distributions are fit with non-Gaussian functions, which differ from Gaussians by a step curve, an S-curve. Even though the distributions are non-Gaussian, the terms “half-width” σ and “mean” η_0 are used for some parameters. For a discussion see Ref. 2.

Definitions

sortηBarMinRANDOM	collect and sort the best alignment angles $\overline{\eta}_{\text{min}}$ from the random runs
σB	an estimate of the half-width of the distribution of $\overline{\eta}_{\text{min}}$
η0B	an estimate of the mean value for the distribution of $\overline{\eta}_{\text{min}}$
hlMin0	histogram list for sorted alignment angles in sortηBarMinRANDOM
hlMin	histogram bar data, *i.e.* {η at midpoint, height}
nlmRandomMin	nonlinear model fit to histogram bar data hlMin
showNLMMin	plot of histogram and the fit function
parametersNLMMin	parameters determined for the fit function
pTableNLMMin	parameter table, has standard error and other stats
σRANDOMmin	width parameter, approximately the half-width at 60.7% of peak
σErrRANDOMmin	standard error in σ
η0RANDOMmin	mean parameter, approximately the value of η at the peak
η0ErrRANDOMmin	standard error in η0

histRandomMin	histogram for alignment angles $\overline{\eta}_{\text{min}}$
ΔR	number of runs in a bin
hlMin	{η at half-bin width, bin height ΔR}

In[55]:=
```
sortηBarMinRANDOM = Sort[ηBarMinRandomData];
σB = nSrc^-1/2 / 4.;
η0B = π / 4. - 1 / nSrc^1/2;
hlMin0 = HistogramList[sortηBarMinRANDOM, {η0B - 5 σB, η0B + 4 σB, 0.31 σB}];
hlMin = Table[{(1 / 2) (hlMin0[[1, i1]] + hlMin0[[1, i1 + 1]]), hlMin0[[2, i1]]},
   {i1, Length[ hlMin0[[2]] ]}];
nlmRandomMin = NonlinearModelFit[hlMin, a (1 + e^(4 (x-x0-b)/b))^-1 Exp[- (1 / 2.) ((x - x0) / b)^2],
   {{a, 1200.}, {b, σB}, {x0, η0B}}, x];
(*A Gaussian modified by a Step-function (1+e^(4 (η-η0-σ)/σ))^-1.*)
```

In[61]:=
```
showNLMMin = Show[{Histogram[sortηBarMinRANDOM, {η0B - 5 σB, η0B + 4 σB, 0.31 σB},
    PlotLabel → "Alignment η̅min Histogram", AxesLabel → {"η̅min", "ΔR"}], Plot[
    Normal[nlmRandomMin], {x, η0B - 6 σB, η0B + 4 σB}, PlotStyle → Purple], ListPlot[hlMin] }]
Print["Figure 4. The histogram is steeper on the π/4 = 0.785 side. A Gaussian
   fit is not approriate. By introducing a step down function, the S-curve
   (1+e^(4 (η-η0-σ)/σ))^-1, one can reduce the π/4 side where the S-curve is small, while
   leaving the Gaussian untouched on the left, where the S-curve is near unity.   "]
```

Out[61]=

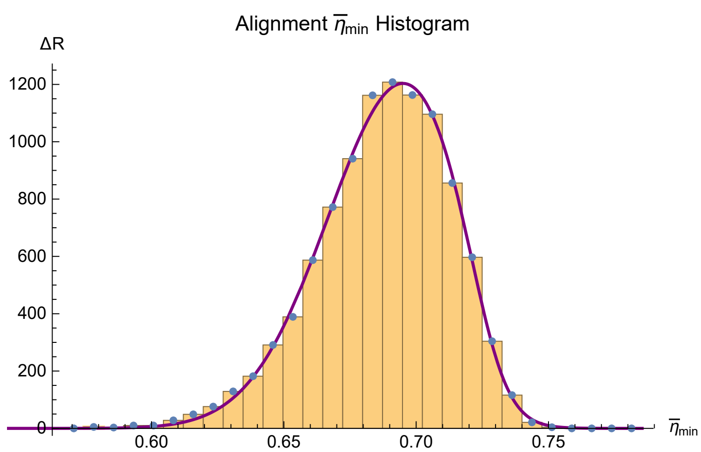


Figure 4. The histogram is steeper on the $\pi/4 = 0.785$ side. A Gaussian fit is not approriate. By introducing a step down function, the S-curve $(1+e^{4\frac{(\eta-\eta 0-\sigma)}{\sigma}})^{-1}$, one can reduce the $\pi/4$ side where the S-curve is small, while leaving the Gaussian untouched on the left, where the S-curve is near unity.

In[63]:=
```
parametersNLMMin = {a, b, x0} /. nlmRandomMin["BestFitParameters"];
pTableNLMMin = nlmRandomMin["ParameterTable"]
{σRANDOMmin, σErrRANDOMmin } =
  {b /. nlmRandomMin["BestFitParameters"], nlmRandomMin["ParameterErrors"][[2]] };
{η0RANDOMmin, η0ErrRANDOMmin} = {x0 /. nlmRandomMin["BestFitParameters"],
   nlmRandomMin["ParameterErrors"][[3]]};
```

Out[64]=

	Estimate	Standard Error	t-Statistic	P-Value
a	1222.99	9.20303	132.89	2.32353×10^{-38}
b	0.0297804	0.000250503	118.883	4.18641×10^{-37}
x0	0.696699	0.000209576	3324.33	1.04862×10^{-74}

Repeat for the avoidance angle $\overline{\eta}_{\max}$. See above for Definitions and adjust the meanings for avoidance, *i.e.* “min”→ “max”.

In[67]:=
```
sortηBarMaxRANDOM = Sort[ηBarMaxRandomData];
σB = nSrc^(-1/2) / 4.;
η0B = π / 4. + 1 / nSrc^(1/2);
histRangeMax = {η0B - 4 σB, η0B + 5 σB, 0.31 σB};
hlMax0 = HistogramList[sortηBarMaxRANDOM, histRangeMax];
hlMax = Table[{(1/2) (hlMax0[[1, i1]] + hlMax0[[1, i1 + 1]]), hlMax0[[2, i1]]},
   {i1, Length[ hlMax0[[2]] ]}];
nlmRandomMax = NonlinearModelFit[hlMax, a (1 + e^(-4 (x-x0+b)/b))^-1 Exp[-(1/2.) ((x - x0) / b)^2],
   {{a, 1200.}, {b, σB}, {x0, η0B}}, x];
(*A Gaussian modified by a Step-function (1+e^(-4 (η-η0+σ)/σ))^-1.*)
```

In[74]:=
```
showNLMMax =
 Show[{Histogram[sortηBarMaxRANDOM, histRangeMax, PlotLabel → "Avoidance Histogram",
    AxesLabel → {"η̄max", "ΔR"}], Plot[Normal[nlmRandomMax],
    {x, η0B - 5 σB, η0B + 5 σB}, PlotStyle → Purple], ListPlot[hlMax] }]
Print["Figure 5. As in Fig. 4, here the Avoidance Histogram is steeper on the π/4
   = 0.785 side. As before with the Alignment Histogram, the step up function
   S-curve  (1+e^(-4 (η-η0+σ)/σ))^-1 is combined with the Gaussian, which reduces the π/4
   side on the left, while leaving the Gaussian untouched on the right.   "]
```

Out[74]=

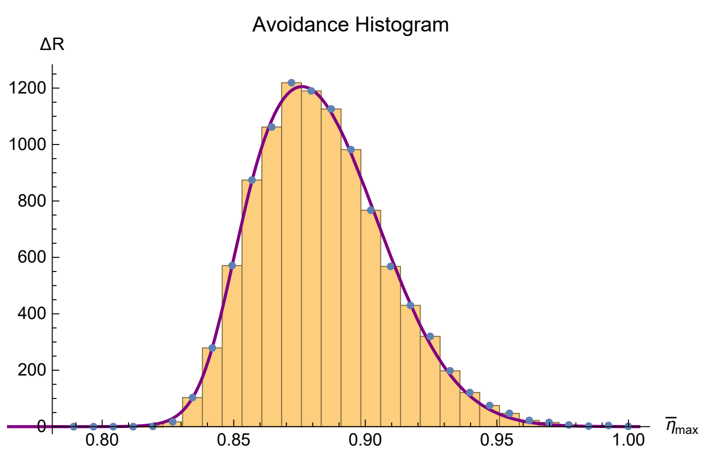


Figure 5. As in Fig. 4, here the Avoidance Histogram is steeper on the $\pi/4 = 0.785$ side. As before with the Alignment Histogram, the step up function S-curve $(1+e^{-4\frac{(\eta-\eta 0+\sigma)}{\sigma}})^{-1}$ is combined with the Gaussian, which reduces the $\pi/4$ side on the left, while leaving the Gaussian untouched on the right.

In[76]:=
```
parametersNLMMax = {a, b, x0} /. nlmRandomMax["BestFitParameters"];
parameterErrorsNLMMax = nlmRandomMax["ParameterErrors"];
pTableNLMMax = nlmRandomMax["ParameterTable"]
 {σRANDOMmax, σErrRANDOMmax } =
  {b /. nlmRandomMax["BestFitParameters"], nlmRandomMax["ParameterErrors"][[2]] };
{η0RANDOMmax, η0ErrRANDOMmax} = {x0 /. nlmRandomMax["BestFitParameters"],
   nlmRandomMax["ParameterErrors"][[3]]};
```

Out[78]=

	Estimate	Standard Error	t-Statistic	P-Value
a	1224.8	7.28957	168.02	5.2542×10^{-41}
b	0.0297779	0.000198109	150.31	9.48322×10^{-40}
x0	0.874236	0.000165742	5274.67	6.42455×10^{-80}

5c. Probability Distributions and Significance Formulas

The histograms in Figs. 4 and 5 are proportional to probability distributions: The height of a bar, ΔR , is the number of runs in a bin of width $\Delta\eta$ centered on η . Thus, the likelihood of η is roughly $\Delta R/R$, where R is the total number of runs $R = \Sigma\,\Delta R$. And the likelihood approximates the probability distribution. The histogram and the probability distribution differ by a constant factor, a normalization constant. See Ref. 2 for more detail.

Definitions:

norm	a constant used to normalize the distribution so the integral of probability is 1.
probMIN0, probMAX0	probability distributions for alignment (MIN) and avoidance (MAX), functions of η, η_0, σ
probMIN, probMAX	probability distributions assuming the best fit values of η_0, σ from the random runs in Sec. 5b above
signiMIN0, signiMAX0	significance as a function of (η, η_0, σ)
signiMIN, signiMAX	significance for the best fit values of η_0, σ from the random runs in Sec. 5b

In[81]:=
```
(* y = ((η - η0)/σ); dy = dη/σ *)
(* The normalization factor "norm" is needed for the probability density *)
```

$$\text{norm} = \left(\frac{1}{(2\pi)^{1/2}} \text{NIntegrate}\left[\left(1 + e^{4\,(y-1)}\right)^{-1} e^{-\frac{y^2}{2}}, \{y, -\infty, \infty\}\right]\right)^{-1};$$

```
norm;(*Constant needed for Eq. (10) and (11) in Ref. 2.*)
```

In[83]:=

$$\text{probMIN0}[\eta\_, \eta0\_, \sigma\_] := \left(\frac{\text{norm}}{\sigma\,(2\pi)^{1/2}}\right)\left(1 + e^{4\,\frac{(\eta-\eta0-\sigma)}{\sigma}}\right)^{-1} e^{-\frac{1}{2}\left(\frac{\eta-\eta0}{\sigma}\right)^2}$$

```
signiMIN0[η_, η0_, σ_] := NIntegrate[probMIN0[η1, η0, σ], {η1, -∞, η}]
```

In[85]:=

$$\text{probMAX0}[\eta\_, \eta0\_, \sigma\_] := \left(\frac{\text{norm}}{\sigma\,(2\pi)^{1/2}}\right)\left(1 + e^{-4\,\frac{(\eta-\eta0+\sigma)}{\sigma}}\right)^{-1} e^{-\frac{1}{2}\left(\frac{\eta-\eta0}{\sigma}\right)^2}$$

```
signiMAX0[η_, η0_, σ_] := NIntegrate[probMAX0[η1, η0, σ], {η1, η, ∞}]
```

The significance signiMIN0[η, η0, σ] is the integral of probMIN0, i.e. signiMIN0 $= \int_{-\infty}^{\eta} P_{MIN}(\eta)\, d\eta$.

The significance signiMAX0[η, η0, σ] is the integral of probMAX0, i.e. signiMAX0 $= \int_{\eta}^{\infty} P_{MAX}(\eta)\, d\eta$.

The following probability distributions and significances make use of the mean η_0 and half-width σ, which are the most likely values according to the random run fitting functions. One expects that the larger the number of random runs, the more accurate the mean η_0 and half-width σ and the more accurate the formulas.

In[87]:=
```
probMIN[η_] := probMIN0[ η, η0RANDOMmin, σRANDOMmin ]
```

In[88]:=
```
signiMIN[η_] := signiMIN0[η, η0RANDOMmin, σRANDOMmin]
```

In[89]:=
```
probMAX[η_] := probMAX0[ η, η0RANDOMmax, σRANDOMmax]
signiMAX[η_] := signiMAX0[η, η0RANDOMmax, σRANDOMmax]
```

6. Results and map using the Best Values ψn of the Polarization Directions

6a. Results Using the Best Values of ψn

"Best" means we used the ψn that were input in Sec. 3. Later on, in Sec. 7, we allow ψn + δψ, where δψ conforms to the uncertainty σψ in the measured values.

Definitions:

vψSrc unit vectors along the polarization directions in the tangent planes of the sources
eNSrc unit vectors along local North in the tangent planes of the sources
eESrc unit vectors along local East in the tangent planes of the sources
jηBarHj {j,$\overline{\eta}$(H)}, where j is the index for grid point H_j and $\overline{\eta}$(H) is the average alignment angle at H_j. See Eq. (1) in the Introduction.
sortjηBarHj {j,$\overline{\eta}$(H)}, rearranged by value of $\overline{\eta}$(H), with smallest angles $\overline{\eta}$(H) first.
jηBarMin {j,$\overline{\eta}$(H)}, the j and $\overline{\eta}$ for the smallest value of $\overline{\eta}$(H) , best alignment
ηBarMin the smallest value of $\overline{\eta}$(H) , measures alignment of the polarization directions
jηBarMax {j,$\overline{\eta}$(H)}, the j and $\overline{\eta}$ for the largest value of $\overline{\eta}$(H) , most avoided
ηBarMax the largest value of $\overline{\eta}$(H) , measures avoidance

nSxψn unit vector, $S_i \times \psi_i$, cross product of the radial vector to the source with the vector in the direction of the polarization
nSxHnj unit vector, $S_i \times H_j$, cross product of the radial vector to the source with the radial vector to the grid point H_j
ηnHj alignment angle between source and grid point H_j, see Fig. 1
ηBarHj alignment angle $\overline{\eta}(H_j)$ between source and grid point H_j, averaged over all sources
jηBarHj {j, $\overline{\eta}(H_j)$ }, the j and $\overline{\eta}$ for grid point H_j

sigηBarMin significance of the smallest alignment angle
sigRangeηBarMin get the range if sigs using the plus/minus values on the parameters η_0, σ from the random runs, *i.e.* η0RANDOMmin, σRANDOMmin
sigSmallηBarMin the smallest of the values in sigRangeηBarMin
sigBigηBarMin the largest of the values in sigRangeηBarMin
sigηBarMax significance of the largest alignment angle (i.e. avoidance)
sigRangeηBarMax get the range if sigs using the plus/minus values on the parameters η_0, σ from the random runs, *i.e.* η0RANDOMmax, σRANDOMmax
sigSmallηBarMax the smallest of the values in sigRangeηBarMax
sigBigηBarMax the largest of the values in sigRangeηBarMax

αHminDegrees	RA of the point H_{min} where $\overline{\eta}$(H) is the smallest
δHminDegrees	dec of the point H_{min} where $\overline{\eta}$(H) is the smallest
αHmaxDegrees	RA of the point H_{max} where $\overline{\eta}$(H) is the largest
δHmaxDegrees	dec of the point H_{max} where $\overline{\eta}$(H) is the largest

In[91]:=

```
(* vψ, eN, eE unit vectors in the tangent plane of each source Si,
pointing along the polarization direction, local North,
and local East, respectively. See Fig. 1.*)
vψSrc = Table[Cos[ψn[[i]]] eN[ αSrc[[i]], δSrc[[i]] ] +
    Sin[ψn[[i]]] eE[ αSrc[[i]], δSrc[[i]] ], {i, nSrc}];
```

In[92]:=

```
(* Analysis using Eq (5) in Ref. 2 to get ηiH, cos(η) = |v̂H.v̂ψ|, then {j,η̄(Hj)}. *)
jηBarHj =
  Table[{j, (1/nSrc) Sum[ArcCos[ Abs[ rGrid[[j]].vψSrc[[i]] / ((rGrid[[j]] - (rGrid[[j]].
          rSrc[[i]]) rSrc[[i]]).(rGrid[[j]] - (rGrid[[j]].rSrc[[i]])
        rSrc[[i]]))^1/2 ] - 0.000001 ] , {i, nSrc}]}, {j, nGrid}];
 sortjηBarHj = Sort[jηBarHj, #1[[2]] < #2[[2]] &];
jηBarMin = sortjηBarHj[[1]]; (*  {j,η̄(Hj)} for smallest η̄(Hj) *)
ηBarMin = jηBarMin[[2]];
jηBarMax = sortjηBarHj[[-1]]; (*  {j,η̄(Hj)} for largest η̄(Hj) *)
ηBarMax = jηBarMax[[2]] ;
```

In[98]:=

```
(*Alternate analysis using Eq (7) in Ref. 2 to get ηiH, cos(η) = |n̂Sxψ.n̂SxH|.*)
(*nSxψn  =  Table[ Sin[ψn[[n]]]eN[αSrc[[n]],δSrc[[n]]]-
   Cos[ψn[[n]]]eE[αSrc[[n]],δSrc[[n]]],    {n,nSrc}];
nSxHnj[j_]:=nSxHnj[j]=Table[ Cross[  rSrc[[n]],rGrid[[j]]  ]/
    (√((Cross[  rSrc[[n]],rGrid[[j]]  ]).(Cross[  rSrc[[n]],rGrid[[j]]  ]))) ,    {n,
    nSrc}];
ηnHj[j_]:=ηnHj[j]=Table[ ArcCos[  Abs[  nSxψn[[n]].nSxHnj[j][[n]]  ] -
     0.000001  ],      {n,nSrc}];
ηBarHj[j_]:=ηBarHj[j]=Sum[ηnHj[j][[n]],{n,nSrc}]/nSrc
   jηBarHj=Table[{j,ηBarHj[j]},{j,Length[grid]}];
sortjηBarHj=Sort[jηBarHj,#1[[2]]<#2[[2]]&];
jηBarMin=sortjηBarHj[[1]];
ηBarMin=jηBarMin[[2]]
  jηBarMax=sortjηBarHj[[-1]];
ηBarMax=jηBarMax[[2]]*)
```

```
In[99]:= (*Significance of the alignment of the polarization directions with hub point Hmin.*)
sigηBarMin = signiMIN[ηBarMin];
sigRangeηBarMin =
  Sort[Partition[Flatten[Table[{signiMIN0[ηBarMin, η0RANDOMmin + γ1 η0ErrRANDOMmin,
        σRANDOMmin + γ2 σErrRANDOMmin], γ1, γ2}, {γ1, -1, 1}, {γ2, -1, 1}] ], 3 ]  ];
{sigRangeηBarMin[[1]], sigRangeηBarMin[[-1]]};
sigSmallηBarMin = sigRangeηBarMin[[1, 1]];
sigBigηBarMin = sigRangeηBarMin[[-1, 1]];
Print["The best value for the significance of alignment is sig. =  ", sigηBarMin,
 ". Using the uncertainties +/- of the η0RANDOMmin, σRANDOMmin, we find that
   the range is from sig. = ", sigSmallηBarMin, " to ", sigBigηBarMin, " . "]
```

The best value for the significance of alignment is sig. = 7.36374×10^-8. Using the uncertainties +/- of the η0RANDOMmin, σRANDOMmin, we find that the range is from sig. = 5.53373×10^-8 to 9.72742×10^-8 .

```
In[105]:= (*Significance of the polarization directions' avoidance of the hub point Hmax.*)
sigηBarMax = signiMAX[ηBarMax];
sigRangeηBarMax =
  Sort[Partition[Flatten[Table[{signiMAX0[ηBarMax, η0RANDOMmax + γ1 η0ErrRANDOMmax,
        σRANDOMmax + γ2 σErrRANDOMmax], γ1, γ2}, {γ1, -1, 1}, {γ2, -1, 1}] ], 3 ]  ];
{sigRangeηBarMax[[1]], sigRangeηBarMax[[-1]]};
sigSmallηBarMax = sigRangeηBarMax[[1, 1]];
sigBigηBarMax = sigRangeηBarMax[[-1, 1]];
Print["The best value for the significance of avoidance is sig. =  ", sigηBarMax,
 ". Using the uncertainties +/- of the η0RANDOMmax, σRANDOMmax, we find that
   the range is from sig. = ", sigSmallηBarMax, " to ", sigBigηBarMax, " . "]
```

The best value for the significance of avoidance is sig. = 0.000124075. Using the uncertainties +/- of the η0RANDOMmax, σRANDOMmax, we find that the range is from sig. = 0.000109936 to 0.000139688 .

```
In[111]:= {jηBarMin, jηBarMax};(* {1. grid#, 2. alignment angle η} at Min and Max η .*)
αHminDegrees0 = grid[[ jηBarMin[[1]] ]][[4]] (360 / (2 π));
δHminDegrees0 = grid[[ jηBarMin[[1]] ]][[5]] (360 / (2 π));
If[(180 < αHminDegrees0 < 361), αHminDegrees = αHminDegrees0 - 180;
  δHminDegrees = -δHminDegrees0 , αHminDegrees = αHminDegrees0;
  δHminDegrees = δHminDegrees0];
αHmaxDegrees0 = grid[[ jηBarMax[[1]] ]][[4]] (360 / (2 π));
δHmaxDegrees0 = grid[[ jηBarMax[[1]] ]][[5]] (360 / (2 π));
If[(180 < αHmaxDegrees0 < 361), αHmaxDegrees = αHmaxDegrees0 - 180;
  δHmaxDegrees = -δHmaxDegrees0 , αHmaxDegrees = αHmaxDegrees0;
  δHmaxDegrees = δHmaxDegrees0];
Print["The alignment hubs Hmin are located at (RA,dec) = ", {αHminDegrees, δHminDegrees },
 " and at ", {αHminDegrees - 180, -δHminDegrees }, " , in degrees"]
Print["The avoidance hubs Hmax are located at (RA,dec) = ", {αHmaxDegrees, δHmaxDegrees },
 " and at ", {αHmaxDegrees - 180, -δHmaxDegrees }, " , in degrees"]
```

The alignment hubs H_{min} are located at (RA,dec) = {121.361, -30.} and at {-58.6389, 30.} , in degrees

The avoidance hubs H_{max} are located at (RA,dec) = {96.3107, 58.} and at {-83.6893, -58.} , in degrees

In[120]:=
```
(*The names "jηBarMin", "jηBarMax" are used again below, so save the current values.*)
{jηBarMinBest, jηBarMaxBest} = {jηBarMin, jηBarMax};
(* jηBar entries: 1. grid# at Hmin or Hmax, 2. alignment angle ηmin or ηmax .*)
```

6b. Plot of the Alignment Angle Function $\overline{\eta}$(H) Using the Best Values ψn

Definitions

αjδjηBarHjTable	{RA$_j$, dec$_j$, $\overline{\eta}$(H)} at each grid point $H = H_j$, in degrees
ηBarHjSmooth	interpolation of αjδjηBarHjTable yields $\overline{\eta}$(H) as a smooth function of the (RA,dec) of H
xyηBarAitoffTable	{x, y, $\overline{\eta}$(x,y)} , where x,y are Aitoff coordinates and $\overline{\eta}$(x,y) is the alignment angle
dηContourPlot	separation of successive contour lines, in degrees
listCP	list contour plot of $\overline{\eta}$(H), based on {x, y, $\overline{\eta}$(x,y)}, *i.e.* xyηBarAitoffTable
xyAitoffSources	{x,y} Aitoff coordinates for the sources' locations on the sphere
mapOfηBar	contour plot of the alignment angle $\overline{\eta}$(H) , adorned with source locations and labels

αH(α,δ) , xH(α,δ) , yH(α,δ) are functions needed when making a 2-D map of the Celestial sphere. The origin xH, yH is centered on $\alpha = \delta = 0$.

Notice the naming conflict: αH(α,δ) is an Aitoff parameter which, in general, differs from the Right Ascension α .

In[121]:=
```
(*The following table αjδjηBarHjTable is interpolated below
 to yield a smooth function of the alignment angle over the sphere.*)
(* Table Entries: 1.  RA at jth grid point (degrees) 2. dec at jth grid
   point (degrees)  3. alignment angle ηBarRgnkj at jth grid point (degrees)*)
αjδjηBarHjTable = (αjδjηBarHjTable0 = {};
   For[j = 1, j ≤ Length[jηBarHj], j++,
    AppendTo[ αjδjηBarHjTable0,  {grid[[j, 4]] * (360. / (2. π)), grid[[j, 5]] * (360. / (2. π)),
      jηBarHj[[j, 2]] * (360. / (2. π))}]   ; If[ 360 ≥ grid[[j, 4]] * (360. / (2. π)) > 354.,
     AppendTo[ αjδjηBarHjTable0,  {grid[[j, 4]] * (360. / (2. π)) - 360.,
       grid[[j, 5]] * (360. / (2. π)), jηBarHj[[j, 2]] * (360. / (2. π))}]   ] ;
    If[ 6. > grid[[j, 4]] * (360. / (2. π)) ≥ 0., AppendTo[ αjδjηBarHjTable0,
      {grid[[j, 4]] * (360. / (2. π)) + 360, grid[[j, 5]] * (360. / (2. π)),
       jηBarHj[[j, 2]] * (360. / (2. π))}]   ]        ];
   αjδjηBarHjTable0);
```

In[122]:=
```
ηBarHjSmooth = Interpolation[αjδjηBarHjTable]
 (*The smooth alignment angle function for the region.*)
```

... Interpolation: Interpolation on unstructured grids is currently only supported for InterpolationOrder->1 or InterpolationOrder->All. Order will be reduced to 1.

Out[122]=
```
InterpolatingFunction[  Domain: {{-5.92, 366.}, {-88., 88.}}
                        Output: scalar  ]
```

The following Aitoff Plot formulas were be found in, for example, Wikipedia; see Ref. 5.

In[123]:=
```
αH[α_, δ_] := αH[α, δ] = ArcCos[Cos[((2. π) / 360.) δ] Cos[((2. π) / 360.) α / 2.]]
(*angles α and δ are in degrees*)
xH[α_, δ_] := xH[α, δ] = (2. Cos[((2. π) / 360.) δ] Sin[((2. π) / 360.) α / 2.]) / Sinc[αH[α, δ]]
yH[α_, δ_] := yH[α, δ] = Sin[((2. π) / 360.) δ] / Sinc[αH[α, δ]]
```

In[126]:=
```
xyηBarAitoffTable = Partition[Flatten[Table[{xH[α - 180, -δ], yH[α - 180, -δ], ηBarHjSmooth[α, δ]},
    {α, 0, 360., 2.}, {δ, -88., 88., 2.}]], 3];
(* The smooth alignment angle function ηBarHjSmooth mapped onto a 2D
 Aitoff projection of the sphere. *)

xyAitoffSources = Table[{xH[ αSrc[[n]] (360 / (2 π)), δSrc[[n]] (360 / (2 π)) ],
   yH[ αSrc[[n]] (360 / (2 π)), δSrc[[n]] (360 / (2 π)) ]}, {n, nSrc}];
 (*The Aitoff coordinates for the sources' locations.*)
```

In[128]:=
```
(* Contour plot of the alignment function ηBarHjSmooth. *)
listCP = ListContourPlot[Union[xyηBarAitoffTable (*,
    {{xH[αHminDegrees,δHminDegrees],yH[αHminDegrees,δHminDegrees],ηBarMin*(360./(2.π))-1.0}},
    {{xH[αHmaxDegrees,δHmaxDegrees],yH[αHmaxDegrees,δHmaxDegrees],
      ηBarMax*(360./(2.π))+1.0}}*)], AspectRatio → 1/2,
  Contours → Table[η, {η, Floor[jηBarMin[[2]] * (360. / (2. π))] + 1,
      Ceiling[jηBarMax[[2]] * (360. / (2. π))] - 1, dηContourPlot}],
  ColorFunction → "TemperatureMap", PlotRange → {{-7, 7}, {-3, 3}}, Axes -> False, Frame → False] ;
```

In[129]:=
```
(*Construct the map of η̅(H).*)
mapOfηBar =
  Show[{listCP, Table[ParametricPlot[{xH[α, δ], yH[α, δ]},
      {δ, -90, 90}, PlotStyle → {Black, Thickness[0.002]}, (*Mesh→{11,5,0}
      (*{23,11,0}*),MeshStyle→Thick,*)PlotPoints → 60], {α, -180, 180, 30}], Table[
     ParametricPlot[{xH[α, δ], yH[α, δ]}, {α, -180, 180}, PlotStyle → {Black, Thickness[0.002]},
      (*Mesh→{11,5,0}(*{23,11,0}*),MeshStyle→Thick,*)PlotPoints → 60], {δ, -60, 60, 30}],
    Graphics[{PointSize[0.007], Text[StyleForm["N", FontSize -> 10, FontWeight -> "Plain"],
       {0, 1.85}], (*Sources S:*)Purple, Point[ xyAitoffSources ],
      Black, Text[StyleForm["Max", FontSize → 8, FontWeight -> "Bold"],
       {xH[ - 180 , 0], yH[0, -60]}], {Arrow[BezierCurve[{{xH[ - 180 , 0], yH[0, -70]}, {-2.3, -2.0},
          {xH[αHmaxDegrees - 180, -δHmaxDegrees], yH[αHmaxDegrees - 180, -δHmaxDegrees]}}]]},
      Text[StyleForm["Min", FontSize → 8, FontWeight -> "Bold"], {xH[ 180 , 0], yH[0, -60]}],
      {Arrow[BezierCurve[{{xH[ 180 , 0], yH[0, -70]}, {2.3, -2.0},
          {xH[αHminDegrees, δHminDegrees], yH[αHminDegrees, δHminDegrees]}}]]},
      Text[StyleForm["Min", FontSize → 8, FontWeight -> "Bold"], {xH[ -180 , 0], yH[0, 60]}],
      {Arrow[BezierCurve[{{xH[ -180 , 0], yH[0, 70]}, {-2.3, 2.0},
          {xH[αHminDegrees - 180, -δHminDegrees], yH[αHminDegrees - 180, -δHminDegrees]}}]]},
      Text[StyleForm["Max", FontSize → 8, FontWeight -> "Bold"], {xH[ 180 , 0], yH[0, 60]}] ,
      {Arrow[BezierCurve[{{xH[ 180 , 0], yH[0, 70]}, {2.3, 2.0}, {xH[αHmaxDegrees, δHmaxDegrees],
           yH[αHmaxDegrees, δHmaxDegrees]}}]]}      }]}, ImageSize → 432];
```

In[130]:=
```
mapOfηBar
Print["Figure 6: The alignment function η̄(H), Eq. (1). The map is centered
   on (RA,dec) = (0°,0°), with contours separated by ", dηContourPlot, "°. "]
Print["Notes: Although somewhat obscured by the distortion needed to plot
   a sphere on a flat surface, the map is symmetric across diameters.
   Diametrically opposite points -H and H have the same alignment angle."]
Print["The sources are located at the dots, shaded ", Purple, " ."]
Print["The best alignment angle (min) is η̄min = ", jηBarMin[[2]] (360. / (2. π)),
 "°, located in the most aligned areas shaded ", Blue, " ."]
Print["The best avoidance angle (max) is η̄max = ", jηBarMax[[2]] (360. / (2. π)),
 "°, located in the least aligned areas shaded ", Red, " ."]
Print["The alignment hubs Hmin and -Hmin are located at (RA,dec) = ",
 {αHminDegrees, δHminDegrees}, " and at ", {αHminDegrees - 180, -δHminDegrees}, " , in degrees."]
Print["The avoidance hubs Hmax and -Hmax are located at (RA,dec) = ",
 {αHmaxDegrees, δHmaxDegrees}, " and at ", {αHmaxDegrees - 180, -δHmaxDegrees}, " , in degrees."]
```

Out[130]=

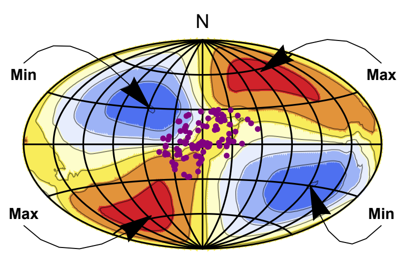


Figure 6: The alignment function $\overline{\eta}(H)$, Eq. (1). The map is centered on (RA,dec) = (0°,0°), with contours separated by 4°.

Notes: Although somewhat obscured by the distortion needed to plot a sphere on a flat surface, the map is symmetric across diameters. Diametrically opposite points −H and H have the same alignment angle.

The sources are located at the dots, shaded  .

The best alignment angle (min) is $\overline{\eta}_{min}$ = 30.8874°, located in the most aligned areas shaded  .

The best avoidance angle (max) is $\overline{\eta}_{max}$ = 56.4281°, located in the least aligned areas shaded  .

The alignment hubs H_{min} and $-H_{min}$ are located at (RA,dec) = {121.361, −30.} and at {−58.6389, 30.} , in degrees.

The avoidance hubs H_{max} and $-H_{max}$ are located at (RA,dec) = {96.3107, 58.} and at {−83.6893, −58.} , in degrees.

In[138]:=
```
(*Export the map "mapOfηBar" as a pdf. The export location can be reset in Sec. 3.*)
(*To activate, remove the remark brackets "(*" and "*)". *)
(*SetDirectory[dataDirectory];
Export["mapOfEtaBar106QSO.pdf",
 Show[mapOfηBar,ImageSize→432],"PDF",ImageSize→{480,Automatic}]*)
```

In[139]:=
```
Print["Statistics of the alignment function η̄(H), Eq. 1, determined using the best
   polarization directions ψn as reported in the catalog, Refs. 6 and 7."]
Print[" "]
Print["The number of sources: N = ", nSrc]
Print["The min alignment angle is ηmin = ", jηBarMin[[2]] * (360. / (2. π)),
 "° , which has a significance of sig. = ", sigηBarMin, ",  plus/minus = + ",
 sigBigηBarMin - sigηBarMin, " and - ", sigηBarMin - sigSmallηBarMin,
 " , giving a range from sig. = ", sigSmallηBarMin, " to ", sigBigηBarMin, " ."]
Print["The max avoidance angle is ηmax = ", jηBarMax[[2]] * (360. / (2. π)),
 "° , which has a significance of sig. = ", sigηBarMax, ",  plus/minus = + ",
 sigBigηBarMax - sigηBarMax, " and - ", sigηBarMax - sigSmallηBarMax,
 " , giving a range from sig. = ", sigSmallηBarMax, " to ", sigBigηBarMax, " ."]
Print["These uncertainties are due to the uncertainties in the constants η0RANDOM and
   σRANDOM found when fitting functions to the random runs results in Sec. 5b."]
```

Statistics of the alignment function $\bar{\eta}(H)$, Eq. 1, determined using the best polarization directions ψn as reported in the catalog, Refs. 6 and 7.

The number of sources: N = 106

The min alignment angle is ηmin = 30.8874° , which has a significance of sig. = 7.36374×10^{-8}, plus/minus = + 2.36368×10^{-8} and - 1.83×10^{-8} , giving a range from sig. = 5.53373×10^{-8} to 9.72742×10^{-8} .

The max avoidance angle is ηmax = 56.4281 ° , which has a significance of sig. = 0.000124075, plus/minus = + 0.0000156127 and - 0.0000141387 , giving a range from sig. = 0.000109936 to 0.000139688 .

These uncertainties are due to the uncertainties in the constants η0RANDOM and σRANDOM found when fitting functions to the random runs results in Sec. 5b.

7. Uncertainty Runs

To determine the uncertainty in the results due to the uncertainty in the measurements, we make many runs with polarization directions conforming to the normal distribution of values determined by the best value ψ and the uncertainty $\sigma\psi$.

7a. Random Run Generator

The catalog entry for B0051+291 is listed in Sec. 3. The polarization position angle is $\psi = 119° \pm 14°$. We call ψn = 119° the best value and $\sigma\psi$ = 14° is the uncertainty. We infer that the likelihood of measured values is normally distributed about the best value with a half-width of $\sigma\psi = 14°$, the width is determined along the curve down by a factor $e^{-1/2}$ from the peak value.

For each run, let the polarization direction ψ for each source be allowed to differ from the best value ψn by an amount $\delta\psi$ chosen according to a Gaussian distribution with mean (best) value ψn and half-width $\sigma\psi$, $\psi = \psi n + \delta\psi$. Both values ψn and $\sigma\psi$ are taken from the catalog data entered as input in Sec. 3c.

Definitions:

rSrcxrGrid unit vector $S_i \times H_j$ in the direction of the cross product of the radial vector to a source with the radial vector to a grid point, $S_i \times H_j$

$\mu = \psi_n$ by convention, the best value ψ_n, input in Sec. 3, is the mean value μ of a Gaussian of half-width $\sigma_{\psi n}$, $\psi \pm \sigma\psi$

$\sigma = \sigma\psi_n$ uncertainty of the measured polarization position angle ψ, an input in Sec. 3
ψData polarization directions $\psi = \psi$n + $\delta\psi$ for each run, $\delta\psi$ consistent with $\sigma\psi$ from catalog. Needed to recover results.
runData collection of data from the uncertainty $\sigma\psi$ runs, see below for entry list
nRunPrint dummy index controlling when TimeUsed and MemoryInUse data are printed
ψSrc a polarization direction ψ for the run. This ψ is moved off the best value ψ_n by an increment determined by the uncertainty $\sigma\psi$
rSrcxψSrc unit vector, $S_i \times \psi_i$, cross product of the radial vector to the source with the vector in the direction of the polarization
jηBarToGrid {j, $\overline{\eta}(H_j)$}, where j is the index # for the grid point H_j and $\overline{\eta}(H_j)$ is the average of the alignment angles for H_j with the sources.
sortjηBarToGrid {j, $\overline{\eta}(H_j)$}, reordered by the value of $\overline{\eta}$(H), with smallest angles $\overline{\eta}$(H) first.

jηBarHj {j,$\overline{\eta}$(H)}, where j is the index for grid point H_j and $\overline{\eta}$(H) is the average alignment angle at H_j. See Eq. (1) in the Introduction.
sortjηBarHj {j,$\overline{\eta}$(H)}, rearranged by value of $\overline{\eta}$(H), with smallest angles $\overline{\eta}$(H) first.
jηBarMin {j,$\overline{\eta}$(H)}, the j and $\overline{\eta}$ for the smallest value of $\overline{\eta}$(H) , best alignment
jηBarMax {j,$\overline{\eta}$(H)}, the j and $\overline{\eta}$ for the largest value of $\overline{\eta}$(H) , most avoided

ηBarMinData values of $\overline{\eta}_{\rm min}$ from uncertainty runs, alignment
ηBarMaxData values of $\overline{\eta}_{\rm max}$ from uncertainty runs, avoidance
HminαData values of RA = α for hub $H_{\rm min}$ from uncertainty runs, alignment
HminδData values of dec = δ for hub $H_{\rm min}$ from uncertainty runs, alignment
HmaxαData values of RA = α for hub $H_{\rm max}$ from uncertainty runs, avoidance
HmaxδData values of dec = δ for hub $H_{\rm max}$ from uncertainty runs, avoidance

Tables:
ψData entries: 1. Run # 2. ψSrc, list of polarization position angles ψ These values are not used in any subsequent calculations. Save memory.
runData entries: 1. Run # 2. {$\overline{\eta}_{\rm min}$, {$\alpha$,$\delta$} at $H_{\rm min}$} 3. {$\overline{\eta}_{\rm max}$, {$\alpha$,$\delta$} at $H_{\rm max}$}

In[145]:=
```
rSrcxrGrid1 = Table[ Cross[ rSrc[[i]], rGrid[[j]] ] ,  {i, nSrc}, {j, nGrid}];
(*first step: raw cross product, not unit vectors*)
rSrcxrGrid = Table[ rSrcxrGrid1[[i, j]] /
   (rSrcxrGrid1[[i, j]].rSrcxrGrid1[[i, j]] + 0.000001)^1/2. ,  {i, nSrc}, {j, nGrid}];
Clear[rSrcxrGrid1];
(*rSrcxrGrid: table of the unit vectors perpendicular to the plane
  of the great circle containing the source Si and the grid point Hj*)
```

In[147]:=
```
(*
μ=ψn;σ=σψn;runData={};ψData={};nRunPrint=0;
For[nRun=1,nRun≤nR,nRun++,
 If[nRun>nRunPrint,Print["At the start of run ",nRun,", the time is ",
   TimeUsed[]," seconds and the memory in use is ",MemoryInUse[]," bytes."];
  nRunPrint=nRunPrint+200];
    ψSrc=Table[RandomVariate[NormalDistribution[μ[[i]],σ[[i]]]],{i,nSrc}];
 (*table of PPA angles ψ for the sources in region j0, in radians*)
 rSrcxψSrc = Table[ Sin[ψSrc[[i]]]eNSrc[[i]]-Cos[ψSrc[[i]]] eESrc[[i]],    {i,nSrc}];
 (*table of the cross product of rSrc and vector in direction of ψSrc,
 a unit vector*)jηBarToGrid = Table[{j,(1/nSrc)Sum[ArcCos[
       Abs[  rSrcxψSrc[[i]].rSrcxrGrid[[i,j]]  ] - 0.000001  ],{i,nSrc}]},{j,nGrid}];
 (*
 {grid point #, value of the alignment angle ηnHj[j] averaged over all sources,
  in radians}*) sortjηBarToGrid=Sort[jηBarToGrid,#1[[2]]<#2[[2]]&];
 (*jηBarToGrid, {j,ηj}, but sorted with the smallest alignment angles first
 *)
 jηBarMin=sortjηBarToGrid[[1]]; (*  {j,ηj}, at the grid point Hj with minimum η̄*)
 jηBarMax=sortjηBarToGrid[[-1]]; (*  {j,ηj},
 at the grid point Hj with maximum η̄*)AppendTo[ψData,{nRun,ψSrc}];
 AppendTo[runData,{nRun,{ jηBarMin[[2]],{αGrid [ [ jηBarMin[[1]]  ]],
     δGrid [[  jηBarMin[[1]]  ]]}},{ jηBarMax[[2]],{αGrid [[  jηBarMax[[1]]  ]],
     δGrid [[  jηBarMax[[1]]  ]]}}} ](*collect data*)    ]
*)
```

Hint: You can save memory if you do not get the “ψData”. The table ψData is not needed in any following calculation.

In[148]:=
```
SetDirectory[dataDirectory];(*Save memory space; ψData is not used below.*)
(*Put[ψData,"20210216psiData.dat"  ] *) (*Save a new "ψData"*)
(*ψData=Get["20210216psiData.dat"]; *) (*Get an old "ψData"*)
```

Hint: Saving “runData” to a file avoids the time it takes to complete the “For” statement. Make the “For” statement into a remark so that it doesn’t run.

In[149]:=
```
 SetDirectory[dataDirectory];
(*Put[runData,"20210216runData.dat"  ] *) (*Save a new "runData".*)
runData = Get["20210207runData.dat"]; (*Get an old "runData".*)
```

In[151]:=
```
Print["The number of runs in the runData table is ", Length[runData], "."]
```

The number of runs in the runData table is 10000.

In[152]:=
```
ηBarMinData = Table[runData[[i1, 2, 1]] , {i1, Length[runData]}];
ηBarMaxData = Table[runData[[i1, 3, 1]] , {i1, Length[runData]}];
HminαData = Table[ runData[[i1, 2, 2, 1]] , {i1, Length[runData]}];
HminδData = Table[runData[[i1, 2, 2, 2]], {i1, Length[runData]}];
HmaxαData = Table[ runData[[i1, 3, 2, 1]] , {i1, Length[runData]}];
HmaxδData = Table[runData[[i1, 3, 2, 2]], {i1, Length[runData]}];
```

7b. Uncertainty in the Best Alignment Angle $\overline{\eta}_{\mathrm{min}}$

This section fits a Gaussian distribution to the $\overline{\eta}_{\mathrm{min}}$ from the uncertainty runs.

Definitions

sortηBarMin	list of $\overline{\eta}_{\min}$ from the data files, sorted small to large
η0B	estimated mean of the Gaussian fit
σB	estimated half-width of the Gaussian fit
histogramRANGE	{min η, max η, $\Delta\eta$} for the histogram
hl0, hl	histogram {η, bin height} tables needed to set up the NonlinearModelFit
lphl	list plot of the histogram table hl
nlmB	non-linear model fit of a Gaussian to the $\overline{\eta}_{\min}$ histogram
showNLMB	plot of Gaussian and histogram
parametersNLMB	amplitude, half-width, and mean of the Gaussian fit
pTableNLMB	table of parameter attributes, including standard error
bestVersusMeanMin	ratio of the $\overline{\eta}_{\min}$ from best values ψn to the mean $\eta 0 = \overline{\eta}_{\min}$ at the peak of the fit

In[158]:=
```
sortηBarMin = Sort[ηBarMinData];
η0B = sortηBarMin[[Floor[(1/2) Length[sortηBarMin]]]]; (*Guess the mean. *)
σB = sortηBarMin[[Floor[(4/5) Length[sortηBarMin]]]] - η0B; (*Guess the width.*)
histogramRANGE = {η0B - 5 σB, η0B + 5 σB, 0.4 σB};
hl0 = HistogramList[sortηBarMin, histogramRANGE];
hl =
 Table[{(1/2) (hl0[[1, i1]] + hl0[[1, i1 + 1]]), hl0[[2, i1]]}, {i1, Length[ hl0[[2]] ]}];
nlmB = NonlinearModelFit[hl, a Exp[-(1/2.) ((x - x0) / b)^2],
  {{a, Length[sortηBarMin / 6]}, {b, σB}, {x0, η0B}}, x]; (*x is ηBarMin*)
```

In[164]:=
```
showNLMB = Show[{Histogram[sortηBarMin, histogramRANGE,
    PlotLabel → "η̄min ", AxesLabel → {"η̄min, radians", "ΔR"}],
   Plot[Normal[nlmB], {x, η0B - 5 σB, η0B + 5 σB}, PlotLabel → "η̄min"],
   ListPlot[hl, PlotLabel → "η̄min"] }]
Print["Figure 7: The Gaussian fit to the alignment angle η̄min histogram, where
   the height is the number of runs ΔR in each bin of width Δη̄min  =  ",
 0.4 σB, " radians, centered on η̄min."]
```

Out[164]=

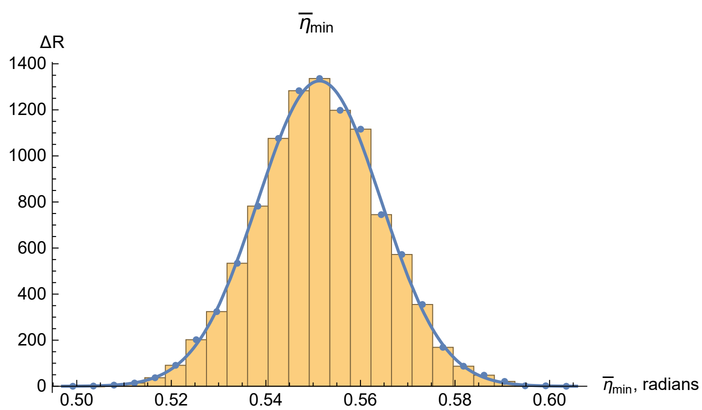


Figure 7: The Gaussian fit to the alignment angle $\overline{\eta}_{\min}$ histogram, where the height is the number of runs ΔR in each bin of width $\Delta\overline{\eta}_{\min}$ = 0.0043482 radians, centered on $\overline{\eta}_{\min}$.

In[166]:=
```
parametersNLMB = {a, b, x0} /. nlmB["BestFitParameters"];
pTableNLMB = nlmB["ParameterTable"]
{σηBarMinFit, ηBarMinFit} = {parametersNLMB[[2]] , parametersNLMB[[3]] };(*radians*)
```

Out[167]=

	Estimate	Standard Error	t-Statistic	P-Value
a	1325.51	13.1489	100.808	8.04799×10^{-31}
b	0.0130625	0.000149625	87.3019	1.89115×10^{-29}
x0	0.551343	0.000149625	3684.85	3.39788×10^{-65}

In[169]:=
```
Print[
 "Results of the Gaussian fit to the uncertainty runs for the alignment angle  η̅min:"]
Print[" "]
Print["For the value of the alignment angle η̅min, we find  η̅min  = ",
 ηBarMinFit (360. / (2. π)), "° ± ", σηBarMinFit (360. / (2. π)), "°." ]
Print["The mean η̅min = ", ηBarMinFit (360. / (2. π)),
 "° has a significance of ", signiMIN[ηBarMinFit], "."]
Print["The value η̅min + ση̅min = ", (ηBarMinFit + σηBarMinFit) (360. / (2. π)),
 "° has a significance of ", signiMIN[ηBarMinFit + σηBarMinFit], " ." ]
Print["The value η̅min - ση̅min = ", (ηBarMinFit - σηBarMinFit) (360. / (2. π)),
 "°  has a significance of ", signiMIN[ηBarMinFit - σηBarMinFit], " ." ]
```

Results of the Gaussian fit to the uncertainty runs for the alignment angle $\bar{\eta}_{min}$:

For the value of the alignment angle $\bar{\eta}_{min}$, we find $\bar{\eta}_{min}$ = 31.5897° ± 0.748427°.

The mean $\bar{\eta}_{min}$ = 31.5897° has a significance of 6.44344×10^{-7}.

The value $\bar{\eta}_{min} + \sigma\bar{\eta}_{min}$ = 32.3381° has a significance of 5.43124×10^{-6} .

The value $\bar{\eta}_{min} - \sigma\bar{\eta}_{min}$ = 30.8412° has a significance of 6.34791×10^{-8} .

7c. Uncertainty in the Largest Avoidance Angle $\bar{\eta}_{max}$

This section fits a Gaussian distribution to the $\bar{\eta}_{max}$ from the uncertainty runs.

Definitions:

The quantities for avoidance (Max) here have similar definitions as the quantities for alignment (Min). See the list of Definitions in Sec. 7b.

In[175]:=
```
sortηBarMax = Sort[ηBarMaxData];
η0MaxB = sortηBarMax [[Floor[(1/2) Length[sortηBarMax ]]]];
σMaxB = sortηBarMax [[Floor[(4/5) Length[sortηBarMax ]]]] - η0MaxB;
histogramRANGEMAX = {η0MaxB - 5 σMaxB, η0MaxB + 5 σMaxB, 0.4 σMaxB};
hl0Max = HistogramList[sortηBarMax, histogramRANGEMAX];
hlMax = Table[{(1/2) (hl0Max[[1, i1]] + hl0Max[[1, i1 + 1]]), hl0Max[[2, i1]]},
   {i1, Length[ hl0Max[[2]] ]}];
 nlmMaxB = NonlinearModelFit[hlMax, a Exp[- (1/2.) ((x - x0) / b)^2],
   {{a, 300.}, {b, σMaxB}, {x0, η0MaxB}}, x];(*x is ηBarMax *)
```

In[182]:=
```
showNLMMaxB = Show[{Histogram[sortηBarMax,
     histogramRANGEMAX, PlotLabel → "η̄max", AxesLabel → {"η̄max, radians", "ΔR"}],
    Plot[Normal[nlmMaxB], {x, η0MaxB - 5 σMaxB, η0MaxB + 5 σMaxB}, PlotLabel → "η̄max"],
    ListPlot[hlMax, PlotLabel → "η̄max"]}]
Print["Figure 8: The Gaussian fit to the avoidance angle
    η̄max histogram. The bins have a width Δη̄max  =  ",
  0.4 σMaxB, " radians, are centered on η̄max, and have a height
    equal to the number of runs ΔR in the bin."]
```

Out[182]=

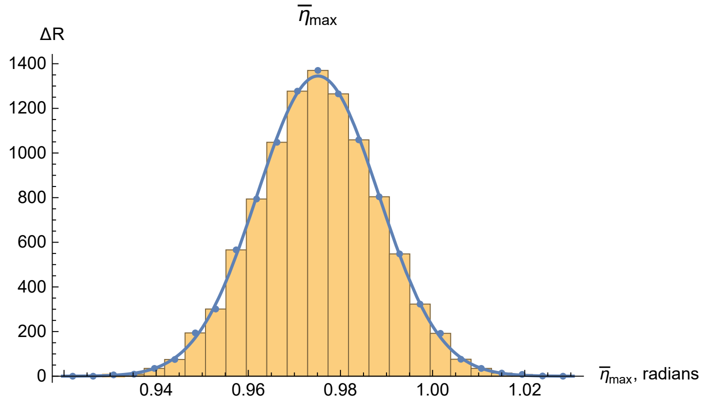


Figure 8: The Gaussian fit to the avoidance angle $\bar{\eta}_{max}$ histogram. The bins have a width $\Delta\bar{\eta}_{max}$ = 0.00443473
radians, are centered on $\bar{\eta}_{max}$, and have a height equal to the number of runs ΔR in the bin.

In[184]:=
```
parametersNLMMaxB = {a, b, x0} /. nlmMaxB["BestFitParameters"];
pTableNLMMaxB = nlmMaxB["ParameterTable"]
{σηBarMaxFit, ηBarMaxFit} = {parametersNLMMaxB[[2]], parametersNLMMaxB[[3]]};
(*radians*)
```

Out[185]=

	Estimate	Standard Error	t-Statistic	P-Value
a	1344.95	6.86385	195.947	3.65413×10^{-37}
b	0.0131474	0.000077477	169.695	8.63457×10^{-36}
x0	0.975124	0.000077477	12 586.	6.23635×10^{-77}

In[187]:=
```
Print[
  "Results of the Gaussian fit to the uncertainty runs for the avoidance angle  η̄max:"]
Print[" "]
Print["For the value of the avoidance angle η̄max, we find   η̄max  = ",
  ηBarMaxFit (360. / (2. π)), "° ± ", σηBarMaxFit (360. / (2. π)), "°."]
Print["The mean η̄max = ", ηBarMaxFit (360. / (2. π)),
  "° has a significance of ", signiMAX[ηBarMaxFit], "."]
Print["The value η̄max + ση̄max = ", (ηBarMaxFit + σηBarMaxFit) (360. / (2. π)),
  "° has a significance of ", signiMAX[ηBarMaxFit + σηBarMaxFit], " ." ]
Print["The value η̄max - ση̄max = ", (ηBarMaxFit - σηBarMaxFit) (360. / (2. π)),
  "°  has a significance of ", signiMAX[ηBarMaxFit - σηBarMaxFit], " ." ]
```

Results of the Gaussian fit to the uncertainty runs for the avoidance angle $\overline{\eta}_{max}$:

For the value of the avoidance angle $\overline{\eta}_{max}$, we find $\overline{\eta}_{max}$ = 55.8705° ± 0.753293°.

The mean $\overline{\eta}_{max}$ = 55.8705° has a significance of 0.000429587.

The value $\overline{\eta}_{max} + \sigma\overline{\eta}_{max}$ = 56.6238° has a significance of 0.0000783435 .

The value $\overline{\eta}_{max} - \sigma\overline{\eta}_{max}$ = 55.1172° has a significance of 0.00196114 .

7d. Location (α,δ) of the Alignment Hubs H_{min}

Each uncertainty run returns an alignment hub H_{min}. In this section, we attempt to find functions that fit the distribution of those hubs.

Issues:

(a) In any one run, the analysis produces an alignment angle $\overline{\eta}$ at each grid point. There can be just one minimum alignment angle $\overline{\eta}_{min}$, and, therefore, just one grid point H_{min} determined. However, by the symmetry across a diameter, the diametrically opposite location $-H_{min}$ should have the same minimum alignment angle, within the accuracy of the computed values. Note that $-H_{min}$ may not be a grid point. So we need to collect the hubs by moving some of them across a diameter.

(b) The spread of near-minimum $\overline{\eta}$ may extend over a large area of the Celestial sphere. See the blue area in Fig. 6. Thus the small variations with the uncertainty runs may produce more than one local minimum of the alignment angle function $\overline{\eta}$(H). There may be several disparate places where hubs H_{min} appear to collect. The plan is to respond reasonably to whatever appears.

(c) Since the hubs are finitely spaced grid points, the cluster of hubs may be so tightly determined that just a handful of grid points are populated. In such cases, the Gaussian fit is not appropriate and estimating the most likely location and the range of likely RAs and decs can be done by inspection.

Definitions

Hminα	RA = α in radians for H_{min}, “0” is raw data, “1” has been worked on, ...
Hminδ	dec = δ in radians for H_{min}, “0” is raw data, “1” has been worked on, ...
HminαAVE	arithmetic average of the RAs, in radians
sortHαMin	list of RA = α for H_{min} from the data files, sorted small to large
μ0αMinB	estimate of the mean value for the RA = α of H_{min}
$\sigma\alpha$MinB	estimate of the half-width of the RA = α pf H_{min}
histogramRangeMin	parameter range for several histograms
hl0Min, hlMin	tables of histogram data needed for plot and fit
nlmαMinB2	two Gaussian fit to the histogram of α for H_{min}
showNLMαMinB2	plot of histogram and the function that fits it
parametersNLMαMinB2	values of the two Gaussians’ parameters
pTableNLMαMinB2	table with values and standard errors of the parameters
$\sigma\alpha$MinFit1	half-width of the larger peak
αMinFit1	value of α at the top of the larger peak
$\sigma\alpha$MinFit2	half-width of the smaller peak
αMinFit2	value of α at the top of the smaller peak
HminαAVE	average over all uncertainty runs of α for H_{min}

Many of the following sections have similarly named quantities with similar definitions.

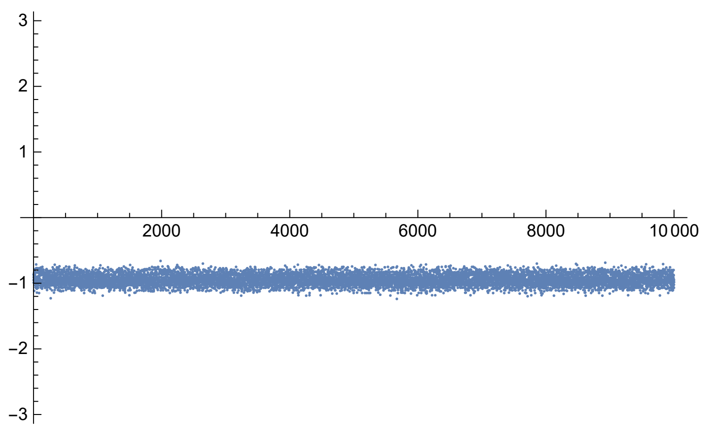
(i) Replace “α” by “δ” for the sections dealing with the uncertainty in dec = δ.

(ii) Replace “min” with “max” in the context of the avoidance hubs $H_{\max}$.

In[193]:=
```
(* Move hubs, if necessary, so that -180°  ≤  α  <  180° *)
Hminα0 = HminαData;
Hminδ0 = HminδData;
HminαBy180n = Round[Hminα0 / π];
Hminα1 = Table[Hminα0[[i1]] - HminαBy180n[[i1]] π , {i1, Length[Hminα0]}];
Hminδ1 = Table[(-1)^HminαBy180n[[i1]] Hminδ0[[i1]] , {i1, Length[Hminδ0]}];
```

In[197]:=
```
ListPlot[Hminα1, PlotRange → {-π, π}]
Print[
 "Figure 9: Most of the RA occur at negative values near RA  =  -1. By the symmetry
   across a diameter, we can move all the hubs to positive values, α  >
   0 band. The move across a diameter changes the sign of the dec = δs."]
```

Out[197]=



```
Figure 9: Most of the RA occur at negative values near RA  =  -1. By
  the symmetry across a diameter, we can move all the hubs to positive values,
  α  >  0 band. The move across a diameter changes the sign of the dec = δs.
```

In[199]:=
```
Hminα = Table[
   If[Hminα1[[i1]] < 0, Hminα1[[i1]] + π, Hminα1[[i1]], "huh?"] , {i1, Length[Hminα1]}];
Hminδ = Table[If[Hminα1[[i1]] < 0, -Hminδ1[[i1]], Hminδ1[[i1]], "huh?"] ,
   {i1, Length[Hminδ1]}];
```

In[201]:=
```
ListPlot[{Sort[Hminα], Sort[Hminδ]},
 PlotLabel → "RA = α and dec = δ for Hmin, radians", AxesLabel → {"Run #", "α,δ"}]
Print["Figure 10: It looks like we can fit Gaussians to the RA
   =  α values, since the values are spread out over many times the
   grid spacing. But the values of dec. = δ  occupy only about 6 grid
   points, it may be better to judge the uncertainty for δ another way."]
```

Out[201]=

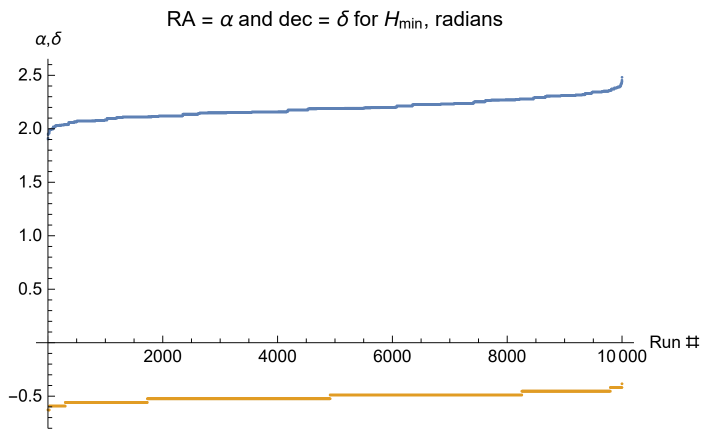


Figure 10: It looks like we can fit Gaussians to the RA = α values, since the values are spread out over many times the grid spacing. But the values of dec. = δ occupy only about 6 grid points, it may be better to judge the uncertainty for δ another way.

Fit a Gaussian, or two, to the histogram for RA = α of H_{min}.

In[203]:=
```
sortHαMin = Sort[Table[Hminα [[i2]], {i2, Length[Hminα]}]];
μ0αMinB = sortHαMin[[ Floor[Length[Hminα] / 2] ]];
σαMinB  = sortHαMin[[Floor[(4 / 5) Length[sortHαMin ]]]] - μ0αMinB;
histogramRangeMin = {μ0αMinB - 5 σαMinB, μ0αMinB + 5 σαMinB, 0.4 σαMinB};
hl0Min = HistogramList[sortHαMin, histogramRangeMin];
{Length[ hl0Min[[1]] ], Length[ hl0Min[[2]] ]};
hlMin = Table[{(1 / 2) (hl0Min[[1, i1]] + hl0Min[[1, i1 + 1]]), hl0Min[[2, i1]]},
   {i1, Length[ hl0Min[[2]] ]}];
nlmαMinB2 = NonlinearModelFit[hlMin, a1 Exp[- (1 / 2.) ((x - x01) / b1)^2] +
    a2 Exp[- (1 / 2.) ((x - x02) / b2)^2], {{a1, 1600.}, {b1, 0.08},
    {x01, 2.2}, {a2, 800.}, {b2, 0.04}, {x02, 2.35}}, x];(*x is α*)
```

In[211]:=
```
showNLMαMinB2 = Show[{Plot[Normal[nlmαMinB2], {x, μ0αMinB - 5 σαMinB, μ0αMinB + 5 σαMinB},
    PlotLabel → "α for H_min", AxesLabel → {"α, radians", "ΔR"},
    PlotRange → {0, 1.1 a1 /. nlmαMinB2["BestFitParameters"]}],
   Histogram[sortHαMin, histogramRangeMin, PlotLabel → "α for H_min"],
   Plot[Normal[nlmαMinB2], {x, μ0αMinB - 5 σαMinB, μ0αMinB + 5 σαMinB},
    PlotLabel → "α for H_min", PlotRange → {0, 700}],
   ListPlot[hlMin, PlotLabel → "α for H_min"] }]
Print["Figure 11: The two-peak Gaussian fit to the H_min hub's RA = α
   histogram. The bin width Δα is  Δα = ", 0.4 σαMinB, " radians."]
```

Out[211]=

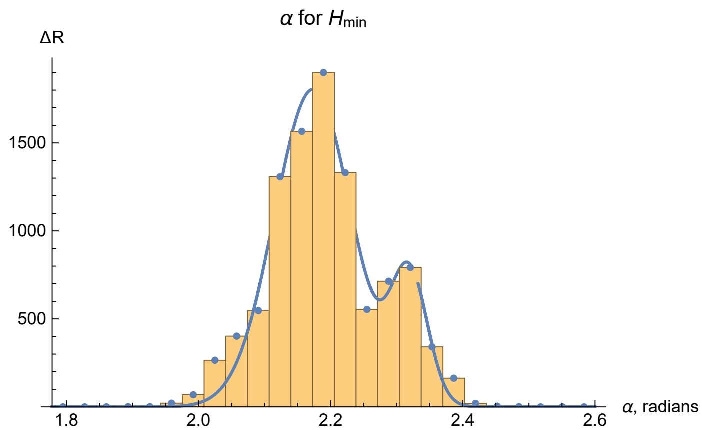


Figure 11: The two-peak Gaussian fit to the H_{min} hub's RA = α histogram. The bin width Δα is Δα = 0.0328535 radians.

In[213]:=
```
parametersNLMαMinB2 = {a1, b1, x01, a2, b2, x02} /. nlmαMinB2["BestFitParameters"];
pTableNLMαMinB2 = nlmαMinB2["ParameterTable"]
{σαMinFit1, αMinFit1} = {parametersNLMαMinB2[[2]] , parametersNLMαMinB2[[3]] };
(*radians*)
{σαMinFit2, αMinFit2} = {parametersNLMαMinB2[[5]] , parametersNLMαMinB2[[6]] };
(*radians*)
```

Out[214]=

	Estimate	Standard Error	t-Statistic	P-Value
a1	1805.08	70.3942	25.6425	3.33296×10^{-16}
b1	0.0584131	0.00319401	18.2883	1.60906×10^{-13}
x01	2.17341	0.00287388	756.265	5.12954×10^{-44}
a2	734.703	98.6679	7.44622	4.78239×10^{-7}
b2	0.0277968	0.00459073	6.05499	7.99152×10^{-6}
x02	2.31896	0.00492096	471.242	4.10247×10^{-40}

In[217]:=
```
Print["The uncertainty runs produce two peaks
   for the RA of H_min , one peak more likely than the other."]
Print["The more likely peak has  α  = ", αMinFit1 (360. / (2. π)), "° ± ",
 σαMinFit1 (360. / (2. π)), "° and the less likely peak has α  = ",
 αMinFit2 (360. / (2. π)), "° ± ", σαMinFit2 (360. / (2. π)) "°."]
```

The uncertainty runs produce two peaks for the RA of H_{min} , one peak more likely than the other.

The more likely peak has α = 124.527° ± 3.34682
 ° and the less likely peak has α = 132.867° ± 1.59264 °.

In[219]:=
```
HminαAVE = (1 / Length[Hminα]) Sum[Hminα[[i4]], {i4, Length[Hminα]}];
(* average α for H_max in radians *)
Print["Also note that, averaging over all runs, the average α for H_min in degrees is ",
 HminαAVE (360 / (2. π)), "° ."]
```

Also note that, averaging over all runs, the average α for H_{min} in degrees is 125.637° .

Next, attempt to fit a Gaussian to the δ for H_{min}. (Spoiler alert: The attempt fails.)

Definitions: For most quantities below for δ, replace "α" with "δ" in the quantities defined above for RA = α.

Hmin$\alpha\delta$	the locations (α,δ) of the hubs H_{min}.
lpHmin	list plot of the locations (α,δ) of the hubs H_{min}.
αiMinj, δiMinj	values needed to draw the uncertainty boxes

In[221]:=
```
sortHδMin = Sort[Table[Hminδ [[i2]], {i2, Length[Hminδ ]}]];
ListPlot[sortHδMin (360. / (2. π)),
 AxesLabel → {"Run ♯", "δ, degrees"}, PlotLabel → "δ for H_min", PlotRange → All]
Print["Figure 12: For H_min, the dec = δ has only 6 distinct values
   separated by the grid spacing 2°. By inspection of this plot, let the ",
 Length[Hminδ ], " values of the uncertainty runs for δ be represented
   by δ = -29° ± 1.5° = -0.506 ± 0.026 radians."]
```

Out[222]=

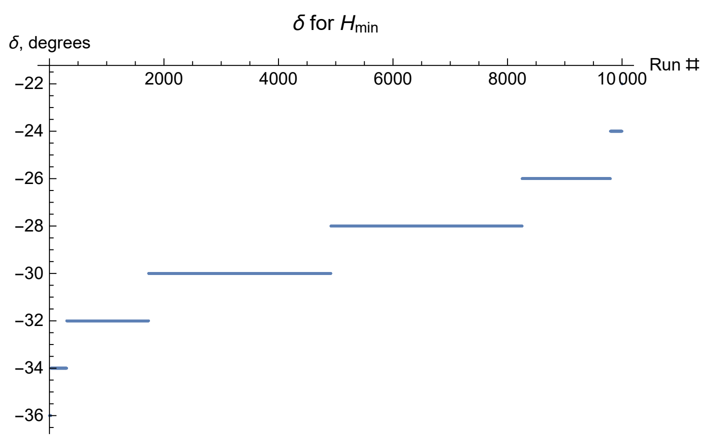


Figure 12: For H_{min}, the dec = δ has only 6 distinct values separated by the grid spacing 2°. By inspection of this plot, let the 10 000 values of the uncertainty runs for δ be represented by δ = −29° ± 1.5° = −0.506 ± 0.026 radians.

In[224]:=
```
{σδMinFit, δMinFit} = {1.5 , -29 } ((2. π) / 360.);(*radians*)
```

In[225]:=
```
HminδAVE = (1 / Length[Hminδ]) Sum[Hminδ[[i4]], {i4, Length[Hminδ]}];
(* average δ for H_max in radians *)
Print["Note that, averaging over all runs, the average δ for H_min in degrees is",
 HminδAVE (360 / (2. π)), "°, averaging over all runs."]
```

Note that, averaging over all runs, the average δ for H_{min} in degrees is
−29.0036°, averaging over all runs.

In[227]:=
```
Print["Allowing the measured PPA ψ = ψn +δψ, with δψ consistent with reported σψ
   uncertainties, produces a value of (RA,dec) = (α,δ) for H_min of  (α, δ)  = ",
 {αMinFit1 (360. / (2. π)), δMinFit (360. / (2. π))}, " ± ",
 {σαMinFit1 (360. / (2. π)), σδMinFit (360. / (2. π))},
 ", in degrees , α according to the Gaussian fit and δ by eye." ]
```

Allowing the measured PPA ψ = ψn +δψ, with δψ consistent with reported σψ uncertainties,
produces a value of (RA,dec) = (α,δ) for H_{min} of (α, δ) = {124.527, −29.}
± {3.34682, 1.5}, in degrees , α according to the Gaussian fit and δ by eye.

In[228]:=
```
Print["The best values ψn produce an alignment hub H_min with (RA,dec) = ",
 {αHminDegrees, δHminDegrees}]
```

The best values ψn produce an alignment hub H_{min} with (RA,dec) = {121.361, −30.}

In[229]:=
```
{σαMinFit1, αMinFit1} = {parametersNLMαMinB2[[2]] , parametersNLMαMinB2[[3]] };
(*radians*)
{σαMinFit2, αMinFit2} = {parametersNLMαMinB2[[5]] , parametersNLMαMinB2[[6]] };
(*radians*)
```

In[231]:=
```
(*Plot the values for H_min. *)
Hminαδ = Sort[Table[{Hminα[[i5]], Hminδ[[i5]]}, {i5, Length[Hminα]}]];
{Hminαδ[[1]], Hminαδ[[-1]]} ;(*radians*)
{Hminαδ[[1]], Hminαδ[[-1]]} (360. / (2. π)) ;(*degrees*)
lpHmin = ListPlot[Hminαδ (360. / (2. π)),
   PlotRange → {{0, 180}, {-90, 90}}, PlotMarkers → Automatic,
   AxesLabel → {"RA, degrees", "dec, degrees"}, PlotLabel → "Locations of the H_min hubs"];
α1Min1 = (αMinFit1 - σαMinFit1) (360. / (2. π));
α2Min1 = (αMinFit1 + σαMinFit1) (360. / (2. π));
δ1Min1 = (δMinFit - σδMinFit) (360. / (2. π));
δ2Min1 = (δMinFit + σδMinFit) (360. / (2. π));
α1Min2 = (αMinFit2 - σαMinFit2) (360. / (2. π));
α2Min2 = (αMinFit2 + σαMinFit2) (360. / (2. π));
δ1Min2 = (δMinFit - σδMinFit) (360. / (2. π));
δ2Min2 = (δMinFit + σδMinFit) (360. / (2. π));
```

In[243]:=
```
Show[{lpHmin,
  Graphics[{Line[{{α1Min1, δ1Min1}, {α1Min1, δ2Min1}, {α2Min1, δ2Min1}, {α2Min1, δ1Min1},
      {α1Min1, δ1Min1}}], Line[{{α1Min2, δ1Min2}, {α1Min2, δ2Min2},
      {α2Min2, δ2Min2}, {α2Min2, δ1Min2}, {α1Min2, δ1Min2}}]}]}]
Print["Figure 13: The locations of the alignment hubs Hmin from the
   uncertainty runs. The boxes outline the most likely locations
   on the left and a less likely cluster on the right."]
```

Out[243]=

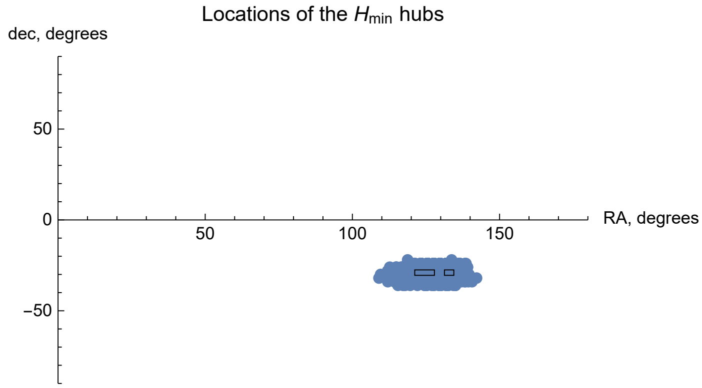


Figure 13: The locations of the alignment hubs H_{min} from the uncertainty runs. The boxes outline the most likely locations on the left and a less likely cluster on the right.

7e. Location of the Avoidance Hubs H_{max}

Find the likelihood of the location (α,δ) of H_{max} in the uncertainty runs. See the comments at the start of Sec. 7d.

Definitions: Explore the definitions for H_{min} at the start of Sec. 7d. Find the similarly named quantity by interchanging Max for Min. Adjust the definition to the present context.

In[245]:=
```
(* Move hubs, if necessary, so that -180° ≤ α < 180° *)
Hmaxα0 = HmaxαData;
Hmaxδ0 = HmaxδData;
HmaxαBy180n = Round[Hmaxα0 / π];
Hmaxα1 = Table[Hmaxα0[[i1]] - HmaxαBy180n[[i1]] π, {i1, Length[Hmaxα0]}];
Hmaxδ1 = Table[(-1)^HmaxαBy180n[[i1]] Hmaxδ0[[i1]], {i1, Length[Hmaxδ0]}];
```

In[249]:=
```
ListPlot[Hmaxα1, PlotRange → {-π, π},
 AxesLabel → {"Run #", "α, radians"}, PlotLabel → "RAs for Hmax"]
Print["Figure 14:  By the symmetry across a diameter, the band at
   positive RA = α can be moved to negative values by deducting π radians
   from each positive value of α. The move changes the sign of  dec = δ."]
```

Out[249]=

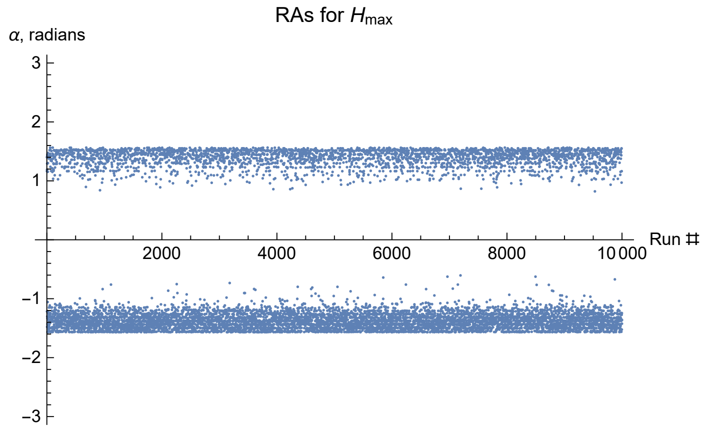


Figure 14: By the symmetry across a diameter, the band at
positive RA = α can be moved to negative values by deducting π radians
from each positive value of α. The move changes the sign of dec = δ.

In[251]:=
```
Hmaxα = Table[
   If[Hmaxα1[[i1]] > 0, Hmaxα1[[i1]] - π, Hmaxα1[[i1]], "huh?"] , {i1, Length[Hmaxα1]}];
Hmaxδ = Table[If[Hmaxα1[[i1]] > 0, -Hmaxδ1[[i1]], Hmaxδ1[[i1]], "ah"] ,
   {i1, Length[Hmaxδ1]}];
```

In[253]:=
```
ListPlot[{Sort[Hmaxα], Sort[Hmaxδ]}, PlotRange → {-π, π},
 PlotLabel → "RA = α and dec = δ for Hmax, radians", AxesLabel → {"Run #"}]
Print["Figure 15: The avoidance hubs found in the uncertainty runs occupy a
   wide range for both α and δ. Both ranges span intervals much larger than
   the grid spacing. So fitting α and δ with Gaussians is appropriate."]
```

Out[253]=

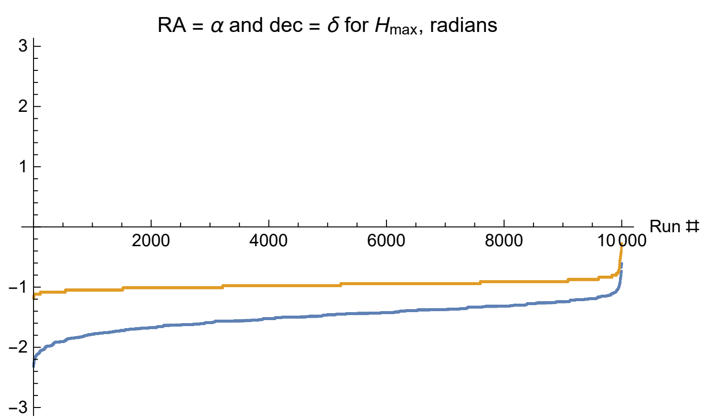


Figure 15: The avoidance hubs found in the uncertainty runs
occupy a wide range for both α and δ. Both ranges span intervals much larger
than the grid spacing. So fitting α and δ with Gaussians is appropriate.

Fit a Gaussian to the α for H_{max}.

In[255]:=
```
sortHαMax = Sort[Table[Hmaxα [[i2]], {i2, Length[Hmaxα]}]];
μ0αMaxB = sortHαMax[[ Floor[Length[Hmaxα] / 2] ]];
σαMaxB  = sortHαMax[[Floor[(4 / 5) Length[sortHαMax ]]]] - μ0αMaxB;
histogramRangeMax = {μ0αMaxB - 10 σαMaxB, μ0αMaxB + 5 σαMaxB, 0.4 σαMaxB};
hl0Max = HistogramList[sortHαMax, histogramRangeMax];
hlMax = Table[{(1 / 2) (hl0Max[[1, i1]] + hl0Max[[1, i1 + 1]]), hl0Max[[2, i1]]},
   {i1, Length[ hl0Max[[2]] ]}];
nlmαMaxB = NonlinearModelFit[hlMax, a Exp[- (1 / 2.) ((x - x0) / b)^2],
   {{a, Length[Hmaxα] / 6}, {b, σαMaxB}, {x0, μ0αMaxB}}, x];(*x is α*)
normalNLMαMaxB = Normal[nlmαMaxB];
```

In[263]:=
```
showNLMαMaxB = Show[{Plot[Normal[nlmαMaxB],
    {x, μ0αMaxB - 10 σαMaxB, μ0αMaxB + 5 σαMaxB}, AxesLabel → {"α, radians", "ΔR"},
    PlotLabel → "α for Hmax", PlotRange → {0, 1.1 a /. nlmαMaxB["BestFitParameters"]}],
   Histogram[sortHαMax, histogramRangeMax, PlotLabel → "α for Hmax"], Plot[
    Normal[nlmαMaxB], {x, μ0αMaxB - 5 σαMaxB, μ0αMaxB + 5 σαMaxB}, PlotLabel → "α for Hmax",
    PlotRange → {0, 700}] , ListPlot[hlMax, PlotLabel → "α for Hmax"]}]
Print["Figure 16: The Gaussian fit to the Hmax hub's RA = α histogram. One could
   argue that as many as four Gaussians could be used to fit the data better,
   two near the peak, one down the left slope and one just before the tail on
   the left side. See Fig. 11, for the similar situation with Hmin. For the
   RAs of Hmax, however, we prefer to stay with the single Gaussian fit. "]
```

Out[263]=

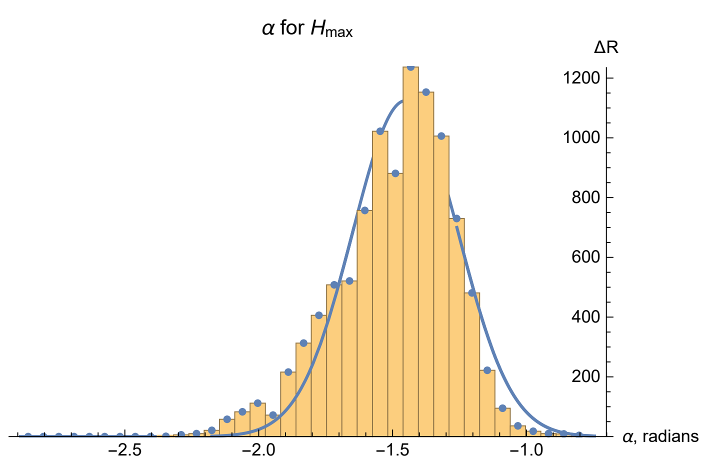


Figure 16: The Gaussian fit to the H_{max} hub's RA = α histogram. One could argue that as many as four Gaussians could be used to fit the data better, two near the peak, one down the left slope and one just before the tail on the left side. See Fig. 11, for the similar situation with H_{min}. For the RAs of H_{max}, however, we prefer to stay with the single Gaussian fit.

In[265]:=
```
parametersNLMαMaxB = {a, b, x0} /. nlmαMaxB["BestFitParameters"];
pTableNLMαMaxB = nlmαMaxB["ParameterTable"]
{σαMaxFit, αMaxFit} = {parametersNLMαMaxB[[2]] , parametersNLMαMaxB[[3]] };
(*Gaussian values, radians*)
```

Out[266]=

	Estimate	Standard Error	t-Statistic	P-Value
a	1124.19	39.1823	28.6914	2.06109×10^{-25}
b	0.197621	0.00795386	24.846	2.21108×10^{-23}
x0	−1.45215	0.00795326	−182.586	1.86614×10^{-52}

In[268]:=
```
Print["The uncertainty runs produce a value of RA = α for Hmax of  α  = ",
 αMaxFit (360. / (2. π)), "° ± ", σαMaxFit (360. / (2. π)), "°." ]
```

The uncertainty runs produce a value of RA = α for H_{max} of α = −83.2022° ± 11.3229°.

In[269]:=
```
HmaxαAVE = (1 / Length[Hmaxα]) Sum[Hmaxα[[i4]], {i4, Length[Hmaxα]}];
(* average α for Hmax in radians *)
Print["Note that, averaging over all runs, the average α for Hmax in degrees is ",
 HmaxαAVE (360 / (2. π)), "° ."]
```

Note that, averaging over all runs, the average α for H_{max} in degrees is −85.5611° .

Fit a Gaussian to the δ for $H_{\max}$.

In[271]:=
```
sortHδMax = Sort[Table[Hmaxδ [[i2]], {i2, Length[Hmaxδ ]}]];
μ0δMaxB = sortHδMax[[ Floor[Length[Hmaxδ ] / 2] ]];
σδMaxB = 0.09;
histogramRangeMax = {μ0δMaxB - 5 σδMaxB, μ0δMaxB + 3 σδMaxB, 0.4 σδMaxB};
hl0Max = HistogramList[sortHδMax, histogramRangeMax];
{Length[ hl0Max[[1]] ], Length[ hl0Max[[2]] ]};
hlδMax = Table[{(1 / 2) (hl0Max[[1, i1]] + hl0Max[[1, i1 + 1]]), hl0Max[[2, i1]]},
   {i1, Length[ hl0Max[[2]] ]}];
nlmδMaxB = NonlinearModelFit[hlδMax, a Exp[- (1 / 2.) ((x - x0) / b)^2],
   {{a, Length[Hmaxδ ] / 6}, {b, σδMaxB}, {x0, μ0δMaxB}}, x];(*x is δ, y is ΔR *)
normalNLMδMaxB = Normal[nlmδMaxB];
```

In[280]:=
```
showNLMδMaxB = Show[{Plot[Normal[nlmδMaxB], {x, μ0δMaxB - 5 σδMaxB, μ0δMaxB + 5 σδMaxB},
    PlotLabel → "δ for Hmax", PlotRange → All, AxesLabel → {"δ, radians", "ΔR"}],
   Histogram[sortHδMax, histogramRangeMax, PlotLabel → "δ for Hmax", PlotRange → {0, 700}],
   ListPlot[hlδMax, PlotLabel → "δ for Hmax"] ,
   Plot[Normal[nlmδMaxB], {x, μ0δMaxB - 5 σδMaxB, μ0δMaxB + 5 σδMaxB},
    PlotLabel → "δ for Hmax", PlotRange → {0, 700}]}]
Print["Figure 17: The Gaussian fit to the Hmax hub's dec = δ
   histogram. The bin width Δδ is Δδ  =  ", 0.4 σδMaxB,
 " radians, which is just a little wider than the grid spacing, dθ = ", dθ, " radians."]
```

Out[280]=

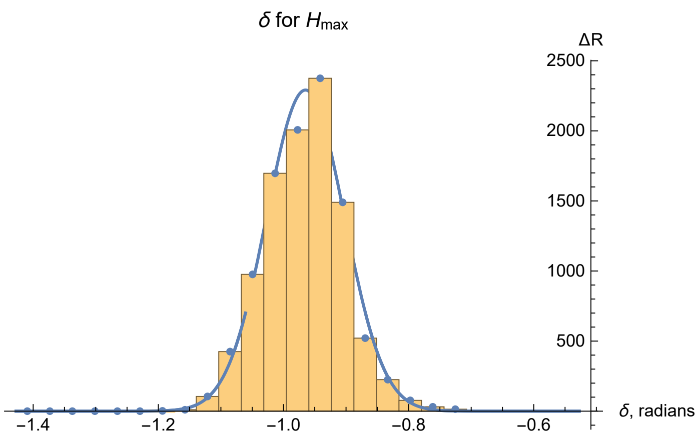


Figure 17: The Gaussian fit to the H_{max} hub's dec = δ histogram. The bin width Δδ is Δδ = 0.036 radians, which is just a little wider than the grid spacing, dθ = 0.0349066 radians.

In[282]:=
```
parametersNLMδMaxB = {a, b, x0} /. nlmδMaxB["BestFitParameters"];
pTableNLMδMaxB = nlmδMaxB["ParameterTable"]
{σδMaxFit, δMaxFit} = {parametersNLMδMaxB[[2]] , parametersNLMδMaxB[[3]] };
(*Gaussian values, radians*)
```

Out[283]=

	Estimate	Standard Error	t-Statistic	P-Value
a	2291.1	68.2788	33.5551	5.60854×10^{-17}
b	0.0622202	0.00214113	29.0595	6.21632×10^{-16}
x0	-0.964806	0.00214113	-450.606	4.21018×10^{-36}

In[285]:=
```
Print["The uncertainty runs produce a value of declination δ for Hmax of δ = ",
 δMaxFit (360. / (2. π)), "° ± ", σδMaxFit (360. / (2. π)), "°."]
```

The uncertainty runs produce a value of declination δ for H_{max} of δ = -55.2793° ± 3.56496°.

In[286]:=
```
HmaxδAVE = (1 / Length[Hmaxδ ]) Sum[Hmaxδ[[i4]], {i4, Length[Hmaxδ ]}];
(* average δ for Hmax in radians *)
Print["Note that, averaging over all runs, the average δ for Hmax is ",
 HmaxδAVE (360 / (2. π)), "° ."]
```

Note that, averaging over all runs, the average δ for H_{max} is -55.2616° .

In[288]:=
```
Print["The uncertainty runs give the location of Hmax as  (α, δ)  = ",
  {αMaxFit (360. / (2. π)), δMaxFit (360. / (2. π))}, " ± ",
  {σαMaxFit (360. / (2. π)), σδMaxFit (360. / (2. π))}, ", in degrees."]
```

The uncertainty runs give the location of H_{max} as (α, δ) =
{-83.2022, -55.2793} ± {11.3229, 3.56496}, in degrees.

In[289]:=
```
(* Plot the values for Hmax. *)
Hmaxαδ = Table[{Hmaxα[[i8]], Hmaxδ[[i8]]}, {i8, Length[Hmaxδ ]}];
{Hmaxαδ[[1]], Hmaxαδ[[-1]]} ;(*radians*)
{Hmaxαδ[[1]], Hmaxαδ[[-1]]} (360. / (2. π)) ;(*degrees*)
lpHmax1 = ListPlot[Hmaxαδ (360. / (2. π)), PlotRange → {{-180, +180}, {-90, 90}},
   PlotMarkers → Automatic, AxesLabel → {"RA, degrees", "dec, degrees"},
   PlotLabel → "Plot of Hmax hubs with the most likely region indicated "];
α1Max = (αMaxFit - σαMaxFit) (360. / (2. π));
α2Max = (αMaxFit + σαMaxFit) (360. / (2. π));
δ1Max = (δMaxFit - σδMaxFit) (360. / (2. π));
δ2Max = (δMaxFit + σδMaxFit) (360. / (2. π));
```

In[297]:=
```
Show[{lpHmax1, Graphics[Line[
    {{α1Max, δ1Max}, {α1Max, δ2Max}, {α2Max, δ2Max}, {α2Max, δ1Max}, {α1Max, δ1Max}}]]}]
Print["Figure 18: The locations of the alignment hubs Hmax from the uncertainty runs. The
   box outlines the most likely locations according to the Gaussian fits for α and δ."]
```

Out[297]=

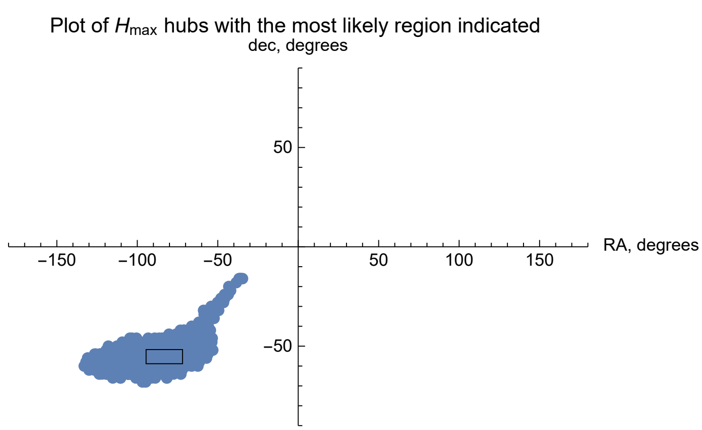


Figure 18: The locations of the alignment hubs H_{max} from the uncertainty runs. The box outlines the most likely locations according to the Gaussian fits for α and δ.

7f. Map of the Hubs for the Uncertainty Runs

In this subsection, we map the locations of the alignment hubs H_{min} and the locations of the avoidance hubs H_{max} , one set for each uncertainty run.

Definitions:

xyAitoffHmin	Aitoff coordinates for the alignment hubs H_{min} from the uncertainty runs
xyAitoffHmax	Aitoff coordinates for the avoidance hubs H_{max} from the uncertainty runs

xyAitoffOppositeHmin	Aitoff coordinates for the alignment hubs $-H_{\min}$ diametrically opposite the $H_{\min}$
xyAitoffOppositeHmax	Aitoff coordinates for the avoidance hubs $-H_{\max}$ diametrically opposite the $H_{\max}$
mapOfσψHminHmax	plot of the alignment and avoidance hubs $H_{\min}$ and $H_{\max}$, respectively

In[299]:=
```
(*The Aitoff coordinates for the hubs Hmin locations.*)
xyAitoffHmin = Table[{xH[ Hminα [[n]] (360 / (2 π)), Hminδ[[n]] (360 / (2 π)) ],
    yH[ Hminα [[n]] (360 / (2 π)), Hminδ[[n]] (360 / (2 π)) ]}, {n, Length[Hminδ ]}];
(*The Aitoff coordinates for the hubs Hmax locations.*)
xyAitoffHmax = Table[{xH[ Hmaxα [[n]] (360 / (2 π)), Hmaxδ[[n]] (360 / (2 π)) ],
    yH[ Hmaxα [[n]] (360 / (2 π)), Hmaxδ[[n]] (360 / (2 π)) ]}, {n, Length[Hminδ ]}];
(*The Aitoff coordinates for the hubs -Hmin locations.*)
xyAitoffOppositeHmin = Table[{xH[ If[0 ≤ Hminα [[n]] (360 / (2 π)) < +180,
      Hminα [[n]] (360 / (2 π)) - 180, If[0 > Hminα [[n]] (360 / (2 π)) > -180,
       Hminα [[n]] (360 / (2 π)) + 180]], -Hminδ[[n]] (360 / (2 π)) ],
    yH[ If[0 ≤ Hminα [[n]] (360 / (2 π)) < +180, Hminα [[n]] (360 / (2 π)) - 180,
      If[0 > Hminα [[n]] (360 / (2 π)) > -180, Hminα [[n]] (360 / (2 π)) + 180]],
     -Hminδ[[n]] (360 / (2 π)) ]}, {n, Length[Hminδ ]}];
(*The Aitoff coordinates for the hubs -Hmax locations.*)
xyAitoffOppositeHmax =
  Table[{xH[ If[0 ≤ Hmaxα [[n]] (360 / (2 π)) < +180, Hmaxα [[n]] (360 / (2 π)) - 180,
      If[0 > Hmaxα [[n]] (360 / (2 π)) > -180, Hmaxα [[n]] (360 / (2 π)) + 180]],
     -Hmaxδ[[n]] (360 / (2 π)) ], yH[ If[0 ≤ Hmaxα [[n]] (360 / (2 π)) < +180,
      Hmaxα [[n]] (360 / (2 π)) - 180, If[0 > Hmaxα [[n]] (360 / (2 π)) > -180,
       Hmaxα [[n]] (360 / (2 π)) + 180]], -Hmaxδ[[n]] (360 / (2 π)) ]}, {n, Length[Hmaxδ ]}];
```

In[303]:=
```
(*Construct the map of Hmin and Hmax hubs with ± regions indicated.*)
mapOfσψHminHmax =
  Show[{Table[
     ParametricPlot[{xH[α, δ], yH[α, δ]}, {δ, -90, 90}, PlotStyle → {Black, Thickness[0.002]},
      PlotPoints → 60, PlotRange → {{-7, 7}, {-3, 3}}, Axes → False], {α, -180, 180, 30}],
    Table[ParametricPlot[{xH[α, δ], yH[α, δ]}, {α, -180, 180},
      PlotStyle → {Black, Thickness[0.002]}, PlotPoints → 60], {δ, -60, 60, 30}], Graphics[
     {PointSize[0.007], Text[StyleForm["N", FontSize -> 10, FontWeight -> "Plain"], {0, 1.85}],
      LightBlue, (*Hmin:*)Point[ xyAitoffHmin ], (*-Hmin:*)Point[ xyAitoffOppositeHmin ],
      LightRed, (*Hmax:*)Point[ xyAitoffHmax ], (*-Hmax:*)Point[ xyAitoffOppositeHmax ],
       Black, Text[StyleForm["Max", FontSize → 8, FontWeight -> "Bold"],
       {xH[ - 180 , 0], yH[0, -60]}], {Arrow[BezierCurve[{{xH[ - 180 , 0], yH[0, -70]}, {-2.3, -2.0},
          {xH[αHmaxDegrees - 180, -δHmaxDegrees], yH[αHmaxDegrees - 180, -δHmaxDegrees]}}]]},
      Text[StyleForm["Min", FontSize → 8, FontWeight -> "Bold"], {xH[ 180 , 0], yH[0, -60]}],
      {Arrow[BezierCurve[{{xH[ 180 , 0], yH[0, -70]}, {2.3, -2.0},
          {xH[αHminDegrees, δHminDegrees], yH[αHminDegrees, δHminDegrees]}}]]},
      Text[StyleForm["Min", FontSize → 8, FontWeight -> "Bold"], {xH[ -180 , 0], yH[0, 60]}],
      {Arrow[BezierCurve[{{xH[ -180 , 0], yH[0, 70]}, {-2.3, 2.0},
          {xH[αHminDegrees - 180, -δHminDegrees], yH[αHminDegrees - 180, -δHminDegrees]}}]]},
      Text[StyleForm["Max", FontSize → 8, FontWeight -> "Bold"], {xH[ 180 , 0], yH[0, 60]}] ,
      {Arrow[BezierCurve[{{xH[ 180 , 0], yH[0, 70]}, {2.3, 2.0},
          {xH[αHmaxDegrees, δHmaxDegrees], yH[αHmaxDegrees, δHmaxDegrees]}}]]}      }],
    Table[ParametricPlot[{xH[α, δ], yH[α, δ]}, {δ, δ1Max, δ2Max},
      PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60], {α, α1Max, α2Max, α2Max - α1Max}],
    Table[ParametricPlot[{xH[α, δ], yH[α, δ]}, {α, α1Max, α2Max},
      PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60], {δ, δ1Max, δ2Max, δ2Max - δ1Max}],
    Table[ParametricPlot[{xH[α, δ], yH[α, δ]}, {δ, -δ2Max, -δ1Max},
      PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60],
     {α, α1Max + 180, α2Max + 180, α2Max - α1Max}],
    Table[ParametricPlot[{xH[α, δ], yH[α, δ]}, {α, α1Max + 180, α2Max + 180},
      PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60], {δ, -δ2Max, -δ1Max, δ2Max - δ1Max}],
    Table[ParametricPlot[{xH[α, δ], yH[α, δ]}, {δ, -δ2Min1, -δ1Min1},
      PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60],
     {α, α1Min1 - 180, α2Min1 - 180, α2Min1 - α1Min1}],
    Table[ParametricPlot[{xH[α, δ], yH[α, δ]}, {α, α1Min1 - 180, α2Min1 - 180},
      PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60],
     {δ, -δ2Min1, -δ1Min1, δ2Min1 - δ1Min1}], Table[ParametricPlot[{xH[α, δ], yH[α, δ]},
      {δ, δ1Min1, δ2Min1}, PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60],
     {α, α1Min1, α2Min1, α2Min1 - α1Min1}], Table[ParametricPlot[{xH[α, δ], yH[α, δ]},
      {α, α1Min1, α2Min1}, PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60],
     {δ, δ1Min1, δ2Min1, δ2Min1 - δ1Min1}]}, ImageSize → 432];
```

In[304]:=
```
mapOfσψHminHmax
Print[
 "Figure 19: The hubs found for the uncertainty runs. The hubs are represented as lightly shaded
   dots, light blue for alignment and pink for avoidance. The arrows point to the hubs found
   with the best values of the polarization directions ψ. Note that the clumps of light
   blue alignment hubs closely follow the areas of convergence (blue) in Fig. 6 and the
   clumps of pink avoidance hubs follow the areas of divergence (red) in Fig. 6. By shifting
   the polarization directions slightly due to experimental uncertainty, the locations
   of the extreme alignment angles shift. But the shift is small, favoring those areas
   that are near the extremes for the best values ψn of the polarization directions.  "]
Print["Notes: (i) The map is centered on (RA,dec) = (0°,0°)."]
Print["(ii) The alignment hubs Hmin and -Hmin are plotted as light blue dots. ", LightBlue]
Print["(iii) The avoidance hubs Hmax and -Hmax are plotted as pink dots. ", LightRed]
Print["(iv) The regions (α,δ) ± (σα,σδ) where the hubs
   H and -H are most likely found are enclosed in purple lines. ", Purple]
```

Out[304]=

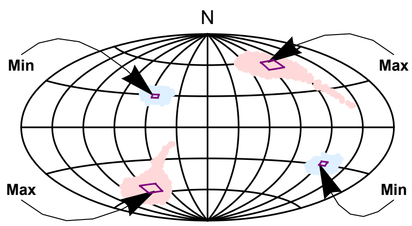


Figure 19: The hubs found for the uncertainty runs. The hubs are represented as lightly shaded dots, light blue for alignment and pink for avoidance. The arrows point to the hubs found with the best values of the polarization directions ψ. Note that the clumps of light blue alignment hubs closely follow the areas of convergence (blue) in Fig. 6 and the clumps of pink avoidance hubs follow the areas of divergence (red) in Fig. 6. By shifting the polarization directions slightly due to experimental uncertainty, the locations of the extreme alignment angles shift. But the shift is small, favoring those areas that are near the extremes for the best values ψn of the polarization directions.

Notes: (i) The map is centered on (RA,dec) = (0°,0°).

(ii) The alignment hubs H_{min} and $-H_{min}$ are plotted as light blue dots.

(iii) The avoidance hubs H_{max} and $-H_{max}$ are plotted as pink dots.

(iv) The regions (α,δ) ± (σα,σδ) where the hubs H and -H are most likely found are enclosed in purple lines.

8. Concluding Remarks

The polarization directions of the 106 QSOs are very well aligned. The minimum alignment angle $\overline{\eta}_{min}$, with a best value $\overline{\eta}_{min}$ of $\overline{\eta}_{min} = 30.9°$, is extremely low for this many QSOs. The likelihood that random polarization directions, *i.e.* the significance, would be better aligned is extremely small. We find in Sec. 6a, that the significance of the best value for $\overline{\eta}_{min}$ is $7.4^{+2.4}_{-1.8} \times 10^{-8}$, or about one in ten million. The plus/minus values are due to the finite number of random runs and not due to experimental uncertainties. In general, the more random runs the smaller these plus/minus values. Random runs do not use any measured polarization directions. The accuracy of the statistics improves with increasing the number of runs, not with more accurate data.

In Sec. 7b, by taking into account the uncertainty in the measured polarization directions, the uncertainty runs produce a range of minimum alignment angles $\overline{\eta}_{\text{min}}$. The histogram of the results, Fig. 7, fits a normal distribution quite well. The mean value of $\overline{\eta}_{\text{min}}$ at the peak of the distribution is $\overline{\eta}_{\text{min}} = 31.6° \pm 0.7°$. One finds the significance of the alignment angle $\overline{\eta}_{\text{min}}$ is $6.4^{+47.9}_{-5.8} \times 10^{-7}$, on the order of one in a few hundred thousand or more.

In contrast with random runs, making more uncertainty runs should not change the +47.9 and -5.8 displayed in the previous paragraph, except perhaps returning somewhat more accurate values. The plus/minus range reflects experimental data listed in the catalog.

By the uncertainty runs, the significance of the alignment most likely has a value of 6.4×10^{-7}, with $1 / 6.4 \times 10^{-7}$ or one in 1.6 million random runs having a better alignment. And, the significance is highly likely to be somewhere in the range from $(6.4 - 5.8) \times 10^{-7} = 0.6 \times 10^{-7}$ to $(6.4 + 47.9) \times 10^{-7} = 54.3 \times 10^{-7}$. Equivalently, the significance range implies that from one in 180 thousand to one in 16 million random runs have better alignment.

The analysis shows that alignment of the polarization directions of these 106 optical QSOs is extremely unlikely to be due to chance.

References

1. R. Shurtleff, the ready-to-run Mathematica version of this notebook is available, for a limited time, at the following URL: https://www.dropbox.com/s/dhuq6uo5k1kq98i/20210205IntermediateKitFor193BestOpticalQSORegions2.nb?dl=0 (2021).
2. R. Shurtleff, "Indirect polarization alignment with points on the sky, the Hub Test" , https://vixra.org/abs/2011.0026 (2020).
3. R. Shurtleff, "Evaluating the Alignment of Astronomical Linear Polarization Data, Intermediate Level Software" , https://vixra.org/abs/2101.0144 (2021).
4. Wolfram Research, Inc., Mathematica, Version 12.1, Champaign, IL (2020).
5. Wikipedia contributors. "Aitoff projection." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 25 May. 2017. Web. 3 Jan. 2018.
6. D. Hutsemekers *et al.*, "Mapping extreme-scale alignments of quasar polarization vectors", A&A 441, 915–930 (2005)
7. The catalog in Ref. 6 is also available online at http://cdsarc.u-strasbg.fr/viz-bin/qcat?J/A+A/441/915
8. D. Hutsemekers *et al.*, "Alignment of quasar polarizations with large-scale structures, A&A 572, A18 (2014)