Schrödinger Equation and Free Particle Wave Function

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Abstract

Using the wave function of a free particle we obtain a solution of the Schrödinger equation for a class of potentials.

1 Accelerating frame of reference

We simpify to one space dimension and time independent potential. Consider an accelerating frame of reference $\mathcal{F}'$ with coordinates x', t' and an inertial frame of reference $\mathcal{F}$ with coordinates x, t. The coordinates of the frames being related by

$$x' = x - \frac{1}{2}at^2 \qquad t' = t \tag{1}$$

Since $dx' = dx$ and position probabilites are the same for $\mathcal{F}'$ and $\mathcal{F}$ we have for the wave function $\psi(x,t)$ with respect to $\mathcal{F}$ and corresponding wave function $\psi'(x',t')$ with respect to $\mathcal{F}'$ that [1]

$$|\psi'(x',t')|^2 = |\psi(x,t)|^2 \tag{2}$$

Consequently there is a real valued function $\beta(x,t)$ such that

$$\psi'(x',t')e^{\frac{i}{\hbar}\beta(x',t')} = \psi(x,t) \tag{3}$$

With respect to $\mathcal{F}$ let the wave function $\psi(x,t)$ satisfy the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}(x,t) = i\hbar\frac{\partial\psi}{\partial t}(x,t) \tag{4}$$

With respect to $\mathcal{F}'$ the potential is $max' + V_0$ where V_0 is a constant hence the wave function $\psi'(x',t')$ satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi'}{\partial x'^2}(x',t') + (max' + V_0)\psi'(x',t') = i\hbar\frac{\partial\psi'}{\partial t'}(x',t') \tag{5}$$

Now

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \qquad \frac{\partial}{\partial t} = -at'\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'} \tag{6}$$

and on substituting (3) in (4) and using (5) and (6) gives

$$\left[\frac{i\hbar}{2m}\frac{\partial^2\beta}{\partial x'^2} - \frac{1}{2m}\left(\frac{\partial\beta}{\partial x'}\right)^2 + max' + V_0 + at'\frac{\partial\beta}{\partial x'} - \frac{\partial\beta}{\partial t'}\right]\psi' + \frac{i\hbar}{m}\left[\frac{\partial\beta}{\partial x'} - mat'\right]\frac{\partial\psi'}{\partial x'} = 0 \tag{7}$$

Let the velocity of $\mathcal{F}'$ with respect to $\mathcal{F}$ be zero for $t' < 0$ and at' for $t' > 0$ hence $\beta(x',t') = 0$ for $t' < 0$. Now ψ and ψ' satisfy Schrödinger equations and so are continuous in time. Consequently β will be continuous in time hence $\beta(x',0) = 0$. We have for $t' \geq 0$ that [2], [3]

$$\beta(x',t') = max't' + V_0t' + \frac{1}{6}ma^2t'^3 \tag{8}$$

is the unique solution of (7) satisfying the initial condition $\beta(x',0) = 0$.

2 Solution of Schrödinger equation

Let $V(x)$ be a smooth potential. Let $\{x_n\}$ with $n \in \mathbb{Z}$ be a set such that the union of sets $[x_n, x_{n+1}]$ is the real line. Let $\delta_n > 0$ be small compared to $x_{n+1} - x_n$. Define the potential $\widehat{V}(x)$ to be the smooth function such that

$$\widehat{V}(x) = \frac{V(x_{n+1}) - V(x_n)}{x_{n+1} - x_n}(x - x_n) + V(x_n) \tag{9}$$

for $x_n + \delta_n < x < x_{n+1} - \delta_n$ and $\widehat{V}(x)$ goes to $V(x)$ as the size of $[x_n, x_{n+1}]$ goes to zero. Define

$$a_n = \frac{1}{m}\frac{d\widehat{V}}{dx}\Big(\frac{x_n + x_{n+1}}{2}\Big) \qquad \widehat{a}(x) = \frac{1}{m}\frac{d\widehat{V}}{dx}(x) \qquad a(x) = \frac{1}{m}\frac{dV}{dx}(x) \tag{10}$$

Require of $\widehat{V}(x)$ that $\hat{a}(x)$ is a nondecreasing function and $\hat{a}(0) = 0$. Let $x' = x'(x,t), t' = t$ be the coordinate transformation associated to the potential $\widehat{V}(x)$ such that the point $(x', 0)$ follows a path $(x(t), t)$ where

$$x(t) = x' + \frac{1}{2}\hat{a}(x')t^2 \tag{11}$$

for $t > 0$ and $x' = x, t' = t$ for $t < 0$. Let $\psi_0(x,t)$ be a free particle wave function that satisfies the Schrödinger equation with zero potential and $\psi(x,t)$ the solution for potential $V(x)$ and $\psi(x,0) = \psi_0(x,0)$. Let $\widehat{\psi}(x,t)$ be the solution of the Schrödinger equation with potential $\widehat{V}(x)$ and $\widehat{\psi}(x,0) = \psi_0(x,0)$. We then have, dropping primes, for $x_n + \delta_n < x < x_{n+1} - \delta_n$ that

$$\widehat{\psi}(x,t) = e^{-\frac{i}{\hbar}[ma_n xt + V_0 t + \frac{1}{6}ma_n^2 t^3]}\psi_0\Big(x + \frac{1}{2}a_n t^2, t\Big) \tag{12}$$

Consquently as the size of all the $[x_n, x_{n+1}]$ go to zero $\widehat{\psi}(x,t)$ converges to $\psi(x,t)$ hence a solution to the Schrödinger equation for potential $V(x)$ and $\psi(x,0) = \psi_0(x,0)$ is

$$\psi(x,t) = e^{-\frac{i}{\hbar}[maxt + V_0 t + \frac{1}{6}ma^2 t^3]}\psi_0\Big(x + \frac{1}{2}at^2, t\Big) \tag{13}$$

References

[1] K. De Paepe, Physics Essays, September 2008

[2] K. De Paepe, Physics Essays, June 2013

[3] A. Colcelli, G. Mussardo, G. Sierra, A. Trombettoni, arXiv, 29 July 2020

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