THE CIRCLE EMBEDDING METHOD AND APPLICATIONS

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ABSTRACT. In this paper we introduce and develop the circle embedding method. We provide applications in the context of problems relating to deciding on the feasibility of partitioning numbers into certain class of integers.

CONTENTS

1. Introduction and Preliminary Results

In this section we recall some well-known results that will partly be needed in this paper. We find some results concerning the distribution of some sequences in arithmetic progression useful in the current paper. First we state the celebrated Szemeredi theorem concerning arithmetic progression. The theorem has both infinite and finite version, but we have considered appropriate to state the finite version.

Theorem 1.1 (Szemeredi). *$\forall\ \epsilon > 0$ and $\forall k \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that if $A \subset \mathbb{N}_n$ [1] satisfies $|A| \geq \epsilon n$, then A contains an arithmetic progression of length k.*

The well-known Green-Tao theorem [4] provides an extension in this direction as

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[1] see the notation in section 3.

Theorem 1.2 (Green-Tao)**.** *Let $\pi(n)$ denotes the number of primes no more than n. If $A \subset \mathbb{P}$ the set of all prime numbers such that*

$$\limsup_{n \longrightarrow \infty} \frac{|A \cap \mathbb{N}_n|}{\pi(n)} > 0$$

then A contains infinitely many arithmetic progressions of length k for any $k > 0$.

In this paper, motivated in part by the binary Goldbach conjecture, we develop a method which we feel might be a valuable resource and a recipe for studying problems concerning partition of numbers in specified subsets of $\mathbb{N}$. The method is very elementary in nature and has parallels with configurations of points on the geometric circle.
Let us suppose that for any $n \in \mathbb{N}$ we can write $n = u + v$ where $u, v \in \mathbb{M} \subset \mathbb{N}$ then the circle embedding method associate each of this summands to points on the circle generated in a certain manner by $n > 2$ and a line joining any such associated points on the circle. This geometric correspondence turns out to useful in our development, as the results obtained in this setting are then transformed back to results concerning the partition of integers. By exploiting this landscape we obtain the following class of results

Theorem 6.2. *There are infinitely many $n \in \mathbb{M}_{a,d}$ [2] with fixed $a, d \in \mathbb{N}$ such that the representation*

$$n = z_1 + z_2$$

where $\mu(z_1) = \mu(z_2) \neq 0$, $z_1, z_2 \in \mathbb{N}$ and μ is the Möbius function defined as

$$\mu(m) = \begin{cases} 1 \quad \text{if} \quad m = 1 \\ 0 \quad \text{if} \quad p^k | m, \ k \in \mathbb{N} \setminus \{1\} \\ (-1)^r \quad \text{if} \quad m = p_1 p_2 \cdots p_r \end{cases}$$

is valid.
The proof follows in section 6.

Conjecture 6.1. (Erdős-Turán) *Let $\mathbb{B} \subset \mathbb{N}$ and consider*

$$r_{\mathbb{B}}(n) := \# \left\{ (a, b) \in \mathbb{B}^2 | \ a + b = n \right\}.$$

If $r_{\mathbb{B}}(n) > 0$ for all sufficiently large values of n, then

$$\limsup_{n \longrightarrow \infty} \ r_{\mathbb{B}}(n) = \infty.$$

The proof of a weaker version follows in section 6.

[2] see (2.5)

2. The Circle of Partition

In this section we introduce the notion of the circle of partition. We study this notion in-depth and explore some potential applications in the following sequel.

Definition 2.1. Let $n \in \mathbb{N}$ and $\mathbb{M} \subset \mathbb{N}$. We denote with

$$\mathcal{C}(n, \mathbb{M}) = \{[x] \mid x, y \in \mathbb{M}, n = x + y\}.$$

The Circle of Partition generated by n with respect to the subset $\mathbb{M}$. We will abbreviate this in the further text as CoP. We call members of $\mathcal{C}(n, \mathbb{M})$ as points and denote them by $[x]$. For the special case $\mathbb{M} = \mathbb{N}$ we denote the CoP shortly as $\mathcal{C}(n)$.

Definition 2.2. We denote the line $\mathbb{L}_{[x],[y]}$ joining the point $[x]$ and $[y]$ as an axis of the CoP $\mathcal{C}(n, \mathbb{M})$ if and only if $x + y = n$. We say the axis point $[y]$ is an axis partner of the axis point $[x]$ and vice versa. We do not distinguish between $\mathbb{L}_{[x],[y]}$ and $\mathbb{L}_{[y],[x]}$, since it is essentially the the same axis. The point $[x] \in \mathcal{C}(n, \mathbb{M})$ such that $2x = n$ is the **center** of the CoP. If it exists then it is their only point which is not an axis point. The line joining any two arbitrary point which are not axes partners on the CoP will be referred to as a **chord** of the CoP. The length of the chord joining the points $[x], [y] \in \mathcal{C}(n, \mathbb{M})$, denoted as $\mathcal{D}([x], [y])$ is given by

$$\mathcal{D}([x], [y]) = |x - y|.$$

It is important to point out that the **median** of the weights of each co-axis point coincides with the center of the underlying CoP if it exists. That is to say, given all the axes of the CoP $\mathcal{C}(n, \mathbb{M})$ as

$$\mathbb{L}_{[u_1],[v_1]}, \mathbb{L}_{[u_2],[v_2]}, \ldots, \mathbb{L}_{[u_k],[v_k]}$$

then the following relations hold

$$\frac{u_1 + v_1}{2} = \frac{u_2 + v_2}{2} = \cdots = \frac{u_k + v_k}{2} = \frac{n}{2}$$

which is equivalent to the conditions for any of the pair of axes $\mathbb{L}_{[u_i],[v_i]}, \mathbb{L}_{[u_j],[v_j]}$ for $1 \leq i, j \leq k$

$$\mathcal{D}([u_i], [u_j]) = \mathcal{D}([v_i], [v_j])$$

and

$$\mathcal{D}([v_j], [u_i]) = \mathcal{D}([u_j], [v_i]).$$

Definition 2.3. Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}(n, \mathbb{M})$ and $\mathcal{C}(m, \mathbb{M})$ be two distinct CoPs for which holds

$$\mathcal{C}(n, \mathbb{M}) \subset \mathcal{C}(m, \mathbb{M}) \tag{2.1}$$

or

$$\mathcal{C}(n, \mathbb{M}) \supset \mathcal{C}(m, \mathbb{M}). \tag{2.2}$$

Then we say the CoPs *admit embedding*. We say the CoPs *admit aligned embedding* if and only if with (2.1) holds $n < m$ and with (2.2) $n > m$ and $\mathcal{C}(n, \mathbb{M}) = \mathcal{C}(m, \mathbb{M})$ holds if and only if $n = m$.

Notations. Let be

$$\mathbb{N}_n = \{m \in \mathbb{N} \mid \; m \leq n\}\,.$$

the **sequence** of the first n natural numbers. Further we will denote

$$||[x]|| := x$$

as the **weight** of the point $[x]$ and correspondingly the weight set of points in the CoP $\mathcal{C}(n, \mathbb{M})$ as $||\mathcal{C}(n, \mathbb{M})||$.

The above language in many ways could be seen as a criterion determining the plausibility of carrying out a partition in a specified set. Indeed this feasibility is trivial if we take the set $\mathbb{M}$ to be the set of natural numbers $\mathbb{N}$. The situation becomes harder if we take the set $\mathbb{M}$ to be a special subset of natural numbers $\mathbb{N}$, as the corresponding CoP $\mathcal{C}(n, \mathbb{M})$ may not always be non-empty for all $n \in \mathbb{N}$. One archetype of problems of this flavour is the binary Goldbach conjecture, when we take the base set $\mathbb{M}$ to be the set of all prime numbers $\mathbb{P}$. One could imagine the same sort of difficulty if we extend our base set to other special subsets of the natural numbers. As such we start by developing the theory assuming the base set of natural numbers $\mathbb{N}$ and latter extend it to other base sets $\mathbb{N}$ equipped with certain important and subtle properties.

Remark 2.1. It is important to notice that a typical CoP need not have a center. In the case of an absence of a center then we say the circle has a deleted center. However all CoPs with even generators have a center. It is easy to see that the CoP $\mathcal{C}(n)$ contains all points whose weights are positive integers from 1 to $n-1$ inclusive:

$$\mathcal{C}(n) = \{[x] \mid \; x \in \mathbb{N}, x < n\}\,.$$

Therefore the CoP $\mathcal{C}(n)$ has $\left\lfloor \frac{n-1}{2} \right\rfloor$ different axes.

Proposition 2.1. *Each axis is uniquely determined by points $[x] \in \mathcal{C}(n, \mathbb{M})$.*

Proof. Let $\mathbb{L}_{[x],[y]}$ be an axis of the CoP $\mathcal{C}(n, \mathbb{M})$. Suppose as well that $\mathbb{L}_{[x],[z]}$ is also an axis with $z \neq y$. Then it follows by Definition 2.2 that we must have $n = x + y = x + z$ and therefore $y = z$. This cannot be and the claim follows immediately. □

Corollary 2.1. *Each point of a CoP $\mathcal{C}(n, \mathbb{M})$ has exactly one axis partner.*

Proof. Let $[x] \in \mathcal{C}(n, \mathbb{M})$ be a point without an axis partner. Then holds for every point $[y] \neq [x]$

$$||[x]|| + ||[y]|| \neq n.$$

This contradiction to the Definition 2.1. Due to Proposition 2.1 the case of more than one axis partners is impossible. This completes the proof. □

Corollary 2.2. *The weights of the points of*

$$\mathcal{C}(n, \mathbb{M}) = \{[x_1], [x_2]. \ldots, [x_k]\}$$

are ***strictly totally ordered****.*

Proof. W.l.o.g. we assume that

$$x_1 = \min\left(x \mid [x] \in \mathcal{C}(n, \mathbb{M})\right) \text{ and} \tag{2.3}$$

$$x_k = \max\left(x \mid [x] \in \mathcal{C}(n, \mathbb{M})\right). \tag{2.4}$$

At first we assume that $x_1 + x_k < n$. Then there is a weight x_i with

$$x_1 < x_i < x_k \text{ and } n = x_1 + x_i.$$
$$\text{Because } x_i < x_k \text{ we get}$$
$$n = x_1 + x_i < x_1 + x_k.$$

This contradicts the assumtion. Now we assume that $x_1 + x_k > n$. Then there is a weight x_i with

$$x_1 < x_i < x_k \text{ and } n = x_i + x_k.$$
$$\text{Because } x_i > x_1 \text{ we get}$$
$$n = x_i + x_k > x_1 + x_k.$$

This also contradicts the assumtion. Therefore remains $x_1 + x_k = n$. Because of (2.3) and (2.4) holds

$$x_1 < x_2 < x_{k-1} < x_k.$$

Now we remove x_1 and x_k out of the consideration and repeat the procedure above with x_2 and x_{k-1} and obtain $x_2 + x_{k-1} = n$ and

$$x_1 < x_2 < x_3 < x_{k-2} < x_{k-1} < x_k.$$

By repeating this procedure for x_i and x_{k+1-i} for $3 \le i \le \left\lfloor \frac{k}{2} \right\rfloor$ we get finally

$$x_1 < x_2 < x_3 < x_4 < \ldots < x_{k-3} < x_{k-2} < x_{k-1} < x_k.$$

□

Proposition 2.2. *Let $\mathcal{C}(n, \mathbb{M})$ and $\mathcal{C}(m, \mathbb{M})$ be two distinct CoPs admitting aligned embedding. Then holds*

$$\mathcal{C}(n, \mathbb{M}) \cup \mathcal{C}(m, \mathbb{M}) \subset \mathcal{C}(n + m, \mathbb{M}).$$

Proof. W.l.o.g. we assume $\mathcal{C}(n, \mathbb{M}) \subset \mathcal{C}(m, \mathbb{M})$. Then holds

$$\begin{aligned} \mathcal{C}(n, \mathbb{M}) \cup \mathcal{C}(m, \mathbb{M}) &= \mathcal{C}(m, \mathbb{M}) \\ &\quad \text{and because of } \textit{admitting aligned embedding} \\ &\subset \mathcal{C}(n + m, \mathbb{M}) \text{ due to } m < n + m. \end{aligned}$$

□

Theorem 2.1. *Let $n \in \mathbb{N}$ and $\mathcal{C}(n)$ be a CoP generated by n. Then $\mathcal{C}(n)$ admits aligned embedding.*

Proof. W.l.o.g. we have to prove for two distinct CoPs

$$\mathcal{C}(n) \subset \mathcal{C}(m) \text{ if and only if } n < m \mid n, m \in \mathbb{N}.$$

First let $n < m$. Then follows that

$$\begin{aligned} \mathcal{C}(n) &= \{[x] \mid x \in \mathbb{N}, x < n\} \\ &\subset \{[x] \mid x \in \mathbb{N}, x < m\} \\ &= \mathcal{C}(m). \end{aligned}$$

Conversely we suppose $\mathcal{C}(n) \subset \mathcal{C}(m)$. Then it follows that

$$\{[x] \mid x \in \mathbb{N}, x < n\} \subset \{[x] \mid x \in \mathbb{N}, x < m\}$$

and it holds $n < m$. □

Now we will see that Theorem 2.1 is always valid for some special subsets $\mathbb{M}$ instead of $\mathbb{N}$, the subsets containing arithmetic progressions. Let be $\mathbb{M}_{a,d} \subset \mathbb{N}$ with

$$\mathbb{M}_{a,d} := \{x \in \mathbb{N} \mid x \equiv a \pmod{d}, d \in \mathbb{N}\} \tag{2.5}$$

and

$$\begin{aligned}\mathcal{C}(n, \mathbb{M}_{a,d}) &= \{[x] \mid x + y = n \wedge x, y \in \mathbb{M}_{a,d}\}, n \in \mathbb{M}_{2a,d} \\ &= \{[x] \mid x \in \mathbb{M}_{a,d} \wedge x \leq n - a\}.\end{aligned}$$

For $x < y \in \mathbb{M}_{a,d}$ holds $y - x \equiv 0 \pmod{d}$. On the other hand holds $x + y \equiv 2a \pmod{d}$, so that $\mathcal{C}(n, \mathbb{M}_{a,d}) = \emptyset$ for $n \notin \mathbb{M}_{2a,d}$.

Theorem 2.2. *Let $n \in \mathbb{M}_{2a,d}$ and $\mathcal{C}(n, \mathbb{M}_{a,d})$ be a CoP generated by n. Then the CoP admits aligned embedding.*

Proof. W.l.o.g. we have to prove

$$\mathcal{C}(n, \mathbb{M}_{a,d}) \subset \mathcal{C}(m, \mathbb{M}_{a,d}) \text{ if and only if } n < m.$$

At first let be $n < m$. Then holds

$$\begin{aligned}\|\mathcal{C}(n, \mathbb{M}_{a,d})\| &= \{k \in \mathbb{M}_{a,d} \mid k \leq n - a\} \\ &\text{and because of } n < m \\ &\subset \{k \in \mathbb{M}_{a,d} \mid k \leq m - a\} \\ &= \|\mathcal{C}(m, \mathbb{M}_{a,d})\|.\end{aligned}$$

On the other hand let be $\mathcal{C}(n, \mathbb{M}_{a,d}) \subset \mathcal{C}(m, \mathbb{M}_{a,d})$. Then holds

$$\begin{aligned}\|\mathcal{C}(n, \mathbb{M}_{a,d})\| &= \{k \in \mathbb{M}_{a,d} \mid k \leq n - a\} \\ &\subset \|\mathcal{C}(m, \mathbb{M}_{a,d})\| \\ &= \{k \in \mathbb{M}_{a,d} \mid k \leq m - a\} \\ &\text{and therefore must be} \\ &n < m.\end{aligned}$$

□

Corollary 2.3. *Let $\mathcal{C}(n, \mathbb{M})$ and $\mathcal{C}(m, \mathbb{M})$ be two distinct CoPs admit align embedding. Then holds*

$$\mathcal{C}(n, \mathbb{M}) \supset \mathcal{C}(m, \mathbb{M}) \text{ if and only if } n > m.$$

Corollary 2.4. *Because of Proposition 2.2 and Theorem 2.2 holds for two distinct CoPs $\mathcal{C}(n, \mathbb{M}_{a,d})$ and $\mathcal{C}(m, \mathbb{M}_{a,d})$* [3]

$$\mathcal{C}(n, \mathbb{M}_{a,d}) \cup \mathcal{C}(m, \mathbb{M}_{a,d}) \subset \mathcal{C}(n + m - 2a, \mathbb{M}_{a,d}).$$

[3] $n + m - 2a$ on the right side in order to get $n + m - 2a \in \mathbb{M}_{2a,d}$ by $n, m \in \mathbb{M}_{2a,d}$.

Remark 2.2. CoPs $\mathcal{C}(n,\mathbb{P})$ with the set of all prime numbers as base set are important examples for CoPs not admitting embedding. The following example demonstrates this scenario.

$$\begin{aligned}\mathcal{C}(20,\mathbb{P}) &= \{[3],[7],[13],[17]\} \text{ but}\\ \mathcal{C}(22,\mathbb{P}) &= \{[3],[5],[11],[17],[19]\}.\end{aligned}$$

Proposition 2.3. *Two distinct CoPs $\mathcal{C}(n,\mathbb{M})$ and $\mathcal{C}(m,\mathbb{M})$ admit* ***aligned*** *embedding if they admit embedding and the first point of both CoPs equals to $[x_o]$.*

Proof. W.l.o.g. we can assume that $\mathcal{C}(n,\mathbb{M}) \subset \mathcal{C}(m,\mathbb{M})$. We have to prove that $n < m$.
By the assumption, we have $|\mathcal{C}(n,\mathbb{M})| < |\mathcal{C}(m,\mathbb{M})|$. Invoking the condition that both CoPs have a common first point $[x_o]$, then between the last points of the CoPs holds

$$\begin{aligned}\|[n-x_o]\| &< \|[m-x_o]\| \text{ and hence}\\ n &< m.\end{aligned}$$

□

Proposition 2.4. *Let*

$$R_{a,d}(n) := \#\{(x,y) \in \mathbb{M}_{a,d}^2 |\ x+y=n, x<y\}$$

then $R_{a,d}(n)$ is a non-decreasing function for all $n \in \mathbb{M}_{2a,d}$.

Proof. Obviously $R_{a,d}(n)$ counts the axes of the CoP $\mathcal{C}(n,\mathbb{M}_{a,d})$ and it holds

$$R_{a,d}(n) = \left\lfloor \frac{|\mathcal{C}(n,\mathbb{M}_{a,d})|}{2} \right\rfloor.$$

By virtue of Theorem 2.2. the CoP $\mathcal{C}(n,\mathbb{M}_{a,d})$ admits aligned embedding. Therefore hold

$$\begin{aligned}\mathcal{C}(n,\mathbb{M}_{a,d}) &\subset \mathcal{C}(m,\mathbb{M}_{a,d}) \text{ for } n<m \in \mathbb{M}_{2a,d}\\ &\text{and hence}\\ R_{a,d}(n) &< R_{a,d}(m).\end{aligned}$$

□

3. The Mass and Moments of Circles of Partition

In this section we introduce and study the notion of the mass of the circle of partition. We then leverage this concept to study the notion of the moment of CoPs. This notion provides the wriggle room to carry out some quantitative methods in our analysis.

Notation. Let us denote the assignment of an axis $\mathbb{L}_{[x],[y]}$ to a CoP $\mathcal{C}(n,\mathbb{M})$ as

$$\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})$$

which means

$$[x],[y] \in \mathcal{C}(n,\mathbb{M}) \text{ and } x+y=n.$$

Definition 3.1. Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}(n,\mathbb{M})$ be a CoP. By the mass of the CoP $\mathcal{C}(n,\mathbb{M})$ we mean the quantity

$$\mathcal{M}\left[\mathcal{C}(n,\mathbb{M})\right] = \left\lfloor \frac{\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})\right\}}{\left\lfloor \frac{n-1}{2} \right\rfloor} \times n \right\rfloor.$$

The definition above formalizes the notion for counting points on a typical CoP. Indeed if we set $\mathbb{M} = \mathbb{N}$ then the mass $\mathcal{M}\left[\mathcal{C}(n,\mathbb{M})\right] = \mathcal{M}\left[\mathcal{C}(n)\right] = n$. Again if we take $\mathbb{M} = \mathbb{P}$, the set of all prime numbers then by the prime number theorem, we have the upper bound for the mass of the CoP $\mathcal{C}(n,\mathbb{P})$

$$\begin{aligned}\mathcal{M}\left[\mathcal{C}(n,\mathbb{P})\right] &= \left\lfloor \frac{\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{P})\right\}}{\left\lfloor \frac{n-1}{2} \right\rfloor} \times n \right\rfloor \\ &\leq \left\lfloor \frac{\frac{\pi(n)}{2}}{\left\lfloor \frac{n-1}{2} \right\rfloor} \times n \right\rfloor \\ &\leq \pi(n) \ll \frac{n}{\log n}.\end{aligned}$$

For some CoPs knowing the mass is an easy counting argument. For instance the mass of the CoP $\mathcal{C}(n,\mathbb{M}_{a,d})$ is given by as

$$\begin{aligned}\mathcal{M}[\mathcal{C}(n,\mathbb{M}_{a,d})] &= 1 + \sum_{\substack{k \leq n-a \\ k-a \equiv 0 \pmod{d} \\ n-2a \equiv 0 \pmod{d}}} 1 \\ &= 1 + \frac{n-2a}{d}.\end{aligned}$$

To the contrary knowing the mass for certain CoPs could also be a terribly non-trivial task. One archetype of this scenario is the mass of the CoP $\mathcal{C}(n,\mathbb{P})$ which in many ways could be zero for some $n \in 2\mathbb{N}$. The binary Goldbach conjecture can be reformulated in this language as follows

Conjecture 3.1 (Goldbach)**.** *Let $\mathbb{P}$ be the set of all prime numbers and $\mathcal{C}(n,\mathbb{P})$ be a CoP for $n \in \mathbb{N}$. Then*

$$\mathcal{M}\left[\mathcal{C}(n,\mathbb{P})\right] > 0$$

for every $n \in 2\mathbb{N}$.

Definition 3.2. Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}(n,\mathbb{M})$ be a CoP with mass $\mathcal{M}[\mathcal{C}(n,\mathbb{M})]$. We denote the moment $\mathcal{V}[\mathcal{C}(n,\mathbb{M})]$ of the CoP $\mathcal{C}(n,\mathbb{M})$ as the quantity

$$\mathcal{V}[\mathcal{C}(n,\mathbb{M})] = \int\limits_{\mathcal{C}(n,\mathbb{M})} \mathcal{M}[\mathcal{C}(n,\mathbb{M})]dn$$

where

$$\int\limits_{\mathcal{C}(n,\mathbb{M})} \mathcal{M}[\mathcal{C}(n,\mathbb{M})]dn := \sum_{\substack{\mathbb{L}_{[a],[b]} \hat{\in} \mathcal{C}(n,\mathbb{M}) \\ ||[a]|| < ||[b]||}} \int\limits_{a}^{b} \mathcal{M}[\mathcal{C}(n,\mathbb{M})]dn.$$

Proposition 3.1. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}(n,\mathbb{M})$ be a CoP with deleted center. Then $\mathcal{V}[\mathcal{C}(n,\mathbb{M})] > 0$ if and only if there exist some constant $K := K(n) > 0$ such that*

$$\frac{2K}{\int\limits_{u_1}^{v_1} \mathcal{M}[\mathcal{C}(n,\mathbb{M})]dn} \leq |\mathcal{C}(n,\mathbb{M})| \leq \frac{2K}{\int\limits_{u_2}^{v_2} \mathcal{M}[\mathcal{C}(n,\mathbb{M})]dn}$$

where $\mathbb{L}_{[u_2],[v_2]}, \mathbb{L}_{[u_1],[v_1]} \hat{\in} \mathcal{C}(n,\mathbb{M})$ with $v_1 = \max\{||\mathcal{C}(n,\mathbb{M})||\}$ and $u_1 = \min\{||\mathcal{C}(n,\mathbb{M})||\}$ with

$$\Big(||[u_2]||, ||[v_2]||\Big) \cap \mathcal{C}(n,\mathbb{M}) = \emptyset.$$

Proof. The result follows by virtue of the inequality

$$\frac{K}{\int\limits_{u_1}^{v_1} \mathcal{M}[\mathcal{C}(n,\mathbb{M})]dn} \leq \#\left\{\mathbb{L}_{[u_i],[v_i]} \hat{\in} \mathcal{C}(n,\mathbb{M})\right\} \leq \frac{K}{\int\limits_{u_2}^{v_2} \mathcal{M}[\mathcal{C}(n,\mathbb{M})]dn}$$

where $\mathbb{L}_{[u_2],[v_2]}, \mathbb{L}_{[u_1],[v_1]} \hat{\in} \mathcal{C}(n,\mathbb{M})$ with $v_1 = \max\{||\mathcal{C}(n,\mathbb{M})||\}$ and $u_1 = \min\{||\mathcal{C}(n,\mathbb{M})||\}$ with

$$\Big(||[u_2]||, ||[v_2]||\Big) \cap \mathcal{C}(n,\mathbb{M}) = \emptyset$$

and $K := K(n) > 0$. □

4. Rotation and Dilation of Circles of Partition

In this section we introduce the notion of the **Rotation** and **Dilation** of CoPs produced by a given generator. We launch the following formal terminology.

Definition 4.1. Let $\mathbb{M} \subseteq \mathbb{N}$ with $n \in \mathbb{N}$ and $\mathcal{C}(n,\mathbb{M})$ be the CoP generated by n. The map

$$\varpi_r : \mathcal{C}(n,\mathbb{M}) \longrightarrow \mathcal{C}^r(n,\mathbb{M})$$

will be the r^{th} level rotation of the CoP $\mathcal{C}(n,\mathbb{M})$ with

$$\begin{aligned}\mathcal{C}^r(n,\mathbb{M}) :=& \{[k] \in \mathcal{C}(n,\mathbb{M}) \mid [x] \in \mathcal{C}(n,\mathbb{M}),\ x + r \equiv k \pmod{n},\ r \in \mathbb{Z}, \\ &\text{if } x + r \equiv 0 \pmod{n} \text{ then } k := (n + r) \operatorname{Mod} n\}.\end{aligned}$$

If the sign is positive then we say the r^{th} level rotation is clockwise. Otherwise, it is an anti-clockwise r^{th} level rotation for $r \neq 0$. However, if we take $r = 0$, then the rotation is trivial and the axes joining points on the CoP remains stable. It is important to say that the result of a rotation must not be necessarily a CoP. Due to the condition $[k] \in \mathcal{C}(n,\mathbb{M})$ it is even possible that the target set is empty. In this case we say that the r^{th} level rotation fails to exist.

Theorem 4.1. *The CoP $\mathcal{C}(n)$ remains invariant under the r^{th} level rotation ϖ_r. That is*

$$\varpi_r : \mathcal{C}(n) \longrightarrow \mathcal{C}(n).$$

Proof. The set of weights of the images of $\mathcal{C}(n)$ is [4]

$$\|\mathcal{C}^r(n)\| = \{r+1, r+2, \ldots, r+n-1\}_n.$$

[4]We denote by $\{a, b, \ldots, z\}_n$ the set $\{a \operatorname{Mod} n, b \operatorname{Mod} n, \ldots, z \operatorname{Mod} n\}$.

The missing value is $(r+n-k)_n$ if $r+n-k \equiv 0 \pmod{n}$. Therefore holds

$$k = (n+r) \operatorname{Mod} n.$$

And this is the substituted value by virtue of the definition. $\square$

If the inequality $-n < r < n$ is valid then we get

$$k = \begin{cases} r & \text{if } r > 0 \\ n - |r| & \text{if } r < 0. \end{cases}$$

Example 4.1. $n = 8, r = +2$
$\|\mathcal{C}(8)\| = \{1,2,3,4,5,6,7\}$.
The critical point is $[6]$ because $6+2 \equiv 0 \pmod 8$. The set of the weights of the images of all points except of $[6]$ is $\{3,4,5,6,7,-,1\}$. Absent is 2.
As image of $[6]$ we set $[(8+2) \operatorname{Mod} 8] = [2]$ and we get as target set
$\|\varpi_3(\mathcal{C}(8))\| = \{3,4,5,6,7,2,1\} \to \{1,2,3,4,5,6,7\} = \|\mathcal{C}(8)\|$.

$n = 8, r = -2$
The critical point is $[2]$ because $2-2 \equiv 0 \pmod 8$. The set of the weights of the images of all points except of $[2]$ is $\{7,-,1,2,3,4,5\}$. Absent is 6.
As image of $[2]$ we set $[(8-2) \operatorname{Mod} 8] = [6]$ and we get as target set
$\|\varpi_3(\mathcal{C}(8))\| = \{7,6,1,2,3,4,5\} \to \{1,2,3,4,5,6,7\} = \|\mathcal{C}(8)\|$.

Proposition 4.1. *Let $\mathcal{C}(n, \mathbb{M}_{a,d})$ be a CoP defined as in (2.5). Then there exists not an r^{th} level rotation for $r \equiv c \pmod{d}$ with $0 < c < d$ and $c \not\equiv 2a \pmod{d}$.*

Proof. W.l.o.g. we let $c \leq n$.
We observe $[n-a-kd]$ is a point of $\mathcal{C}(n, \mathbb{M}_{a,d})$ for $k = 0(1)\frac{n-2a}{d}$ [5]. By applying the rotation ϖ_r its weight will be transformed to

$$\begin{aligned}(n-a-kd+c) \operatorname{Mod} n &= (c-a-kd) \operatorname{Mod} n \text{ and because of } c \leq n \\ &= c-a-kd \\ &\equiv (c-a) \pmod{d} \text{ and because of } c \not\equiv 2a \pmod{d} \\ &\not\equiv a \pmod{d}.\end{aligned}$$

Hence all rotated points of $\mathcal{C}(n, \mathbb{M}_{a,d})$ are not points of $\mathcal{C}(n, \mathbb{M}_{a,d})$ and therefore the target set of the rotation is an **empty set**. $\square$

Proposition 4.2. *Let $\mathcal{C}(n, \mathbb{M}_{a,d})$ be a CoP defined as in (2.5). Then $\mathcal{C}(n, \mathbb{M}_{a,d})$ remains invariant under the r^{th} level rotation ϖ_r provided $d = 2a$ and $r \equiv 0 \pmod{d}$.*

Proof. First we recall that $n \equiv 2a \pmod{d}$. Under the assumption $d = 2a$ it certainly follows that $n \equiv 0 \pmod{d}$. Now, let $(x+r) \operatorname{Mod} n = c$ be the weight of a rotated point $[x]$. Then it is easy to see that the following congruence condition is valid

$$\begin{aligned}x + r &\equiv c \pmod{n} \text{ and because } n \equiv 0 \pmod{d} \\ &\equiv c \pmod{d}.\end{aligned}$$

[5]Because of $n \in \mathbb{M}_{2a,d}$ is it a positive integer.

On the other hand the congruence conditions $x \equiv a \,(\mathrm{mod}\, d)$ and $r \equiv 0 \,(\mathrm{mod}\, d)$ imply

$$x + r \equiv a \,(\mathrm{mod}\, d)\,.$$

Hence we have $a = c$ and $x + r \equiv a \,(\mathrm{mod}\, d)$. Therefore all image points $\mathcal{C}(n, \mathbb{M}_{a,d})$ are members of $\mathbb{M}_{a,d}$ and less than n. In principle all image points of the r^{th} level rotation of the CoP $\mathcal{C}(n, \mathbb{M}_{a,d})$ are again points of the CoP $\mathcal{C}(n, \mathbb{M}_{a,d})$. This proves the claim that CoPs of the form $\mathcal{C}(n, \mathbb{M}_{a,d})$ remains invariant under some r^{th} level rotation with special conditions. □

Example 4.2. $n = 24$, $a = 2$, $d = 4$, $r = 4$
$\|\mathcal{C}(24, \mathbb{M}_{2,4})\| = \{2, 6, 10, 14, 18, 22\}$. Then is
$\|\varpi_4 \left(\mathcal{C}(24, \mathbb{M}_{2,4})\right)\| = \{6, 10, 14, 18, 22, 2\} \rightarrow \{2, 6, 10, 14, 18, 22\}$.

Corollary 4.1. *For conditions espoused in Proposition 4.1 and of Proposition 4.2 the r^{th} level rotation of a CoP $\mathcal{C}(n, \mathbb{M}_{a,d})$ results in a set which is a real subset of $\mathcal{C}(n, \mathbb{M}_{a,d})$.*

Definition 4.2. Let $\mathbb{M} \subseteq \mathbb{N}$ with $n \in \mathbb{N}$ and $\mathcal{C}(n, \mathbb{M})$ be the CoP generated by n. The map

$$\delta_r : \mathcal{C}(n, \mathbb{M}) \longrightarrow \mathcal{C}_r(n, \mathbb{M})$$

will be the r^{th} scale dilation of the CoP $\mathcal{C}(n, \mathbb{M})$ with

$$\mathcal{C}_r(n, \mathbb{M}) := \{[x] \in \mathcal{C}(n + r, \mathbb{M}) \mid \; r \in \mathbb{Z},\; n + r > 1\}\,.$$

If the sign is positive then we say the r^{th} scale dilation is an expansion. Otherwise, it is an r^{th} scale compression for $r \neq 0$. However if we take $r = 0$, then the dilation is a trivial dilation and the CoP remains invariant under the dilation.

Remark 4.1. It is important to note that if the base set is taken to be the set of natural numbers $\mathbb{N}$, then the image set of dilation collapses to the following

$$\begin{aligned} \delta_r\big(\mathcal{C}(n)\big) &:= \mathcal{C}_r(n) \\ &= \{[x] \mid x \in \mathbb{N}_{n+r-1},\; r \in \mathbb{Z},\; n + r > 1\} \\ &= \mathcal{C}(n + r). \end{aligned} \tag{4.1}$$

Additionally, it is important to point out that in case $r < 0$ some points of $\mathcal{C}(n)$ have the same image where as in the case $r > 0$ some points of $\mathcal{C}(n)$ have more than one image.

As it happens, dilation at any scale between CoPs have the natural tendency of translating the generator of the source CoP by the size of the scale of the dilation. However it is somewhat difficult to define dilation on individual points in a given CoP. Any perceived dilation map could manifestly work on a typical CoP but it may proved handicapped for some other CoPs. In the sense that some points may poke outside the target CoP under this fixed dilation. In light of this anomaly, we ask the following questions

Question 4.1. Let $\mathbb{M} \subseteq \mathbb{N}$. Does there exists a well-defined dilation

$$\delta_r : \mathcal{C}(n, \mathbb{M}) \longrightarrow \mathcal{C}(m, \mathbb{M})$$

on each $[x] \in \mathcal{C}(n, \mathbb{M})$ for all CoPs?

Put it differently, Question 4.1 asks if there exists a fixed map that assigns each points in a typical CoP to its target CoP in a sufficiently uniform way. That is to say, the map we seek should avoid the subtleties as espoused in our earlier discussion.

Theorem 4.2. *Let $n, m \in \mathbb{N}$, $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}(n,\mathbb{M})$ be a CoP admitting aligned embedding. Then there exists some dilation δ_r such that*

$$\delta_r : \mathcal{C}(n,\mathbb{M}) \longrightarrow \mathcal{C}(m,\mathbb{M}).$$

Proof. It is evident that for $m = n$ the trivial dilation δ_0 meets the claim. For the case $m \neq n$ we break the proof into several cases. The case r is positive and the case it is negative. Let δ_r be any dilation for $r > 0$ and suppose for any two CoP $\mathcal{C}(n,\mathbb{M})$ and $\mathcal{C}(m,\mathbb{M})$ with $\mathcal{C}(m,\mathbb{M}) \subset \mathcal{C}(n,\mathbb{M})$ there exists no dilation associating them. By virtue of the property that the CoPs admitting embedding exactly one of the following embedding holds

$$\delta_r\big(\mathcal{C}(m,\mathbb{M})\big) \subset \mathcal{C}(n,\mathbb{M}) \text{ or } \mathcal{C}(n,\mathbb{M}) \subset \delta_r\big(\mathcal{C}(m,\mathbb{M})\big).$$

We analyze each of these sub-cases. First let us assume that $\delta_r\big(\mathcal{C}(m,\mathbb{M})\big) \subset \mathcal{C}(n,\mathbb{M})$. It follows that there exists some CoP $\mathcal{C}(s,\mathbb{M})$ with $\delta_r(\mathcal{C}(m,\mathbb{M})) \subseteq \mathcal{C}(s,\mathbb{M})$ such that $\mathcal{C}(s,\mathbb{M}) \subset \mathcal{C}(n,\mathbb{M})$. Since there exists no dilation between CoPs the following proper embedding must necessarily hold

$$\delta_r\big(\mathcal{C}(m,\mathbb{M})\big) \subset \mathcal{C}(s,\mathbb{M}) \subset \mathcal{C}(n,\mathbb{M}).$$

Again there exists some CoP $\mathcal{C}(t,\mathbb{M})$ with $\delta_r(\mathcal{C}(m,\mathbb{M})) \subseteq \mathcal{C}(t,\mathbb{M})$ such that $\mathcal{C}(t,\mathbb{M}) \subset \mathcal{C}(s,\mathbb{M})$. Then under the underlying assumption that there exists no dilation between CoPs, we obtain the following proper embedding

$$\delta_r(\mathcal{C}(m,\mathbb{M})) \subset \mathcal{C}(t,\mathbb{M}) \subset \mathcal{C}(s,\mathbb{M}) \subset \mathcal{C}(n,\mathbb{M}).$$

By repeating the argument in this manner, we obtain the following infinite descending chains of covers of the smallest CoP

$$\mathcal{C}(m+r,\mathbb{M}) := \delta_r\big(\mathcal{C}(m,\mathbb{M})\big) \subset \cdots \subset \mathcal{C}(t,\mathbb{M}) \subset \mathcal{C}(s,\mathbb{M}) \subset \mathcal{C}(n,\mathbb{M}).$$

Because the CoPs admit aligned embedding we obtain the infinite descending sequence of positive integers towards the generator $m + r$ of the last CoP

$$n > s > t > \cdots > \cdots > m + r.$$

This is absurd, thereby ending the proof of the first sub-case. We now turn to the case $\mathcal{C}(n,\mathbb{M}) \subset \delta_r(\mathcal{C}(m,\mathbb{M}))$. Then in a similar fashion there must exist some CoP $\mathcal{C}(t,\mathbb{M})$ with $\mathcal{C}(t,\mathbb{M}) \subseteq \delta_r(\mathcal{C}(m,\mathbb{M}))$ such that $\mathcal{C}(n,\mathbb{M}) \subset \mathcal{C}(t,\mathbb{M})$. Then under the assumption that there exists no dilation between CoP, we have the following embedding

$$\mathcal{C}(n,\mathbb{M}) \subset \mathcal{C}(t,\mathbb{M}) \subset \delta_r(\mathcal{C}(m,\mathbb{M})).$$

Again there exists some CoP $\mathcal{C}(s,\mathbb{M})$ with $\mathcal{C}(s,\mathbb{M}) \subseteq \delta_r(\mathcal{C}(m,\mathbb{M}))$ such that $\mathcal{C}(t,\mathbb{M}) \subset \mathcal{C}(s,\mathbb{M})$. Under the assumption that there exists no dilation between CoP, we have the following embedding

$$\mathcal{C}(n,\mathbb{M}) \subset \mathcal{C}(t,\mathbb{M}) \subset \mathcal{C}(s,\mathbb{M}) \subset \delta_r(\mathcal{C}(m,\mathbb{M})).$$

By repeating this argument indefinitely we obtain the following infinite sequence of embedding

$$\mathcal{C}(n,\mathbb{M}) \subset \mathcal{C}(t,\mathbb{M}) \subset \mathcal{C}(s,\mathbb{M}) \cdots \subset \delta_r(\mathcal{C}(m,\mathbb{M})) := \mathcal{C}(m+r,\mathbb{M}).$$

By virtue of the CoPs admitting aligned embedding, we obtain an infinite ascending sequence of positive integers towards the generator of the last CoP in the chain

$$n < t < s < \cdots < m + r.$$

This is absurdity, since we cannot have positive integers approaching a fixed positive integer for infinite amount of time. This completes the proof for the case $r > 0$. We now turn to the case $r < 0$ for any two CoP $\mathcal{C}(m,\mathbb{M}), \mathcal{C}(n,\mathbb{M})$ with $\mathcal{C}(n,\mathbb{M}) \subset \mathcal{C}(m,\mathbb{M})$. Under the main assumption exactly one of the following embedding must hold

$$\delta_r\big(\mathcal{C}(m,\mathbb{M})\big) \subset \mathcal{C}(n,\mathbb{M}) \text{ or } \mathcal{C}(n,\mathbb{M}) \subset \delta_r\big(\mathcal{C}(m,\mathbb{M})\big).$$

A similar analysis could be carried out for each of the above cases. □

Corollary 4.2. *Because of Theorem 2.1 the CoP $\mathcal{C}(n)$ admits aligned embedding and there is the dilation $\delta_1 : \mathcal{C}(n) \longrightarrow \mathcal{C}(n+1)$ with*

$$\delta_1([x]) := \begin{cases} [x] & \textit{for } \ 1 \leq x \leq n-1 \\ [n] & \textit{additional for } x = 1 \end{cases} \tag{4.2}$$

that can produce an infinite ascending chain of CoPs

$$\mathcal{C}(n) \subset \mathcal{C}(n+1) \subset \mathcal{C}(n+2) \subset \cdots.$$

It is easy to see that the assignment of $[n]$ as also an image of $[1]$ is not the only possibility. Also possible would be $[n]$ as the image of $[2] \ldots [n-1]$. In all cases we would have a correct point-to-point mapping. Hence a subset of the cross set $\mathcal{C}(n) \times \mathcal{C}(n+1)$ for which holds:

- for each point of $\mathcal{C}(n)$ there is at least one image point of $\mathcal{C}(n+1)$ and
- for each image point of $\mathcal{C}(n+1)$ there is only one preimage point of $\mathcal{C}(n)$

is not a well-defined pointwise definition of the map $\mathcal{C}(n) \longrightarrow \mathcal{C}(n+1)$ because there are several such subsets.

Corollary 4.3. *In light of Theorem 2.2 the CoP $\mathcal{C}(n,\mathbb{M}_{a,d})$ admits aligned embedding and there is the dilation $\delta_d : \mathcal{C}(n,\mathbb{M}_{a,d}) \longrightarrow \mathcal{C}(n+d,\mathbb{M}_{a,d})$ with*

$$\delta_d([x]) := \begin{cases} [x] & \textit{for } a \leq x \leq n-a \\ [n-a+d] & \textit{additional for } x = a \end{cases}$$

that can generate an infinite ascending chain of CoPs

$$\mathcal{C}(n,\mathbb{M}_{a,d}) \subset \mathcal{C}(n+d,\mathbb{M}_{a,d}) \subset \mathcal{C}(n+2d,\mathbb{M}_{a,d}) \subset \cdots.$$

5. **Stable and Unstable Points on the Circle of Partition**

In this section we launch the notion of stability of a sequence under a given dilation.

Definition 5.1. Let $\Theta(n)$ be a subsequence of $\mathbb{N}_n$ and suppose the CoP $\mathcal{C}(n,\mathbb{M}) \neq \emptyset$. Let $\mathbb{L}_{[x],[y]}$ be an axis of the CoP $\mathcal{C}(n,\mathbb{M})$ with $x, y \in \Theta(n)$. Then we say the point $[x] \in \mathcal{C}(n,\mathbb{M})$ is **stable** relative to the subsequence $\Theta(n)$ under the r^{th} level rotation $\varpi_r : \mathcal{C}(n,\mathbb{M}) \longrightarrow \mathcal{C}(n,\mathbb{M})$ if $||\varpi_r([x])|| \in \Theta(n)$ and $\exists z \in \Theta(n)$ such that $\mathbb{L}_{[\varpi_r([x])],[z]}$ is also an axis of the CoP $\mathcal{C}(n,\mathbb{M})$. We say the subsequence $\Theta(n)$ is **stable** under the r^{th} level rotation ϖ_r if all points in $[x] \in \mathcal{C}(n,\mathbb{M})$ with $x \in \Theta(n)$ are stable.

Definition 5.2. Let $\Theta(n)$ be a subsequence of $\mathbb{N}_n$ and suppose the CoP $\mathcal{C}(n,\mathbb{M}) \neq \emptyset$. Let $\mathbb{L}_{[x],[y]}$ be an axis of the CoP $\mathcal{C}(n,\mathbb{M})$ with $x, y \in \Theta(n)$. Then we say the point $[x] \in \mathcal{C}(n,\mathbb{M})$ is **stable** relative to the subsequence $\Theta(n)$ under the r^{th} scale dilation $\delta_r : \mathcal{C}(n,\mathbb{M}) \longrightarrow \mathcal{C}(s,\mathbb{M})$ if $||\delta_r([x])|| \in \Theta(n)$ and $\exists z \in \Theta(n)$ such that $\mathbb{L}_{[\delta_r([x])],[z]}$ is also an axis of the CoP $\mathcal{C}(s,\mathbb{M})$. We say the subsequence $\Theta(n)$ is **stable** under the r^{th} scale dilation δ_r if all points in $[x] \in \mathcal{C}(n,\mathbb{M})$ with $x \in \Theta(n)$ are stable.

Next we establish an important result in the special case where the base set is the set $\mathbb{N}$ of natural numbers.

Proposition 5.1. *Let $\Theta(n) = \mathbb{N}_{n-1}$ and let $\delta_r : \mathcal{C}(n) \longrightarrow \mathcal{C}(m)$ be a dilation. Then the subsequence $\Theta(n)$ is stable if and only if $n \geq m$.*

Proof. In the case $m = n$ then the dilation is trivial and the claim is trivially true. Suppose the sequence $\Theta(n)$ is stable under the dilation

$$\delta_r : \mathcal{C}(n) \longrightarrow \mathcal{C}(m)$$

and assume to the contrary that $n < m$. Then the dilation is an expansion. It follows that for all $[x] \in \mathcal{C}(n)$ with $x \in \Theta(n)$ there exists $z \in \Theta(n)$ such that $z + ||\delta_r([x])|| = m$. Under the assumption $n < m$ and by virtue of Theorem 2.1 we have the embedding $\mathcal{C}(n) \subset \mathcal{C}(m)$ and for all $x \in \Theta(n)$ holds $[x] \in \mathcal{C}(n)$ and $1 + x \leq n < m$. There exist some $[y] \in \mathcal{C}(n)$ such that $\delta_r([y]) = [1]$ but there exists no $z \in \Theta(n)$ such that $1 + z = m$. It follows that the point $[y]$ is not a stable point under δ_r. This contradicts the claim that $\Theta(n)$ is stable and so $n < m$ is impossible. Conversely let us suppose that $m < n$ and consider the dilation

$$\delta_r : \mathcal{C}(n) \longrightarrow \mathcal{C}(m).$$

We note that for any point $[x] \in \mathcal{C}(n)$ there exist some $k < m < n$ such that $||\delta_r([x])|| + k = m$. Because $k \in \mathbb{N}_{n-1} = \Theta(n)$ it follows that the subsequence $\Theta(n)$ is stable under any dilation δ_r. ☐

Next we show that any consecutive subsequence of $\mathbb{N}_n$ containing none of its degenerate terms must be stable under the simple dilation. We formalize this assertion in the following results.

Proposition 5.2. *Let $\Theta(n) := \{x, x+1, \ldots, n-x, n-x+1\}$ be a subsequence of $\mathbb{N}_n$ for any $1 < x < \frac{n}{2}$ and $\delta_r : \mathcal{C}(n) \longrightarrow \mathcal{C}(n+1)$ be an expansion. Then $\Theta(n)$ is stable under the expansion δ_r.*

Proof. For any point $[x] \in \mathcal{C}(n)$ we see that $\mathbb{L}_{[x],[n-x]}$ is an axis of the CoP. By enforcing $1 < x < \frac{n}{2}$, then we observe that the dilation $\delta_1 : \mathcal{C}(n) \longrightarrow \mathcal{C}(n+1)$ with

$$\delta_1([x]) := \begin{cases} [x] & \text{for } \ 1 \leq x \leq n-1 \\ [n] & \text{additional for } x = 1 \end{cases} \tag{5.1}$$

is achievable. It follows that for each $1 < x < \frac{n}{2}$ the line $\mathbb{L}_{[x],[n-x+1]}$ is also an axis of the CoP $\mathcal{C}(n+1)$. This proves that $\Theta(n)$ is stable under the dilation δ_r. □

6. **The Density of Points on the Circle of Partition**

In this section we introduce the notion of density of points on CoP $\mathcal{C}(n, \mathbb{M})$ for $\mathbb{M} \subseteq \mathbb{N}$. We launch the following language in that regard.

Definition 6.1. Let be $\mathbb{H} \subset \mathbb{N}$. Then the quantity

$$\mathcal{D}(\mathbb{H}) = \lim_{n\to\infty} \frac{|\mathbb{H} \cap \mathbb{N}_n|}{n}$$

denotes the density of $\mathbb{H}$.

Definition 6.2. Let $\mathcal{C}(n, \mathbb{M})$ be CoP with $\mathbb{M} \subset \mathbb{N}$ and $n \in \mathbb{N}$. Suppose $\mathbb{H} \subset \mathbb{M}$ then by the density of points $[x] \in \mathcal{C}(n, \mathbb{M})$ such that $x \in \mathbb{H}$, denoted $\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty,\mathbb{M})})$, we mean the quantity

$$\mathcal{D}\left(\mathbb{H}_{\mathcal{C}(\infty,\mathbb{M})}\right) = \lim_{n\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M}) |\ \{x,y\} \cap \mathbb{H} \neq \emptyset\}}{\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})\right\}}.$$

Proposition 6.1. *Let $\mathbb{H} \subset \mathbb{M}$ with $\mathbb{M} \subseteq \mathbb{N}$ and suppose $\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty,\mathbb{M})})$ exists. Then the following properties hold:*

(i) $\mathcal{D}(\mathbb{M}_{\mathcal{C}(\infty,\mathbb{M})}) = 1$ *and* $\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty,\mathbb{M})}) \leq 1$.

(ii)
$$1 - \lim_{n\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M}\setminus\mathbb{H})\}}{\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})\right\}} = \mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty,\mathbb{M})})$$
.

(iii) *If the $|\mathbb{H}| < \infty$ then $\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty,\mathbb{M})}) = 0$.*

Proof. It is easy to see that **Property** (i) and **Property** (iii) are both easy consequences of the definition of density of points on the CoP $\mathcal{C}(n, \mathbb{M})$. We establish **Property** (ii), which is the less obvious case. We observe by the uniqueness of the axes of CoPs that we can write

$$\begin{aligned}
1 &= \lim_{n\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})\}}{\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})\right\}} \\
&= \lim_{n\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M}) |\ x \in \mathbb{H}\ , y \in \mathbb{M}\setminus\mathbb{H}\}}{\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})\right\}} \\
&+ \lim_{n\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{H})\}}{\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})\right\}} + \lim_{n\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M}\setminus\mathbb{H})\}}{\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})\right\}} \\
&= \mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty,\mathbb{M})}) + \lim_{n\longrightarrow\infty} \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M}\setminus\mathbb{H})\}}{\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})\right\}}
\end{aligned}$$

and (ii) follows immediately. □

Lemma 6.1. *Let $\mathcal{C}(n,\mathbb{P})$ be a CoP, where $\mathbb{P}$ is the set of all prime numbers. If $\mathbb{A}\subset\mathbb{P}$ then the inequality holds*

$$\mathcal{D}(\mathbb{A}_{\mathcal{C}(\infty,\mathbb{P})})\geq \lim_{n\longrightarrow\infty}\frac{\left\lfloor\frac{|\mathbb{A}\cap\mathbb{N}_n|}{2}\right\rfloor}{\left\lfloor\frac{\pi(n)-1}{2}\right\rfloor}$$

where $\pi(n)$ counts the number of primes no more than n.

Proof. The inequality is easily obtained from the upper bound of the cardinality of the axes of the CoP $\mathcal{C}(n,\mathbb{P})$

$$\#\left\{\mathbb{L}_{[x],[y]}\,\hat{\in}\,\mathcal{C}(n,\mathbb{P})\right\}\leq\left\lfloor\frac{\pi(n)-1}{2}\right\rfloor$$

and the lower bound of the cardinality of the axes of the CoP

$$\#\left\{\mathbb{L}_{[x],[y]}\,\hat{\in}\,\mathcal{C}(n,\mathbb{P})|\ \{x,y\}\cap\mathbb{A}\neq\emptyset\right\}\geq\left\lfloor\frac{|\mathbb{A}\cap\mathbb{N}_n|}{2}\right\rfloor$$

by virtue of the configuration of CoPs. □

Proposition 6.2. *Let $\mathcal{C}(n)$ with $n\in\mathbb{N}$ be a CoP and $\mathbb{H}\subset\mathbb{N}$. Then the following inequality holds*

$$\lim_{n\longrightarrow\infty}\frac{\left\lfloor\frac{|\mathbb{H}\cap\mathbb{N}_n|}{2}\right\rfloor}{\left\lfloor\frac{n-1}{2}\right\rfloor}\leq\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty)})\leq\lim_{n\longrightarrow\infty}\frac{|\mathbb{H}\cap\mathbb{N}_n|}{\left\lfloor\frac{n-1}{2}\right\rfloor}.$$

Proof. The upper is obtained from a configuration where no two points $[x],[y]\in\mathcal{C}(n)$ such that $x,y\in\mathbb{H}$ lie on the same axis of the CoP. The lower bound however follows from a configuration where any two points $[x],[y]\in\mathcal{C}(n)$ with $x,y\in\mathbb{H}$ are joined by an axis of the CoP. □

It is important to notice that the same result also hold if we replace the set of natural numbers $\mathbb{N}$ with any special subset $\mathbb{M}$. Next we transfer the notion of the density of a sequence to the density of corresponding points on the CoP $\mathcal{C}(n)$. This notion will play a crucial role in our latter developments.

Proposition 6.3. *Let $\epsilon\in(0,1]$ and $\mathbb{H}$ be a sequence with $\mathbb{H}\subset\mathbb{N}$ and $\mathcal{C}(n)$ be a CoP. Then $\mathcal{D}(\mathbb{H})\geq\epsilon$ if and only if $\mathcal{D}\left(\mathbb{H}_{\mathcal{C}(\infty)}\right)\geq\epsilon$.*

Proof. The result follows by exploiting the inequality in Proposition 6.2 □

Proposition 6.4. *Let $\mathbb{H}$ be a sequence with $\mathbb{H}\subset\mathbb{N}$. For $\epsilon\in(0,1]$ and any $k\in\mathbb{N}$ if*

$$|\mathbb{H}\cap\mathbb{N}_n|\geq n\epsilon$$

and the common difference of arithmetic progressions in $\left(\mathbb{N}\setminus\mathbb{H}\right)\cap\mathbb{N}_n$ are different from those in $\mathbb{H}\cap\mathbb{N}_n$, then there exists some rotation ϖ_r such that the CoP $\mathcal{C}(n)$ contains at least $(k-1)$ stable points $[x]$ for $x\in\mathbb{H}\cap\mathbb{N}_n$.

Proof. Suppose $\mathbb{H} \subset \mathbb{N}$ with the underlying conditions, then by Theorem 1.1 the sequence $\mathbb{H}$ contains fairly long arithmetic progressions of length k. We enumerate them as follows

$$x, x+s, x+2s, \ldots, x+(k-1)s$$

for $s \in \mathbb{N}$. It follows that the corresponding points on the CoP $\mathcal{C}(n)$, namely

$$[x], [x+s], [x+2s], \ldots, [x+(k-1)s] \in \mathcal{C}(n)$$

are equally spaced and the chord joining two of these adjacent points are of equal distance. Similarly points on the other end of the axis are equally spaced and the chords joining any of these two adjacent points are of equal distance s. Let us enumerate them as follows

$$[n-x], [n-x-s], [n-x-2s], \ldots, [n-x-(k-1)s] \in \mathcal{C}(n).$$

Apply the rotation ϖ_r by choosing $r = s$ then we have

$$\varpi_s([x]), \varpi_s([x+s]), \ldots, \varpi_s([x+(k-1)s]).$$

The image of these points under the rotation is given by

$$[x+s], [x+2s], \ldots, [x+(k-1)s], [x+ks].$$

Since the point $[x+ks]$ a priori was not on any of the axes considered at least $(k-1)$ points on these axes will be transferred to their immediate next point on an axis containing all points $[x]$ with $x \in \mathbb{H} \cap \mathbb{N}_n$. Similarly under the rotation the corresponding images of the points on the other half of the CoP lying on the same axis with these points have the images

$$\varpi_s([n-x]), \varpi_s([n-x-s]), \ldots, \varpi_s([n-x-(k-1)s])$$

which we can recast as

$$[n-x-s], [n-x-2s], \ldots, [n-x-(k-1)s], [n-x-ks].$$

At least $(k-1)$ of these points are points on the previous axis and they lying on the same axis with the points on the other half of the CoP. Since the sequence

$$n-x-s, n-x-2s, \ldots, n-x-ks$$

are in arithmetic progression, it follows by the assumption

$$n-x-s, n-x-2s, \ldots, n-x-ks \in \mathbb{H} \cap \mathbb{N}_n.$$

This completes the proof. □

In the accompanying proof we will make use of degenerate and non-degenerate points of a given set of points on a CoP. However intricate the proof might seem to be, it can be pinned down to just a simple principle. The highly dense nature of the sequence allows us to break their components into several boxes. The closest components in each of these boxes are equidistant from each other. The residue which are not dense will be thrown away into another box whose components are very sparse. We then translate a component by their gap if it ever happens to be in some dense box at the same time live on the same axis with other component. This forces the second component to also belong to some dense box. If the component on the same axis with another component does not belong to the dense box, then the components and the associated components must live in the sparse box. We can then move them into the dense box and repeat the arguments. We make these terminologies more precise in the following definitions and then present our argument.

Definition 6.3. Let $\mathcal{P} \subseteq \mathcal{C}(n, \mathbb{M})$ with $\mathbb{M} \subseteq \mathbb{N}$. Then a point $[x] \in \mathcal{P}$ is a degenerate point if the line joining the point $[x]$ to the centre (resp. deleted centre) of the CoP $\mathcal{C}(n, \mathbb{M})$ is a boundary of the largest sector induced by the points in $\mathcal{P}$. Otherwise, we say it is a non-degenerate point in $\mathcal{P}$.

Theorem 6.1. *Let $\mathbb{H} \subset \mathbb{N}$ and suppose that $\mathcal{C}(n, \mathbb{H}) \neq \emptyset$. If for any $\epsilon \in (0,1]$ holds*

$$|\mathbb{H} \cap \mathbb{N}_n| \geq n\epsilon$$

with

$$\lim_{n \longrightarrow \infty} \frac{|(\mathbb{N} \setminus \mathbb{H}) \cap \mathbb{N}_n|}{n} < \mathcal{D}(\mathbb{H})$$

then there exists a dilation $\delta_r : \mathcal{C}(n, \mathbb{H}) \longrightarrow \mathcal{C}(n+r, \mathbb{H})$ such that

$$\mathcal{C}(n+r, \mathbb{H}) \neq \emptyset.$$

Proof. Under the assumption $|\mathbb{H} \cap \mathbb{N}_n| \geq n\epsilon$ for any $\epsilon \in (0,1]$, then $\mathbb{H}$ contains fairly long arithmetic progressions. Let us enumerate them as follows

$$\mathbb{G}_1 = \{x_1 + kd_1 \in \mathbb{H}\}_{k=0}^{s_1; s_1 \geq 1}.$$

Let us consider the residual set

$$\mathbb{G}_2 = \mathbb{H} \setminus \{x_1 + kd_1 \in \mathbb{H}\}_{k=0}^{s_1; s_1 \geq 1}.$$

Then we can partition the sequence $\mathbb{H}$ in the following way

$$\mathbb{H} = \mathbb{G}_1 \cup \mathbb{G}_2.$$

If $\mathbb{G}_2$ is still dense then we can repeat this process and obtain further a partition of $\mathbb{H}$ into three subsequence

$$\mathbb{H} = \mathbb{G}_1 \cup \mathbb{G}_2 \cup \mathbb{G}_3.$$

By induction, we can partition the sequence $\mathbb{H}$ in the following way

$$\mathbb{H} = \bigcup_{i=1}^{m} \mathbb{G}_i \cup \mathbb{T}$$

where

$$\lim_{n \longrightarrow \infty} \frac{|\mathbb{T} \cap \mathbb{N}_n|}{n} = 0$$

and $\mathbb{G}_i = \{x_i + kd_i \in \mathbb{H}\}_{k=0}^{s_i; s_i \geq 1}$. Now it suffices to work with the corresponding points on the CoP $\mathcal{C}(n, \mathbb{H})$. Since by assumption $\mathcal{C}(n, \mathbb{H}) \neq \emptyset$, It follows that there exist some axes $\mathbb{L}_{[a],[b]} \hat{\in} \mathcal{C}(n, \mathbb{H})$. Now let us suppose that

$$[b] \notin \bigcup_{i=1}^{m} \{[x_i + kd_i] \in \mathcal{C}(n, \mathbb{H})\}_{k=0}^{s_i; s_i \geq 1}$$

for $b \in \mathbb{H}$, then it follows that no two adjacent chords of equal length joining points in

$$\bigcup_{i=1}^{m} \{[x_i + kd_i] \in \mathcal{C}(n, \mathbb{H})\}_{k=0}^{s_i; s_i \geq 1}$$

contains the point $[b]$. Let us suppose on the contrary that

$$[a] \in \bigcup_{i=1}^{m} \{[x_i + kd_i] \in \mathcal{C}(n, \mathbb{H})\}_{k=0}^{s_i; s_i \geq 1}$$

then it follows that $[a] \in \{[x_i + kd_i] \in \mathcal{C}(n, \mathbb{H})\}_{k=0}^{s_i; s_i \geq 1}$ for some $1 \leq i \leq m$. We consider two cases. The case $[a]$ is a degenerate point in the set and the case it is non-degenerate point in the set. If $[a]$ is a degenerate point in the set $\{[x_i + kd_i] \in \mathcal{C}(n, \mathbb{H})\}_{k=0}^{s_i; s_i \geq 1}$, in particular, $[a]$ is the first point in the set. Then it follows that the following points

$$[a], [x_i + d_i], [x_i + 2d_i], \ldots [x_i + sd_i]$$

are equally spaced with $b = n - x_i$. It follows that b is contained in the arithmetic progression

$$n - x_i, n - x_i - d_i, \ldots, n - x_i - sd_i$$

which contradicts the assumption that $[b]$ cannot lie on at least one of any two adjacent chords of equal length. Otherwise

$$n - x_i, n - x_i - d_i, \ldots, n - x_i - sd_i \in \Bigg(\mathbb{N} \setminus \mathbb{K}\Bigg) \cap \mathbb{N}_n$$

and it follows that each point in the set $\mathbb{K}^* = \bigcup_{i=1}^{m} \{[x_i + kd_i] \in \mathcal{C}(n, \mathbb{H})\}_{k=0}^{s_i; s_i \geq 1}$ uniquely generates an element in the set $\Bigg(\mathbb{N} \setminus \mathbb{K}\Bigg) \cap \mathbb{N}_n$. It follows that

$$\mathcal{D}(\mathbb{K}_{\mathcal{C}(\infty)}) = \mathcal{D}\left(\mathbb{H}_{\mathcal{C}(\infty)}\right) \leq \mathcal{D}\left((\mathbb{N} \setminus \mathbb{K})_{\mathcal{C}(\infty)}\right) = \mathcal{D}\left((\mathbb{N} \setminus \mathbb{H})_{\mathcal{C}(\infty)}\right),$$

where $\mathbb{K}$ is the corresponding weight set of $\mathbb{K}^*$. This contradicts the minimality of the density $\mathcal{D}\left(\mathbb{N} \setminus \mathbb{H}_{\mathcal{C}(\infty)}\right)$ by virtue of the scale of the density of the set $\mathbb{N} \setminus \mathbb{H}$. For the case $[a] = [x_i + sd_i]$, then we obtain the a priori arithmetic progression with $b = n - x_i - sd_i$. The corresponding point $[b]$ also violates the required specification. If the point $[a] \in \{[x_i + kd_i] \in \mathcal{C}(n, \mathbb{H})\}_{k=0}^{s_i; s_i \geq 1}$ is a a non-degenerate point, then $a = x_i + jd_i$ for some $0 < j < s$. The same analysis can be carried out to yield a contradiction. Now for the case

$$[a] \in \bigcup_{i=1}^{m} \{[x_i + kd_i] \in \mathcal{C}(n, \mathbb{H})\}_{k=0}^{s_i; s_i \geq 1}$$

then we choose the dilation δ_r with $r = d_j$ such that $[b] \in \{[x_j + kd_j] \in \mathcal{C}(n, \mathbb{H})\}_{k=0}^{s_i; s_i \geq 1}$ for $r < 0$ if $[b]$ is the last degenerate point in the set and $r > 0$ if $[b]$ is the first degenerate point or a non-degenerate point in the set, so that we have

$$\mathbb{L}_{[a],[b+d_j]} \hat{\in} \mathcal{C}(n + d_j, \mathbb{H}).$$

This completes the first part of the proof. For the second part let us assume that for the axis $\mathbb{L}_{[a],[b]}$ of $\mathcal{C}(n, \mathbb{H})$, then

$$[a] \notin \bigcup_{i=1}^{m} \{[x_i + kd_i] \in \mathcal{C}(n, \mathbb{H})\}_{k=0}^{s_i; s_i \geq 1}$$

then it must necessarily be that

$$[a] \in \mathbb{T}^*$$

where $\mathbb{T}^*$ is the corresponding point set of elements in $\mathbb{T}$. Since

$$|\mathbb{T}^*| < \left|\bigcup_{i=1}^{m}\{[x_i+kd_i]\in\mathcal{C}(n,\mathbb{H})\}_{k=0}^{s_i;s_i\geq 1}\right|,$$

there exists some rotation ϖ_t such that the point $\varpi_t([a]) \in \bigcup_{i=1}^{m}\{[x_i+kd_i]\in\mathcal{C}(n,\mathbb{H})\}_{k=0}^{s_i;s_i\geq 1}$. In particular

$$\varpi_t([a]) \in \{[x_j+kd_j]\in\mathcal{C}(n,\mathbb{H})\}_{k=0}^{s_i;s_i\geq 1}$$

for some $1 \leq j \leq m$. It follows there must exist a point

$$[v] \in \bigcup_{i=1}^{m}\{[x_i+kd_i]\in\mathcal{C}(n,\mathbb{H})\}_{k=0}^{s_i;s_i\geq 1}$$

such that $\mathbb{L}_{[v],[\varpi_t([a])]}$ is an axis of the CoP $\mathcal{C}(n,\mathbb{H})$, by virtue of the previous arguments. Otherwise, we discard this choice of point and scout for a point with such property by varying the scale of the rotation ϖ_t. The proof is completed by choosing the dilation δ_r such that $r = d_j$ for $r < 0$ if $\varpi_t([a])]$ is the last degenerate point in the set and $r > 0$ if $\varpi_t([a])]$ is the first degenerate point or a non-degenerate point in the set, so that $\mathbb{L}_{[v],[||\varpi_t([a])||+d_j]}$ is an axis of the CoP

$$\mathcal{C}(n+d_j,\mathbb{H}).$$

□

Theorem 6.2. *There are infinitely many $n \in \mathbb{M}_{a,d}$ with fixed $a, d \in \mathbb{N}$ such that the representation*

$$n = z_1 + z_2$$

where $\mu(z_1) = \mu(z_2) \neq 0$, $z_1, z_2 \in \mathbb{N}$ and μ is the Möbius function defined as

$$\mu(m) = \begin{cases} 1 & \text{if} \quad m = 1 \\ 0 & \text{if} \quad p^k|m,\ k \in \mathbb{N}\setminus\{1\} \\ (-1)^r & \text{if} \quad m = p_1p_2\cdots p_r \end{cases}$$

is valid.

Proof. The set of square-free integers

$$\mathcal{Q} := \{m \in \mathbb{N} : \mu(m) \neq 0\}$$

has natural density $\frac{6}{\pi^2}$ [1, 2]. For n large enough there exists some fixed $N_0 > n$ such that the representation is valid

$$N_o = z_1 + z_2$$

with $\mu(z_1), \mu(z_2) \neq 0$. Invoking Theorem 6.1 there exist some $t \in \mathbb{N}$ such that the representation is valid

$$N_t := N_o + t = v_1 + v_2$$

with $\mu(v_1) = \mu(v_2) \neq 0$. The result follows by an upwards induction in this manner.

□

Corollary 6.1. *There are infinitely many $n \in \mathbb{M}_{a,d}$ with fixed $a, d \in \mathbb{N}$ such that the representation*

$$n = z_1 + z_2$$

with $\gcd(z_1, z_2) = 1$ *and* $z_1, z_2 \in \mathbb{N}$ *is valid.*

Proof. The set

$$\mathcal{R} := \{(m, n) : \gcd(m, n) = 1,\ 1 \leq m < n\}$$

has natural density $\mathcal{D}(\mathcal{R}) = \frac{6}{\pi^2}$ with relatively small density for the residual set [2]. The result follows by adapting a similar reasoning in Theorem 6.2. ☐

It is worth recognizing that we can obtain an analogous formulation of Theorem 6.1 for the primes by virtue of Theorem 1.2. We state the result as follows

Theorem 6.3. *Let $\pi(n)$ denotes the number of primes no more than n. If $\mathbb{A} \subset \mathbb{P}$ the set of all prime numbers such that*

$$\limsup_{n \longrightarrow \infty} \frac{|\mathbb{A} \cap \mathbb{N}_n|}{\pi(n)} > 0$$

with

$$\lim_{n \longrightarrow \infty} \frac{|(\mathbb{P} \setminus \mathbb{A}) \cap \mathbb{N}_n|}{\pi(n)} < \lim_{n \longrightarrow \infty} \frac{|\mathbb{A} \cap \mathbb{N}_n|}{\pi(n)}$$

then there exists a dilation $\delta_r : \mathcal{C}(n, \mathbb{A}) \longrightarrow \mathcal{C}(n + r, \mathbb{A})$ such that

$$\mathcal{C}(n + r, \mathbb{A}) \neq \emptyset.$$

Proof. We keep the conditions, replace n with $\pi(n)$ in the bottom expression and $\mathbb{H}$ with $\mathbb{A}$ and repeat the same argument as espoused in Theorem 6.1 tied with the computation of density of the point with weights in $\mathbb{A}$ on the CoP $\mathcal{C}(n, \mathbb{P})$ by applying Lemma 6.1. ☐

Conjecture 6.1 (Erdős-Turán)**.** *Let $\mathbb{B} \subset \mathbb{N}$ and consider*

$$r_{\mathbb{B}}(n) := \#\left\{(a, b) \in \mathbb{B}^2 |\ a + b = n\right\}.$$

If $r_{\mathbb{B}}(n) > 0$ for all sufficiently large values of n, then

$$\limsup_{n \longrightarrow \infty}\ r_{\mathbb{B}}(n) = \infty.$$

This version, which one might think of as the **binary** version of the Erdős-Turan conjecture, can be reformulated in the language of CoPs as follows:

Conjecture 6.2 (Erdős-Turán)**.** *Let $\mathbb{B} \subset \mathbb{N}$ and consider*

$$\mathcal{G}_{\mathbb{B}}(n) = \#\left\{\mathbb{L}_{[a],[b]} \hat{\in} \mathcal{C}(n, \mathbb{B})\right\}.$$

If $\mathcal{G}_{\mathbb{B}}(n) > 0$ for all sufficiently large values of n, then

$$\limsup_{n \longrightarrow \infty} \mathcal{G}_{\mathbb{B}}(n) = \infty.$$

Remark 6.1. Though we are nowhere near the proof of this conjecture, we prove a weaker version by imposing some suitable conditions. The result is encapsulated in the following theorem.

Theorem 6.4. *Let* $\mathbb{B} \subset \mathbb{N}$ *with*

$$\lim_{n\longrightarrow\infty}\frac{|\mathbb{B}\cap\mathbb{N}_n|}{n}>0$$

such that

$$\#\left\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n)|\ x\in\mathbb{N}\setminus\mathbb{B},\ y\in\mathbb{B}\right\}\leq\#\left\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n,\mathbb{B})\right\}.$$

If $\mathcal{G}_{\mathbb{B}}(n)=\#\left\{\mathbb{L}_{[a],[b]}\hat{\in}\mathcal{C}(n,\mathbb{B})\right\}>0$ *for all sufficiently large values of* n*, then*

$$\limsup_{n\longrightarrow\infty}\mathcal{G}_{\mathbb{B}}(n)=\infty.$$

Proof. Suppose $\mathbb{B}\subset\mathbb{N}$ and let $\mathcal{G}_{\mathbb{B}}(n)>0$ for all sufficiently large values of n. Suppose to the contrary that

$$\limsup_{n\longrightarrow\infty}\mathcal{G}_{\mathbb{B}}(n)<\infty.$$

Consider the CoP $\mathcal{C}(n,\mathbb{B})$, then we note that by the uniqueness of axes of CoPs we can compute the density of points $[x]\in\mathcal{C}(n)$ with $||[x]||\in\mathbb{B}$ in the following way

$$\begin{aligned}\mathcal{D}(\mathbb{B}_{\mathcal{C}(\infty)})&=\lim_{n\longrightarrow\infty}\frac{\#\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n)|\{x,y\}\cap\mathbb{B}\neq\emptyset\}}{\lfloor\frac{n-1}{2}\rfloor}\\&=\lim_{n\longrightarrow\infty}\frac{\#\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n)|\ x\in\mathbb{N}\setminus\mathbb{B},\ y\in\mathbb{B}\}}{\lfloor\frac{n-1}{2}\rfloor}+\lim_{n\longrightarrow\infty}\frac{\#\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n,\mathbb{B})\}}{\lfloor\frac{n-1}{2}\rfloor}\\&\leq 2\lim_{n\longrightarrow\infty}\frac{\#\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n,\mathbb{B})\}}{\lfloor\frac{n-1}{2}\rfloor}\\&=0\end{aligned}$$

by virtue of the earlier assumption. By applying Proposition 6.2, it follows that

$$\lim_{n\longrightarrow\infty}\frac{\left\lfloor\frac{|\mathbb{B}\cap\mathbb{N}_n|}{2}\right\rfloor}{\lfloor\frac{n-1}{2}\rfloor}=0.$$

It follows that $\mathcal{D}(\mathbb{B})=0$, thereby contradicting the requirement of the statement. ◻

7. Special Maps of Circles of Partition

In this section we introduce and study the notion of several special maps of circles of partition. We launch more formally the following languages.

Definition 7.1. Let $\mathbb{M}\subseteq\mathbb{N}$ and $\mathcal{C}(n,\mathbb{M})\neq\emptyset$ be a CoP containing the axis $\mathbb{L}_{[a],[b]}$. By the **flipping** of the CoP $\mathcal{C}(n,\mathbb{M})$ along the so called flipping axis $\mathbb{L}_{[a],[b]}$, we mean the map

$$\vartheta_{[a],[b]}:\mathcal{C}(n,\mathbb{M})\longrightarrow\mathcal{C}(m,\mathbb{M})$$

with $\vartheta_{[a],[b]}([a])=[a]$ and $\vartheta_{[a],[b]}([b])=[b]$ such that for any two $[x],[y]\in\mathcal{C}(n,\mathbb{M})$ with $[x],[y]\neq[a],[b]$ holds

$$\|\vartheta_{[a],[b]}([x])\|+\|\vartheta_{[a],[b]}([y])\|\neq n$$

We say the CoP $\mathcal{C}(n,\mathbb{M})$ is susceptible to flipping if there exists such a map.

Example 7.1. Let be $\mathbb{M} = \mathbb{P}$ and $n = 20$. The CoP $\mathcal{C}(20, \mathbb{P})$ is the set $\{[3], [7], [13], [17]$ with two axes $\mathbb{L}_{[3],[17]}$ and $\mathbb{L}_{[7],[13]}$. Then the map

$$\vartheta_{[3],[17]} : \mathcal{C}(20, \mathbb{P}) \longrightarrow \mathcal{C}(22, \mathbb{P})$$

with $\mathcal{C}(22, \mathbb{P}) = \{[3], [5], [11], [17], [19]\}$ is a flipping of $\mathcal{C}(20, \mathbb{P})$ along the axis $\mathbb{L}_{[3],[17]}$ if f.i.

$$\begin{aligned}\vartheta_{[3],[17]}([3]) &= [3]\\ \vartheta_{[3],[17]}([7]) &= [5]\\ \vartheta_{[3],[17]}([13]) &= [11] \text{ and } [19]\\ \vartheta_{[3],[17]}([17]) &= [17].\end{aligned}$$

Hence we get $\|[5]\| + \|[11]\| = 16 \neq 20$ or $\|[5]\| + \|[19]\| = 24 \neq 20$.

Vice versa there are no axis points of $\mathcal{C}(22, \mathbb{P})$ that are also points of $\mathcal{C}(20, \mathbb{P})$. Hence there exists no flipping from $\mathcal{C}(22, \mathbb{P})$ to $\mathcal{C}(20, \mathbb{P})$ along an axis of $\mathcal{C}(22, \mathbb{P})$.

Proposition 7.1. *Let $\mathbb{M}_{a,d}$ be defined as in (2.5) with $0 < a \leq d$. Then the CoP $\mathcal{C}(n, \mathbb{M}_{a,d})$ is susceptible to flipping if and only if $n > m$.*

Proof. We must regard that in order to get $\mathcal{C}(n, \mathbb{M}_{a,d}) \neq \emptyset$ it must be $n \in \mathbb{M}_{2a,d}$. Then is $n - a \in \mathbb{M}_{a,d}$. The same is valid for $\mathcal{C}(m, \mathbb{M}_{a,d})$.
We assume that $n > m$. Then holds with Corollary 2.3

$$\mathcal{C}(n, \mathbb{M}_{a,d}) \supset \mathcal{C}(m, \mathbb{M}_{a,d}).$$

Due to $n \in \mathbb{M}_{2a,d}$ holds $\frac{n-2a}{d} \in \mathbb{N}$. The weights of $\mathcal{C}(n, \mathbb{M}_{a,d})$ are

$$\|\mathcal{C}(n, \mathbb{M}_{a,d})\| = \left\{ a + k \cdot d \mid k = 0, 1, 2, \ldots, \frac{n-2a}{d} \right\}.$$

Hence $\mathcal{C}(n, \mathbb{M}_{a,d})$ has

$$l_n = \frac{n-2a}{d} + 1 \text{ members.}$$

This is in accordance with the general counting function for CoPs:

$$\begin{aligned}|\mathcal{C}(n, \mathbb{M}_{a,d})| &= 1 + \sum_{\substack{1 \leq x \leq n-a \\ x \equiv a \pmod d}} 1\\ &= 1 + \frac{n-2a}{d}.\end{aligned}$$

The addition of 1 is required because the counting starts with 0. Now we must distinguish two cases

rC: The CoP $\mathcal{C}(n, \mathbb{M}_{a,d})$ has a real center.
dC: The CoP $\mathcal{C}(n, \mathbb{M}_{a,d})$ has a deleted center.

In the case rC holds l_n is odd and l_n is even in the other case. Now we choose the axis $\mathbb{L}_{[u],[v]}$ of the CoP $\mathcal{C}(n, \mathbb{M}_{a,d})$ as the flipping axis which is the closest one to the center of the CoP. The weights of $[u], [v]$ are $u = v = \frac{n}{2}$ for the case rC and $u = \frac{n-d}{2}$, $v = \frac{n+d}{2}$ in the other case. In order to satisfy the requirements

$$\vartheta_{[u],[v]}([u]) = [u] \text{ and } \vartheta_{[u],[v]}([v]) = [v]$$

the last point of $\mathcal{C}(m, \mathbb{M}_{a,d})$ should be $[v]$. Due to Corollary 2.1 we get for m as the sum of the weights of the first and the last member of CoP $\mathcal{C}(m, \mathbb{M}_{a,d})$

$$m = \begin{cases} a + \dfrac{n}{2} & \text{for rC} \\ a + \dfrac{n+d}{2} & \text{for dC.} \end{cases} \tag{7.1}$$

Analogously to $\mathcal{C}(n, \mathbb{M}_{a,d})$ holds for the number of members of $\mathcal{C}(m, \mathbb{M}_{a,d})$

$$\begin{aligned} l_m - 1 &:= \sum_{\substack{1 \leq x \leq m-a \\ x \equiv a \,(\mathrm{mod}\, d)}} 1 = \frac{m-2a}{d} \\ &= \begin{cases} \dfrac{a + \frac{n}{2} - 2a}{d} = \dfrac{n-2a}{2d} = \dfrac{l_n - 1}{2} & \text{for rC} \\ \dfrac{a + \frac{n+d}{2} - 2a}{d} = \dfrac{n-2a}{2d} + \dfrac{1}{2} = \dfrac{l_n - 1}{2} + \dfrac{1}{2} & \text{for dC.} \end{cases} \end{aligned}$$

Hence we obtain for both cases

$$l_m = \left\lceil \frac{l_n - 1}{2} \right\rceil + 1 = \left\lfloor \frac{l_n}{2} \right\rfloor + 1.$$

All these fulfills the following map

$$\begin{aligned} \vartheta_{[u],[v]}(x) &= a + k(x) \cdot d \text{ with} \\ \frac{x-a}{d} &\equiv k(x) \,(\mathrm{mod}\, l_m)\,. \end{aligned}$$

The heaviest point of CoP $\mathcal{C}(m, \mathbb{M}_{a,d})$ is $[m-a]$. In the case rC the flipping axis is $\mathbb{L}_{[u],[v]}$ with $u = v = \frac{n}{2}$ and we get with (7.1)

$$\left\| \vartheta_{[u],[v]} \left(\left[\frac{n}{2} \right] \right) \right\| = m - a = \frac{n}{2}.$$

Hence the requirements $\|\vartheta_{[u],[v]}([u])\| = u = \frac{n}{2}$ and $\|\vartheta_{[u],[v]}([v])\| = v = \frac{n}{2}$ are fulfilled. In the case dC we get with (7.1)

$$\left\| \vartheta_{[u],[v]} \left(\left[\frac{n+d}{2} \right] \right) \right\| = m - a = \frac{n+d}{2} = v.$$

And therefore is $u = v - d = \frac{n-d}{2}$ and for each two points $[x], [y] \in \mathcal{C}(n, \mathbb{M}_{a,d})$ with $[x], [y] \neq [u], [v]$ holds

$$\|\vartheta_{[u],[v]}([x])\| + \|\vartheta_{[u],[v]}([y])\| < n$$

because $\vartheta_{[u],[v]}([u]) = [u]$ and $\vartheta_{[u],[v]}([v]) = [v]$ are the two heaviest points of $\mathcal{C}(m, \mathbb{M}_{a,d})$ in case dC respectively is the heaviest point of $\mathcal{C}(m, \mathbb{M}_{a,d})$ in rC with the sum of weights of the two heaviest points $\leq n$. The weight sum of all others is lesser. Thereby the first part of the claim is proven.

If on the other hand holds $n \leq m$ then the source CoP is a subset of the target CoP. All axis points of $\mathcal{C}(n, \mathbb{M}_{a,d})$ are identically mapped into $\mathcal{C}(m, \mathbb{M}_{a,d})$. And for all these $\vartheta_{[u],[v]}([x])$ and $\vartheta_{[u],[v]}([y])$ from any axis $\mathbb{L}_{[x],[y]}$ of $\mathcal{C}(n, \mathbb{M}_{a,d}$ holds

$$\|\vartheta_{[u],[v]}([x])\| + \|\vartheta_{[u],[v]}([y])\| = n.$$

This is a contradiction to the requirements of the claim. □

Remark 7.1. Note that due to $\mathbb{M}_{1,1} = \mathbb{N}$ this statement also holds for each CoP $\mathcal{C}(n)$.

Proposition 7.2. *The chosen axis closest to the center of the CoP $\mathcal{C}(n, \mathbb{M}_{a,d})$ is the only one for flipping along an axis in the case of $\mathbb{M} = \mathbb{M}_{a,d}$.*

Proof. For all axes $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{M}_{a,d})$ holds [6]

$$x \leq \frac{n}{2} \leq y.$$

Therefore there is no axis $\mathbb{L}_{[x],[y]}$ with $y < \frac{n}{2}$. For the chosen axis $\mathbb{L}_{[u],[v]}$ closest to the center of $\mathcal{C}(n, \mathbb{M}_{a,d})$ holds

$$\frac{n-d}{2} \leq \|[u]\| \leq \|[v]\| \leq \frac{n+d}{2}.$$

The only opposite of this are axes $\mathbb{L}_{[w],[z]}$ with $\|[w]\| < \frac{n-d}{2}$ and its axis partner with $\|[z]\| > \frac{n+d}{2}$. Then between $[w]$ and $[z]$ there is at least one axis $\mathbb{L}_{[x],[y]}$ with $w < x \leq y < z$ and $x + y = n$. This is a contradiction to the requirements of flipping along the axes $\mathbb{L}_{[w],[z]}$. Hence only the axis $\mathbb{L}_{[u],[v]}$ with

rC: $\|[u]\| = \|[v]\| = \dfrac{n}{2}$

dC: $\|[u]\| = \dfrac{n-d}{2}, \|[v]\| = \dfrac{n+d}{2}$

satisfies the requirements of a flipping axis. □

It is quite suggestive from this proposition the notion of flipping of CoPs under $\mathbb{M} = \mathbb{M}_{a,d}$ can be thought of as the process of slicing a circle into two equal half and overturning one half to lie perfectly on top of the other half, thereby forming a geometric structure akin to the semi-circle.

Example 7.2. We choose $a = 2, d = 4$ and hence $\mathbb{M} = \mathbb{M}_{2,4}$. Then with $n = 20$ is

$$\begin{aligned} \|\mathcal{C}(20, \mathbb{M}_{2,4})\| &= \{2, 6, 10, 14, 18\}, \\ l_n &= \frac{20 - 2 \cdot 2}{4} + 1 = 5, \\ l_m &= \left\lfloor \frac{5}{2} \right\rfloor + 1 = 3 \text{ and} \\ m &= 2 + \frac{20}{2} = 12 \end{aligned}$$

with the flipping axis $\mathbb{L}_{[10],[10]}$. Hence is

$$\begin{aligned} \|\vartheta_{[10],[10]}\left(\mathcal{C}(20, \mathbb{M}_{2,4})\right)\| &= \|\mathcal{C}(12, \mathbb{M}_{2,4})\| \\ &= \{2, 6, 10\}. \end{aligned}$$

All weight sums of any two members of $\{[2], [6], [10]\} \setminus \{10\}$ are less than 20. If we would take $\mathbb{L}_{[14],[6]}$ as flipping axis we would obtain as target set

$$\mathcal{C}(16, \mathbb{M}_{2,4}) = \{[2], [6], [10], [14]\}.$$

And here would be possible out of $\{[6], [14]\}$ one weight sum contradicting to the requirements:

$$10 + 10 = 20.$$

---

[6] W.l.o.g. we assume $x \leq y$ for all axes $\mathbb{L}_{[x],[y]}$.

Now we introduce and study the concept of filtration of the CoPs. At first we deal with the filtration along an axis.

Definition 7.2. Let $\mathbb{M} \subseteq \mathbb{N}$ with the corresponding CoP $\mathcal{C}(n, \mathbb{M})$ containing the axis $\mathbb{L}_{[x],[y]}$. By the **filtration** of the CoP $\mathcal{C}(n, \mathbb{M})$ along the filtration axis $\mathbb{L}_{[x],[y]}$ we mean the map

$$\Phi_{[x],[y]} : \mathcal{C}(n, \mathbb{M}) \longrightarrow \mathcal{C}(m, \mathbb{M})$$

such that $[x], [y] \notin \mathcal{C}(m, \mathbb{M})$ for some $m \in \mathbb{N} \setminus \{1\}$ and there exists the so called co-axis $\mathbb{L}_{[u],[v]}$ of $\mathcal{C}(n, \mathbb{M})$ so that $\mathbb{L}_{[u],[a]}$ and $\mathbb{L}_{[v],[b]}$ are axes of $\mathcal{C}(m, \mathbb{M})$ for some $[a], [b] \in \mathbb{M}$. We say the CoP $\mathcal{C}(n, \mathbb{M})$ is susceptible to filtration if there exists such a map.

Also here an example will demonstrate this special map.

Example 7.3. Let be again $\mathbb{M} = \mathbb{P}$ and $n = 20$. Then the map

$$\Phi_{[7],[13]} : \mathcal{C}(20, \mathbb{P}) \longrightarrow \mathcal{C}(22, \mathbb{P})$$

is a filtration of $\mathcal{C}(20, \mathbb{P})$ along the filtration axis $\mathbb{L}_{[7],[13]}$ due to the target CoP

$$\mathcal{C}(22, \mathbb{P}) = \{[3], [5], [11], [17], [19]\}$$

contains the axes $\mathbb{L}_{[3],[19]}$ and $\mathbb{L}_{[17],[5]}$ where $\mathbb{L}_{[3],[17]}$ is the co-axis of $\mathcal{C}(20, \mathbb{P})$.

Proposition 7.3. *The CoP $\mathcal{C}(n, \mathbb{M})$ admits aligned embedding is* ***not*** *susceptible to filtration along an axis.*

Proof. The claim is true if one of the following statements holds

(A) The CoP $\mathcal{C}(n, \mathbb{M})$ has no filtration axis.
(B) The CoP $\mathcal{C}(n, \mathbb{M})$ has no co-axis

We suppose at first $n \leq m$. Then holds with Theorem 2.2

$$\mathcal{C}(n, \mathbb{M}) \subseteq \mathcal{C}(m, \mathbb{M}).$$

Then the images of all axis points of the source CoP are points of the target CoP. Hence there is no filtration axis (A).
Now we look for $m < n$. In this case holds with Corollary 2.3

$$\mathcal{C}(n, \mathbb{M}) \supset \mathcal{C}(m, \mathbb{M}).$$

At first let be $m < \frac{n}{2}$. In this case the images of the end points of all axes of $\mathcal{C}(n, \mathbb{M})$ do not exist in $\mathcal{C}(m, \mathbb{M})$. Hence there is no co-axis (B).
At last we look for $\frac{n}{2} \leq m < n$. In this case the images of the begin points of all axes of $\mathcal{C}(n, \mathbb{M})$ are points of $\mathcal{C}(m, \mathbb{M})$. Hence there is no filtration axis (A). ☐

Definition 7.3. Let $\mathbb{M} \subseteq \mathbb{N}$ with the corresponding CoP $\mathcal{C}(n, \mathbb{M})$ containing the axis $\mathbb{L}_{[x],[y]}$. By the **reduction** of the CoP $\mathcal{C}(n, \mathbb{M})$ in the base set $\mathbb{M}$ we mean the map

$$\phi_{[x],[y]} : \mathcal{C}(n, \mathbb{M}) \longrightarrow \mathcal{C}(n, \mathbb{M}^*)$$

with $\mathbb{M}^* := \mathbb{M} \setminus \{x, y\}$. We say the CoP $\mathcal{C}(n, \mathbb{M})$ is susceptible to reduction if there exists such a map.

Proposition 7.4. *Let $\mathbb{M}_{a,d}$ be defined as in (2.5). Then the CoP $\mathcal{C}(n, \mathbb{M}_{a,d})$ is susceptible to reduction.*

Proof. W.l.o.g. we suppose $x < y$ and take

$$\phi_{[x],[y]}([u]) = \begin{cases} [u] & \text{if } u \neq x \text{ and } u \neq y \\ [u+d] & \text{if } u = x \\ [u-d] & \text{if } u = y \end{cases}$$

for all points $[u] \in \mathcal{C}(n, \mathbb{M}_{a,d})$. Due to all members of $\mathbb{M}_{a,d}$ have the same distance d it holds that if $u \in \mathbb{M}_{a,d}$ then is also $u \pm d \in \mathbb{M}_{a,d}$ and

$$\|\phi_{[x],[y]}([x])\| + \|\phi_{[x],[y]}([y])\| = x + d + y - d = n$$

because $\mathbb{L}_{[x],[y]}$ is an axis of $\mathcal{C}(n, \mathbb{M}_{a,d})$. □

Due to $\mathbb{M}_{1,1} = \mathbb{N}$ this proposition holds for $\mathcal{C}(n)$ too.

8. **Open and Connected Circles of Partition**

In this section we introduce the notion of open CoP. We first launch the notion of a path connecting CoP and examine in-depth the concept of connected CoPs and their interplay with other notions launched thus far.

Definition 8.1. Let $\mathbb{M} \subseteq \mathbb{N}$ with $\mathcal{C}(n, \mathbb{M}) \neq \emptyset$ and $\mathcal{C}(s, \mathbb{M}) \neq \emptyset$ be any two distinct CoPs. Then by the path joining the CoP $\mathcal{C}(n, \mathbb{M})$ to the CoP $\mathcal{C}(s, \mathbb{M})$ we mean the line joining $[x] \in \mathcal{C}(n, \mathbb{M})$ to $[y] \in \mathcal{C}(s, \mathbb{M})$, denoted as $\mathcal{L}_{[x],[y]}$, such that $\mathcal{L}_{[x],[y]}$ is an axis of the CoP $\mathcal{C}(s, \mathbb{M})$

$$\mathcal{L}_{[x],[y]} = \mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(s, \mathbb{M}).$$

We say the CoP $\mathcal{C}(n, \mathbb{M})$ is connected to the CoP $\mathcal{C}(s, \mathbb{M})$ if there exists such a path.
We say the CoP $\mathcal{C}(n, \mathbb{M})$ is **strongly connected** to some CoP $\mathcal{C}(m, \mathbb{M})$ if the connection exists for all possible dilations

$$\delta_r : \mathcal{C}(n, \mathbb{M}) \longrightarrow \mathcal{C}(s, \mathbb{M}) \text{ by } s = n + r.$$

with $\delta_r([x]) = [y]$. We say the CoP $\mathcal{C}(n, \mathbb{M})$ is fully connected to the CoP $\mathcal{C}(s, \mathbb{M})$ if there exists such a path for each $[x] \in \mathcal{C}(n, \mathbb{M})$.

Proposition 8.1. *Let $\mathbb{M} \subseteq \mathbb{N}$ with $\mathcal{C}(n, \mathbb{M}) \neq \emptyset$ and $\mathcal{C}(s, \mathbb{M}) \neq \emptyset$ be any two distinct CoPs with a common point $[x]$. Then and only then $\mathcal{C}(n, \mathbb{M})$ is connected to $\mathcal{C}(s, \mathbb{M})$.*

Proof. Since $[x] \in \mathcal{C}(s, \mathbb{M})$ there must be an axis $\mathbb{L}_{[x],[s-x]} \hat{\in} \mathcal{C}(s, \mathbb{M})$. Since $[x] \in \mathcal{C}(n, \mathbb{M})$ there exists the path $\mathcal{L}_{[x],[s-x]}$. Hence $\mathcal{C}(n, \mathbb{M})$ is connected to $\mathcal{C}(s, \mathbb{M})$.
If otherwise there exists such a path $\mathcal{L}_{[x],[y]}$ with a fixed $[x] \in \mathcal{C}(n, \mathbb{M})$ and any $[y] \in \mathcal{C}(s, \mathbb{M})$ such that $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(s, \mathbb{M})$ then it must certainly be that $[y] = [s-x]$ and $[x]$ is also a point of $\mathcal{C}(n, \mathbb{M})$. □

Theorem 8.1. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}(n, \mathbb{M})$ be any CoP admits aligned embedding. Then $\mathcal{C}(n, \mathbb{M})$ is strongly connected to some CoP $\mathcal{C}(m, \mathbb{M})$ admits aligned embedding.*

Proof. We assume that $\mathcal{C}(n, \mathbb{M})$ is not strongly connected to any $\mathcal{C}(m, \mathbb{M})$, by virtue of the definition. Invoking the virtue the CoPs admit aligned embedding, we can assume $\mathcal{C}(n, \mathbb{M}) \subset \mathcal{C}(s, \mathbb{M})$. The line $\mathbb{L}_{[x],[n-x]}$ is an axis of $\mathcal{C}(n, \mathbb{M})$ for any $[x] \in \mathcal{C}(n, \mathbb{M})$. It follows that $\mathbb{L}_{[x],[s-x]}$ is also an axis of the CoP $\mathcal{C}(s, \mathbb{M})$. Since no two CoPs are strongly connected and because of Theorem 4.2 there exists some dilation $\delta_{r_1} : \mathcal{C}(n, \mathbb{M}) \longrightarrow \mathcal{C}(s, \mathbb{M})$ such that $[s-x] \neq \delta_{r_1}([x])$ for each $[x] \in \mathcal{C}(n, \mathbb{M})$. Let us produce a line $\mathcal{L}_{[x],[\delta_{r_1}([x])]}$ by joining $[x]$ to $\delta_{r_1}([x])$. Now, we can certainly partition these lines as axes of large and small CoPs relative to the CoP $\mathcal{C}(s)$ as below

$$\{\mathbb{L}_{[x],\delta_{r_1}([x])} \hat{\in} \mathcal{C}(v,\mathbb{M})|\ n < v \leq s-1\} \bigcup \{\mathbb{L}_{[x],\delta_{r_1}([x])} \hat{\in} \mathcal{C}(k,\mathbb{M})|\ k > s\}.$$

Let us now pick arbitrarily a small CoP relative to the CoP $\mathcal{C}(s,\mathbb{M})$ and large relative to the CoP $\mathcal{C}(n,\mathbb{M})$. That is we pick a CoP $\mathcal{C}(v,\mathbb{M})$ from the first set arbitrarily. Then we obtain the strict embedding

$$\mathcal{C}(n,\mathbb{M}) \subset \mathcal{C}(v,\mathbb{M}) \subset \mathcal{C}(s,\mathbb{M}).$$

Otherwise the CoP $\mathcal{C}(n,\mathbb{M})$ will have the axis $\mathbb{L}_{[x],[\delta_{r_1}([x])]}$, which will contradict our assumption. Under the assumption that no two CoPs are strongly connected, it follows that there exist some dilation

$$\delta_{r_2} : \mathcal{C}(n,\mathbb{M}) \longrightarrow \mathcal{C}(v,\mathbb{M})$$

such that for each $[x] \in \mathcal{C}(n,\mathbb{M})$ then $\delta_{r_2}([x]) \neq [v-x]$. By repeating the argument in this manner under the assumption that no two CoPs are connected we obtain the following infinite embedding into the CoP $\mathcal{C}(n,\mathbb{M})$ as follows

$$\mathcal{C}(n,\mathbb{M}) \subset \cdots \subset \mathcal{C}(t,\mathbb{M}) \subset \mathcal{C}(v,\mathbb{M}) \subset \mathcal{C}(s,\mathbb{M})$$

and we have the following infinite descending sequence of generators toward the generator n

$$n < \cdots < t < v < s.$$

This is absurd, thereby ending the proof of the claim. □

Corollary 8.1. *Let $\mathcal{C}(n,\mathbb{M})$ and $\mathcal{C}(n,\mathbb{M})$ be two CoPs admit aligned embedding. If holds $n < m$ then $\mathcal{C}(n,\mathbb{M})$ is fully connected to the CoP $\mathcal{C}(m,\mathbb{M})$.*

Proof. Due to Theorem 2.2 holds

$$\mathcal{C}(n,\mathbb{M}) \subset \mathcal{C}(m,\mathbb{M}).$$

Hence each point of $\mathcal{C}(n,\mathbb{M})$ is also a point of $\mathcal{C}(m,\mathbb{M})$. Because of Proposition 8.1, it follows that $\mathcal{C}(n,\mathbb{M})$ is connected to $\mathcal{C}(m,\mathbb{M})$ for each point $[x] \in \mathcal{C}(n,\mathbb{M})$. Hence $\mathcal{C}(n,\mathbb{M})$ is fully connected to $\mathcal{C}(m,\mathbb{M})$ □

Definition 8.2. Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}(n,\mathbb{M})$ be a CoP with $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})$. Then by the open CoP induced by the point $[x],[y]$, we mean the exclusion $\mathcal{C}(n,\mathbb{M}) \setminus [x],[y]$. We call the points $[x],[y]$ the gates to the interior of the open CoP. We denote the induced open CoP by $\widehat{\mathcal{C}(n,\mathbb{M})}_{[x],[y]} \subset \mathcal{C}(n,\mathbb{M})$. We say the CoPs $\mathcal{C}(s,\mathbb{M})$ and $\mathcal{C}(n,\mathbb{M})$ forms a two-member community if and only if there is a path joining the gate $[x],[y]$ of $\widehat{\mathcal{C}(n,\mathbb{M})}_{[x],[y]}$ to the CoP $\mathcal{C}(s,\mathbb{M})$.

9. Children, Offspring and Family Induced by Circles of Partition

In this section we introduce the notion of children, the offspring and the family induced by a typical CoP. We relate this notion to the notion of connected CoPs.

Definition 9.1. Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}(n,\mathbb{M}) \neq \emptyset$ and let $\{\mathbb{L}_{[u_i],[v_i]}\}_{i=1}^{N;N\geq 2}$ for some $N \geq 2$ be the set of all the axes. Then we say the CoP $\mathcal{C}(s,\mathbb{M})$ is a **child** of the CoP $\mathcal{C}(n,\mathbb{M})$ if there exist some axes $\mathbb{L}_{[u_k],[v_k]}, \mathbb{L}_{[u_j],[v_j]} \in \{\mathbb{L}_{[u_i],[v_i]}\}_{i=1}^{N;N\geq 2}$ such that at least one of $\mathbb{L}_{[u_k],[u_j]}, \mathbb{L}_{[u_k],[v_j]}, \mathbb{L}_{[v_k],[u_j]}, \mathbb{L}_{[v_k],[v_j]}$ is an axis of the child CoP $\mathcal{C}(s,\mathbb{M})$. This axis forms the **principal axis** of the child CoP. We call the collection of all

CoPs generated in this manner the **offspring** of the **parent** CoP $\mathcal{C}(n,\mathbb{M})$. The parent CoP $\mathcal{C}(n,\mathbb{M})$ together with its offspring forms a **complete family** of CoPs. The size of the family of CoPs is the number of CoPs in the family. A subset of a family is said to be an **incomplete family** of CoPs.

Example 9.1. Let us consider the CoP with $\|\mathcal{C}(20,\mathbb{P})\| = \{3,7,13,17\}$ with axes $\mathbb{L}_{[3],[17]}$ and $\mathbb{L}_{[7],[13]}$. We consider the following axes $\mathbb{L}_{[3],[7]}, \mathbb{L}_{[3],[13]}, \mathbb{L}_{[7],[17]}, \mathbb{L}_{[13],[17]}$. These axes correspond to the following CoPs

$$\mathcal{C}(10,\mathbb{P}),\ \mathcal{C}(16,\mathbb{P}),\ \mathcal{C}(24,\mathbb{P}),\ \mathcal{C}(30,\mathbb{P}).$$

Hence we obtain a complete family of CoPs of size 5.

Proposition 9.1. *Let $\mathcal{C}(n,\mathbb{M})$ a non-empty CoP. Then each axis point $[x]$ together with a point $[u]$ of another axis of $\mathcal{C}(n,\mathbb{M})$ generates a child $\mathcal{C}(s,\mathbb{M})$ of the parent $\mathcal{C}(n,\mathbb{M})$ with $s = \|[x]\| + \|[u]\|$.*

Proof. Let $\mathbb{L}_{[x],[y]}$ and $\mathbb{L}_{[u],[v]}$ be two axes of $\mathcal{C}(n,\mathbb{M})$. Appealing to Proposition 2.1, we have

$$\|[x]\| + \|[u]\| = s \neq n.$$

Hence $[x]$ and $[u]$ form the axis $\mathbb{L}_{[x],[v]} \hat{\in} \mathcal{C}(s,\mathbb{M})$ and $\mathcal{C}(s,\mathbb{M})$ is a child of $\mathcal{C}(n,\mathbb{M})$. □

Proposition 9.2. *Let $n \in \mathbb{N}$, $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}(n,\mathbb{M})$ be a CoP admitting aligned embedding. If holds $|\mathcal{C}(n,\mathbb{M})| \geq 4$ then the CoP $\mathcal{C}(n,\mathbb{M})$ admits an infinite chain of its descendants.*

Proof. Due to $|\mathcal{C}(n,\mathbb{M})| \geq 4$ there is an axis point $[u] \in \mathcal{C}(n,\mathbb{M})$ with

$$u > \min\left(\|[w]\| \mid [w] \in \mathcal{C}(n,\mathbb{M})\right)$$

and a point $[v] \in \mathcal{C}(n,\mathbb{M})$ of another axis with $u + v = m > n$. It follows that there exists an axis $\mathbb{L}_{[u],[v]} \hat{\in} \mathcal{C}(m,\mathbb{M})$. Ergo holds $[u] \in \mathcal{C}(m,\mathbb{M})$. Appealing to Proposition 9.1 the CoP $\mathcal{C}(m,\mathbb{M})$ is a child of the CoP $\mathcal{C}(n,\mathbb{M})$. Since $m > n$ and $\mathcal{C}(n,\mathbb{M})$ admits aligned embedding, it holds

$$\mathcal{C}(n,\mathbb{M}) \subset \mathcal{C}(m,\mathbb{M}).$$

Now we choose a point $[w]$ of $\mathcal{C}(m,\mathbb{M})$ and the latter changes its role to be a parent. With the same procedure as above we produce an axis $\mathbb{L}_{[u],[w]} \hat{\in} \mathcal{C}(r,\mathbb{M})$ with $[u],[w] \in \mathcal{C}(r,\mathbb{M})$ and

$$\mathcal{C}(n,\mathbb{M}) \subset \mathcal{C}(m,\mathbb{M}) \subset \mathcal{C}(r,\mathbb{M}).$$

This procedure can be repeated infinitely many often. We obtain an infinite chain of descendants of the CoP $\mathcal{C}(n,\mathbb{M})$ as its prime father. □

Proposition 9.3. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}(n,\mathbb{M})$ be a parent of a complete family. Then $\mathcal{C}(n,\mathbb{M})$ partitions the offspring into two incomplete families of equal sizes.*

Proof. In virtue of Proposition 9.1 two points of distinct axes of the CoP $\mathcal{C}(n,\mathbb{M})$ generates a child of it. Let

$$\mathbb{L}_{[u],[v]}, \mathbb{L}_{[x],[y]} \mid u < x$$

two arbitrary axes of $\mathcal{C}(n,\mathbb{M})$. Because $[u],[v]$ and $[x],[y]$ are axis points holds

$$\begin{aligned}&n = u+v = x+y \text{ and therefore}\\&v = x-u+y \text{ and because of } x>u\\&v>y\end{aligned}$$

Hence we get

$$\begin{aligned}&u<x<y<v \text{ and therefore}\\&s_1 := u+x < s_2 := u+y < n = x+y \text{ and}\\&t_1 := v+y > t_2 := v+x > n = v+u\end{aligned}$$

and a chain of children

$$\begin{aligned}\mathcal{C}(s_1,\mathbb{M}),\mathcal{C}(s_2,\mathbb{M}),\mathcal{C}(n,\mathbb{M}),\mathcal{C}(t_1,\mathbb{M}),\mathcal{C}(t_2,\mathbb{M}) \text{ with}\\ s_1<s_2<n<t_1<t_2.\end{aligned}$$

Therefore holds that for all two axes 4 children are generated, two on the left side of $\mathcal{C}(n,\mathbb{M})$ and two on the right side in a chain of children. Because $\mathcal{C}(n,\mathbb{M})$ for all two axes is located in the middle of the chain, the parent CoP $\mathcal{C}(n,\mathbb{M})$ partitions its offspring in two halves, the incomplete families of equal sizes. □

Proposition 9.4. *If the parent CoP admits embedding then their children admit aligned embedding.*

Proof. We look at the last proof and choose $[u]$ as the first point of the parent CoP $\mathcal{C}(n,\mathbb{M})$

$$u := \min\left(w \in \|\mathcal{C}(n,\mathbb{M})\|\right).$$

Then holds

$$\begin{aligned}&[u]\in\mathcal{C}(s_1,\mathbb{M}) \text{ and } [u]\in\mathcal{C}(s_2,\mathbb{M}) \text{ and}\\&\max\left(w\in\|\mathcal{C}(s_1,\mathbb{M})\|\right) = x < y = \max\left(w\in\|\mathcal{C}(s_2,\mathbb{M})\|\right)\\&\text{and hence}\\&\mathcal{C}(s_1,\mathbb{M})\subset\mathcal{C}(s_2,\mathbb{M}) \text{ under } s_1<s_2.\end{aligned}$$

Because $\mathcal{C}(n,\mathbb{M})$ admits embedding holds

$$\begin{aligned}\mathcal{C}(s_1,\mathbb{M})\subset\mathcal{C}(s_2,\mathbb{M})\subset\mathcal{C}(n,\mathbb{M})\subset\mathcal{C}(t_1,\mathbb{M})\subset\mathcal{C}(t_2,\mathbb{M}) \text{ under}\\ s_1<s_2<n<t_1<t_2.\end{aligned}$$

□

Corollary 9.1. *If the CoP $\mathcal{C}(n,\mathbb{M})$ admits embedding, then it follows by appealing to Proposition 9.4 and Proposition 9.3 for its complete family*

$$\mathcal{C}(s_1,\mathbb{M})\subset\ldots\subset\mathcal{C}(s_k,\mathbb{M})\subset\mathcal{C}(n,\mathbb{M})\subset\mathcal{C}(t_1,\mathbb{M})\subset\ldots\subset\mathcal{C}(t_k,\mathbb{M})$$

provided $\mathcal{C}(n,\mathbb{M})$ has $2k$ children.

Theorem 9.1. *Let $\mathbb{M}\subseteq\mathbb{N}$ and $\mathcal{C}(n,\mathbb{M})$ be a CoP with $|\mathcal{C}(n,\mathbb{M})|=k$. Then the number of children in the family with parent $\mathcal{C}(n,\mathbb{M})$ satisfies the upper bound*

$$\leq 2\left\lfloor\frac{k}{2}\right\rfloor\left(\left\lfloor\frac{k}{2}\right\rfloor-1\right)$$

and the lower bound

$$\geq 2(n_a - 2) = 4\left(\left\lfloor \frac{k}{2} \right\rfloor - 1\right) \text{ with } n_a = 2\left\lfloor \frac{k}{2} \right\rfloor.$$

Proof. At first we prove the upper bound. The CoP $\mathcal{C}(n,\mathbb{M})$ with $|\mathcal{C}(n,\mathbb{M})| = k$ contains $\lfloor \frac{k}{2} \rfloor$ different axes. Each axis contains two points of the parent $\mathcal{C}(n,\mathbb{M})$ and determines children with at most $\lfloor \frac{k}{2} \rfloor - 1$ number of axes. The upper bound follows from this counting argument.
Now we prove the lower bound. In virtue of Corollary 2.2 the weights of the points of $\mathcal{C}(n,\mathbb{M})$ are strictly totally ordered. Now we remove from this sequence the weight of the center if it exists. It remains $n_a = 2\left\lfloor \frac{k}{2} \right\rfloor$ weights. We enumerate them as

$$x_1 < x_2 < \ldots < x_{n_a-1} < x_{n_a}$$

and form the following sequences

$$s_1 := x_1 + x_2 < s_2 := x_1 + x_3 < \ldots < s_{n_a-2} := x_1 + x_{n_a-1} < x_1 + x_{n_a} = n$$

and

$$t_1 := x_{n_a} + x_{n_a-1} > t_2 := x_{n_a} + x_{n_a-2} > \ldots > t_{n_a-2} := x_{na} + x_2 > x_{n_a} + x_1 = n.$$

Hence we obtain

$$s_1 < \ldots < s_{n_a-2} < t_1 < \ldots < t_{n_a-2}$$

and have at least $2(n_a - 2)$ different generators for children of $\mathcal{C}(n,\mathbb{M})$. □

Remark 9.1. Next we launch an important result that will certainly have significant offshoots throughout our studies. Very roughly, It tells us that we can always partition any complete family into incomplete families with equal dilation between the members.

Lemma 9.1 (Regularity lemma)**.** *The offspring of a CoP $\mathcal{C}(n,\mathbb{M})$ can be partitioned into incomplete families with equal scale dilation between sequence of successive embedding.*

Proof. If there exist no embedding among the children of the parent $\mathcal{C}(n,\mathbb{M})$, then obviously we have a partition into a one member incomplete family and the dilation in each family is trivial. Let us assume $\mathcal{C}(s_1,\mathbb{M}) \subset \mathcal{C}(s_2,\mathbb{M}) \subset \cdots \subset \mathcal{C}(s_k,\mathbb{M})$ for $k \geq 2$ be a sequence of children of the parent $\mathcal{C}(n,\mathbb{M})$ with equal scale dilation between successive embedding. If the sequence is all of the children of the parent $\mathcal{C}(n,\mathbb{M})$ then the parent $\mathcal{C}(n,\mathbb{M})$ must be contained in the embedding. Suppose the parent $\mathcal{C}(n,\mathbb{M})$ is outside the embedding then we have two possible cases

$$\mathcal{C}(s_1,\mathbb{M}) \subset \mathcal{C}(s_2,\mathbb{M}) \subset \cdots \subset \mathcal{C}(s_k,\mathbb{M}) \subset \mathcal{C}(n,\mathbb{M})$$

or

$$\mathcal{C}(n,\mathbb{M}) \subset \mathcal{C}(s_1,\mathbb{M}) \subset \mathcal{C}(s_2,\mathbb{M}) \subset \cdots \subset \mathcal{C}(s_k,\mathbb{M}).$$

In the first case there must exists a point of the parent $\mathcal{C}(n,\mathbb{M})$ that is not contained in all the children. This contradicts the fact the CoP $\mathcal{C}(n,\mathbb{M})$ is a parent of all its children. In the second case there must exist a point on all children that is not contained in the parent $\mathcal{C}(n,\mathbb{M})$. This is also absurd since points on each child must be a point on the parent $\mathcal{C}(n,\mathbb{M})$. Now let us insert the parent $\mathcal{C}(n,\mathbb{M})$ into the a priori sequence of embedding. Next let us remove from the chain produced the parent $\mathcal{C}(n,\mathbb{M})$ with two closest children. Then we obtain a partition of collection of children in the embedding into two sub-chains of embedding with equal scale dilation between successive children, those to the left of the children closest to the parent $\mathcal{C}(n,\mathbb{M})$ and to the right of the children closest to the parent $\mathcal{C}(n,\mathbb{M})$. For the sequence removed from the a priori sequence of children given below

$$\mathcal{C}(s_i,\mathbb{M}) \subset \mathcal{C}(n,\mathbb{M}) \subset \mathcal{C}(s_{i+1},\mathbb{M})$$

we remove the parent $\mathcal{C}(n,\mathbb{M})$ and we obtain a third partition of offspring with equal scale dilation

$$\mathcal{C}(s_i,\mathbb{M}) \subset \mathcal{C}(s_{i+1},\mathbb{M}).$$

For the case where not all children are contained in the a priori embedding, then we have already obtained a partition of collection of children into an incomplete family with equal scale dilation between successive members. The remaining collection of children can also be partitioned into incomplete families by choosing an embedding with equal scale dilation between successive children. □

Definition 9.2. Let

$$\mathcal{O} = \bigcup_{i\geq 2} \mathcal{F}_i$$

be a partition of the offspring of the parent $\mathcal{C}(n,\mathbb{M})$ into incomplete families $\mathcal{F}_i$. Then we say the the partition is irreducible if no embedding into the parent overlaps the embedding of children $\mathcal{C}(s,\mathbb{M}) \in \mathcal{F}_i$ for $i \geq 2$.

Theorem 9.2. *Let $\mathbb{M} \subset \mathbb{N}$ and let $\mathcal{C}(n,\mathbb{M})$ be a parent with $|\mathcal{C}(n,\mathbb{M})| = k$ and admits embedding. Let $\mathcal{O}$ be the offspring of the parent $\mathcal{C}(n,\mathbb{M})$ and*

$$\mathcal{O} = \bigcup_{i\geq 2} \mathcal{F}_i$$

be an irreducible partition into incomplete families $\mathcal{F}_i$ such that for a fixed $r > 0$ there exist a dilation δ_r between successive children of the embedding in each $\mathcal{F}_i$. Then

$$|\mathcal{F}_i| \leq \left\lfloor \frac{k}{2} \right\rfloor \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 \right).$$

Proof. Let $\mathcal{C}(n,\mathbb{M})$ be a parent and construct its offspring. Let us consider the sequence of embedding with equal scale dilation between successive children

$$\mathcal{C}(s_1,\mathbb{M}) \subset \mathcal{C}(s_2,\mathbb{M}) \subset \cdots \subset \mathcal{C}(s_k,\mathbb{M}).$$

Let us assume all the children are contained in the chain and insert the parent $\mathcal{C}(n,\mathbb{M})$. By invoking Proposition 9.3 there are as many children above and below the parent $\mathcal{C}(n,\mathbb{M})$ in the chain. Finally let us remove the parent from the chain and we obtain a partition into two incomplete family with equal scale dilation between successive children in their embedding. The size of each of the incomplete family is at most

$$\leq \left\lfloor \frac{k}{2} \right\rfloor \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 \right).$$

In the case where not all children are members of the a priori chain then inserting and removing from the chain the parent $\mathcal{C}(n,\mathbb{M})$ cuts down on the expected size of the incomplete family with such property. □

Theorem 9.3. *The number of pairs of connected children in any complete family is lower bounded by*

$$\geq \frac{n_a(n_a-2)(n_a-3)}{2} = 2\left\lfloor\frac{k}{2}\right\rfloor\left(\left\lfloor\frac{k}{2}\right\rfloor-1\right)(n_a-3) \geq 2\left\lfloor\frac{k}{2}\right\rfloor\left(\left\lfloor\frac{k}{2}\right\rfloor-1\right)$$

if the parent CoP has n_a axis points and $n_a = 2\left\lfloor\frac{k}{2}\right\rfloor > 3$.

Proof. In virtue of Proposition 8.1 two CoPs are connected if and only if they have a common point. And the children are generated by pairs of points on different axes. Each such point $[x]$ of the parent CoP occurs therefore in $n_a - 2$ children at least. Hence there are $\frac{(n_a-2)(n_a-3)}{2}$ pairs of children containing the point $[x]$. There are n_a axis points. Therefore this number of pairs must be multiplied by n_a. This results the formula of the lower bound.

□

In comparison with Theorem 9.1 we observe that the number of pairs of connected children of a complete family is always greater or equal to the number of its children. From the proof of Theorem 9.3 we see that each child is connected with another child of the same family.

Example 9.2. We take as parent CoP
$\mathcal{C}(22,\mathbb{P}) = \{[3],[5],[11],[17],[19]\} \to k = 5, n_a = 4$. In virtue of Theorem 9.1. it has maximal

$$2 \cdot 2 \cdot 1 = 4$$

children and in virtue of Theorem 9.3 at least

$$\frac{4 \cdot 2 \cdot 1}{2} = 4$$

pairs of connected children. As children we get

$$\begin{aligned}
\mathcal{C}(8,\mathbb{P}) &= \{[\mathbf{3}],[\mathbf{5}]\} \\
\mathcal{C}(20,\mathbb{P}) &= \{[\mathbf{3}],[7],[13],[\mathbf{17}]\} \\
&-----------\\
\mathcal{C}(24,\mathbb{P}) &= \{[\mathbf{5}],[7],[11],[13],[17],[\mathbf{19}]\} \\
\mathcal{C}(36,\mathbb{P}) &= \{[5],[7],[13],[\mathbf{17}],[\mathbf{19}],[23],[29],[31]\}.
\end{aligned}$$

We see that [3] occurs in the children 2 times. With it there is 1 pair of children containing the point [3]. [5] occurs 3 times and is hence contained in 3 pairs. [17] occurs 3 times too and [19] occurs 2 times and is contained in 1 pair. All together we have $8 > 6$ pairs of connected children with respect to the points of the parent CoP. But we see that more than these points are common points in the offset. Hence there are 6 further pairs of connected children. The principal axes are marked as boldface.
The CoP $\mathcal{C}(24,\mathbb{P})$ contains 6 axis points and has therefore at most 12 children with 36 pairs of connected children at least.

Conjecture 9.1. *Let $\mathbb{M} \subset \mathbb{N}$ and let $\mathcal{C}(n,\mathbb{M})$ be a parent with $|\mathcal{C}(n,\mathbb{M})| = k$. Let $\mathcal{O}$ be the offspring of the parent $\mathcal{C}(n,\mathbb{M})$ and*

$$\mathcal{O} = \bigcup_{i\geq 2}^{N} \mathcal{F}_i$$

be an irreducible partition into incomplete families $\mathcal{F}_i$ such that for a fixed $r > 0$ there exist a dilation δ_r between successive children of the embedding in each $\mathcal{F}_i$. Then

$$|\mathcal{F}_i| = \frac{|\mathcal{O}|}{N}$$

where N is the number of such incomplete families in the partition.

Theorem 9.4. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}(n,\mathbb{M})$ and $\mathcal{C}(m,\mathbb{M})$ be two CoPs admitting aligned embedding. Without loss of generality we assume*

$$\mathcal{C}(n,\mathbb{M}) \subset \mathcal{C}(m,\mathbb{M}).$$

Then $\mathcal{C}(n,\mathbb{M})$ is a child of $\mathcal{C}(m,\mathbb{M})$. If there is a chord $\mathcal{L}_{[x],[y]}$ of $\mathcal{C}(n,\mathbb{M})$ with $x+y=m$ then $\mathcal{C}(m,\mathbb{M})$ is also a child of $\mathcal{C}(n,\mathbb{M})$. Additionally the complete family $\hat{\mathcal{O}}_n$ of $\mathcal{C}(n,\mathbb{M})$ is a subset of the complete family $\hat{\mathcal{O}}_m$ of $\mathcal{C}(m,\mathbb{M})$.

Proof. Due to $\mathcal{C}(n,\mathbb{M}) \subset \mathcal{C}(m,\mathbb{M})$ by virtue of Definition 2.3 hold $n < m$ and

$$\min\left(x \mid [x] \in \mathcal{C}(n,\mathbb{M})\right) = \min\left(u \mid [u] \in \mathcal{C}(m,\mathbb{M})\right).$$

All chords $\mathcal{L}_{[x],[y]}$ of $\mathcal{C}(n,\mathbb{M})$ are also chords of $\mathcal{C}(m,\mathbb{M})$ excluding the chords between points $[x],[y] \in \mathcal{C}(n,\mathbb{M})$ with $x+y=m$. By exploiting the underlying embedding, we notice that all chords of $\mathcal{C}(m,\mathbb{M})$ which are axes of $\mathcal{C}(n,\mathbb{M})$ generate all the same child, the CoP $\mathcal{C}(n,\mathbb{M})$. Hence the CoP $\mathcal{C}(n,\mathbb{M})$ is a child of the CoP $\mathcal{C}(m,\mathbb{M})$, and if there is no chord $\mathcal{L}_{[x],[y]}$ of $\mathcal{C}(n,\mathbb{M})$ with $x+y=m$ then all children of $\mathcal{C}(n,\mathbb{M})$ are children of $\mathcal{C}(m,\mathbb{M})$ too. Hence the complete family $\hat{\mathcal{O}}_n$ is a subset of the complete family $\hat{\mathcal{O}}_m$ in this case.
If such a chord of $\mathcal{C}(n,\mathbb{M})$ exists then this chord is an axis of $\mathcal{C}(m,\mathbb{M})$, so that $\mathcal{C}(m,\mathbb{M})$ is a child of $\mathcal{C}(n,\mathbb{M})$. Because the parents belong to its complete family holds that the complete family $\hat{\mathcal{O}}_n$ is a subset of $\hat{\mathcal{O}}_m$ in this case too. ☐

10. Isomorphic Circles of Partition

In this section we introduce and study the notion of isomorphism between CoPs.

Definition 10.1. Let $\mathbb{M} \subseteq \mathbb{N}$ and let $\mathcal{C}(n,\mathbb{M})$ and $\mathcal{C}(m,\mathbb{M})$ be parents with the complete families $\hat{\mathcal{O}}_n$ and $\hat{\mathcal{O}}_m$, respectively. Then we say the parents $\mathcal{C}(n,\mathbb{M})$ and $\mathcal{C}(m,\mathbb{M})$ are isomorphic if

$$\hat{\mathcal{O}}_m \cap \hat{\mathcal{O}}_n \neq \emptyset.$$

We call the number $|\hat{\mathcal{O}}_m \cap \hat{\mathcal{O}}_n|$ the degree of isomorphism. We denote this isomorphism by $\mathcal{C}(n,\mathbb{M}) \cong \mathcal{C}(m,\mathbb{M})$. We say the degree of isomorphism is high if at least one of the following equalities holds

$$\frac{|\hat{\mathcal{O}}_m \cap \hat{\mathcal{O}}_n|}{|\hat{\mathcal{O}}_n|} = 1$$

or

$$\frac{|\hat{\mathcal{O}}_m \cap \hat{\mathcal{O}}_n|}{|\hat{\mathcal{O}}_m|} = 1.$$

Otherwise, we say the degree of isomorphism is low.

Theorem 10.1. *Let $\mathcal{C}(n,\mathbb{M})$ and $\mathcal{C}(m,\mathbb{M})$ be two parent CoPs admitting aligned embedding. Then holds*

$$\mathcal{C}(n,\mathbb{M}) \cong \mathcal{C}(m,\mathbb{M})$$

with a high degree.

Proof. Without loss of generality let us assume that $\mathcal{C}(n,\mathbb{M}) \subset \mathcal{C}(m,\mathbb{M})$. Then by virtue of Theorem 9.4 all children of $\mathcal{C}(n,\mathbb{M})$ are children of $\mathcal{C}(m,\mathbb{M})$ too. Hence holds $\hat{\mathcal{O}}_n \subset \hat{\mathcal{O}}_m$ and therefore

$$\frac{|\hat{\mathcal{O}}_n \cap \hat{\mathcal{O}}_m|}{|\hat{\mathcal{O}}_n|} = 1.$$

□

Corollary 10.1. *Let*

$$R_{a,d}(n) := \#\left\{(x,y) \in \mathbb{M}_{a,d}^2 |\ x+y=n, x<y\right\}$$

then $R_{a,d}(n)$ is non-decreasing for all $n \in \mathbb{M}_{2a,d}$.

Proof. By virtue of Theorem 2.2 the CoP $\mathcal{C}(n,\mathbb{M}_{a,d})$ admits aligned embedding. Let $n, n+t \in \mathbb{M}_{2a,d}$ with $n < n+t$. Let us assume to the contrary that

$$\#\left\{(x,y) \in \mathbb{M}_{a,d}^2 |\ x+y=n+t, x<y\right\} < \#\left\{(x,y) \in \mathbb{M}_{a,d}^2 |\ x+y=n, x<y\right\}$$

for all $t \in \mathbb{N}$. Then by virtue of Theorem 10.1, it follows that each $[x] \in \mathcal{C}(n+t,\mathbb{M}_{a,d})$ is such that $[x] \in \mathcal{C}(n,\mathbb{M}_{a,d})$. It follows that there exists no axis $\mathbb{L}_{[u],[v]} \hat{\in} \mathcal{C}(n+t,\mathbb{M}_{a,d})$ such that $||[u]||, ||[v]|| \in (n, n+t)$ for all $t \in \mathbb{N}$. This is absurd, thereby ending the proof of the claim. □

It is worth pointing out that this result could yet be obtained from several perspective (see Proposition 2.4). The only reason we have chosen to adopt this proof strategy is to illustrate the relevance of the notion of isomorphism between CoPs.

Theorem 10.2. *Let $\mathbb{P}$ be the set of prime numbers. Then there exist infinitely many parents $\mathcal{C}(n,\mathbb{P})$ and $\mathcal{C}(m,\mathbb{P})$ such that $\mathcal{C}(n,\mathbb{P}) \cong \mathcal{C}(m,\mathbb{P})$ with low degree.*

Proof. Suppose to the contrary that there are only finitely many such parent CoPs and that $\mathcal{C}(n_o,\mathbb{P}) \cong \mathcal{C}(m_o,\mathbb{P})$ with low degree where n_o, m_o are the greatest generators for such CoPs. Without loss of generality we can assume $n_o < m_o$. Then we have for all generators $n, m > m_o$ only pairs of parent CoPs $\mathcal{C}(n,\mathbb{P}) \cong \mathcal{C}(m,\mathbb{P})$ with high degree. Without loss of generality, let us assume that

$$\#\left\{\mathbb{L}_{[p],[q]} \hat{\in} \mathcal{C}(m,\mathbb{P})\right\} \leq \#\left\{\mathbb{L}_{[p],[q]} \hat{\in} \mathcal{C}(n,\mathbb{P})\right\}$$

then it follows that

$$\frac{|\hat{\mathcal{O}}_m \cap \hat{\mathcal{O}}_n|}{|\hat{\mathcal{O}}_m|} = 1$$

so that for each $[p] \in \mathcal{C}(m,\mathbb{P})$ then $[p] \in \mathcal{C}(n,\mathbb{P})$. Let us suppose that for all $t \in \mathbb{N}$ then

$$\#\left\{\mathbb{L}_{[p],[q]} \hat{\in} \mathcal{C}(m+2t,\mathbb{P})\right\} < \#\left\{\mathbb{L}_{[p],[q]} \hat{\in} \mathcal{C}(m,\mathbb{P})\right\}$$

so that by the **high** degree of isomorphism we have

$$\frac{|\hat{\mathcal{O}}_{m+2t} \cap \hat{\mathcal{O}}_m|}{|\hat{\mathcal{O}}_{m+2t}|} = 1.$$

It follows that each $[p] \in \mathcal{C}(m+2t, \mathbb{P})$ is such that $[p] \in \mathcal{C}(m, \mathbb{P})$. It follows that there exists no axis $\mathbb{L}_{[u],[v]} \hat{\in} \mathcal{C}(m+2t, \mathbb{P})$ such that $||[u]||, ||[v]|| \in (m, m+2t)$ for all $t \in \mathbb{N}$. This is absurd. Now we can order the cardinality of axes of the CoPs

$$\#\left\{\mathbb{L}_{[p],[q]} \hat{\in} \mathcal{C}(m, \mathbb{P})\right\} \leq \#\left\{\mathbb{L}_{[p],[q]} \hat{\in} \mathcal{C}(m+2, \mathbb{P})\right\} \leq \cdots \leq \#\left\{\mathbb{L}_{[p],[q]} \hat{\in} \mathcal{C}(m+2s, \mathbb{P})\right\}$$

for all $s \geq 1$ with $s \in \mathbb{N}$. Let $\mathbb{L}_{[p],[q]} \hat{\in} \mathcal{C}(m, \mathbb{P})$ then it follows that $\mathbb{L}_{[p+2],[q]} \hat{\in} \mathcal{C}(m+2, \mathbb{P})$, $\mathbb{L}_{[p+4],[q]} \hat{\in} \mathcal{C}(m+4, \mathbb{P}), \ldots, \mathbb{L}_{[p+2s],[q]} \hat{\in} \mathcal{C}(m+2s, \mathbb{P})$ for $s \geq 2$ with $s \in \mathbb{N}$ by the virtue of **high** degree isomorphism between CoPs. It follows that each term of the infinite sequence

$$p < p+2 < p+4 < \cdots < p+2s < \cdots$$

for all $s \geq 2$ with $s \in \mathbb{N}$ must be prime. This is absurd, hence the supposed finite cardinality of CoPs with low degree cannot be true. □

The language of **isomorphism** could be a good enough tool for studying the Hardy-Littlewood prime tuple conjecture or what is now known as the **first Hardy-Littlewood conjecture**. In fact the conjecture can be stated in the language of isomorphism in the following manner:

Conjecture 10.1 (Hardy-Littlewood)**.** *Let $\mathbb{P}$ be the set of all prime numbers. Then there exist infinitely many $n, t \in \mathbb{N}$ such that*

$$\mathcal{C}(n, \mathbb{P}) \cong \mathcal{C}(n+2t, \mathbb{P})$$

with high degree.

To see why this formulation yields the **first Hardy-Littlewood Conjecture**, we notice that the **high** degree isomophism between the CoPs implies that for $t \in \mathbb{N}$ sufficiently large then

$$\#\left\{\mathbb{L}_{[p],[q]} \hat{\in} \mathcal{C}(n, \mathbb{P})\right\} \leq \#\left\{\mathbb{L}_{[p],[q]} \hat{\in} \mathcal{C}(n+2t, \mathbb{P})\right\}$$

so that for each axis $\mathbb{L}_{[p],[q]} \hat{\in} \mathcal{C}(n, \mathbb{P})$ then $\mathbb{L}_{[p+2t],[q]} \hat{\in} \mathcal{C}(n+2t, \mathbb{P})$. It follows that for all $[p_1], \ldots, [p_k] \in \mathcal{C}(n, \mathbb{P})$ then $[p_1+2t], \ldots, [p_k+2t] \in \mathcal{C}(n+2t, \mathbb{P})$ so that for the tuple $(p_1-1, p_2-1, \ldots, p_k-1) \in \mathbb{N}^k$, then

$$(p_1-1+r, p_2-1+r, \ldots, p_k-1+r) \in \mathbb{P}^k$$

where by choice $r = 2t+1$. We note that the case $t = 1$ in the **conjecture** is the Twin Prime Conjecture.

Corollary 10.2. *There are infinitely many CoPs of the forms $\mathcal{C}(n, \mathbb{P})$, $\mathcal{C}(m, \mathbb{P})$ such that none of the following embedding holds*

$$\mathcal{C}(n, \mathbb{P}) \subset \mathcal{C}(m, \mathbb{P}) \quad \text{and} \quad \mathcal{C}(m, \mathbb{P}) \subset \mathcal{C}(n, \mathbb{P}).$$

Proof. The proof is an easy consequence of Theorem 10.2. Assume to the contrary that there are finitely many CoPs of the forms $\mathcal{C}(n,\mathbb{P})$ that fails to admit an embedding. The upshot will be that there are infinitely many chains of CoPs of the forms $\mathcal{C}(n,\mathbb{P})$ and $\mathcal{C}(m,\mathbb{P})$ with consecutive even generators such that $\mathcal{C}(m,\mathbb{P}) \cong \mathcal{C}(n,\mathbb{P})$ with a high degree. Then the proof technique in Theorem 10.2 can then be adapted by taking n and m sufficiently large. □

11. Ascending, Descending and Stationary Circles of Partition

In this section we introduce the notion of **ascending**, **descending** and **stationary** CoPs between generators. We formalize this notion in the following language.

Definition 11.1. Let $\mathbb{M} \subset \mathbb{N}$ with $\mathcal{C}(n,\mathbb{M})$ be a CoP. Then we say the CoP $\mathcal{C}(n,\mathbb{M})$ is **ascending** from n to the **spot** m if for $n < m$ holds

$$\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})\right\} < \#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m,\mathbb{M})\right\}.$$

Similarly, we say it is **descending** from n to the **spot** m if for $n < m$ then

$$\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})\right\} > \#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m,\mathbb{M})\right\}.$$

We say it is **globally** ascending (resp. descending) if at $\forall m \in \mathbb{N}$ it is ascending (resp. descending). We say the CoP $\mathcal{C}(n,\mathbb{M})$ is **stationary** from n to the **spot** m if for $n < m$ then

$$\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})\right\} = \#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m,\mathbb{M})\right\}.$$

Similarly, we say it is **globally stationary** if it is stationary at all spots $m \in \mathbb{N}$. If the CoP $\mathcal{C}(n,\mathbb{M})$ is neither globally ascending, descending nor stationary, then we say it is globally **oscillatory**.

Proposition 11.1. *The CoP $\mathcal{C}(n,\mathbb{M}_{a,d})$ is globally ascending for all $n \in \mathbb{M}_{2a,d}$ with fixed $a,d \in \mathbb{N}$.*

Proof. Pick arbitrarily $m \in \mathbb{M}_{2a,d}$ such that $m > n$ and consider the CoP $\mathcal{C}(m,\mathbb{M}_{a,d})$. Since CoPs with base set $\mathbb{M}_{a,d}$ are isomorphic with high degree, it follows that

$$\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M}_{a,d})\right\} < \#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m,\mathbb{M}_{a,d})\right\}$$

and the claim follows since m was chosen arbitrarily in $\mathbb{M}_{2a,d}$. □

Theorem 11.1. *Let $\mathbb{H} \subset \mathbb{N}$ and $\mathcal{C}(n,\mathbb{H})$ be a CoP. If*

$$\lim_{n\longrightarrow\infty} \frac{|\mathbb{H}\cap\mathbb{N}_n|}{n} > 0$$

with

$$\lim_{n\longrightarrow\infty} \frac{|(\mathbb{N}\setminus\mathbb{H})\cap\mathbb{N}_n|}{n} < \frac{1}{2}\lim_{n\longrightarrow\infty} \frac{|\mathbb{H}\cap\mathbb{N}_n|}{n}$$

then $\mathcal{C}(n,\mathbb{H})$ is ascending at infinitely many spots.

Proof. Let $\mathcal{C}(n,\mathbb{H})$ be a CoP and assume to the contrary that there are finitely many spots at which it is ascending. Let us name and arrange the spots as follows $m_1 < m_2 < \cdots < m_k$. It follows that

$$\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{H})\right\} \geq \#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m_{k+1},\mathbb{H})\right\} \geq \cdots \#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m_{k+i},\mathbb{H})\right\} \geq \cdots$$

for all $i \geq 1$. The upshot is that

$$\lim_{n\longrightarrow\infty}\frac{\#\left\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n,\mathbb{H})\right\}}{\lfloor\frac{n-1}{2}\rfloor}=0.$$

Next, by virtue of uniqueness of axes of CoPs, we can compute the density of points with weight in $\mathbb{H}$ on the CoP $\mathcal{C}(n)$ as follows

$$\begin{aligned}\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty)}) &= \lim_{n\longrightarrow\infty}\frac{\#\left\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n)|\ \{x,y\}\cap\mathbb{H}\neq\emptyset\right\}}{\lfloor\frac{n-1}{2}\rfloor}\\ &= \lim_{n\longrightarrow\infty}\frac{\#\left\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n)|\ x\in\mathbb{H},\ y\in\mathbb{N}\setminus\mathbb{H}\right\}}{\lfloor\frac{n-1}{2}\rfloor}+\lim_{n\longrightarrow\infty}\frac{\#\left\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n,\mathbb{H})\right\}}{\lfloor\frac{n-1}{2}\rfloor}\\ &= \lim_{n\longrightarrow\infty}\frac{\#\left\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n)|\ x\in\mathbb{H},\ y\in\mathbb{N}\setminus\mathbb{H}\right\}}{\lfloor\frac{n-1}{2}\rfloor}\\ &\leq \lim_{n\longrightarrow\infty}\frac{|(\mathbb{N}\setminus\mathbb{H})\cap\mathbb{N}_n|}{\lfloor\frac{n-1}{2}\rfloor}\\ &\leq 2\lim_{n\longrightarrow\infty}\frac{|(\mathbb{N}\setminus\mathbb{H})\cap\mathbb{N}_n|}{n}.\end{aligned}$$

Invoking Proposition 6.2, we have the inequality

$$\lim_{n\longrightarrow\infty}\frac{|\mathbb{H}\cap\mathbb{N}_n|}{n}\leq 2\lim_{n\longrightarrow\infty}\frac{|(\mathbb{N}\setminus\mathbb{H})\cap\mathbb{N}_n|}{n}.$$

This, however, violates the requirement of the statement, thereby ending the proof. □

Remark 11.1. Next we obtain from this result another weak variant of the Erdős-Turán conjecture. Roughly speaking, it purports very dense sequences sufficiently qualifies to be an additive base.

Corollary 11.1. *Let $\mathbb{H}\subset\mathbb{N}$ with $\mathcal{D}(\mathbb{H})>0$ such that $\mathcal{D}(\mathbb{N}\setminus\mathbb{H})<\frac{1}{2}\mathcal{D}(\mathbb{H})$. If*

$$r_{\mathbb{H}}(n):=\#\left\{(a,b)\in\mathbb{H}^2|\ a+b=n\right\}$$

then $\lim_{n\longrightarrow\infty} r_{\mathbb{H}}(n)=\infty$.

12. Compatible and Incompatible Circles of Partition

In this section we introduce the notion of compatibility and incompatibility of circles of partition. We launch the following formal language.

Definition 12.1. Let $\mathbb{M}\subseteq\mathbb{N}$ and $\mathcal{C}(n,\mathbb{M})$ and $\mathcal{C}(m,\mathbb{M})$ be any two CoPs. Then we say the CoPs $\mathcal{C}(n,\mathbb{M})$ and $\mathcal{C}(m,\mathbb{M})$ are **compatible** if there exists some CoP $\mathcal{C}(r,\mathbb{M})$ satisfying

$$\mathcal{C}(n,\mathbb{M})\cup\mathcal{C}(m,\mathbb{M})\subseteq\mathcal{C}(r,\mathbb{M})$$

such that for each $[x]\in\mathcal{C}(n,\mathbb{M})\cup\mathcal{C}(m,\mathbb{M})$ with $2x\neq n$ there exist some $[y]\in\mathcal{C}(n,\mathbb{M})\cup\mathcal{C}(m,\mathbb{M})$ so that

$$\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(r,\mathbb{M}).$$

We denote the compatibility by $\mathcal{C}(n,\mathbb{M})\diamond\mathcal{C}(m,\mathbb{M})$. We call the CoP $\mathcal{C}(r,\mathbb{M})$ the **cover** of this compatibility.

Proposition 12.1. *Let $\mathcal{C}(n,\mathbb{M})$ and $\mathcal{C}(m,\mathbb{M})$ be any two CoPs admitting aligned embedding. Then $\mathcal{C}(n,\mathbb{M}) \diamond \mathcal{C}(m,\mathbb{M})$.*

Proof. The result follows very easily by virtue of the feasibility of the embedding

$$\mathcal{C}(n,\mathbb{M}) \subseteq \mathcal{C}(m,\mathbb{M})$$

or

$$\mathcal{C}(m,\mathbb{M}) \subseteq \mathcal{C}(n,\mathbb{M})$$

according as $n \leq m$ or $m \leq n$. □

Theorem 12.1. *Let $\mathbb{M} \subseteq \mathbb{N}$. Then there exists no CoPs of the forms $\mathcal{C}(n,\mathbb{M})$ and $\mathcal{C}(m,\mathbb{M})$ with all axes points concentrated at their center and additionally that $\mathcal{C}(n,\mathbb{M}) \cap \mathcal{C}(m,\mathbb{M}) = \emptyset$ for $|\mathcal{C}(n,\mathbb{M})| > 2$ and $|\mathcal{C}(m,\mathbb{M})| > 2$ with*

$$\#\left\{\mathbb{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(n,\mathbb{M})\right\} \neq \#\left\{\mathbb{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(m,\mathbb{M})\right\}$$

such that

$$\mathcal{C}(n,\mathbb{M}) \diamond \mathcal{C}(m,\mathbb{M})$$

with a cover whose axes points are away from the center.

Proof. Let us suppose there exists at least a pair of CoPs of the form $\mathcal{C}(n,\mathbb{M})$ and $\mathcal{C}(m,\mathbb{M})$ with $m \neq n$ such that $\mathcal{C}(n,\mathbb{M}) \cap \mathcal{C}(m,\mathbb{M}) = \emptyset$ for $|\mathcal{C}(n,\mathbb{M})|, |\mathcal{C}(m,\mathbb{M})| > 2$ and additionally that

$$\#\left\{\mathbb{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(n,\mathbb{M})\right\} \neq \#\left\{\mathbb{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(m,\mathbb{M})\right\}$$

so that $\mathcal{C}(n,\mathbb{M}) \diamond \mathcal{C}(m,\mathbb{M})$. It follows that there exists some CoP $\mathcal{C}(s,\mathbb{M})$ such that

$$\mathcal{C}(n,\mathbb{M}) \cup \mathcal{C}(m,\mathbb{M}) \subseteq \mathcal{C}(s,\mathbb{M})$$

so that for each $[x] \in \mathcal{C}(n,\mathbb{M}) \cup \mathcal{C}(m,\mathbb{M})$ there exists some $[y] \in \mathcal{C}(n,\mathbb{M}) \cup \mathcal{C}(m,\mathbb{M})$ such that

$$\mathbb{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(s,\mathbb{M}).$$

Under the conditions

$$\#\left\{\mathbb{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(n,\mathbb{M})\right\} \neq \#\left\{\mathbb{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(m,\mathbb{M})\right\}$$

and

$$\mathcal{C}(n,\mathbb{M}) \cap \mathcal{C}(m,\mathbb{M}) = \emptyset$$

it follows from the **pigeon-hole** principle and the **uniqueness** of the axes of CoPs there exists some $\mathbb{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(s,\mathbb{M})$ such that $[x],[y] \in \mathcal{C}(n,\mathbb{M})$ or $[x],[y] \in \mathcal{C}(m,\mathbb{M})$. Without loss of generality let us assume that $[x],[y] \in \mathcal{C}(n,\mathbb{M})$. By virtue of the embedding

$$\mathcal{C}(n,\mathbb{M}) \subset \mathcal{C}(s,\mathbb{M})$$

the line $\mathcal{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(n,\mathbb{M})$ is such that $\mathcal{L}_{[x],[y]} \neq \mathbb{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(n,\mathbb{M})$. It follows that the line $\mathcal{L}_{[x],[y]}$ must be a chord in $\mathcal{C}(n,\mathbb{M})$ and $\mathcal{C}(s,\mathbb{M})$ must be a **child** of the parent $\mathcal{C}(n,\mathbb{M})$. Now let us locate all the remaining **chords** $\mathcal{L}_{[u],[v]} \neq \mathcal{L}_{[x],[y]}$ in the parent $\mathcal{C}(n,\mathbb{M})$. We claim that each chord $\mathcal{L}_{[u],[v]}$ must be an axis of the child

$\mathcal{C}(s,\mathbb{M})$. Let us assume to the contrary that some chord $\mathcal{L}_{[u],[v]} \hat{\in} \mathcal{C}(n,\mathbb{M})$ is also a chord in the child $\mathcal{C}(s,\mathbb{M})$. Then there exist some axes

$$\mathbb{L}_{[u],[a]}, \mathbb{L}_{[v],[b]} \hat{\in} \mathcal{C}(n,\mathbb{M}).$$

By virtue of the underlying embedding, it follows that the lines

$$\mathbb{L}_{[u],[a]}, \mathbb{L}_{[v],[b]}$$

cannot be axes of the CoP $\mathcal{C}(s,\mathbb{M})$ so that $\mathbb{L}_{[u],[a]}$ and $\mathbb{L}_{[v],[b]}$ are chords in $\mathcal{C}(s,\mathbb{M})$ with

$$\mathcal{D}([u],[b]) = \mathcal{D}([v],[a]). \tag{12.1}$$

It follows that at least one of $\mathcal{L}_{[u],[b]}$ and $\mathcal{L}_{[v],[a]}$ must be chords in $\mathcal{C}(s,\mathbb{M})$. Otherwise, it would mean both lines $\mathcal{L}_{[u],[b]} = \mathbb{L}_{[u],[b]} \hat{\in} \mathcal{C}(s,\mathbb{M})$ and $\mathcal{L}_{[v],[a]} = \mathbb{L}_{[v],[a]} \hat{\in} \mathcal{C}(s,\mathbb{M})$, which in relation to (12.1) is absurd for axes points of CoPs. Without loss of generality let us assume $\mathcal{L}_{[u],[b]}$ is a chord then so is $\mathcal{L}_{[v],[a]}$ under the condition $\mathbb{L}_{[u],[a]}, \mathbb{L}_{[v],[b]} \hat{\in} \mathcal{C}(n,\mathbb{M})$. Otherwise it would imply the chord $\mathcal{L}_{[a],[v]}$ must be an axis of $\mathcal{C}(s,\mathbb{M})$. Since all the axes points of $\mathcal{C}(n,\mathbb{M})$ are concentrated around the center, it certainly follows that

$$\frac{n}{2} = \frac{a+u}{2} \approx a \quad \text{and} \quad \frac{n}{2} = \frac{a+u}{2} \approx u \tag{12.2}$$

and

$$\frac{n}{2} = \frac{b+v}{2} \approx b \quad \text{and} \quad \frac{n}{2} = \frac{b+v}{2} \approx v \tag{12.3}$$

so that we have $a \approx b \approx u \approx v$ and we deduce that the co-axis point $[a],[v]$ of the cover CoP $\mathcal{C}(s,\mathbb{M})$ is close to the center by the relation

$$\frac{s}{2} = \frac{a+v}{2} \approx a \approx v$$

which contradicts the requirement of the proximity of the axes points of the cover $\mathcal{C}(s,\mathbb{M})$. It follows that $\mathcal{L}_{[u],[v]}$ and $\mathcal{L}_{[a],[b]}$ are also chords in $\mathcal{C}(s,\mathbb{M})$ with

$$\mathcal{D}([u],[v]) = \mathcal{D}([a],[b]) \tag{12.4}$$

since the lines $\mathbb{L}_{[u],[a]}, \mathbb{L}_{[v],[b]} \hat{\in} \mathcal{C}(n,\mathbb{M})$ tied with the embedding $\mathcal{C}(n,\mathbb{M}) \subset \mathcal{C}(s,\mathbb{M})$. It follows from (12.1) and (12.4)

$$\mathbb{L}_{[u],[a]}, \mathbb{L}_{[v],[b]} \hat{\in} \mathcal{C}(s,\mathbb{M})$$

so that $n = u + a = v + b = s$ and $\mathcal{C}(n,\mathbb{M}) = \mathcal{C}(s,\mathbb{M})$, thereby contradicting the embedding

$$\mathcal{C}(n,\mathbb{M}) \subset \mathcal{C}(s,\mathbb{M}).$$

Thus each **chord** in $\mathcal{C}(n,\mathbb{M})$ must be an axis of the **child** $\mathcal{C}(s,\mathbb{M})$. The upshot is that the parent has only one child $\mathcal{C}(s,\mathbb{M})$, which is impossible since $|\mathcal{C}(n,\mathbb{M})| > 2$. □

Conjecture 12.1. *Let $\mathcal{C}(n,\mathbb{M})$ and $\mathcal{C}(m,\mathbb{M})$ be parents CoPs with the offspring $\mathcal{O}_n$ and $\mathcal{O}_m$, respectively. Then $\mathcal{C}(n,\mathbb{M}) \diamond \mathcal{C}(m,\mathbb{M})$ if and only if there exists some $\mathcal{C}(s,\mathbb{M}) \in \mathcal{O}_m$ and $\mathcal{C}(t,\mathbb{M}) \in \mathcal{O}_n$ such that*

$$\mathcal{C}(s,\mathbb{M}) \diamond \mathcal{C}(t,\mathbb{M}).$$

Conjecture 12.1 could have several ramifications if it turns out to be true. Yet we believe it is very hard to establish as we found it far-fetched with the current tools developed thus far. Any progress on this conjecture would require an expansion on the notion of compatibility and their interplay with other concepts.

13. The l^{th} Fold Energy of Circles of Partition

In this section we introduce and study the notion of the l^{th} fold energy of CoPs and exploit some applications in this context.

Definition 13.1. Let $\mathbb{M} \subset \mathbb{N}$ and $\mathcal{C}(n, \mathbb{M})$ be a CoP. Then by the l^{th}-fold energy of the CoP $\mathcal{C}(n, \mathbb{M})$, we mean the quantity

$$\mathcal{E}(l, \mathbb{M}) := \sum_{n=3}^{\infty} \frac{\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n^l, \mathbb{M})\right\}}{\left\lfloor \frac{n^l-1}{2} \right\rfloor}$$

for a fixed $l \in \mathbb{N}$.

It is important to remark that the l^{th} energy of a typical CoP $\mathcal{C}(n, \mathbb{M})$ could either be infinite or finite. In that latter case it certainly should have a finite value. To that effect we state the following proposition.

Proposition 13.1. *Let $\mathbb{J}^l \subset \mathbb{N}$ be the set of all l^{th} powers. Then $\mathcal{E}(l, \mathbb{J}^l) < \infty$ for all $l \geq 3$ and $\mathcal{E}(2, \mathbb{J}^2) = \infty$.*

Proof. Let $l \geq 3$ be fixed and consider the CoP $\mathcal{C}(n^l, \mathbb{J}^l)$, where $\mathbb{J}^l \subset \mathbb{N}$ is the set of all l^{th} powers. Then it follows from the configuration of CoPs the following inequality

$$\begin{aligned}\mathcal{E}(l, \mathbb{J}^l) &= \sum_{n=3}^{\infty} \frac{\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n^l, \mathbb{J}^l)\right\}}{\left\lfloor \frac{n^l-1}{2} \right\rfloor} \\ &\leq \sum_{n=3}^{\infty} \frac{\frac{n}{2}}{\left\lfloor \frac{n^l-1}{2} \right\rfloor} \\ &\ll \sum_{n=3}^{\infty} \frac{1}{n^{l-1}} < \infty \end{aligned} \tag{13.1}$$

for all $l \geq 3$. □

Proposition 13.2. *Let $\mathbb{M} \subset \mathbb{N}$ and $\mathcal{C}(n, \mathbb{M})$ be a CoP. If $\mathcal{E}(l, \mathbb{M}) = \infty$ for $l \geq 2$, then $\mathcal{C}(n^l, \mathbb{M})$ is ascending at infinitely many spots.*

Proof. Let $\mathcal{E}(l, \mathbb{M}) = \infty$ and assume to the contrary that the CoP $\mathcal{C}(n^l, \mathbb{M})$ is ascending at finitely many spots. Then it follows that

$$\lim_{n \longrightarrow \infty} \#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n^l, \mathbb{M})\} < \infty.$$

This implies $\mathcal{E}(l, \mathbb{M}) < \infty$, thereby contradicting the requirement of the statement. □

Theorem 13.1. *Let $\mathbb{J} \subset \mathbb{N}$ be the set of all squares and*

$$\mathbb{R}_2(n^2) := \#\left\{(a^2, b^2) \in \mathbb{J} |\ a^2 + b^2 = n^2\right\}$$

then

$$\lim_{n \longrightarrow \infty} \mathbb{R}_2(n^2) = \infty$$

Proof. The equation $x^2 + y^2 = z^2$ definitely has a solution in the positive integers. That is there exists at least a triple (a, b, c) such that

$$a^2 + b^2 = c^2.$$

It follows that there are infinitely many triples of the forms

$$(a, b, c), (2a, 2b, 2c), (3a, 3b, 3c) \ldots, (ta, tb, tc) \ldots$$

for all $t \in \mathbb{N}$ satisfying the equation $x^2 + y^2 = z^2$. It follows that for k sufficiently large, we will have the following inequality [7]

$$\sum_{n=3}^{k} \frac{\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n^2, \mathbb{J})\right\}}{\left\lfloor \frac{n^2-1}{2} \right\rfloor} = \mathcal{P}(k) \sum_{n=3}^{k} \frac{\frac{n}{2}}{\left\lfloor \frac{n^2-1}{2} \right\rfloor}$$
$$\gg_k \sum_{n=3}^{k} \frac{1}{n}$$

By taking limits on both side as $k \longrightarrow \infty$, we have

$$\mathcal{E}_2(2, \mathbb{J}) = \sum_{n=3}^{\infty} \frac{\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n^2, \mathbb{J})\right\}}{\left\lfloor \frac{n^2-1}{2} \right\rfloor}$$
$$\gg \sum_{n=3}^{\infty} \frac{\frac{n}{2}}{\left\lfloor \frac{n^2-1}{2} \right\rfloor}$$
$$\gg \sum_{n=3}^{\infty} \frac{1}{n}.$$

It follows that $\mathcal{E}_2(2, \mathbb{J}) = \infty$ and by invoking Proposition 13.2, it follows that the CoP $\mathcal{C}(n^2, \mathbb{J})$ is ascending at infinitely many spots. This completes the proof of the theorem. □

Remark 13.1. In the spirit of attacking the **binary** Goldbach conjecture, we obtain variants of Theorem 13.1 for the primes $\mathbb{P}$ without resorting to the notion of density of CoPs. It is important to remark that the notion of density as used in the previous section has no real content in this regard, since the primes have density zero by virtue of the prime number theorem. The main tool used to tighten up our result concerns that of Estermann [5], that almost all even number can be written as the sum of two prime numbers.

Theorem 13.2. *Let $\mathbb{P}$ be the set of all prime numbers and let*

$$r_2(n^2) = \#\left\{(p, q) \in \mathbb{P}^2 |\ p + q = n^2\right\}$$

then

$$\lim_{n \longrightarrow \infty} r_2(n^2) = \infty.$$

[7] $\mathcal{P}(k)$ plays the role of a constant, but depending on k.

Proof. We compute a lower bound of the single-fold energy of the CoP $\mathcal{C}(n^2,\mathbb{P})$. By virtue of the result [5], we can write for k large enough the inequality

$$\sum_{n=3}^{k}\frac{\#\left\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n^2,\mathbb{P})\right\}}{\left\lfloor\frac{n^2-1}{2}\right\rfloor}=\mathcal{P}(k)\sum_{n=3}^{k}\frac{\frac{\pi(n^2)}{2}}{\left\lfloor\frac{n^2-1}{2}\right\rfloor}$$
$$\gg_k\sum_{n=3}^{k}\frac{1}{\log n}$$

By taking limits on both sides of the equation we obtain the single-fold energy of the CoP $\mathcal{C}(n^2,\mathbb{P})$

$$\mathcal{E}(2,\mathbb{P})=\sum_{n=3}^{\infty}\frac{\#\left\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n^2,\mathbb{P})\right\}}{\left\lfloor\frac{n^2-1}{2}\right\rfloor}$$
$$\gg\sum_{n=3}^{\infty}\frac{1}{\log n}.$$

The upshot is that $\mathcal{E}(2,\mathbb{P})=\infty$ and by appealing to Proposition 13.2, we have

$$\lim_{n\longrightarrow\infty}\#\left\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n^2,\mathbb{P})\right\}=\infty$$

thereby ending the proof of the claim. □

Corollary 13.1. *Let* $\mathbb{P}$ *be the set of all prime numbers and let*

$$r_2(n)=\#\left\{(p,q)\in\mathbb{P}^2|\ p+q=n\right\}$$

then

$$\lim_{n\longrightarrow\infty}r_2(n)=\infty.$$

Proof. The result follows very easily since $n\equiv n^2\equiv 0 \pmod 2$. □

The result established ascertains the very notion that the number of representation of even numbers - Even numbers so represented as the sum of two prime numbers - increases as the n increases without bounds. Next we launch an important criterion for investigating the status of the weight of CoPs. We make this assertion formal in the following proposition.

Proposition 13.3. *Let* $\mathbb{M}\subseteq\mathbb{N}$ *and* $\mathcal{C}(n,\mathbb{M})$ *be a non-empty CoP. Then* $\mathcal{E}(1,\mathbb{M})=\infty$ *if and only if*

$$\#\left\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n,\mathbb{M})\right\}>0$$

for most $n\in\mathbb{N}$.

Proof. We basically compute the 1-fold energy of the CoP $\mathcal{C}(n,\mathbb{M})$ as

$$\mathcal{E}(1,\mathbb{M})=\sum_{n=3}^{\infty}\frac{\#\left\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n,\mathbb{M})\right\}}{\left\lfloor\frac{n-1}{2}\right\rfloor}.$$

The result follows very easily since the series considered is divergent, so that if the counting function for the number of axes of the CoP $\mathcal{C}(n,\mathbb{M})$ is positive for finitely many generators then the series will converge, which is an absurdity given our presumption. The converse is also an easy argument, thereby ending the proof. □

Theorem 13.3. *Let $\mathbb{B} \subset \mathbb{N}$ with $\#\{n \leq x|\ n \in \mathbb{B}\} \sim x^{1-\epsilon}$ for any $0 < \epsilon \leq \frac{1}{2}$ and consider*

$$\mathcal{G}_{\mathbb{B}}(n) = \#\left\{\mathbb{L}_{[a],[b]} \hat{\in} \mathcal{C}(n, \mathbb{B})\right\}.$$

If $\mathcal{G}_{\mathbb{B}}(n) > 0$ for all sufficiently large values of n, then

$$\limsup_{n \longrightarrow \infty} \mathcal{G}_{\mathbb{B}}(n) = \infty.$$

Proof. First we compute the two fold energy $\mathcal{E}(2, \mathbb{B})$ of the CoP $\mathcal{C}(n^2, \mathbb{B})$. Since $\mathcal{G}_{\mathbb{B}}(n) > 0$ for all sufficiently large values of n, It follows that for k large enough there exist some constant $\mathcal{L} = \mathcal{L}(k) > 0$ such that

$$\sum_{n=3}^{k} \frac{\mathcal{G}_{\mathbb{B}}(n^2)}{\left\lfloor \frac{n^2-1}{2} \right\rfloor} = \mathcal{L}(k)(1 + o(1)) \sum_{n=3}^{k} \frac{\left\lfloor \frac{n^{2-2\epsilon}-1}{2} \right\rfloor}{\left\lfloor \frac{n^2-1}{2} \right\rfloor}$$

$$\gg_k \sum_{n=3}^{k} \frac{1}{n^{2\epsilon}}.$$

By taking limits on both sides as $k \longrightarrow \infty$ and noting that $0 < \epsilon \leq \frac{1}{2}$, we deduce $\mathcal{E}(2, \mathbb{B}) = \infty$. Appealing to Proposition 13.2, it follows that

$$\limsup_{n \longrightarrow \infty} \mathcal{G}_{\mathbb{B}}(n^2) = \infty.$$

Since $\left\{n^2 \in \mathbb{N}\right\} \subset \{n \in \mathbb{N}\}$, it follows that $\limsup_{n \longrightarrow \infty} \mathcal{G}_{\mathbb{B}}(n) = \infty$. ☐

Let $\mathbb{B}$ be an additive base of order 2, then it is well-known that

$$\#\{n \leq x|\ n \in \mathbb{B}\} \geq \sqrt{x}.$$

In line with this tied with Theorem 13.3 the solution to the Erdős-Turán additive bases conjecture is an easy consequence.

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