Metric Conventions and the FLRW Metric

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Abstract

We investigate the flat space time Friedmann Lemaitre Robertson Walker model of cosmology in the light of two metric signatures:$(-,+,+,+)$ and $(+,-,-,-)$. We arrive at the interesting fact that with the $(+,-,-,-)$signature both the Ricci scalar and the energy component of the stress energy tensor are negative for all points no cosmological time.

Introduction

The flat space time Friedmann Lemaitre Robertson Walker model of cosmology has been explored in the light of two metric signatures:$(-,+,+,+)$ and $(+,-,-,-)$. We arrive at the interesting fact that with the $(+,-,-,-)$ signature both the Ricci scalar and the energy component of the stress energy tensor are negative for all points no cosmological time.

The Metric and some Associated Properties

Metric[general form]

$$ds^2 = g_{tt}c^2dt^2 + g_{xx}dx^2 + g_{yy}dy^2 + g_{zz}dz^2 \quad (1)$$

The coefficients g_{xx}, g_{yy} and g_{zz} always have the same sign. This sign is opposite to that of g_{tt}

We also have

$$g^{\alpha i}g_{i\beta} = \delta^{\alpha}{}_{\beta} \quad (2)$$

The above implies that for the orthogonal systems

$$g^{\alpha i}g_{i\beta} = \delta^{\alpha}{}_{\beta}$$

$$g^{\alpha\alpha}g_{\alpha\alpha} = 1$$

with no summation on α

$$g^{\alpha\alpha} = \frac{1}{g_{\alpha\alpha}} \quad (3)$$

With that information we proceed with the conventions

We have by definition[1]

$$g^{\alpha\beta} = \frac{Cofactor\ of\ g_{\alpha\beta}}{\det g} \quad (4)$$

where matrix $g = [g_{\alpha\beta}]_{4\times 4}$

From (4) by applying the rules for determinants we can prove (2)

There is also another important observation that we might make. If the sign of each $g_{\alpha\beta}$ reverses the sign of $\det g$ will not change since g is a 4×4 matrix. We may infer that if the sign of each $g_{\alpha\beta}$ reverses then the sign of cofactor of $g_{\alpha\beta}$ will reverse since this cofactor corresponds to a 3×3 matrix. We will make use of this property later.

Regarding the metric we consider two conventions and their impact on FLRW metric

With Signature 1

Signature:$(-,+,+,+)$

$$ds^2 = -|g_{tt}|c^2dt^2 + |g_{xx}|dx^2 + |g_{yy}|dy^2 + |g_{zz}|dz^2 \quad (5)$$

$ds^2 = -c^2d\tau^2$;if $ds^2 = c^2d\tau^2$then proper time interval becomes imaginary for time like paths

FLRW metric[2] for the flat space time model of cosmology in this convention reads

$$ds^2 = -c^2dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) \quad (6)$$

we have the following results[3]

$$R_{tt} = -3\frac{\ddot{a}}{a}; R_{xx} = R_{yy} = R_{zz} = c^{-2}(a\ddot{a} + 2\dot{a}^2) \ (7.1)$$

We set c=1

Therefore

$$R_{tt} = -3\frac{\ddot{a}}{a}; R_{xx} = R_{yy} = R_{zz} = (a\ddot{a} + 2\dot{a}^2) \quad (7.2)$$

$$R = g^{\mu\mu}R_{\mu\mu} \ (8)$$

$$R = g^{tt}R_{tt} + g^{xx}R_{xx} + g^{yy}R_{yy} + g^{zz}R_{zz}$$

$$R = -R_{tt} + \frac{1}{a^2}\left(R_{xx} + R_{yy} + R_{zz}\right)$$

$$R = -\left(-3\frac{\ddot{a}}{a}\right) + \frac{3}{a^2}(a\ddot{a} + 2\dot{a}^2)$$

The Ricci scalar reads

$$R = \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \quad (9)$$

$$R_{tt} - \frac{1}{2}Rg_{tt} = 8\pi G T_{tt}; c = 1$$

$$-3\frac{\ddot{a}}{a} - \frac{1}{2}6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)(-1) = 8\pi G T_{tt}$$

$$6\frac{\dot{a}^2}{a^2} = 8\pi G T_{tt}$$

$$T_{tt} > 0 \quad (10)$$

With Signature2

Signature:$(+,-,-,-)$

$$ds^2 = |g_{tt}|c^2dt^2 - |g_{xx}|dx^2 - |g_{yy}|dy^2 - |g_{zz}|dz^2 \quad (11)$$

$ds^2 = c^2 d\tau^2$;

Christoffel symbols

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}g^{\alpha s}\left[\frac{\partial g_{\beta s}}{\partial x^{\alpha}} + \frac{\partial g_{\alpha s}}{\partial x^{\beta}} - \frac{\partial g_{\alpha\beta}}{\partial x^{s}}\right] \quad (12)$$

If convention changes then the sign of each $g_{\alpha\beta}$ and $g^{\alpha\beta}$ reverses . This has been already discussed. The sign of $\Gamma^{\alpha}{}_{\beta\gamma}$ does not change. The Riemann tensor and the Ricci tensor remain unaltered by convention . But what about the sign of the Ricci scalar and the stressenergy tensor? Let us check

Friedmann metric in this convention

$$ds^2 = -c^2dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) \quad (13)$$

Since the Ricci tensor components remain unchanged[discussed earlier] we have

$$R_{tt} = -3\frac{\ddot{a}}{a}; R_{xx} = R_{yy} = R_{zz} = c^{-2}(a\ddot{a} + 2\dot{a}^2) \ \ (14.1)$$

Setting c to unity

$$R_{tt} = -3\frac{\ddot{a}}{a}; R_{xx} = R_{yy} = R_{zz} = (a\ddot{a} + 2\dot{a}^2) \ \ (14.2)$$

$$R = g^{\mu\mu}R_{\mu\mu}$$

$$R = g^{tt}R_{tt} + g^{xx}R_{xx} + g^{yy}R_{yy} + g^{zz}R_{zz}$$

$$R = R_{tt} - \frac{1}{a^2}\left(R_{xx} + R_{yy} + R_{zz}\right)$$

$$R = \left(-3\frac{\ddot{a}}{a}\right) - \frac{3}{a^2}(a\ddot{a} + 2\dot{a}^2)$$

then

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)$$

$$R < 0 \ \ (15)$$

From then field equations

$$R_{tt} - \frac{1}{2}Rg_{tt} = 8\pi G T_{tt}$$

$$-3\frac{\ddot{a}}{a} - \frac{1}{2}6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = 8\pi G T_{tt}$$

$$T_{tt} < 0 \ \ (16)$$

Convention changes the sign and the value of T_{tt}

Conclusion

With the signature $(+,-,-,-)$ the Ricci scalar and T_{tt} are negative while with the signature $(-,+,+,+)$the Ricci scalar and T_{tt} are positive.

References

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