“Primeless” Sieves for Primes and for Prime Pairs with Gap 2m

Sally Myers Moite

ABSTRACT Numbers of form 6N – 1 and 6N + 1 factor into numbers of the same form. This observation provides elimination sieves for numbers N that lead to primes and prime pairs. The sieves do not explicitly reference primes.

Introduction. All primes except 2 and 3 are of the form 6N – 1 or 6N + 1. Also, if any numbers 6N – 1 or 6N + 1 factor, their factors are (6c +- 1) * (6d +-1). Sequences of numbers N that give primes or twin or cousin prime pairs appear in the Online Encyclopedia of Integer Sequences[1]. In particular, the sequence A067611[2] gives numbers 6cd +-c +- d, which are the numbers N for which 6N – 1 and 6N + 1 are not both prime. This paper lists the sieves for prime gaps 6k + 2, 6k – 2, and 6k in matrix form. It includes worksheets that apply these sieves to numbers N = 1 to 68.

Twin Primes and Gap 8

Twin primes sieve matrix. Twin primes other than 3 and 5 are of the form 6N – 1 and 6N + 1. If the number 6N – 1 factors, it factors as (6c – 1) * (6d + 1) or (6c + 1) * (6d – 1) which give equations N = 6cd + c – d or N = 6cd – c + d. If the number 6N + 1 factors, it factors as (6c – 1) * (6d – 1) or (6c + 1) * (6d + 1) which give equations N = 6cd – c – d or 6cd + c + d. Thus, if 6N – 1 and 6N + 1 are prime, N cannot be of the form 6cd +- c +- d.

The numbers 6cd +- c +- d, for c, d positive integers, can be formed into 2 X 2 blocks.

6cd – c – d   6cd + c – d
6cd – c + d   6cd + c + d

or

(6c – 1)d – c   (6c – 1)d + c
(6c + 1)d – c   (6c + 1)d + c

These blocks give the sieve matrix below in which alternate rows are multiples of 6c – 1 increased or decreased by c, and multiples of 6c + 1 increased or decreased by c.

4, 6, 9, 11, 14, 16, 19, 21, 24, 26, ...
6, 8, 13, 15, 20, 22, 27, 29, 34, 36, ...
9, 13, 20, 24, 31, 35, 42, 46, 53, 57, ...
11, 15, 24, 28, 37, 41, 50, 54, 63, 67, ...
14, 20, 31, 37, 48, 54, 65, 71, 82, 88, ...
16, 22, 35, 41, 54, 60, 73, 79, 92, 98, ...
19, 27, 42, 50, 65, 73, 88, 96, 111, 119, ...
21, 29, 46, 54, 71, 79, 96, 104, 121, 129, ...
24, 34, 53, 63, 82, 92, 111, 121, 140, 150, ...
26, 36, 57, 67, 88, 98, 119, 129, 150, 160, ...

[1] OEIS, oeis.org, A046953, A046954, A002822, A056956.
[2] Ibid.

...

Note that, for example, the third row (or column) contains numbers that differ by 2 from multiples of 11 = 6*2 - 1, and the eighth row contains numbers that differ by 4 from multiples of 25 = 6*4 + 1.

A formula for this matrix is
a(m, n) = 6*floor((m+1)/2)*floor((n+1)/2) + ((-1)^n)*floor((m+1)/2) + ((-1)^m)*floor((n+1)/2).

Figure 1 shows a worksheet that sieves the numbers 1 to 68. Multiples of 6c – 1 and 6c + 1 are marked with a dot, numbers eliminated by the sieve are marked by X. The underlined numbers have no X in their column and give rise to twin primes.

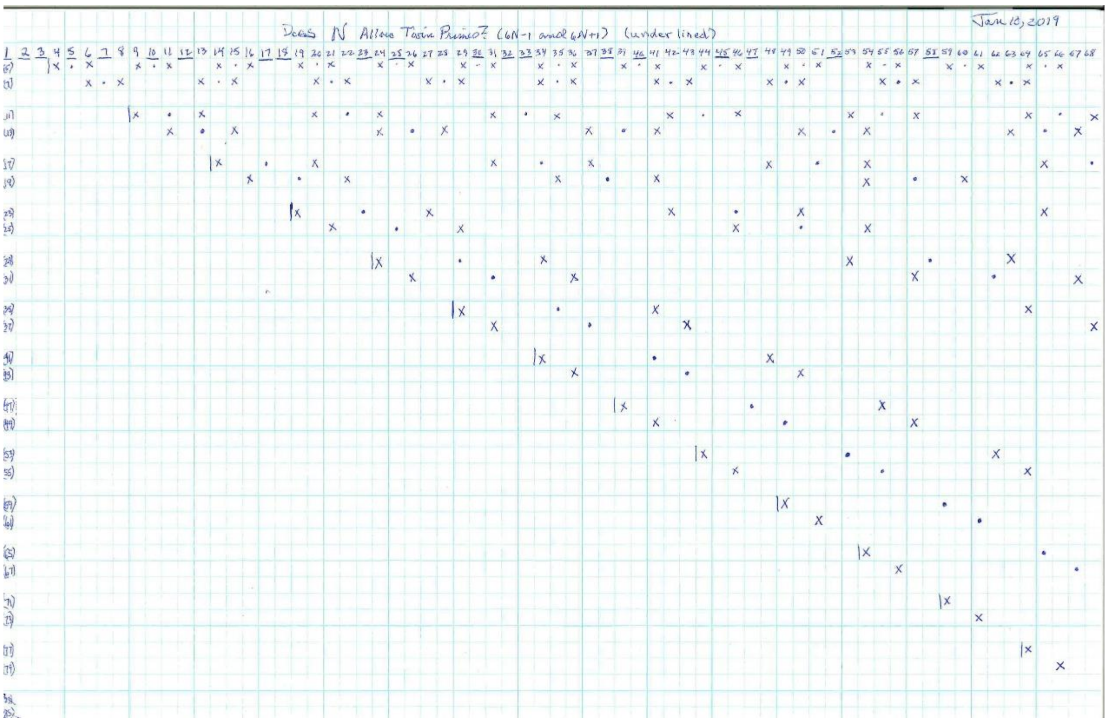

Figure 1 - Worksheet for twin primes

Gap 8 sieve matrix. Primes with gap 8 except for 3 and 11 can be written 6N – 1 and 6N + 7. The sieve array for gap 8 consists of 2 X 2 blocks, for c >= 1, d >= 1, which are

6cd - c - d - 1   6cd + c - d
6cd - c + d   6cd + c + d - 1

or

(6c - 1)d - c - 1   (6c - 1)d + c
(6c + 1)d - c   (6c + 1)d + c – 1.

The sieve matrix begins

3, 6, 8, 11, 13, 16, 18, 21, 23, 26, ...

6, 7, 13, 14, 20, 21, 27, 28, 34, 35, ...
8, 13, 19, 24, 30, 35, 41, 46, 52, 57, ...
11, 14, 24, 27, 37, 40, 50, 53, 63, 66, ...
13, 20, 30, 37, 47, 54, 64, 71, 81, 88, ...
16, 21, 35, 40, 54, 59, 73, 78, 92, 97, ...
18, 27, 41, 50, 64, 73, 87, 96, 110, 119, ...
21, 28, 46, 53, 71, 78, 96, 103, 121, 128, ...
23, 34, 52, 63, 81, 92, 110, 121, 139, 150, ...
26, 35, 57, 66, 88, 97, 119, 128, 150, 159, ...
...

A formula is

a(m,n) = 6*floor((m+1)/2)*floor((n+1)/2) + ((-1)^n)*floor((m+1)/2) + ((-1)^m)*floor((n+1)/2) - (m+n+1) mod 2, m,n >= 1.

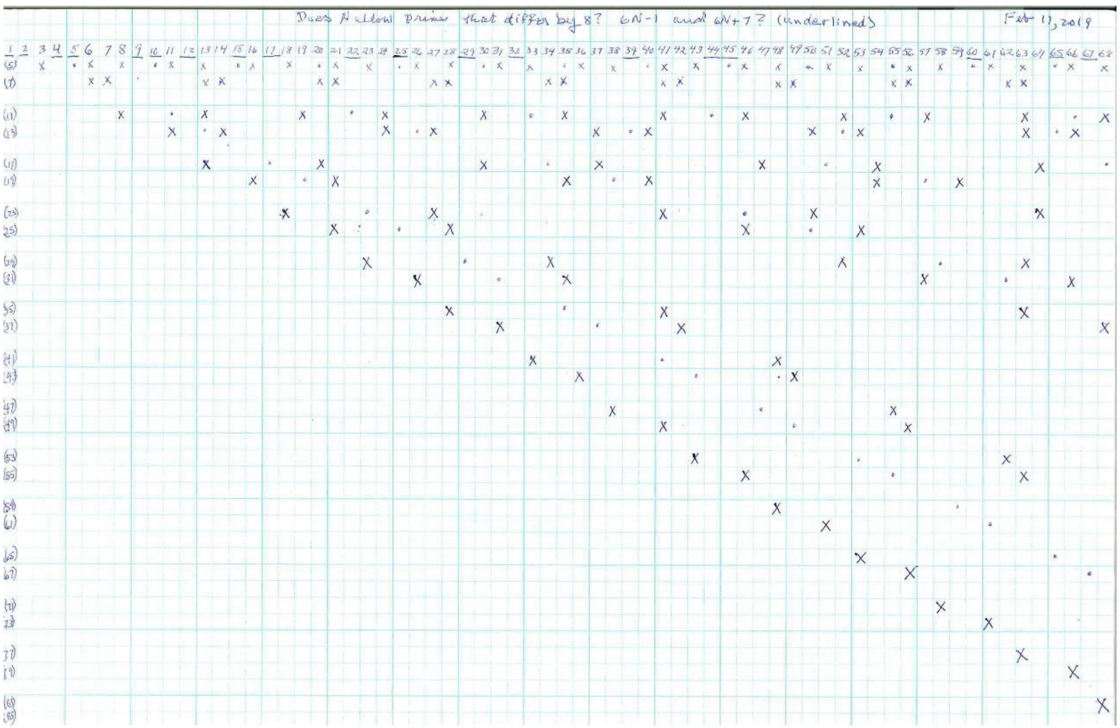

Figure 2 - Worksheet for prime pairs with gap 8.

Gap 6k + 2 matrices. Prime pairs with gap 6k + 2 are of the form 6N – 1 and 6N + 6k + 1. The sieve array consists of 2 X 2 blocks, for c >= 1, d >= 1, which are

6cd - c - d - k   6cd + c - d
6cd - c + d   6cd + c + d - k

or

(6c - 1)d - c - k   (6c - 1)d + c
(6c + 1)d - c   (6c + 1)d + c – k

These are sieves for twin primes when k = 0, and for prime pairs with gap 8 when k = 1.

Cousin Primes

Cousin primes sieve matrix. Cousin primes, prime pairs with gap 4, except for 3 and 7, are of the form 6N + 1 and 6N + 5. The gap 4 sieve array consists of 2 X 2 blocks, for c >= 1, d >= 1, which are

$$\begin{array}{ll} 6cd - c - d & 6cd + c - d - 1 \\ 6cd - c + d - 1 & 6cd + c + d \end{array}$$

or

$$\begin{array}{ll} (6c - 1)d - c & (6c - 1)d + c - 1 \\ (6c + 1)d - c - 1 & (6c + 1)d + c. \end{array}$$

The sieve array begins

```
  4,   5,   9,  10,  14,  15,  19,  20,  24,  25, ...
  5,   8,  12,  15,  19,  22,  26,  29,  33,  36, ...
  9,  12,  20,  23,  31,  34,  42,  45,  53,  56, ...
 10,  15,  23,  28,  36,  41,  49,  54,  62,  67, ...
 14,  19,  31,  36,  48,  53,  65,  70,  82,  87, ...
 15,  22,  34,  41,  53,  60,  72,  79,  91,  98, ...
 19,  26,  42,  49,  65,  72,  88,  95, 111, 118, ...
 20,  29,  45,  54,  70,  79,  95, 104, 120, 129, ...
 24,  33,  53,  62,  82,  91, 111, 120, 140, 149, ...
 25,  36,  56,  67,  87,  98, 118, 129, 149, 160, ...
 ...
```

A formula is

a(m,n) = 6*floor((m+1)/2)*floor((n+1)/2) + ((-1)^n)*floor((m+1)/2) + ((-1)^m)*floor((n+1)/2) - (m+n) mod 2, m,n >= 1.

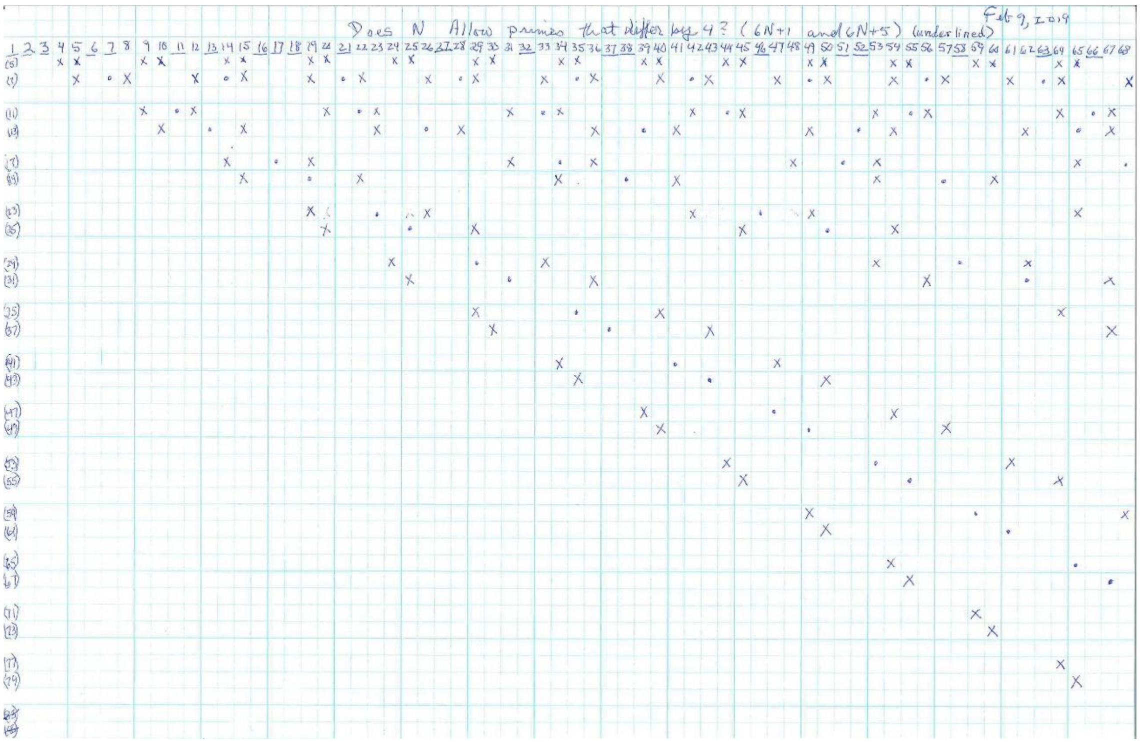

Figure 3 - Worksheet for cousin primes.

Gap 6k – 2 matrices. Prime pairs with gap 6k – 2 are of the form 6N + 1 and 6N + 6k – 1. The gap 6k – 2 sieve array consists of 2 X 2 blocks, for c >= 1, d >= 1, which are

$$\begin{matrix} 6cd - c - d & 6cd + c - d - k \\ 6cd - c + d - k & 6cd + c + d \end{matrix}$$

or

$$\begin{matrix} (6c - 1)d - c & (6c - 1)d + c - k \\ (6c + 1)d - c - k & (6c + 1)d + c. \end{matrix}$$

For k = 1, this gives the sieve array for cousin primes.

Sexy Primes

Sexy primes 6N – 1 and 6N + 5 matrix. One type of pair with gap 6 is of the form 6N – 1 and 6N + 5. The sieve array consists of 2 X 2 blocks, for c >= 1, d >= 1, which are

$$\begin{matrix} 6cd + c - d - 1 & 6cd + c - d \\ 6cd - c + d - 1 & 6cd - c + d \end{matrix}$$

or

$$\begin{matrix} (6c - 1)d + c - 1 & (6c - 1)d + c \\ (6c + 1)d - c - 1 & (6c + 1)d - c. \end{matrix}$$

The sieve matrix begins

5, 6, 10, 11, 15, 16, 20, 21, 25, 26, ...
5, 6, 12, 13, 19, 20, 26, 27, 33, 34, ...
12, 13, 23, 24, 34, 35, 45, 46, 56, 57, ...
10, 11, 23, 24, 36, 37, 49, 50, 62, 63, ...
19, 20, 36, 37, 53, 54, 70, 71, 87, 88, ...
15, 16, 34, 35, 53, 54, 72, 73, 91, 92, ...
26, 27, 49, 50, 72, 73, 95, 96, 118, 119, ...
20, 21, 45, 46, 70, 71, 95, 96, 120, 121, ...
33, 34, 62, 63, 91, 92, 120, 121, 149, 150, ...
25, 26, 56, 57, 87, 88, 118, 119, 149, 150, ...
...

A formula for this array is

a(m,n) = 6*floor((m+1)/2)*floor((n+1)/2) + ((-1)^(m+1))*floor((m+1)/2) + ((-1)^m)*floor((n+1)/2) - n mod 2, m,n >= 1.

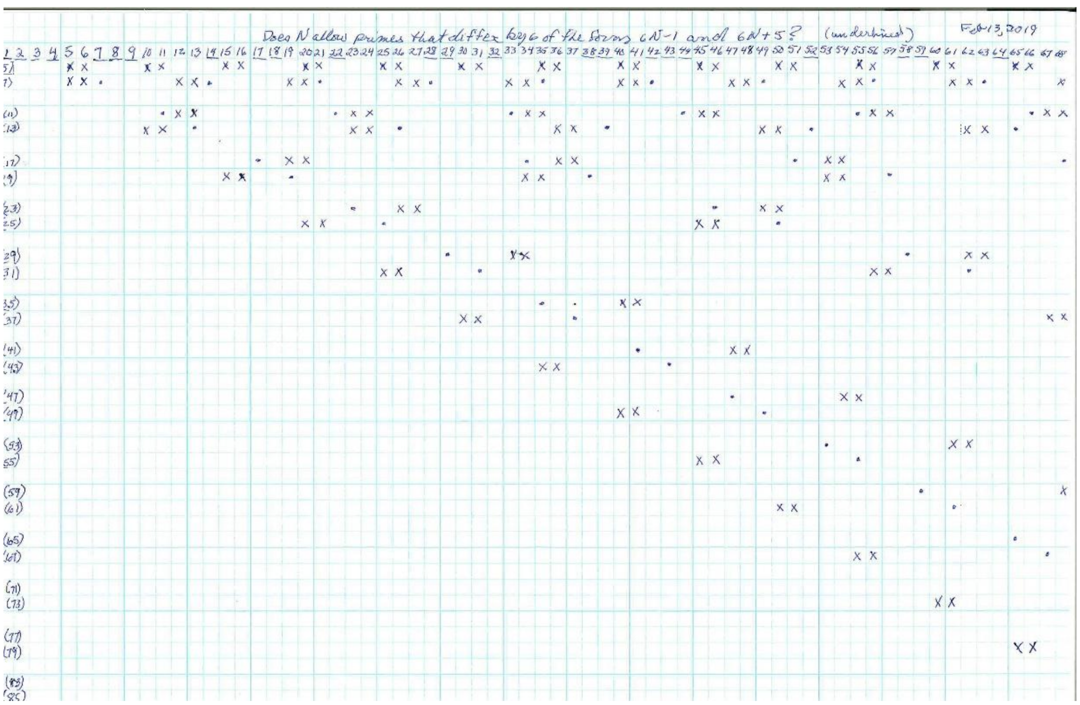

Figure 4 Worksheet for primes 6N - 1 and 6N + 1.

Gap 6k matrices of 6N – 1 type, and 6N – 1 primes. Some prime pairs with gap 6k are of the form 6N – 1 and 6N + 6k – 1. The sieve array consists of 2 X 2 blocks, for c >= 1, d >= 1, which are

6cd + c - d - k   6cd + c - d
6cd - c + d - k   6cd - c + d

or

(6c - 1)d + c – k   (6c - 1)d + c
(6c + 1)d - c - k   (6c + 1)d - c.

These are sieves for primes of form 6N – 1 when k = 0, and for prime pairs with gap 6 above when k = 1.

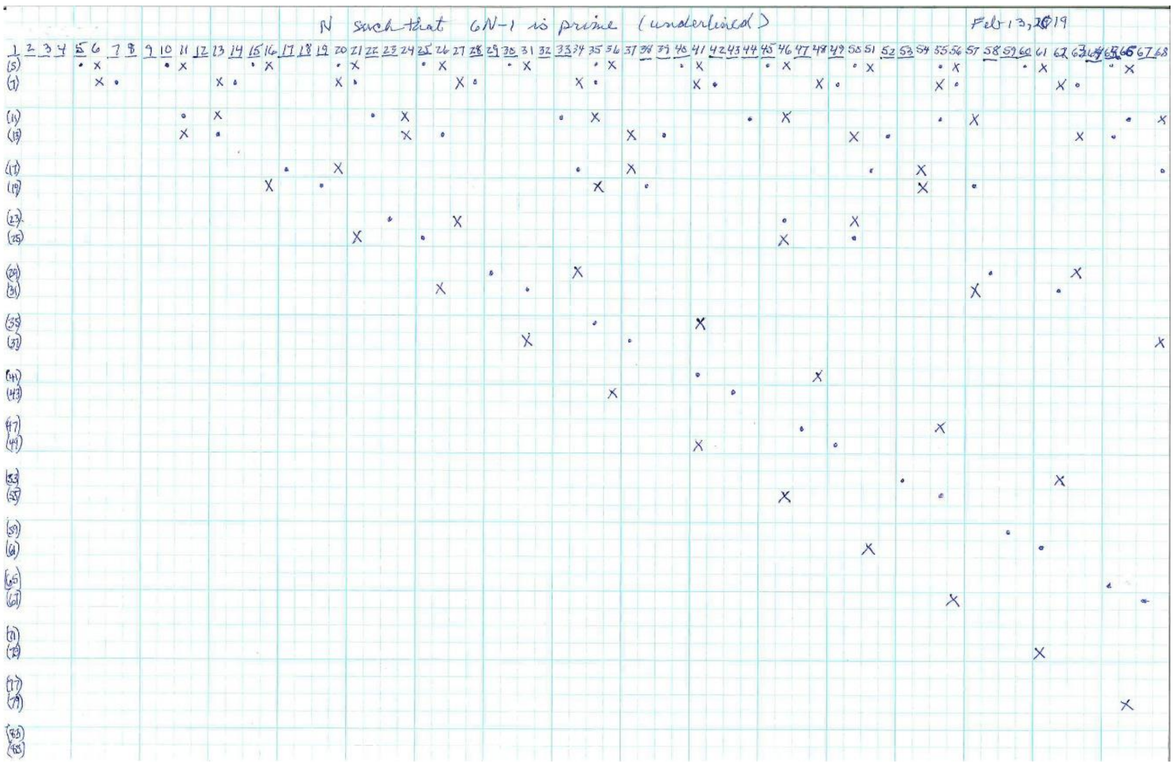

Figure 5 Worksheet for primes 6N - 1.

Sexy primes 6N + 1 and 6N + 7 matrix. The other type of pair with gap 6 is of the form 6N + 1 and 6N + 7. The sieve array for these pairs consists of 2 X 2 blocks, for c >= 1, d >= 1, which are

6cd - c - d - 1   6cd - c - d
6cd + c + d - 1   6cd + c + d

or

(6c - 1)d - c - 1   (6c - 1)d - c
(6c + 1)d + c - 1   (6c + 1)d + c.

The sieve matrix is

3, 4, 8, 9, 13, 14, 18, 19, 23, 24, ...
7, 8, 14, 15, 21, 22, 28, 29, 35, 36, ...
8, 9, 19, 20, 30, 31, 41, 42, 52, 53, ...
14, 15, 27, 28, 40, 41, 53, 54, 66, 67, ...
13, 14, 30, 31, 47, 48, 64, 65, 81, 82, ...
21, 22, 40, 41, 59, 60, 78, 79, 97, 98, ...

18, 19, 41, 42, 64, 65, 87, 88, 110, 111, ...
28, 29, 53, 54, 78, 79, 103, 104, 128, 129, ...
23, 24, 52, 53, 81, 82, 110, 111, 139, 140, ...
35, 36, 66, 67, 97, 98, 128, 129, 159, 160, ...
...

A formula for the matrix is

a(m,n) = 6*floor((m+1)/2)*floor((n+1)/2) + ((-1)^m)*floor((m+1)/2) + ((-1)^m)*floor((n+1)/2) - n mod 2, m,n >= 1.

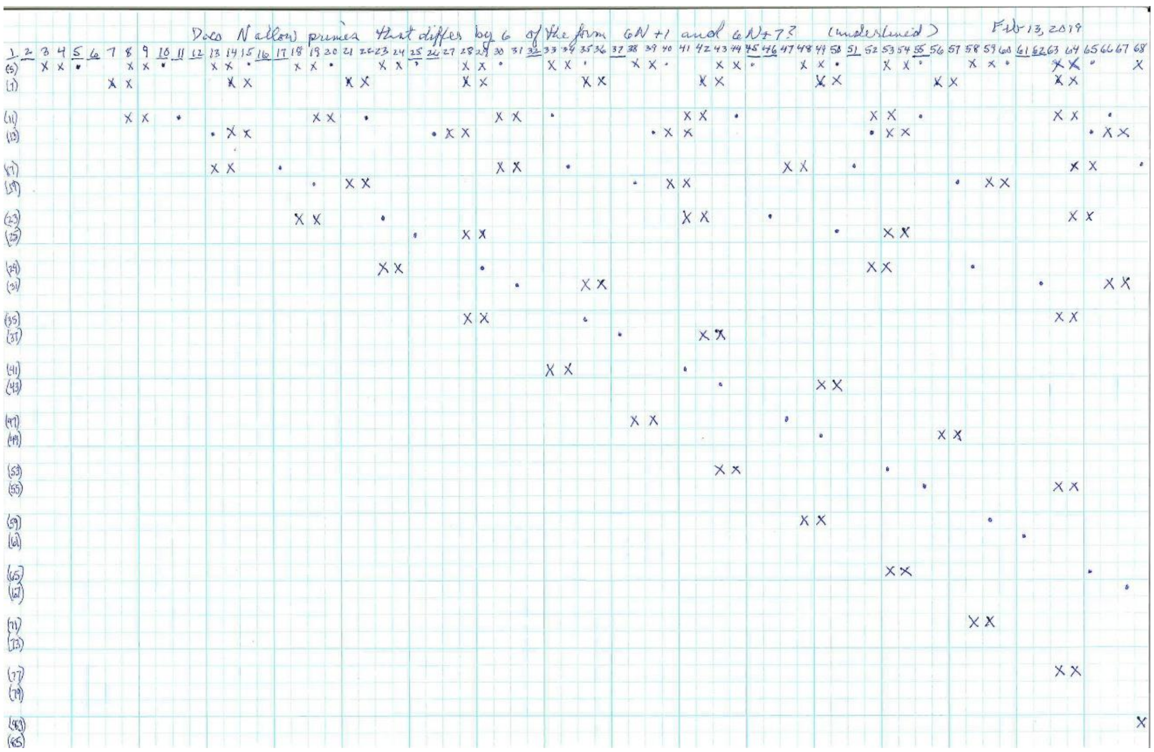

Figure 6 - Worksheet for pairs 6N + 1 and 6N + 7.

Gap 6k matrices of 6N + 1 type, and 6N + 1 primes. Some prime pairs with gap 6k are of the form 6N + 1 and 6N + 6k + 1. The sieve array for these pairs consists of 2 X 2 blocks, for c >= 1, d >= 1, which are

6cd - c - d - k   6cd - c - d
6cd + c + d - k   6cd + c + d

or

(6c - 1)d - c - k   (6c - 1)d - c
(6c + 1)d + c - k   (6c + 1)d + c.

These are sieves for primes of form 6N + 1 when k = 0, and for prime pairs with gap 6 above when k = 1.

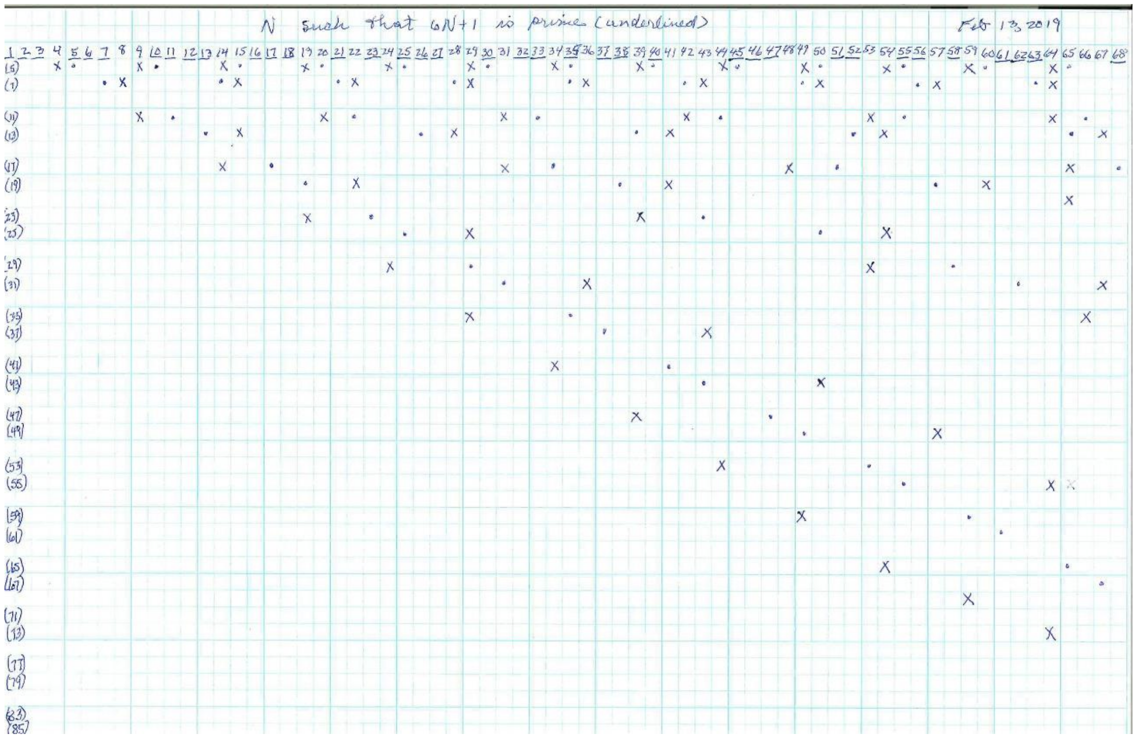

Figure 7 - Worksheet for primes 6N + 1.

Note. The matrix for twin primes appears in OEIS[3]. My thanks to the editors at OEIS for improvements to the writeups for the other matrices, which ultimately were not accepted by OEIS.

Bibliography

Lampret, S. (2014). Sieving 2m-Prime Pairs. *Notes on Number Theory and Discrete Mathematics, 20*(3), 54–60.

[3] OEIS, oeis.org, A323674.