# A class of singular binomial eigenvalue equations

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## Abstract

This paper shows that a class of binomial equation can be solved as an eigenvalue problem. The major finding is that the solution can be computed in terms of elementary functions.

## Introduction

Consider the singular binomial equation

$$u''(x)+\left[\frac{a}{x^2}+bx^{2q}\right]u(x)=0 \tag{1}$$

with $a$, $b$, $q$, are arbitrary parameters.

The conditions of the existence of the Liouvillian solution of the kind of equation (1) were discuted in [1]. However, it seems that the solution of this equation has not been computed in terms of elementary functions in the literature. Also, its study as an eigenvalue problem is not available in literature. Therefore no one can answer the question: Can the solution of the corresponding eigenvalue problem be computed in terms of elementary functions? This work assumes such a prediction. One may see that such a solution is of a great importance since the analytical properties of elementary functions are well known and widely studied in the literature. Moreover, they are easy to be used in practical application. To do so, the required point transformation [2, 3] is formulated and applied to map the constant coefficient equation into a general class of equations so that under the convenient choice of parameters it is clearly shown that equation (1) belongs to the previous class of equations (section 2). The explicit solution is given (section 3) and (1) is mathematically shown to be equivalent to a Sturm-Liouville eigenvalue problem for which the solution is computed in terms of elementary functions over the interval [0, L] (Section 4).

## 2-Point canonical transformation of the constant coefficient equation

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Let

$$y''(\tau) + c\,y(\tau) = 0 \tag{2}$$

be the constant coefficient equation, where prime means differentiation with respect to $\tau$, $c$ is arbitrary parameter.

Consider as a general formulation, the following point transformation [2, 3]

$$y(\tau) = f(u)\varphi^m(x), \qquad \tau = g^{\ell}(x) \tag{3}$$

where the exponents $\ell$ and $m$ are arbitrary parameters.

By application of (3), (1) may give the desired general class of equations.

Using (3) one may compute

$$y'(\tau) = \frac{f'(u)}{\ell}\frac{g^{1-\ell}(x)}{g'(x)}\varphi^m(x)\frac{du}{dx} + \frac{m}{\ell}f(u)\varphi'(x)\varphi^{m-1}\frac{g^{1-\ell}(x)}{g'(x)} \tag{4}$$

from which, it follows after a few mathematical manipulations

$$\begin{aligned} y''(\tau) = {} & \frac{f'(u)}{\ell^2}\varphi^m(x)\frac{g^{2(1-\ell)}(x)}{g'^2(x)}\frac{d^2u}{dx^2} + \frac{f''(u)}{\ell^2}\varphi^m(x)\frac{g^{2(1-\ell)}(x)}{g'^2(x)}\left(\frac{du}{dx}\right)^2 + \\ & \left[\frac{2m}{\ell^2}f'(u)\varphi'(x)\varphi^{m-1}(x)\frac{g^{2(1-\ell)}(x)}{g'^2(x)} + \frac{f'(u)}{\ell^2}\varphi^m(x)\frac{g^{(1-2\ell)}(x)}{g'(x)}\left((1-\ell) - \frac{g''(x)g(x)}{g'^2(x)}\right)\right]\frac{du}{dx} + \\ & \frac{m}{\ell^2}f(u)\varphi''(x)\varphi^{m-1}(x)\frac{g^{2(1-\ell)}(x)}{g'^2(x)} + \frac{m(m-1)}{\ell^2}f(u)\varphi'^2(x)\varphi^{m-2}(x)\frac{g^{2(1-\ell)}(x)}{g'^2(x)} + \\ & \frac{m}{\ell^2}f(u)\varphi'(x)\varphi^{m-1}(x)\frac{g^{(1-2\ell)}(x)}{g'(x)}\left((1-\ell) - \frac{g''(x)g(x)}{g'^2(x)}\right) \end{aligned} \tag{5}$$

so that (2) is easily rewriting in this form

$$\begin{aligned} & \frac{d^2u}{dx^2} + \frac{f''(u)}{f'(u)}\left(\frac{du}{dx}\right)^2 + \left((1-\ell)\frac{g'(x)}{g(x)} - \frac{g''(x)}{g'(x)} + 2m\frac{\varphi'(x)}{\varphi(x)}\right)\frac{du}{dx} + \\ & m\frac{f(u)}{f'(u)}\left[\frac{\varphi''(x)}{\varphi(x)} + (m-1)\left(\frac{\varphi'(x)}{\varphi(x)}\right)^2 + \frac{\varphi'(x)}{\varphi(x)}\left((1-\ell)\frac{g'(x)}{g(x)} - \frac{g''(x)}{g'(x)}\right) + \frac{c\,\ell^2}{m}\frac{g'^2(x)}{g^{2(1-\ell)}(x)}\right] = 0 \end{aligned} \tag{6}$$

(6) is desired general class of equations.

Now from (6), one can deduce (1).

By setting $f(u) = u(x)$, $\ell = 1$, $g'(x) = \varphi(x)$, (6) becomes

$$u''(x)+\left(2m-1\right)\frac{\varphi'(x)}{\varphi(x)}u'(x)+\left[m(m-2)\frac{\varphi'^2(x)}{\varphi^2(x)}+m\frac{\varphi''(x)}{\varphi(x)}+c\,\varphi^2(x)\right]u(x)=0 \tag{7}$$

For $m=\frac{1}{2}$, $\varphi(x)=x^q$, (7) is mapped into

$$u''(x)+\left[-\frac{1}{2}q\left(1+\frac{1}{2}q\right)\frac{1}{x^2}+c\,x^{2q}\right]u(x)=0 \tag{8}$$

Setting $-\frac{1}{2}q\left(1+\frac{1}{2}q\right)=a$, $b=c$, in (8) leads to (1). Thus (1) belongs to the class of (6).

**3-Exact and explicit solution of (8)**

Since $f(u)=u(x)$, $\ell=1$, $m=\frac{1}{2}$, $g'(x)=\varphi(x)=x^q$, (3) becomes:

$$y(\tau)=u(x)\,x^{\frac{q}{2}}, \qquad \tau=\frac{1}{q+1}x^{q+1} \tag{9}$$

from which

$$u(x)=x^{-\frac{q}{2}}y(\tau) \tag{10}$$

since $y(\tau)$ is a solution of (2). Then

$$y(\tau)=A\sin\left(\sqrt{c}\,\tau\right)+B\cos\left(\sqrt{c}\,\tau\right) \tag{11}$$

where $c\succ 0$;

Hence

$$u(x)=x^{-\frac{q}{2}}\left[A\sin\left(\sqrt{c}\,\frac{x^{q+1}}{q+1}\right)+B\cos\left(\sqrt{c}\,\frac{x^{q+1}}{q+1}\right)\right] \tag{12}$$

**4-Eigenvalue problem**

The confined equation (8) over the interval [0, L] may be considered as a Sturm-Liouville eigenvalue problem. In the other word, the problem is to compute $u(x)$ such that $u(x)\in L^2(\left[0,L\right])$. In this purpose, let us consider (12), setting $B=0$, yields

$$u(x) = x^{-\frac{q}{2}}\left[A\sin\left(\sqrt{c}\,\frac{x^{q+1}}{q+1}\right)\right] \tag{13}$$

As $u(x)$ must satisfy the boundary conditions $u(0) = u(L) = 0$, it is required that $-1 \prec q \prec 0$, and $L^{-\frac{q}{2}} A\sin\left(\sqrt{c}\,\frac{L^{q+1}}{q+1}\right) = 0$, which leads to the quantization of

$$c = \frac{n\,(q+1)^2 \pi^2}{L^{q+1}} \tag{14}$$

So $c$ can be considered as the spectral parameter for the eigenvalue problem studied.

## References


[1] M. Setoyonagi, ''Liouvillian solutions of the differential equation $y''+S(x)y = 0$ with $S(x)$ binomial'', Proceedings of the American Mathematical Society, Vol. 100, N°4, 1987.

[2] Y. J. F. Kpomahou, J. Akande, D. K. K. Adjaï, L. H. Koudahoun and M. D. Monsia, Elementary functions and Schrödinger equations, (2018) Math.Phys.,viXra.org/1804.0475v1.pdf.

[3] J. Akande, D. K. K. Adjaï, and M. D. Monsia, On Schrödinger equations equivalent to constant coefficient equations, (2018) Math. Phys.,viXra.org/1804. 0222v1.pdf.