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# New Equations Derived from the Navier-Stokes Equations for the Description of the Motion of Viscous incompressible Fluids with a Proposed Solution*

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**Abstract** This note represents an attempt to give a solution of Navier-Stokes equations under the assumptions $(A)$ of the problem as described by the Clay Mathematics Institute [2]. After elimination of the pressure, we obtain the fundamental equations function of the velocity vector $u$ and vorticity vector $\Omega = curl(u)$, then we deduce the new equations for the description of the motion of viscous incompressible fluids, derived from the Navier-Stokes equations, given by:

$$\nu \Delta \Omega - \frac{\partial \Omega}{\partial t} = 0$$
$$\Delta p = -\sum_{i=1}^{i=3}\sum_{j=1}^{j=3} \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

Then, we give a proof of that the solutions of the Navier-Stokes equations $u$ and $p$ are smooth functions and $u$ verifies the condition of bounded energy.

**Keywords** Prime numbers · Fermat's Last Theorem · Diophantine equations.

**Mathematics Subject Classification (2010)** 11AXX · 11D41

***To the memory of my Father who taught me arithmetic.***

* The idea of the title was inspired from the title of the supplement of the book of O.A. Ladyzhenskaya [1].

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## 1 Introduction

As it was described in the paper cited above, the Euler and Navier-Stokes equations describe the motion of a fluid in $R^n$ ($n = 2$ or 3). These equations are to be solved for an unknown velocity vector $u(x,t) = (u_i(x,t), u_2(x,t), \dots, u_n(x,t))^T \in R^n$ and pressure $p(x,t) \in R$ defined for position $x \in R^n$ and time $t \geq 0$.

Here we are concerned with incompressible fluids filling all of $R^n$. The Navier-Stokes equations are given by:

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t) \quad i \in \{1,.,n\} \ (x \in R^n,\ t \geq 0) \quad (1)$$

$$divu = \sum_{i=1}^{i=n} \frac{\partial u_i}{\partial x_i} = 0 \ \ (x \in R^n,\ t \geq 0) \quad (2)$$

with the initial conditions:

$$u(x,0) = u^o(x) \ \ (x \in R^n) \quad (3)$$

where $u^o(x)$ a given vector function of class $C^\infty$, $f_i(x,t)$ are the components of a given external force (e.g gravity), $\nu$ is a positive coefficient (viscosity), and $\Delta$ is the Laplacian in the space variables. Euler equations are equations (1) (2) (3) with $\nu = 0$.

## 2 The Navier-Stokes Equations

We try to present a solution to the Navier-Stokes equations following assumptions $(A)$ as described in [2] that summarized here:

* $(A)$ **Existence and smooth solutions $\in R^3$ the Navier-Stokes equations:**

- Take $\nu > 0$. Let $u^0(x)$ a smooth function such that $div(u^0(x)) = 0$ and satisfying:

$$||\partial^\delta_{x_j} u^0(x)|| \leq C_{\delta K}(1 + ||x||)^{-K} \ on \ R^3 \quad \forall \delta,\, K \quad (4)$$

- Take $f \equiv 0$. Then show that there are functions $p(x,t), u(x,t)$ of class $C^\infty$ on $R^3 \times [0, +\infty)$ satisfying (1),(2),(3),(4) and:

$$\int_{R^3} ||u(x,t)||^2 dx < C \ ,\forall t \geq 0, \ \ \text{(bounded energy)} \quad (5)$$

We consider the Navier-Stokes equations in this case, we take $\nu > 0$ and $f_i \equiv 0$, then equations (1) are written for $n = 3$ as :

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} + u_3\frac{\partial u_1}{\partial z} - \nu\Delta u_1 = -\frac{\partial p}{\partial x} \tag{6}$$

$$\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_2}{\partial y} + u_3\frac{\partial u_2}{\partial z} - \nu\Delta u_2 = -\frac{\partial p}{\partial y} \tag{7}$$

$$\frac{\partial u_3}{\partial t} + u_1\frac{\partial u_3}{\partial x} + u_2\frac{\partial u_3}{\partial y} + u_3\frac{\partial u_3}{\partial z} - \nu\Delta u_3 = -\frac{\partial p}{\partial z} \tag{8}$$

Let:

$$A(u) = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \\ \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & \frac{\partial u_3}{\partial z} \end{pmatrix} \tag{9}$$

The equations (6-7-8) can be written under vectorial form:

$$\frac{\partial u}{\partial t} + A(u).u = \nu\Delta u - gradp \tag{10}$$

Let $\Omega$ the vector $curl(u)$, then:

$$\Omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{vmatrix} \partial_x \\ \partial_y \\ \partial_z \end{vmatrix} \wedge \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix} = \begin{pmatrix} \partial_y u_3 - \partial_z u_2 \\ \partial_z u_1 - \partial_x u_3 \\ \partial_x u_2 - \partial_y u_1 \end{pmatrix} \tag{11}$$

Taking the curl of the both members of (10), then, equation (10) becomes as follows:

$$\boxed{A(u).\Omega - A(\Omega).u = \nu\Delta\Omega - \frac{\partial \Omega}{\partial t}} \tag{12}$$

where:

$$A(\Omega) = \begin{pmatrix} \frac{\partial \omega_1}{\partial x} & \frac{\partial \omega_1}{\partial y} & \frac{\partial \omega_1}{\partial z} \\ \frac{\partial \omega_2}{\partial x} & \frac{\partial \omega_2}{\partial y} & \frac{\partial \omega_2}{\partial z} \\ \frac{\partial \omega_3}{\partial x} & \frac{\partial \omega_3}{\partial y} & \frac{\partial \omega_3}{\partial z} \end{pmatrix} \tag{13}$$

The equations (12) are the fundamental equations of this study. These are nonlinear partial differential equations of the third order. Their resolutions are the solutions of the Navier-Stokes equations.

## 3 The Study of The Fundamental Equations (12)

### 3.1 A New Fundamental Equations of the Navier-Stokes Equations

We re-write the equations (12):

$$A(u).\Omega - A(\Omega).u = \nu\Delta\Omega - \frac{\partial\Omega}{\partial t}$$

We can also write it :

$$A(-u).(-\Omega) - A(-\Omega).(-u) = \nu\Delta\Omega - \frac{\partial\Omega}{\partial t} \tag{14}$$

As $u$ and $\Omega$ are not independent variables, we have $curl(-u) = -curl(u) = -\Omega$, we obtain :

$$A(-u).(-\Omega) - A(-\Omega).(-u) = \nu\Delta(-\Omega) - \frac{\partial(-\Omega)}{\partial t} \tag{15}$$

Comparing the last two equations (14-15), we arrive to:

$$\nu\Delta\Omega - \frac{\partial\Omega}{\partial t} = \nu\Delta(-\Omega) - \frac{\partial(-\Omega)}{\partial t} = -\left(\nu\Delta\Omega - \frac{\partial\Omega}{\partial t}\right)$$

Hence:

$$\nu\Delta\Omega - \frac{\partial\Omega}{\partial t} = 0 \tag{16}$$

From the equation (12), we get necessary that:

$$A(u).\Omega - A(\Omega).u = 0 \tag{17}$$

The first new fundamental equation is (16), from it we will obtain $u(x,t)$. Taking the divergence of the both members of equation (10), we obtain the known equation determining $p(x,t)$ :

$$\Delta p = -\sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j} \cdot \frac{\partial u_j}{\partial x_i} \tag{18}$$

It is therefore the new fundamental differential system:

$$\boxed{\begin{cases} \nu\Delta\Omega - \dfrac{\partial\Omega}{\partial t} = 0 \Longrightarrow \ u \\ \\ \Delta p = -\displaystyle\sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j} \cdot \frac{\partial u_j}{\partial x_i} \Longrightarrow \ p \end{cases}} \tag{19}$$

## 4 Resolution of the equations (19)

From the first equation of (19), we can write that:

$$curl(\nu \Delta u - \frac{\partial u}{\partial t}) = 0 \tag{20}$$

then:

Case 1- $\nu \Delta u - \frac{\partial u}{\partial t} \equiv 0\,(x \in R^n,\ t \geq 0)$;

Case 2- $\nu \Delta u - \frac{\partial u}{\partial t} = K(t)$ with $K$ is a vector function depending only of $t$.

### 4.1 Resolution of the equations (19) case 1

Let the change of variables:

$$x = \nu X \tag{21}$$
$$y = \nu Y \tag{22}$$
$$z = \nu Z \tag{23}$$
$$t = \nu T \tag{24}$$
$$u(x, y, z, t) = U(X, Y, Z, T) \tag{25}$$
$$p(x, y, z, t) = P(X, Y, Z, T) \tag{26}$$

Then:

$$\partial_x u dx + \partial_y u dy + \partial_z u dz + \partial_t u dt = \partial_X U dX + \partial_Y U dY + \partial_Z U dZ + \partial_T U dT$$
$$\nu(\partial_x u dX + \partial_y u dY + \partial_z u dZ + \partial_t u dT) = \partial_X U dX + \partial_Y U dY + \partial_Z U dZ + \partial_T U dT$$
$$\partial_x u = \frac{1}{\nu}\partial_X U,\ \partial_y u = \frac{1}{\nu}\partial_Y U,\ \partial_z u = \frac{1}{\nu}\partial_Z U,\ \partial_t u = \frac{1}{\nu}\partial_T U \tag{27}$$

Then the equation

$$\frac{\partial u}{\partial t} - \nu \Delta u = 0$$

becomes:

$$\boxed{\frac{\partial U}{\partial T} - \Delta U = 0} \tag{28}$$

This is the heat equation!

*4.1.1 Resolution of the Equation (28)*

Noting that $U^0(X,Y,Z) = U^0(\boldsymbol{X}) = U(X,Y,Z,0) = u(x,y,z,0) = u^0(x,y,z)$, then the solution of (28) with $T \geq 0$ satisfying:

$$U \in R^3 \text{ and of class } C^\infty(R^3 \times [0,+\infty)) \tag{29}$$
$$U(\boldsymbol{X},0) = U^0(\boldsymbol{X}) \tag{30}$$

is given by [3]:

$$U(\boldsymbol{X},T) = \frac{1}{2\sqrt{\pi}} \int_{R^3} \frac{U^0(\alpha,\beta,)}{\sqrt{T}} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{4T}} dV \tag{31}$$

where $dV = d\alpha d\beta d$ and $U(\boldsymbol{X},T)$ is unique with $U(\boldsymbol{X},0) = U^0(\boldsymbol{X})$, then $u$ is unique.

We denote:

$$\boldsymbol{X} = (X,Y,Z)^T \tag{32}$$
$$\Gamma = (\alpha,\beta,)^T \tag{33}$$

Then, we can write the norm of $U(\boldsymbol{X},T)$ as:

$$||U(\boldsymbol{X},T)|| \leq \frac{e^{-\dfrac{X^2+Y^2+Z^2}{4T}}}{2\sqrt{\pi T}} \int_{R^3} ||U^0(\alpha,\beta,)|| e^{-\dfrac{(||\Gamma||^2 - 2\Gamma.\boldsymbol{X})}{4T}} dV \tag{34}$$

The presence of the term $e^{-\dfrac{X^2+Y^2+Z^2}{4T}}$ implies that if $||X|| \longrightarrow +\infty$, $||U(\boldsymbol{X},T)|| \longrightarrow 0$ fast enough [4]. Then, for $t$ fixed, $||u(x,y,z,t)|| \longrightarrow 0$ when $\sqrt{x^2+y^2+z^2} \longrightarrow +\infty$, hence, from now, we assume that we are dealing only with such rapidly decreasing velocities.

*4.1.2 Expression of U*

We have:

$$U_1 = \frac{1}{2\sqrt{\pi}} \int_{R^3} \frac{U_1^0(\alpha,\beta,)}{\sqrt{T}} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{4T}} dV \tag{35}$$

$$U_2 = \frac{1}{2\sqrt{\pi}} \int_{R^3} \frac{U_2^0(\alpha,\beta,)}{\sqrt{T}} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{4T}} dV \tag{36}$$

$$U_3 = \frac{1}{2\sqrt{\pi}} \int_{R^3} \frac{U_3^0(\alpha,\beta,)}{\sqrt{T}} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{4T}} dV \tag{37}$$

*4.1.3 Checking $div(U) = 0$*

Let us calculate $\partial_X U_1$, we get:

$$\frac{\partial U_1}{\partial X} = \frac{-1}{4\sqrt{\pi}} \int_{R^3} \frac{(X-\alpha)U_1^0(\alpha,\beta,)}{T\sqrt{T}} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{4T}} dV \tag{38}$$

We can write the above expression as follows:

$$\frac{\partial U_1}{\partial X} = \frac{-1}{2\sqrt{\pi T}} \int_{R^2} d\beta d \int_{\alpha=-\infty}^{\alpha=+\infty} U_1^0(\alpha,\beta,) \frac{\partial}{\partial \alpha} \left( e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{4T}} \right) d\alpha \tag{39}$$

Now we do an integration by parts, we get:

$$\frac{\partial U_1}{\partial X} = \frac{-1}{2\sqrt{\pi T}} \int_{R^2} d\beta d \left[ U_1^0(\alpha,\beta,).e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{4T}} \right]_{\alpha=-\infty}^{\alpha=+\infty} +$$
$$\frac{1}{2\sqrt{\pi T}} \int_{R^2} d\beta d \int_{\alpha=-\infty}^{\alpha=+\infty} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{4T}} \frac{\partial U_1^0(\alpha,\beta,)}{\partial \alpha}.d\alpha \tag{40}$$

Taking into account the assumption that:

$$||\partial_{X_j}^{\delta} U^0(\boldsymbol{X})|| \leq \nu C_{\delta K}(1+\nu||\boldsymbol{X}||)^{-K} \; on \; R^3 \quad \forall \delta, \, K \tag{41}$$

where $X_j$ denotes one of the coordinates $X, Y, Z$, and choosing $K > 1$ and $\delta = 0$, we obtain :

$$||U^0(\boldsymbol{X})|| \leq C_{0K}(1+\nu||\boldsymbol{X}||)^{-K} \tag{42}$$

and the first term of the right member of (40) is zero. Then:

$$\frac{\partial U_1}{\partial X} = \frac{1}{2\sqrt{\pi T}} \int_{R^2} d\beta d \int_{\alpha=-\infty}^{\alpha=+\infty} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{4T}} \frac{\partial U_1^0(\alpha,\beta,)}{\partial \alpha}.d\alpha \tag{43}$$

or:

$$\frac{\partial U_1}{\partial X} = \frac{1}{2\sqrt{\pi T}} \int_{R^3} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{4T}} \frac{\partial U_1^0(\alpha,\beta,)}{\partial \alpha}.dV \tag{44}$$

As a result:

$$div(U) = \sum_{X_j} \frac{\partial U_j}{\partial X_j} = \frac{1}{2\sqrt{\pi T}} \int_{R^3} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{4T}} \sum_{\alpha_j} \frac{\partial U_j^0(\alpha,\beta,)}{\partial \alpha}.dV = 0 \tag{45}$$

because $U^0(\alpha,\beta,)$ satisfies $div(U^0) = \sum_{\alpha_j} \frac{\partial U_j^0(\alpha,\beta,)}{\partial \alpha_j} = 0$.

*4.1.4 Estimation of* $\int_{R^3} ||U(\boldsymbol{X},T)||^2 dV$

We have:

$$||U(\boldsymbol{X},T)||^2 = \sum_i U_i^2 = \frac{1}{4\pi T}\left\|\int_{R^3} U^0(\alpha,\beta,).e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{4T}} dV\right\|^2$$

$$\leq \frac{1}{4\pi T}\int_{R^3} \left\|U^0(\alpha,\beta,)\right\|^2 .e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{2T}} dV \qquad (46)$$

Using the condition (42):

$$||U^0(\boldsymbol{X})|| \leq C_{0K}(1+\nu||\boldsymbol{X}||)^{-K}$$

We obtain as a result:

$$||U(\boldsymbol{X},T)||^2 \leq \frac{C_{0K}^2}{4\pi T}\int_{R^3} \frac{e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{2T}}}{(1+\nu||\sqrt{\alpha^2+\beta^2+^2}||)^{2K}} d\alpha d\beta d \qquad (47)$$

Let us now majorize $\int_{R^3} ||u(\boldsymbol{x},t)||^2 dxdydz$ :

$$\int_{R3} ||u(\boldsymbol{x},t)||^2 dxdydz = \int_{R3} ||U(\boldsymbol{X},T)||^2 dxdydz = \nu^3 \int_{R3} ||U(\boldsymbol{X},T)||^2 dXdYdZ$$

$$\leq \frac{\nu^3 C_{0K}^2}{4\pi T}\int_{R^3}\left[\int_{R^3} \frac{e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{2T}}}{(1+\nu||\sqrt{\alpha^2+\beta^2+^2}||)^{2K}} d\alpha d\beta d\right] dXdYdZ \qquad (48)$$

As the integral $\int_{R^3} e^{-X^2-Y^2-Z^2} dXdYdZ < +\infty$, we can permute the two triple integrals of the above equation. Let:

$$\tau_0 = \frac{\nu^3 C_{0K}^2}{4\pi} \qquad (49)$$

we obtain:

$$\int_{R3} ||u(\boldsymbol{x},t)||^2 dxdydz \leq \frac{\tau_0}{T}\int_{R^3}\left[\int_{R^3} e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{2T}} dXdYdZ\right].\frac{d\alpha d\beta d}{(1+\nu||\sqrt{\alpha^2+\beta^2+^2}||)^{2K}} \qquad (50)$$

Let:

$$I = \int_{R^3} e^{-\frac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{2T}} dXdYdZ \qquad (51)$$

and let the following change of variables:

$$\begin{cases} X = \frac{X-\alpha}{\sqrt{2T}} \Longrightarrow dX = \sqrt{2T}dX \quad and \; X^2 = \frac{(X-\alpha)^2}{2T} \\ Y = \frac{Y-\beta}{\sqrt{2T}} \Longrightarrow dY = \sqrt{2T}dY \quad and \; Y^2 = \frac{(Y-\beta)^2}{2T} \\ Z = \frac{Z-}{\sqrt{2T}} \Longrightarrow dZ = \sqrt{2T}dZ \quad and \; Z^2 = \frac{(Z-)^2}{2T} \end{cases} \tag{52}$$

$I$ is written as:

$$I = (\sqrt{2T})^3 \left[\int_{-\infty}^{+\infty} e^{-X^2} dX\right]^3 = 2T\sqrt{2T}\left[2\int_0^{+\infty} e^{-\xi^2} d\xi\right]^3 = 2T\sqrt{T}.\pi\sqrt{\pi} = 2\pi T\sqrt{\pi T} \tag{53}$$

using the formula $2\int_0^{+\infty} e^{-\xi^2} d\xi = \sqrt{\pi}$. Then the equation (50) becomes:

$$\int_{R3} ||u(\boldsymbol{x},t)||^2 dxdydz \leq 2\tau_0 \pi\sqrt{\pi T}\int_{R^3} \frac{d\alpha d\beta d}{(1+\nu||\sqrt{\alpha^2+\beta^2+^2}||)^{2K}} \tag{54}$$

Let us now:

$$B = \int_{R^3} \frac{d\alpha d\beta d}{(1+\nu||\sqrt{\alpha^2+\beta^2+^2}||)^{2K}} \tag{55}$$

and we use the spherical coordinates:

$$\begin{cases} \alpha = rsin\theta cos\varphi \\ \beta = rsin\theta sin\varphi \\ = rcos\theta \end{cases} \tag{56}$$

the form of the volume $d\alpha d\beta d = r^2 sin\theta dr d\theta d\varphi$ and $B$ becomes:

$$B = \int_{\theta=0}^{\theta=\pi} sin\theta d\theta \int_{\varphi=0}^{\varphi=2\pi} d\varphi \int_0^r \frac{r^2 dr}{(1+\nu r)^{2K}} = 4\pi \int_0^r \frac{r^2 dr}{(1+\nu r)^{2K}} \tag{57}$$

We take $K = 2$, the integral $B$ is convergent when $r \to +\infty$. Let:

$$F = lim_{r\to+\infty} \int_0^r \frac{r^2 dr}{(1+\nu r)^4} = \int_0^{+\infty} \frac{r^2 dr}{(1+\nu r)^4} = \int_0^1 \frac{r^2 dr}{(1+\nu r)^4} + \int_1^{+\infty} \frac{r^2 dr}{(1+\nu r)^4} \tag{58}$$

But :

$$\int_0^1 \frac{r^2 dr}{(1+\nu r)^4} < \int_0^1 r^2 dr = \left[\frac{r^3}{3}\right]_0^1 = \frac{1}{3} \tag{59}$$

We calculate now $\int_1^{+\infty} \frac{r^2 dr}{(1+\nu r)^4}$. Let the change of variables:

$$\xi = 1 + \nu r \Rightarrow r = \frac{\xi - 1}{\nu} \Rightarrow dr = \frac{d\xi}{\nu} \tag{60}$$

then:

$$\int_1^{+\infty} \frac{r^2 dr}{(1+\nu r)^4} = \frac{1}{\nu^3} \int_{1+\nu}^{+\infty} \frac{\xi^2 - 2\xi + 1}{\xi^4} d\xi = l(\nu) \; avec \quad l(\nu) = \frac{3\nu 2 + 9\nu + 5}{\nu^3(1+\nu)^3} \tag{61}$$

As a result:

$$B < 4\pi(\frac{1}{3} + l(\nu)) \tag{62}$$

Hence the important result:

$$\int_{R3} ||u(\boldsymbol{x},t)||^2 dxdydz < 8\tau_0\pi^2\sqrt{\pi T}\left(\frac{1}{3} + l(\nu)\right) \tag{63}$$

or:

$$\boxed{\int_{R3} ||u(\boldsymbol{x},t)||^2 dxdydz < +\infty \quad \forall t} \tag{64}$$

let:

$$\boxed{\int_{R3} ||U(\boldsymbol{X},T)||^2 dXdYdZ < +\infty \quad \forall T} \tag{65}$$

because:

$$\int_{R3} ||U(\boldsymbol{X},T)||^2 dXdYdZ = \frac{1}{\nu^3}\int_{R3} ||u(\boldsymbol{x},t)||^2 dxdydz$$

*4.1.5 The expression of partial derivatives of $U(X,T)$*

We begin with the first partial derivative $\partial_X$ of the first component of $U(X,T)$: it is given by the equation (44):

$$\frac{\partial U_1}{\partial X} = \frac{1}{2\sqrt{\pi T}}\int_{R^3} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{4T}}\frac{\partial U_1^0(\alpha,\beta,)}{\partial \alpha}.dV$$

Let us calculate $\dfrac{\partial^2 U_1}{\partial X^2}$. We obtain:

$$\begin{aligned}
\frac{\partial^2 U_1}{\partial X^2} &= \frac{-1}{4T\sqrt{\pi T}}\int_{R^3}(X-\alpha)e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{4T}}\frac{\partial U_1^0(\alpha,\beta,)}{\partial \alpha}.dV \\
&= \frac{-1}{2\sqrt{\pi T}}\int_{R^3}\frac{\partial}{\partial\alpha}e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{4T}}.\frac{\partial U_1^0(\alpha,\beta,)}{\partial \alpha}.dV \\
= \frac{-1}{2\sqrt{\pi T}}\int_{R^2}d\beta d&\left[\frac{\partial}{\partial\alpha}U_1^0(\alpha,\beta,).e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{4T}}\right]_{\alpha=-\infty}^{\alpha=+\infty} + \\
\frac{1}{2\sqrt{\pi T}}\int_{R^2}d\beta d&\int_{\alpha=-\infty}^{\alpha=+\infty} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{4T}}\frac{\partial^2 U_1^0(\alpha,\beta,)}{\partial \alpha^2}.d\alpha
\end{aligned} \tag{66}$$

Taking into account the assumption (41), we obtain:

$$\frac{\partial^2 U_1}{\partial X^2} = \frac{1}{2\sqrt{\pi T}} \int_{R^3} e^{-\dfrac{(X-\alpha)^2+(Y-\beta)^2+(Z-)^2}{4T}} \frac{\partial^2 U_1^0(\alpha,\beta,)}{\partial \alpha^2} .d\alpha d\beta d \tag{67}$$

Using the same assumption cited above, we obtain that $\left\| \dfrac{\partial^2 U_1}{\partial X^2} \right\| \longrightarrow 0$ when $||\boldsymbol{X}|| \longrightarrow +\infty$. Then for $t$ fixed $||\partial_x u(x,y,z,t)|| \longrightarrow 0$ if $\sqrt{x^2+y^2+z^2} \longrightarrow +\infty$. We easily verify this property for the derivatives of $u(x,y,z,t)$ concerning the spatial coordinates of all order, with $t$ fixed.

*4.1.6 The expression of $p(x,y,z,t)$*

We rewrite equation (10):

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} - \nu \Delta u_i = -\frac{\partial p}{\partial x_i}$$

It can be written under vectorial form:

$$\nabla p = \nu \Delta u - \frac{\partial u}{\partial t} - A(u).u \tag{68}$$

with the matrix $A(u)$ given by (9). As $\nu \Delta u - \dfrac{\partial u}{\partial t} = 0$, then the equation (68) becomes:

$$\nabla p = -A(u).u \tag{69}$$

As $u \in R^3$ and of class $C^\infty(R^3 \times [0,+\infty))$, $\partial_i p$ are of class $C^\infty(R^3 \times [0,+\infty)) \Longrightarrow p(x,y,z,t)$ is also of class $C^\infty(R^3 \times [0,+\infty))$.

With the variables $X,Y,Z,T$, the pressure verifies the equation:

$$\Delta P = -\frac{\partial U_i}{\partial X_j} . \frac{\partial U_j}{\partial X_i} \tag{70}$$

we denote:

$$H = H(X,Y,Z,T) = \frac{\partial U_i}{\partial X_j} . \frac{\partial U_j}{\partial X_i} \tag{71}$$

The equation (70) becomes:

$$\Delta P = -H \tag{72}$$

It is the Poisson equation.

**Definition 1** The function :

$$\Phi(\boldsymbol{X}) = \frac{1}{4\pi ||\boldsymbol{X}||} \tag{73}$$

defined for $||\boldsymbol{X}|| \in R^3$, $\boldsymbol{X} \neq \boldsymbol{O}$ is the fundamental solution of Laplace equation.

The solution of Poisson equation (72) is given by [5]:

$$P = P(X,Y,Z,T) = P(\boldsymbol{X},T) = \frac{1}{4\pi}\int_{R^3} \frac{1}{||\boldsymbol{X}-\boldsymbol{Q}||} H(\boldsymbol{Q})d\boldsymbol{Q} \tag{74}$$

where $\boldsymbol{Q} = (X',Y',Z')^T \in R^3$ and $d\boldsymbol{Q} = dX'dY'dZ'$ the volume form.

From equation (51), we can write for example, the first component of $\nabla p$ :

$$\frac{\partial p}{\partial x} = -\sum_j u_j \frac{\partial u_1}{\partial x_j} \tag{75}$$

Using the new variables, we obtain:

$$\frac{\partial P}{\partial x} = -\sum_j U_j \frac{\partial U_1}{\partial X_j} \Longrightarrow P = -\sum_i \int_0^X U_i(\alpha,Y,Z,T)\frac{\partial U_1(\alpha,Y,Z,T)}{\partial \alpha_i} d\alpha \tag{76}$$

Then:

$$|P| \leq \sum_i |X||U_i(X,Y,Z,T)| \left|\frac{\partial U_1(X,Y,Z,T)}{\partial X_i}\right| \leq 3||\boldsymbol{X}||.||\boldsymbol{U}||.\left\|\frac{\partial \boldsymbol{U}(X,Y,Z,T)}{\partial X_i}\right\| \tag{77}$$

As seen above, $||\boldsymbol{U}||$ and $\left\|\frac{\partial \boldsymbol{U}(X,Y,Z,T)}{\partial X_i}\right\|$ tend to zero if $||\boldsymbol{X} = \sqrt{X^2+Y^2+Z^2}|| \longrightarrow +\infty$. With the presence of the term $e^{-||\boldsymbol{X}||^2}$ in the expression of the vectors $\boldsymbol{U}$ and its first derivative $\partial_X \boldsymbol{U}$, $||\boldsymbol{X}||.||\boldsymbol{U}||.\left\|\frac{\partial \boldsymbol{U}(X,Y,Z,T)}{\partial X_i}\right\|$ tend to zero as $||\boldsymbol{X}|| \longrightarrow +\infty$. Then $|P| \longrightarrow 0$.

Again, from equation (51), we can write for the vector $\nabla p$ :

$$||\nabla p|| = \sqrt{\sum_j \left(\frac{\partial p}{\partial x_j}\right)^2} \leq ||A(u)||.||u|| \tag{78}$$

Taking $||A(u)|| = max\left\|\frac{\partial u_i}{\partial x_j}\right\|$, then:

$$\left|\frac{\partial p}{\partial x_i}\right| \leq ||\nabla p|| \leq max\left\|\frac{\partial u_i}{\partial x_j}\right\|.||u(x,y,z,t)|| \tag{79}$$

As seeing in paragraph 411, for $t$ fixed, $||u(x,y,z,t)||$ and $||\partial_{x_i} u(x,y,z,t)||$ tend to zero as $\sqrt{x^2+y^2+z^2} \to +\infty$. We easily verify this property for the derivatives of $p$ concerning the spatial coordinates of all order, with $t$ fixed.

Let us study $lim_{\boldsymbol{X}\longrightarrow+\infty}\frac{\partial P}{\partial T}$. With the variables $X, Y, Z, T$, we have for example:

$$\frac{\partial p}{\partial x}=-\sum_i u_i\frac{\partial u_1}{\partial x_i}\Longrightarrow\frac{\partial P}{\partial X}=-\sum_i U_i\frac{\partial U_1}{\partial X_i}\Longrightarrow P=-\sum_i\int_0^X U_i(\alpha,\beta,,T)\frac{\partial U_1(\alpha,\beta,,T)}{\partial\alpha_i}d\alpha \tag{80}$$

We calculate $\partial_T P(X, Y, Z, T)$, we obtain:

$$\frac{\partial P}{\partial T}=-\sum_i\int_0^X\left(\frac{\partial U_i}{\partial T}.\frac{\partial U_1}{\partial\alpha_i}+U_i\frac{\partial^2 U_1}{\partial\alpha_i\partial T}\right)d\alpha \tag{81}$$

We suppose that $X>0$, then:

$$\left|\frac{\partial P}{\partial T}\right|\leq\sum_i\left(\left|X.\frac{\partial U_i}{\partial T}.\frac{\partial U_1}{\partial\alpha_i}\right|+\left|U_i.X.\frac{\partial^2 U_1}{\partial\alpha_i\partial T}\right|\right) \tag{82}$$

The presence of $e^{\frac{X^2+Y^2+Z^2}{4T}}$ in the bounded expression of the six terms of the right member of the above inequality gives that $lim\left|\frac{\partial P}{\partial T}\right|\longrightarrow 0$ when $\sqrt{X^2+Y^2+Z^2}\longrightarrow+\infty$. We verify easily that the derivatives $\partial^\delta_{X,Y,Z,T}P$ of all orders, for $T$ fixed, tend to zero as $\sqrt{X^2+Y^2+Z^2}\longrightarrow+\infty$.

We have given a proof of smooth solutions $u(x, y, z, t), p(x, y, z, t)$ of Navier-Stokes equations, defined for $(x, y, z)\in R^3$ and $t\in[0,\tau)$ for any $\tau\in R$.

### 4.2 Resolution of the equations (19) case 2

With the new variables $X, Y, Z, T$ the equation of case 2 is written as:

$$\Delta\overline{U}-\frac{\partial\overline{U}}{\partial T}=\overline{K}(T) \tag{83}$$

with $\overline{K}(T)=\nu K(t)$. We put $\overline{U}=U-\int_0^T\overline{K}(\tau)d\tau$, then the new function $U$ verifies:

$$\Delta U-\frac{\partial U}{\partial T}=0 \tag{84}$$

The solution of (83) is the function $\overline{U}=U-\int_0^T\overline{K}(\tau)d\tau$ where $U$ is the solution of the case 1 studied above. The function $\overline{U}$ verifies the same remarks studies above as $U$.

## 5 Conclusion

In this work, we have obtained a solution $u$ that verifies the conditions (A) of existence and smooth solutions $\in R^3$ of the Navier-Stokes equation. It remains the study of the cases:

- $u = \lambda\Omega$ , with $\lambda$ is a function of $(x, y, z, t)$;
- there is a scalar function $\varphi(x, y, z)$ and $u \wedge \Omega = grad\varphi$.

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