Real Numbers in the Neighborhood of Infinity

by

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Abstract

We demonstrate the existence of a broad class of real numbers which are not elements of any number field: those in the neighborhood of infinity. After considering the reals and the affinely extended reals we prove that numbers in the neighborhood of infinity are ordinary real numbers. As an application in complex analysis, we show that the Riemann zeta function has infinitely many non-trivial zeros off the critical line in the neighborhood of infinity.

§1. Real Numbers

Definition 1.1. A real number $x \in \mathbb{R}$ is a cut in the real number line.

Definition 1.2. A cut in a line separates one line into two pieces.

Remark. A number is a cut in a line. A line is defined a priori. All lines can be cut so all lines are number lines. A given line is the real line by definition. A real number separates the real number line into a set of "larger" real numbers and a set of "smaller" real numbers.

Definition 1.3. The real numbers are defined in interval notation as

$$\mathbb{R} \quad \equiv \quad (-\infty, \infty) \quad .$$

Definition 1.4. Call real numbers in the neighborhood of the origin $x \in \mathbb{R}_0$. Define them such that

$$x \in \mathbb{R}_0 \qquad \Longleftrightarrow \qquad x \in \mathbb{R} \quad , \quad \text{and} \quad -n < x < n \quad ,$$

for some $n \in \mathbb{N}$.

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Definition 1.5. Call real numbers in the neighborhood of infinity $x \in \mathbb{R}_\infty$. Define them as all real numbers except for real numbers in the neighborhood of the origin:

$$\mathbb{R}_\infty \quad \equiv \quad \mathbb{R}/\mathbb{R}_0 \quad .$$

Definition 1.6. For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have the property

$$\lim_{x \to 0^\pm} \frac{1}{x} = \text{diverges} \quad , \qquad \text{and} \qquad \lim_{n \to \infty} \sum_{k=1}^{n} k = \text{diverges} \quad .$$

§2. Affinely Extended Real Numbers

Definition 2.1. Define two affinely extended real numbers $\pm\infty$ such that for $x \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\lim_{x \to 0^\pm} \frac{1}{x} = \pm\infty \quad , \qquad \text{and} \qquad \lim_{n \to \infty} \sum_{k=1}^{n} k = \infty \quad .$$

Definition 2.2. The set of all affinely extended real numbers is

$$\overline{\mathbb{R}} \quad \equiv \quad \mathbb{R} \cup \{\pm\infty\} \quad .$$

Definition 2.3. In $\overline{\mathbb{R}}$, $\pm\infty$ are such that the limit any sequence of real numbers which diverges in $\mathbb{R}$ is equal to ∞ or $-\infty$.

Theorem 2.4. *If $x \in \overline{\mathbb{R}}$ and $x \neq \pm\infty$ then $x \in \mathbb{R}$.*

Proof. Proof follows from Definition 2.2. $\square$

Definition 2.5. The affinely extended real numbers are defined in interval notation as

$$\overline{\mathbb{R}} \quad \equiv \quad [-\infty, \infty] \quad .$$

Definition 2.6. Infinity is such that

$$\infty - \infty = \text{undefined} \quad , \qquad \text{and} \qquad \frac{\infty}{\infty} = \text{undefined} \quad .$$

Definition 2.7. Infinity does not have the distributive property of multiplication. For two non-zero real numbers a and b we have

$$\infty \cdot (b + a) = \infty \quad .$$

To the contrary, we write for two orthogonal unit vectors $\hat{e}_1$ and $\hat{e}_2$

$$\infty \cdot (\hat{e}_1 + \hat{e}_2) = \text{undefined} \quad .$$

Remark. In Definition 2.7 the former case gives the appearance of a distributive property because we can sum $b + a$ and then use the multiplicative absorptive property of ∞ to obtain a simplified result but we cannot do so in the latter case which is only mentioned in anticipation of the expression $\infty \cdot (x + iy)$ which appears in Section 5.

§3. Modified Infinity

Definition 3.1. When the $\pm\infty$ symbols appear as $\pm\widehat{\infty}$, let the hat be an instruction to delay the additive absorption of $\pm\infty$ indefinitely or until such delay causes a contradiction. At that time the hat must be removed. The hat is inserted and removed by choice except in the case where it invokes a contradiction and must be removed by definition.

Example 3.2. An example of a statement in which the hat does not invoke a contradiction and may be left in place is

$$x = \widehat{\infty} - b \ \ .$$

Example 3.3. An example of a statement in which the hat invokes a contradiction and may not be left in place is given by two sequences

$$x_n = \sum_{k=1}^{n} k \ \ , \qquad \text{and} \qquad y_n = c_0 + \sum_{k=1}^{n} k \ \ ,$$

where $n \in \mathbb{N}$ and c_0 is some non-zero real number. Since ∞ and $\widehat{\infty}$ are the same number we can use Definitions 2.1 and 3.1 to write

$$\lim_{n \to \infty} x_n = \infty = \widehat{\infty} \ \ , \qquad \text{and} \qquad \lim_{n \to \infty} y_n = \infty = \widehat{\infty} \ \ .$$

We may also write, however,

$$\lim_{n \to \infty} y_n = \lim_{n \to \infty} c_0 + \lim_{n \to \infty} x_n = c_0 + \widehat{\infty} \ \ .$$

This delivers an equality

$$\widehat{\infty} = c_0 + \widehat{\infty} \ \ ,$$

which contradicts the delayed additive absorption of $\widehat{\infty}$. At this point, we must cease to delay additive absorption by removing the hat. Then

$$\infty = c_0 + \infty \ \ ,$$

demonstrates the usual additive absorptive property of infinity and there is no contradiction.

Definition 3.4. $\widehat{\infty}$ is such that for any non-zero $b \in \mathbb{R}_0$

$$\pm\widehat{\infty} + b = b \pm \widehat{\infty}$$
$$\pm\widehat{\infty} - b = -b \pm \widehat{\infty}$$
$$\pm\widehat{\infty} + \left(-b \right) = \pm\widehat{\infty} - b$$
$$\pm\widehat{\infty} + b = \pm\widehat{\infty} - \left(-b \right)$$
$$-\left(\pm\widehat{\infty} \right) = \mp\widehat{\infty}$$
$$\pm\widehat{\infty} \cdot b = b \cdot \pm\widehat{\infty} = \begin{cases} \pm\widehat{\infty} & \text{if} \quad b > 0 \\ \mp\widehat{\infty} & \text{if} \quad b < 0 \end{cases}$$
$$\frac{\pm\widehat{\infty}}{b} = \begin{cases} \pm\widehat{\infty} & \text{if} \quad b > 0 \\ \mp\widehat{\infty} & \text{if} \quad b < 0 \end{cases}$$
$$\frac{b}{\pm\widehat{\infty}} = 0 \quad .$$

Definition 3.5. $\widehat{\infty}$ is such that

$$\pm\widehat{\infty} + 0 = 0 \pm \widehat{\infty} = \pm\widehat{\infty} - 0 = \text{undefined}$$
$$\pm\widehat{\infty} \cdot 0 = 0 \cdot \pm\widehat{\infty} = \text{undefined}$$
$$\frac{\pm\widehat{\infty}}{0} = \text{undefined}$$
$$\frac{0}{\pm\widehat{\infty}} = 0 \quad .$$

Remark. We will revisit the lack of an additive identity in Example 4.11.

Definition 3.6. $\pm\infty$ has all the properties assigned to $\pm\widehat{\infty}$ in Definitions 3.4 and 3.5 plus the additive absorptive operation for non-zero $b \in \mathbb{R}_0$

$$\pm\infty \pm b = \pm\infty \mp b = \pm\infty \quad ,$$

such that b is an additive identity of ∞.

§4. Real Numbers in the Neighborhood of Infinity

Definition 4.1. Let $\widehat{\mathbb{R}}$ be the set of all numbers of the form

$$x = \pm\left(\widehat{\infty} - b \right) \quad , \qquad \text{where} \qquad b \in \mathbb{R}_0 \quad , \quad b > 0 \quad ,$$

Definition 4.2. The ordering of $\widehat{\mathbb{R}}$ numbers is such that

$$
\begin{aligned}
\pm\left(\widehat{\infty} - b\right) = \pm\left(\widehat{\infty} - a\right) &\qquad \Longleftrightarrow \qquad a = b \\
\left(\widehat{\infty} - b\right) > \left(\widehat{\infty} - a\right) &\qquad \Longleftrightarrow \qquad a > b \\
-\left(\widehat{\infty} - b\right) > -\left(\widehat{\infty} - a\right) &\qquad \Longleftrightarrow \qquad a < b \\
\left(\widehat{\infty} - b\right) > -\left(\widehat{\infty} - a\right) &\qquad .
\end{aligned}
$$

Theorem 4.3. *All numbers $x \in \widehat{\mathbb{R}}$ are cuts in the affinely extended real number line, i.e.: they are affinely extended real numbers.*

Proof. Definition 1.2 requires that a cut separates one line into two pieces. Observe that

$$
\begin{aligned}
\overline{\mathbb{R}}/(\widehat{\infty} - b) &\equiv [-\infty, \widehat{\infty} - b) \cup (\widehat{\infty} - b, \infty] \\
\overline{\mathbb{R}}/\left(-\widehat{\infty} + b\right) &\equiv [-\infty, -\widehat{\infty} + b) \cup (-\widehat{\infty} + b, \infty] \quad .
\end{aligned}
$$

$\square$

Main Theorem 4.4. *All numbers $x \in \widehat{\mathbb{R}}$ are real numbers.*

Proof. If a number is an affinely extended real number $x \in \overline{\mathbb{R}}$ and $x \neq \pm\infty$ then by Theorem 2.4 we have $x \in \mathbb{R}$. In the absence of additive absorption,

$$
\pm\left(\widehat{\infty} - b\right) \neq \pm\widehat{\infty} = \pm\infty \quad ,
$$

because it is the definition of $\widehat{\mathbb{R}}$ that $b \neq 0$. Also note that

$$
\mathbb{R}/(\widehat{\infty} - b) \quad \equiv \quad (-\infty, \widehat{\infty} - b) \cup (\widehat{\infty} - b, \infty) \quad .
$$

All numbers $x \in \widehat{\mathbb{R}}$ satisfy Definition 1.1. $\square$

Theorem 4.5. *All numbers $x \in \widehat{\mathbb{R}}$ are real numbers in the neighborhood of infinity $x \in \mathbb{R}_\infty$.*

Proof. We have shown in Main Theorem 4.4 that

$$
\widehat{\mathbb{R}} \subset \mathbb{R} \quad ,
$$

so we will satisfy Definition 1.5 if we show

$$
\widehat{\mathbb{R}} \cap \mathbb{R}_0 \quad \equiv \quad \emptyset \quad .
$$

Definition 1.4 requires that elements of $\mathbb{R}_0$ satisfy

$$
-n < x < n \quad ,
$$

so assume

$$-n < \pm\left(\widehat{\infty} - b\right) < n \quad.$$

Since $b \in \mathbb{R}_0$ we know it has an additive inverse. Add or subtract b to obtain

$$-n + b < \widehat{\infty} < n + b \quad, \qquad \text{and} \qquad -n - b < -\widehat{\infty} < n - b \quad.$$

We obtain a contradiction as ∞ cannot be less than the sum of two finite numbers and $-\infty$ cannot be greater than the difference of two finite numbers. All $\widehat{\mathbb{R}}$ numbers satisfy the definition of $\mathbb{R}_\infty$. $\qquad\square$

Remark. The remaining definitions in the section define the arithmetic operations for $\widehat{\mathbb{R}}$ numbers. The purpose in defining these operations is to supplement the canonical operations for $\mathbb{R}_0$ and $\infty \sim \widehat{\infty}$. Every $\widehat{\mathbb{R}}$ number can be decomposed and its pieces manipulated separately but the main purpose of defining special operations for $\widehat{\mathbb{R}}$ is to define new operations for expressions which are undefined under the arithmetic operations of $\mathbb{R}_0$ and ∞ alone or whose structure vanishes under additive absorption.

Definition 4.6. The arithmetic operations of $\widehat{\mathbb{R}}$ numbers with $\mathbb{R}_0$ numbers are

$$-\left(\widehat{\infty} - b\right) = -\widehat{\infty} + b$$

$$-\left(-\widehat{\infty} + b\right) = \widehat{\infty} - b$$

$$\pm\left(\widehat{\infty} - b\right) + x = x \pm \left(\widehat{\infty} - b\right) = \begin{cases} \pm\widehat{\infty} \mp \left(b - x\right) & \text{if} \quad b \neq x \\ \pm\widehat{\infty} & \text{if} \quad b = x \end{cases}$$

$$\pm\left(\widehat{\infty} - b\right) \cdot x = x \cdot \pm\left(\widehat{\infty} - b\right) = \begin{cases} \pm\left(\widehat{\infty} - xb\right) & \text{if} \quad x \neq 0 \\ \text{undefined} & \text{if} \quad x = 0 \end{cases}$$

$$\frac{\pm\left(\widehat{\infty} - b\right)}{x} = \begin{cases} \pm\widehat{\infty} \mp \dfrac{b}{x} & \text{if} \quad x \neq 0 \\ \text{undefined} & \text{if} \quad x = 0 \end{cases}$$

$$\frac{x}{\pm\left(\widehat{\infty} - b\right)} = 0 \quad.$$

Theorem 4.7. *The quotient of a number $x \in \mathbb{R}_0$ divided by a number $y \in \widehat{\mathbb{R}}$ is identically zero.*

Proof. Let z be any non-zero real number such that

$$\frac{x}{y} = z \quad.$$

Since $||x|| < ||y||$, we have $||z|| < 1$ which implies $z \in \mathbb{R}_0$. All $\mathbb{R}_0$ numbers have a multiplicative inverse. We find, therefore, that

$$\frac{x}{zy} = 1 \qquad \Longleftrightarrow \qquad x = zy \quad .$$

The hat on $\widehat{\infty}$ only suppresses additive absorption so

$$zy = z \cdot \pm\big(\widehat{\infty} - b\big) = \pm\big(\widehat{\infty} - zb\big) \quad .$$

This delivers a contradiction because it requires that x is a real number in the neighborhood of infinity while we have already defined it to be a real number in the neighborhood of the origin. Therefore, the only possible numerical value for x/y is 0. $\qquad\square$

Definition 4.8. The arithmetic operations of $\widehat{\mathbb{R}}$ numbers with $\widehat{\mathbb{R}}$ numbers are

$$\pm\big(\widehat{\infty} - b\big) \pm \big(\widehat{\infty} - a\big) = \begin{cases} \pm\widehat{\infty} \mp \big(b + a\big) & \text{if} \quad a \neq -b \\ \pm\widehat{\infty} & \text{if} \quad a = -b \end{cases}$$

$$\pm\big(\widehat{\infty} - b\big) \mp \big(\widehat{\infty} - a\big) = \pm\big(a - b\big)$$

$$\big(\widehat{\infty} - b\big)\big(\widehat{\infty} - a\big) = \text{undefined}$$

$$\frac{\widehat{\infty} - b}{\widehat{\infty} - a} = \text{undefined} \quad .$$

Theorem 4.9. *Products of the form $\widehat{\mathbb{R}} \cdot \widehat{\mathbb{R}}$ are undefined.*

Proof. $\widehat{\infty}$ is not endowed with the distributive property of multiplication so

$$\big(\widehat{\infty} - b\big)\big(\widehat{\infty} - a\big) = \widehat{\infty}\big(\widehat{\infty} - a\big) - b\big(\widehat{\infty} - a\big) = \text{undefined} \quad .$$

$\qquad\square$

Theorem 4.10. *Quotients of the form $\widehat{\mathbb{R}}/\widehat{\mathbb{R}}$ are undefined.*

Proof. Observe that

$$\frac{\widehat{\infty} - b}{\widehat{\infty} - a} = \frac{\widehat{\infty}}{\widehat{\infty} - a} - \frac{b}{\widehat{\infty} - a} \quad .$$

Insert the multiplicative identity into the first term so that

$$\frac{\widehat{\infty} \cdot 1}{\widehat{\infty} - a} = \widehat{\infty}\left(\frac{1}{\widehat{\infty} - a}\right) = \widehat{\infty} \cdot 0 = \text{undefined} \quad .$$

$\qquad\square$

Remark. Although

$$\left(\widehat{\infty} - b\right) - \left(\widehat{\infty} - a\right) = a - b \quad,$$

implies the existence of an additive inverse for every $\widehat{\mathbb{R}}$ number, this does not imply an additive inverse for ∞ because the case of $a = b = 0$ is ruled out by the definition of $\widehat{\mathbb{R}}$.

Example 4.11. Definition 3.5 states that infinity does not have an additive identity element. An example motivating this condition is given by the limit

$$\lim_{x \to \infty} \left(x^2 - x\right) = \infty$$
$$\lim_{x \to \infty} \left(x^2 - x\right) = \lim_{x \to \infty} x^2 - \lim_{x \to \infty} x = \infty - \infty \quad.$$

which is usually used to demonstrate the lack of an additive inverse for ∞. If infinity is bestowed with an additive inverse then we obtain a contradiction $\infty = 0$. The expression $\infty - \infty$ is thus undefined. If we added the hats to infinity then we could insert the additive identity to write

$$\infty = \widehat{\infty} - \widehat{\infty} = \widehat{\infty} - \widehat{\infty} + 0 = \widehat{\infty} - \widehat{\infty} + 1 - 1 = \left(\widehat{\infty} - 1\right) - \left(\widehat{\infty} - 1\right) = 0 \quad.$$

We see that unhatted infinity likewise cannot have zero as an additive identity because we could write

$$\infty = \infty - \infty = \infty - \infty + \left(1 - 1\right) = \widehat{\infty} - \widehat{\infty} + 1 - 1 = \left(\widehat{\infty} - 1\right) - \left(\widehat{\infty} - 1\right) = 0 \quad.$$

where we have simply chosen not to do the additive absorptive operation within the freedom afforded to the order of algebraic operations. This example confirms that the only difference between ∞ and $\widehat{\infty}$ is an instruction to delay additive absorption for the latter.

Remark. The expressions ∞ and $\widehat{\infty}$ are perfectly well defined but $\infty + 0$ and $\widehat{\infty} + 0$ are examples of an undefined composition. Since, ∞ is not an $\widehat{\mathbb{R}}$ number, this property cannot create problems for the algebra of $\widehat{\mathbb{R}}$ numbers. Essentially, we have traded the zero additive identity element of infinity for the freedom to add and subtract $\widehat{\mathbb{R}}$ numbers.

Theorem 4.12. *An $\widehat{\mathbb{R}}$ number does not have a multiplicative inverse.*

Proof. Assume

$$x\left(\widehat{\infty} - b\right) = 1 \quad.$$

If $x \in \widehat{\mathbb{R}}$ then the operation is undefined. If $x \in \mathbb{R}_0$ then we obtain

$$\widehat{\infty} - xb = 1 \quad.$$

This is a contradiction because it requires that 1 is a number in the neighborhood of infinity. If $x \in \mathbb{R}_\infty$ is a positive number less than every positive $\widehat{\mathbb{R}}$ number then we find that the product or two real numbers larger than one is equal to one. This is another contradiction. If x is a negative number then its product with a positive real number must be negative. $\square$

Remark. Since real numbers in the neighborhood infinity do not always have a multiplicative inverse such numbers cannot be elements of number fields. The common practice of using number fields as a generalized proxy for all numbers, therefore, should be considered to have a narrower scope of valid application than is commonly understood.

§5. Complex Numbers

Definition 5.1. A number is a complex number $z \in \mathbb{C}$ if and only if

$$ z = x + iy \ , \qquad \text{where} \qquad x, y \in \mathbb{R} \ , \quad i = \sqrt{-1} \ . $$

Definition 5.2. As ∞ does not absorb -1 in 1D, in 2D we have the condition that infinity does not absorb -1 or $\pm i$.

Definition 5.3. As the extended real line has two distinct infinites, the extended complex plane has four: $\{+\infty, +i\infty, -\infty, -i\infty\}$.

Definition 5.4. The affinely extended complex plane is

$$ \overline{\mathbb{C}} \quad \equiv \quad \mathbb{C} \cup \{\pm\infty\} \cup \{\pm i\infty\} \ . $$

Definition 5.5. The multiplicative operations for $\pm\infty$ and $\pm i\infty$ with i are

$$ \pm\infty \cdot i = \pm i\infty $$
$$ \pm i\infty \cdot i = \mp\infty \ . $$

Remark. The non-distributive property of $\pm\infty$ (Definition 2.7) was practically redundant in 1D but for $z \in \mathbb{C}$ this feature gains significance.

Definition 5.6. The multiplicative operations for $\pm\infty$ with complex numbers $z \in \mathbb{C}$ are

$$\pm\infty \cdot z = z \cdot \pm\infty = \begin{cases} \pm\infty & \text{if} & \text{Re}(z) > 0 & \text{and} & \text{Im}(z) = 0 \\ \mp\infty & \text{if} & \text{Re}(z) < 0 & \text{and} & \text{Im}(z) = 0 \\ \pm i\infty & \text{if} & \text{Im}(z) > 0 & \text{and} & \text{Re}(z) = 0 \\ \mp i\infty & \text{if} & \text{Im}(z) < 0 & \text{and} & \text{Re}(z) = 0 \\ \text{undefined} & \text{if} & \text{Im}(z) \neq 0 & \text{and} & \text{Re}(z) \neq 0 \\ \text{undefined} & \text{if} & z = 0 \end{cases} .$$

Definition 5.7. The multiplicative operations for $\pm i\infty$ with complex numbers $z \in \mathbb{C}$ are

$$\pm i\infty \cdot z = z \cdot \pm i\infty = \begin{cases} \pm i\infty & \text{if} & \text{Re}(z) > 0 & \text{and} & \text{Im}(z) = 0 \\ \mp i\infty & \text{if} & \text{Re}(z) < 0 & \text{and} & \text{Im}(z) = 0 \\ \mp\infty & \text{if} & \text{Im}(z) > 0 & \text{and} & \text{Re}(z) = 0 \\ \pm\infty & \text{if} & \text{Im}(z) < 0 & \text{and} & \text{Re}(z) = 0 \\ \text{undefined} & \text{if} & \text{Im}(z) \neq 0 & \text{and} & \text{Re}(z) \neq 0 \\ \text{undefined} & \text{if} & z = 0 \end{cases} .$$

Definition 5.8. The arithmetic operations for complex numbers $z \in \mathbb{C}$ whose real and/or imaginary parts are $\widehat{\mathbb{R}}$ numbers follow directly from the other definitions.

Corollary 5.9. *The Riemann zeta function has infinitely many non-trivial zeros off the critical line.*

Proof. Consider a number $z_0 \in \mathbb{C}$ such that

$$z_0 = -\left(\widehat{\infty} - b\right) + iy_0 \quad, \qquad \text{where} \qquad b, y_0 \in \mathbb{R}_0 \quad.$$

Observe that

$$\zeta(z_0) = \prod_{p|primes} \frac{1}{1 - p^{(\widehat{\infty}-b)-iy_0}}$$

$$= \frac{1}{1 - P^{(\widehat{\infty}-b)-iy_0}} \left(\prod_{\substack{p|primes \\ p \neq P}} \frac{1}{1 - p^{(\widehat{\infty}-b)-iy_0}} \right)$$

$$= \frac{1}{1 - \frac{1}{P^b} P^{\widehat{\infty}} \left[\cos(y_0 \ln P) - i \sin(y_0 \ln P) \right]} \left(\prod_{\substack{p|primes \\ p \neq P}} \frac{1}{1 - p^{(\widehat{\infty}-b)-iy_0}} \right) \quad.$$

Let $y_0 \ln P = 2n\pi$ for some prime P and $n \in \mathbb{N}$. Then

$$\zeta(z_0) = \frac{1}{1 - \widehat{\infty}} \left(\prod_{\substack{p \mid primes \\ p \neq p'}} \frac{1}{1 - p^{(\widehat{\infty} - b) - iy_0}} \right) \ .$$

By Theorem 4.7, we find that

$$\zeta(z_0) = 0 \ .$$

Since z_0 neither is on the critical line nor the real axis, the Riemann hypothesis is false. $\qquad\square$