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Search for New Numbers.

CHAPTER I

I.1 Introduction.

In 1984, in the first issue for that year of the Science and Life magazine, I read the article "Non-Standard Analysis" by V. Uspensky, in which the notion of hyperreal numbers had been introduced. And, already somewhere in June 1984, I thought – "There was a time when they were not known about. And now learned. So maybe there are other numbers that we don't know now, and then we'll find out. And is it possible not to wait for this, but somehow to find out about these unknown now numbers by myself?".

I.2 The help of dialectics.

Dialectics says, that the subject should be studied in development. In this case, the subject is numbers. And their development can be traced back to my own experience - what numbers, and in what order I studied at school.

I.3 Numbers, studied at school.

First we studied numbers: 1, 2, 3, ... - they are called natural. Then zero and negative numbers. Natural, zero and negative are called integers.

Then there were fractional numbers. Integers and fractional numbers are called rational numbers.

After them we studied the irrational numbers. Rational numbers together with irrational numbers are called real numbers.

They were followed by imaginary numbers. Real numbers together with imaginary numbers are called complex numbers.

I.4 The direct operations, studied in school.

There are many operations, studied in the school. But the main ones are addition, multiplication, exponentiation. They will be called "direct operations". We introduce the following notation: [1] – addition, [2] – multiplication, [3] - exponentiation. Also, we will introduce the conventional symbols of these three "direct operations": [1] – "+", [2] – "*", [3] – "^".

I.5 The first direct operation.

Take any two natural numbers: a, b. And apply to them the first direct operation. The result is also a natural number– c: a [1] b = c ($a + b = c$). (I.5.1)

[1]	1	2	3	4	b
1	2	3	4	5	
2	3	4	5	6	
3	4	5	6	7	
4	5	6	7	8	
a					c

Table I.1

I.6 The second direct operation.

Take any two integers: a, b. And apply to them the second direct operation. The result will also be an integer c: a [2] b = c ($a * b = c$). (I.6.1)

[2]	1	2	3	b
1	1	2	3	
2	2	4	6	
3	3	6	9	
a				c

Table I.2

I.7 The third direct operation.

Take any two natural numbers: a, b . And apply to them the third direct operation. The result is also a natural number - c : $a [3] b = c$ ($a \wedge b = c$). (I.7.1)

[3]	1	2	3	b
1	1	1	1	
2	2	4	8	
3	3	9	27	
a				c

Table I.3

I.8 The definition of inverse operations.

Let a direct operation n be defined on some set of numbers:

$$a [n] b = c; \text{ (I.8.1)}$$

(Here a, b are numbers from this set. They are called operands.)

If in (I.8.1) we rearrange the first operand " a " with the result " c ", we get:

$$c [n]1 b = a; \text{ (I.8.2)}$$

This is the definition of the first inverse operation for n direct operation – $[n]1$.

If in (I.8.1) we rearrange the second operand " b " with the result " c ", we get:

$$a [n]2 c = b; \text{ (I.8.3)}$$

This is the definition of the second inverse operation for n direct operation – $[n]2$.

I.9 The first inverse operation for the first direct operation.

Now for $[1]1$ we build Table I.4 from Table I.1 (using (I.8.1) and (I.8.2)):

[1]1]	1	2	3	4	b
2	1				
3	2	1			
4	3	2	1		
5	4	3	2	1	
6		4	3	2	
7			4	3	
8				4	
c					a

Table I.4

We see from Table I.4, that $[1]1]$ coincides with "-". That is:

$$a = c [1]1] \quad b = c - b; \quad (I.9.1)$$

But not all the cells in Table I.4 are filled in yet. Let's try to find such natural numbers **c** and **b**, so that **a** too was a natural number.

And that's what we get:

[1]1]	1	2	3	4	b
2	1				
3	2	1			
4	3	2	1		
5	4	3	2	1	
6	5	4	3	2	
7	6	5	4	3	
8	7	6	5	4	
c					a

Table I.5

There are no numbers in the upper right corner of Table I.5 among natural numbers. These numbers: 2 - 2, 3 - 3, 4 - 4, 2 - 3, 3 - 4, 2 - 4. That is, operation $[1]1]$ is not equal to operation $[1]$ on the set of natural numbers. To achieve equality with operation $[1]$, it is necessary to move to a wider range of numbers than natural

numbers. To integers. These are natural numbers, zero, and negative numbers. Then any pair of integers **c** and **b** for operation $[1]1$ gives an integer. And for operation $[1]$ any pair of integers **a** and **b** also gives an integer. And then operations $[1]$ and $[1]1$ become equal. And empty cells in Table 5 can be filled with new numbers: 0, -1, -2. Then we get Table I.6:

$[1]1$	1	2	3	4	b
2	1	0	-1	-2	
3	2	1	0	-1	
4	3	2	1	0	
5	4	3	2	1	
6	5	4	3	2	
7	6	5	4	3	
8	7	6	5	4	
c					a

Table I.6

I.10 The second inverse operation for the first direct operation.

Now, for $[1]2$, let's build the Table I.7 from the Table I.1 (using (I.8.1) and (I.8.3)):

$[1]2$	2	3	4	5	6	7	8	c
1	1	2	3	4				
2		1	2	3	4			
3			1	2	3	4		
4				1	2	3	4	
a								b

Table I.7

From (I.8.2), (I.8.3), Table I.4, and Table I.7, we see that

$$b = a [1]2 \quad c = c [1]1 \quad a = c - a; \quad (I.10.1)$$

To populate the Table I.7 completely, you have numbers a, c be taken from the integers. Then b will also be an integer. And operation [1]2 becomes equal with operation [1]. And then we get Table I.8:

[1]2]	2	3	4	5	6	7	8	c
1	1	2	3	4	5	6	7	
2	0	1	2	3	4	5	6	
3	-1	0	1	2	3	4	5	
4	-2	-1	0	1	2	3	4	
a								b

Table I.8

We now have:

$$x [1] y = x + y; (I.10.2)$$

$$x [1]1 y = x - y; (I.10.3)$$

$$x [1]2 y = y - x; (I.10.4)$$

I.11 Connection of operation [1] with operation [2].

The operation of addition is taught at school first. Then move on to the study of the second operation - multiplication. Let's look at some examples of these operations.

$$3 + 3 = 3 * 2 = 6; (I.11.1)$$

$$3 + 3 + 3 = 3 * 3 = 9; (I.11.2)$$

$$2 + 2 + 2 + 2 = 2 * 4 = 8; (I.11.3)$$

We rewrite these examples in the new notation:

$$3 [1] 3 = 3 [2] 2 = 6; (I.11.4)$$

$$3 [1] 3 [1] 3 = 3 [2] 3 = 9; (I.11.5)$$

$$2 [1] 2 [1] 2 [1] 2 = 2 [2] 4; (I.11.6)$$

The multiplication operation is formed as follows: to the left of the multiplication operation, the summand is written, and to the right of it, the number of these summands is written.

I.12 The first inverse operation for the second direct operation.

Second direct operation [2]:

$$a [2] b = c; (I.12.1)$$

From (I.8.2) for $n = 2$ we obtain a definition for $[2]1$:

$$c [2]1 b = a; (I.12.2)$$

Using Table I.2, we obtain Table I.9 for $[2]1$:

[2]1	1	2	3	b
1	1			
2	2	1		
3	3		1	
4		2		
5				
6		3	2	
7				
8				
9			3	
c				a

Table I.9

Table I.9 shows that $[2]1$ is a division “/”:

$$a = c [2]1 b = c / b; (I.12.3)$$

But not all the cells in Table I.9 are filled in yet. Let's look for **c** and **b** among integers so that **a** is also integer. And we get Table I.10:

[2]1]	1	2	3	b
1	1			
2	2	1		
3	3		1	
4	4	2		
5	5			
6	6	3	2	
7	7			
8	8	4		
9	9		3	
c				a

Table I.10

For empty cells in Table I.10 there were no numbers among integers. That is, operation $[2]1]$ is not equal to operation $[2]$ on the set of integers. Equality of these operations can be achieved by introducing new fractional numbers. Then all empty cells in Table I.10 can be filled with new fractional numbers. And we get Table I.11:

[2]1]	1	2	3	b
1	1	$1/2$	$1/3$	
2	2	1	$2/3$	
3	3	$3/2$	1	
4	4	2	$4/3$	
5	5	$5/2$	$5/3$	
6	6	3	2	
7	7	$7/2$	$7/3$	
8	8	4	$8/3$	
9	9	$9/2$	3	
c				a

Table I.11

Integers together with fractional numbers are called rational numbers. And for any pair of rational numbers **a**, **b** in (I.12.1) there

will be a rational number **c**. Also for any pair of rational numbers **c** and **b** in (I.12.2) there will be a rational number **a**. That is, operations [2] and [2]1] became equal on the set of rational numbers.

I.13 The second inverse operation for the second direct operation.

Second direct operation [2] of (I.8.1) at $n = 2$:

$$a [2] b = c; \text{ (I.13.1)}$$

From (I.8.3) for $n = 2$ we obtain a definition for [2]2]:

$$a [2]2] c = b; \text{ (I.13.2)}$$

Using Table I.2, we obtain Table I.12 for [2]2]:

[2]2]	1	2	3	4	5	6	7	8	9	c
1	1	2	3							
2		1		2		3				
3			1			2			3	
a										b

Table I.12

This table shows that $b = c / a$; (I.13.3)

But not all the cells in Table I.12 are filled in yet. Let's look for **c** and **a** among integers so that **b** is also integer. And we get table I.13:

[2]2]	1	2	3	4	5	6	7	8	9	c
1	1	2	3	4	5	6	7	8	9	
2		1		2		3		4		
3			1			2			3	
a										b

Table I.13

For empty cells in Table I.13 there were no numbers among integers. That is, operation $[2]2]$ is not equal to operation $[2]$ on the set of integers. Equality of these operations can be achieved by introducing new fractional numbers. Then all empty cells in Table I.13 can be filled with new fractional numbers. And we get Table I.14:

$[2]2]$	1	2	3	4	5	6	7	8	9	c
1	1	2	3	4	5	6	7	8	9	
2	$1/2$	1	$3/2$	2	$5/2$	3	$7/2$	4	$9/2$	
3	$1/3$	$2/3$	1	$4/3$	$5/3$	2	$7/3$	$8/3$	3	
a										b

Table I.14

Integers together with fractional numbers are called rational numbers. And for any pair of rational numbers **a**, **b** in (I.13.1) there will be a rational number **c**. Also for any pair of rational numbers **a** and **c** in (I.13.2) there will be a rational number **b**. That is, operations $[2]$ and $[2]2]$ became equal on the set of rational numbers.

Given (I.12.3), (I.13.2), (I.13.3) we obtain:

$$b = a [2]2] \quad c = c / a = c [2]1] \quad a; \quad (I.13.4)$$

We now have:

$$x [2] y = x * y; \quad (I.13.5)$$

$$x [2]1] y = x / y; \quad (I.13.6)$$

$$x [2]2] y = y / x; \quad (I.13.7)$$

I.14 Connection of operation $[2]$ with operation $[3]$.

The operation of exponentiation - [3] is studied in school after multiplication – [2]. These operations are closely related to each other. Here are some examples of this connection:

$$3 * 3 = 3^2 = 9; (I.14.1)$$

$$3 * 3 * 3 = 3^3 = 27; (I.14.2)$$

$$2 * 2 * 2 * 2 = 2^4 = 16; (I.14.3)$$

We rewrite these examples in the new notation:

$$3 [2] 3 = 3 [3] 2 = 9; (I.14.4)$$

$$3 [2] 3 [2] 3 = 3 [3] 3 = 27; (I.14.5)$$

$$2 [2] 2 [2] 2 [2] 2 = 2 [3] 4 = 16; (I.14.6)$$

The exponentiation operation is formed as follows: the factor is written to the left of the exponentiation operation sign, and the number of these factors is written to the right of this sign.

I.15 The first reverse operation for the third direct operation.

Now we will build for [3]1 Table I.15 from the Table I.3 (using (I.8.1) and (I.8.2)):

[3]1	1	2	3	b
1	1	1	1	
2	2			
3	3			
4		2		
8			2	
9		3		
27			3	
c				a

Table I.15

Table I.15 shows that [3]1] coincides with the root:

a = root of degree **b** of **c**; (I.15.1)

Not all cells in Table I.15 are occupied. Let's try to occupy them with rational numbers. Then we get Table I.16:

[3]1]	1	2	3	b
1	1	1	1	
2	2			
3	3			
4	4	2		
8	8		2	
9	9	3		
27	27		3	
c				a

Table I.16

Not all the cells of the Table I.16 were filled with rational numbers. That is, operation [3]1] is not equal to operation [3] on the set of rational numbers. We have to introduce new numbers here – the square root of 2, the root of the third degree of 2, the square root of 3, the root of the third degree of 3, and so on. These numbers are called irrational. They – together with rational numbers – are called real numbers. And now all the empty cells in Table I.16 are filled.

And now we will build Table I.17:

[3]1]	1	2	3	b
-1	-1		-1	
1	1	1	1	
c				a

Table I.17

One cell in the top row of Table I.17 could not be filled not only with a rational, but even with a real number. That is, operation

$[3]1]$ is still unequal to operation $[3]$ even on the set of real numbers.

Let's introduce another number - the root of the second degree from -1. It is called "imaginary unit", and is usually denoted by the letter i :

$$-1 [3]1] 2 = i; (I.15.2)$$

$$i^2 = -1 < 0; (I.15.3)$$

If the imaginary unit is multiplied by the real number b , you get another new imaginary number. That is, if it is raised to the second degree, it will also be a negative number:

$$(i * b)^2 = -(b * b) < 0; (I.15.4)$$

If real numbers are added with imaginary numbers, then complex numbers will be obtained.

Now we can fill an empty cell in the top row of Table I.17 with the imaginary unit i and get Table I.18:

$[3]1]$	1	2	3	b
-1	-1	i	-1	
1	1	1	1	
c				a

Table I.18

And now – after the introduction of irrational numbers, and imaginary numbers – operation $[3]1]$ became equal to operation $[3]$: For any complex numbers c, b for operation $[3]1]$ there is always a complex number a . And for any complex a, b for operation $[3]$ there is always a complex number c .

I.16 The second reverse operation for the third direct operation.

From (I.8.1), (I.8.3) for $n = 3$ we have definitions of $[3]$ and $[3]2$ operations:

$$a [3] b = c; \text{ (I.16.1)}$$

$$a [3]2 c = b; \text{ (I.16.2)}$$

Based on these definitions and Table I.3, we construct a Table I.19:

[3]2	1	2	3	4	8	9	27	c
1	1							
2		1		2	3			
3			1			2	3	
a								b

Table I.19

Table I.19 shows that operation $[3]2$ coincides with the logarithm:

$$b = \text{logarithm}(c) \text{ for base } a; \text{ (I.16.3)}$$

Let's try to fill in the empty cells in Table I.19 using complex numbers. The result is presented in Table I.20:

[3]2	1	2	3	4	8	9	27	c
1	1	?	?	?	?	?	?	
2	0	1	+	2	3	+	+	
3	0	+	1	+	+	2	3	
a								b

Table I.20

The "+" sign marks the cells, where the known numbers are located – the real numbers, only the cells are too small to fit these numbers. But zero fit, and it's standing in two cells.

Those cells, where no known complex number can stand, are marked with sing "?". It is possible, that there is a new class of numbers.

I.17 Numbers with "?".

We introduce into consideration the new numbers through Table I.21:

[3]2	1	2	3	4	8	9	27	c
1	1	?2	?3	?4	?8	?9	?27	
2	0	1	+	2	3	+	+	
3	0	+	1	+	+	2	3	
a								b

Table I.21

Numbers ?2,?3,?4, and so on - are defined through operation [3] as follows:

$$1 [3] ?2 = 1 \wedge ?2 = 2; (I.17.1)$$

$$1 [3] ?3 = 1 \wedge ?3 = 3; (I.17.2)$$

$$1 [3] ?4 = 1 \wedge ?4 = 4; (I.17.3)$$

$$1 [3] ?a = 1 \wedge ?a = a; (I.17.4)$$

Here **a** is any complex number. And the number ?a will be called a "question number".

I.18 Operation [1] for the two question numbers.

Let's define the addition operation [1] for any two question numbers ?a, ?b:

$$?a + ?b = ?c; (I.18.1)$$

$$1 \wedge (?a + ?b) = (1 \wedge ?a) * (1 \wedge ?b) = a * b = 1 \wedge ?(a * b); (I.18.2)$$

$$?a + ?b = ?(a * b); (I.18.3)$$

$$?a [1] ?b = ?(a [2] b) = ?c; (I.18.4)$$

According to this formula, we build a Table I.22:

[1]	?1	?2	?3	b
?1	?1	?2	?3	
?2	?2	?4	?6	
?3	?3	?6	?9	
a				c

Table I.22

I.19 The first inverse operation [1]1 for the first direct operation [1] on the set of question numbers.

For the question numbers ?a, ?b, ?c for operation [1] in Table I.22, construct operation [1]1 and display it in Table I.23:

[1]1	?1	?2	?3	b
?1	?1			
?2	?2	?1		
?3	?3		?1	
?4		?2		
?6		?3	?2	
?9			?3	
c				a

Table I.23

Let's define the subtraction operation " - " [1]1 for any two question numbers ?c, ?b:

$$1 \wedge (?c - ?b) = (1 \wedge ?c) * (1 \wedge (-?b)) = c / (1 \wedge ?b) = c / b =$$

$$= 1 \wedge ?(c / b); (I.19.1)$$

$$?c - ?b = ?(c / b); (I.19.2)$$

$$?c [1]1] ?b = ?c - ?b = ?(c / b); (I.19.3)$$

According to this formula for operation " - " [1]1], we fill in the empty cells in Table I.23 and get Table I.24:

[1]1]	?1	?2	?3	b
?1	?1	?(1/2)	?(1/3)	
?2	?2	?1	?(2/3)	
?3	?3	?(3/2)	?1	
?4	?4	?2	?(4/3)	
?6	?6	?3	?2	
?9	?9	?(9/2)	?3	
c				a

Table I.24

I.20 The second inverse operation [1]2] for the first direct operation [1] on the set of question numbers.

According to Table I.22, moving **b** and **c**, we get Table I.25:

[1]2]	?1	?2	?3	?4	?6	?9	c
?1	?1	?2	?3				
?2		?1		?2	?3		
?3			?1		?2	?3	
a							b

Table I.25

From the comparison of Table I.25 with Table I.23 (where the difference is: $?c - ?b = ?a$) Table I.25 shows the difference:

$$?c - ?a = ?b; (I.20.1)$$

From (I.19.2) – after replacement ?b on ?a - we get:

$$?c - ?a = ?\{c / a\}; \text{ (I.20.2)}$$

$$?b = ?a [1]2] ?c = ?c [1]1] ?a = ?c - ?a = ?(c / a); \text{ (I.20.3)}$$

And now we can fill in the Table I.25 empty cells and get the Table I.26:

[1]2]	?1	?2	?3	?4	?6	?9	?c
?1	?1	?2	?3	?4	?6	?9	
?2	?(1/2)	?1	?(3/2)	?2	?3	?(9/2)	
?3	?(1/3)	?(2/3)	?1	?(4/3)	?2	?3	
?a							?b

Table I.26

So we have:

$$?x [1] ?y = ?(x * y); \text{ (I.20.4)}$$

$$?x [1]1] ?y = ?(x / y); \text{ (I.20.5)}$$

$$?x [1]2] ?y = ?(y / x); \text{ (I.20.6)}$$

I.21 Operation [2] for the two question numbers.

We define the multiplication operation [2] for two any question numbers ?a, ?b:

$$?a [2] ?b = ?a * ?b = ?c; \text{ (I.21.1)}$$

$$1 \wedge (?a [2] ?b) = (1 \wedge ?a) \wedge ?b = a \wedge ?b = 1 \wedge (?a \wedge ?b) = 1 \wedge ?c; \text{ (I.21.2)}$$

$$?c = ?(a \wedge ?b); \text{ (I.21.3)}$$

$$c = a \wedge ?b; \text{ (I.21.4)}$$

$$?a [2] ?b = ?(a [3] ?b); \text{ (I.21.5)}$$

According to the formula (I.21.5), we build a Table I.27:

[2]	?1	?2	?3	?b
?1	?1	?2	?3	
?2	?(2 [3] ?1)	?(2 [3] ?2)	?(2 [3] ?3)	
?3	?(3 [3] ?1)	?(3 [3] ?2)	?(3 [3] ?3)	
?a				?c

Table I.27

?2 increases 1: 1 [3] ?2 = 2;

?1/2 reduces 1: 1 [3] ?(1/2) = 1/2;

?1 saves 1: 1 [3] ?1 = 1;

What follows ?1 = 1; (I.21.6)

With this in mind, the Table I.27 becomes the Table I.28:

[2]	?1	?2	?3	?b
?1	?1	?2	?3	
?2	?2	?(2 [3] ?2)	?(2 [3] ?3)	
?3	?3	?(3 [3] ?2)	?(3 [3] ?3)	
?a				?c

Table I.28

Let's introduce $a|b$:

$a|b = a \wedge ?b$; (I.21.7)

$a|b = a [3] ?b$; (I.21.8)

After that, the Table I.28 will take the form of the Table I.29:

[2]	?1	?2	?3	?b
?1	?1	?2	?3	
?2	?2	?(2 2)	?(2 3)	
?3	?3	?(3 2)	?(3 3)	
?a				?c

Table I.29

Here's a useful formula:

$$(?a) * b = ?(a \wedge b); (I.21.9)$$

$$(?a) [2] b = ?(a [3] b); (I.21.10)$$

We'll get it out.:

$$1 \wedge ((?a) * b) = (1 \wedge ?a) \wedge b = a \wedge b = 1 \wedge ?(a \wedge b); (I.21.11)$$

Got. We'll be using it soon. In the meantime, build a logarithm:

$$a = b \wedge \log(b|a); (I.21.12)$$

$$a = b \wedge \{\log(a) \text{ base } b\}$$

$$\text{For complex } b, d \text{ we have: } d * (?b) = (?b) * d; (I.21.13)$$

But this formula – (I.21.13) we have not checked for correctness.

And here is another useful formula:

$$a|b = b|a; (I.21.14)$$

We will deal with its conclusion:

$$(21.7) + (21.12) + (21.13) + (21.9) = (I.21.14)$$

$$a|b = a \wedge ?b = b \wedge \{(\log(b|a)) * ?b\} = (I.21.13) =$$

$$= b \wedge \{(?b * \log(b|a))\} = (I.21.9) =$$

$$= b \wedge ?(b \wedge \log(b|a)) =$$

$$= b \wedge ?a = b|a;$$

Apply to Table I.29 equation (I.21.14), and we obtain

Table I.30:

[2]	?1	?2	?3	?b
?1	?1	?2	?3	
?2	?2	?(2 2)	?(2 3)	
?3	?3	?(2 3)	?(3 3)	
?a				?c

Table I.30

I.22 The first inverse operation [2] 1] for the second direct operation [2] on the set of question numbers.

(I.21.1): ?a [2] ?b = ?c;

?a = ?c [2]1] ?b; (I.22.1)

(I.21.4): $c = a \wedge b$;

From here we have:

a = root of degree **?b** from **c** ; (I.22.2)

(I.22.1) + (I.22.2) = (I.22.3):

?a = ?c [2]1] ?b = ?(root of degree **?b** from **c**); (I.22.3)

Let's switch places in Table I.30 **?a** and **?c** and get Table I.31:

[2]1]	?1	?2	?3	?b
?1	?1			
?2	?2	?1		
?3	?3		?1	
?(2 2)		?2		
?(2 3)		?3	?2	
?(3 3)			?3	
?c				?a

Table I.31

Next, consider the three empty cells (?b = ?1 = 1):

$?a = ?(\text{root of degree } 1 \text{ from } 2|2); (I.22.4)$

$?a = ?(\text{root of degree } 1 \text{ from } 2|3); (I.22.5)$

$?a = ?(\text{root of degree } 1 \text{ from } 3|3); (I.22.6)$

Table I.31 takes the form of Table I.32:

[2]1	?1	?2	?3	?b
?1	1			
?2	?2	1		
?3	?3		1	
?(2 2)	?(2 2)	?2		
?(2 3)	?(2 3)	?3	?2	
?(3 3)	?(3 3)		?3	
?c				?a

Table 32

$(I.22.3): ?a = ?c [2]1 \quad ?b = ?(\text{root of degree } ?b \text{ from } c);$

Substitute (I.22.3) in Table I.32 and get Table I.33:

[2]1	?1	?2	?3	?b
?1	1	1+	2+	
?2	?2	1	3+	
?3	?3	4+	1	
?(2 2)	?(2 2)	?2	5+	
?(2 3)	?(2 3)	?3	?2	
?(3 3)	?(3 3)	6+	?3	
?c				?a

Table I.33

Because of the closeness I had to enter the symbols:

$1+ = ?(\text{root of degree } ?2 \text{ of } 1); (I.22.4)$

$2+ = ?(\text{root of degree } ?2 \text{ of } 1); (I.22.5)$

$$3+ = ?(\text{root of degree } ?2 \text{ of } 2); (I.22.6)$$

$$4+ = ?(\text{root of degree } ?2 \text{ of } 3); (I.22.7)$$

$$5+ = ?(\text{root of degree } ?2 \text{ of } 2|2); (I.22.8)$$

$$6+ = ?(\text{root of degree } ?2 \text{ of } 3|3); (I.22.9)$$

I.23 The second inverse operation [2]2 for the second direct operation [2] on the set of question numbers.

$$(I.21.1): ?a [2] ?b = ?c;$$

$$?b = ?a [2]2 ?c; (I.23.1)$$

$$?b = \log(c) \text{ on the basis of } a; (I.23.2)$$

$$?a [2]2 ?c = \log(c) \text{ on the basis of } a = ?b; (I.23.3)$$

Let's move **?b** and **?c** in Table I.30. Then we get Table I.34:

[2]2	?1	?2	?3	?(2 2)	?(2 3)	?(3 3)	?c
?1	?1	?2	?3				
?2		?1		?2	?3		
?3			?1		?2	?3	
?a							?b

Table I.34

Let's look at the three empty cells in the top row of Table I.34:

Substitute in (I.23.3):

$$a = 1; c = 2|2; \text{ and we get: } ?b = ?(2|2); (I.23.4)$$

To verify the correctness of this, raise 1 to the power (I.23.3) and use (I.23.4):

$$1^{\wedge}(\log(2|2) \text{ base } 1) = 1^{\wedge}?(2|2) = 2|2;$$

$$a = 1; c = 2|3; \text{ and we get: } ?b = ?(2|3); (\text{I.23.5})$$

$$a = 1; c = 3|3; \text{ and we get: } ?b = ?(3|3); (\text{I.23.6})$$

Substituting this into the Table I.34 to get a Table I.35:

[2 2]	?1	?2	?3	?(2 2)	?(2 3)	?(3 3)	?c
?1	?1	?2	?3	?(2 2)	?(2 3)	?(3 3)	
?2		?1		?2	?3		
?3			?1		?2	?3	
?a							?b

Table I.35

The remaining six empty cells in Table I.35 are filled with numbers using (I.23.3), and we get Table I.36:

[2 2]	?1	?2	?3	?(2 2)	?(2 3)	?(3 3)	?c
?1	1	?2	?3	?(2 2)	?(2 3)	?(3 3)	
?2	1+	1	4+	?2	?3	6+	
?3	2+	3+	1	5+	?2	?3	
?a							?b

Table I.36

Here, the numbers that do not fit in these six cells are indicated by the numbers with the plus sign to the right of the number:

$$?a [2|2] ?c = \log(c) \text{ on the basis of } a = ?b; (\text{I.23.3})$$

$$1+ = \text{logarithm}(1) \text{ for base } 2 = 0; (\text{I.23.7})$$

$$2+ = \text{logarithm}(1) \text{ for base } 3 = 0; (\text{I.23.8})$$

$$3+ = \text{logarithm}(2) \text{ for base } 3; (\text{I.23.9})$$

$$4+ = \text{logarithm}(3) \text{ for base } 2; (\text{I.23.10})$$

$5+ = \text{logarithm } (2/2) \text{ for base } 3; (I.23.11)$

$6+ = \text{logarithm } (3/3) \text{ on base } 2; (I.23.12)$

Given (I.23.7) and (I.23.8) we obtain Table I.37:

[2]2	?1	?2	?3	?2 2	?2 3	?3 3	?c
?1	1	?2	?3	?2 2	?2 3	?3 3	
?2	0	1	4+	?2	?3	6+	
?3	0	3+	1	5+	?2	?3	
?a							?b

Table I.37

I.24 Operation [3] for the two question numbers.

$?a [3] ?b = c; (I.24.1)$

По формуле (I.24.1), с учётом $?1 = 1$, построим Таблицу I.38:

[3]	?1	?2	?3	?b
?1	1	2	3	
?2	?2	?2 [3] ?2	?2 [3] ?3	
?3	?3	?3 [3] ?2	?3 [3] ?3	
?a				c

Table I.38

I.25 More about question numbers.

Let us return to the definition of question numbers. If **a** is any number (complex or question),

$1 [3] ?a = 1 \wedge ?a = a; (I.25.1)$

$?a = \log(a) \text{ on the basis of } 1; (I.25.2)$

It can be seen, that the question mark is the operator:

$$\?() = \log () \text{ on the basis of } 1 ; (I.25.3)$$

It can be applied to the same number multiple times:

$$b = \?(?(a)); c = \?(?(?(b))); (I.25.4)$$

And more – from (I.25.1) it follows that at $a > 1$, $?a$ is one of infinitely large numbers. And at $a < 1$, $?a$ is one of infinitely small numbers.

Question numbers can be used in formulas along with complex numbers. They can be multiplied by the imaginary unit.

Here is an example when the question number is represented as the sum of the imaginary question, and the actual question summands:

$$\?(-2) = \?(e) * i * PI + \?(2); (I.25.5)$$

Check this formula by raising 1 to the power of (I.25.5):

$$1 [3] \?(-2) = 1 [3] \{ \?(e) * i * PI + \?2 \}$$

$$-2 = \{ (1 [3] \?e) [3] (i * PI) \} * (1 [3] \?2)$$

$$-2 = \{ e [3] (i * PI) \} * 2$$

$$-2 = \{ -1 \} * 2$$

$$-2 = -2$$

Here $PI = 3.14$, and i is the imaginary unit, and $e = 2.7$.

I.26 Building new direct operations.

So we traced the process of the emergence of new numbers, starting with natural to complex. And found that they arose at the introduction of reverse operations to direct operations. The introduction of subtraction gave zero and negative numbers. Introduction of division gave the fractional numbers. The introduction of the root gave irrational and imaginary numbers. Their addition gave complex numbers. This pattern was confirmed for all numbers. And if someone wants to build new numbers, you can use this pattern. But the fact is that all reverse operations have already been used. And all direct operations have already been used.

If we were given a new direct operation, we would build a pair of inverse operations for it, and then apply the found pattern to build new numbers. Let us turn to dialectics again. It says that the subject should be studied in development. In this case, the subject is direct operations. And their development can be traced back to our own experience - what direct operations, and in what order were studied at school.

At the very beginning, we have already mentioned all the currently known direct operations, and numbered them in the following order: addition – [1], multiplication – [2], exponentiation – [3]. In this order they have been studied in school. We are now interested in the rules of building new direct operations on the basis of previous direct operations.

In point I.11 the construction of multiplication operation - [2] by addition operation – [1] is considered. And at the end of this paragraph, this rule is summarized. Here it is:

"The multiplication operation is formed as follows: to the left of the multiplication operation, the summand is written, and to the right of it, the number of these summands is written.»

In paragraph 14 we consider the construction of the operation of exponentiation – [3] by operation of multiplication [2]. And at the end of this paragraph, this rule is summarized. Here it is:

"The exponentiation operation is formed as follows: the cofactor is written to the left of the exponentiation operation sign, and the number of these cofactors is written to the right of this sign.»

These two definitions can be summarized as follows:

Operation [n+1] is formed by operation [n] as follows: operand for operation [n] is written to the left of [n+1], and the number of these operands is written to the right of [n+1].

I.27 Construction of the fourth direct operation - [4].

Let's build some examples to obtain [4] by [3]:

$$2 [4] 2 = 2 [3] 2 = 4; (I.27.1)$$

$$3 [4] 2 = 3 [3] 3 = 27; (I.27.2)$$

$$2 [4] 4 = 2 [3] 2 [3] 2 [3] 2 = X; (I.27.3)$$

In the last formula there are two options – the account in the right-hand side of the equation to lead a left-to-right (get 256), or from right to left (get 65 536). How to choose the right option?

Shall we use the question number ?2. Is it an infinitely large number (1 [3] ?2 = 2):

$$2 [4] ?2 = 2 [3] 2 [3] 2 [3] \dots ; (I.27.4)$$

In this formula, there is only one option – from left to right. The second option (from right to left) is not possible because the number of operands on the right side (I.27.4) is infinite and there

is no right operand. Of course, this is a special example, but the right option should be universal. Therefore, the equation (I.27.3) takes the form:

$$2 [4] 4 = ((2 [3] 2) [3] 2) [3] 2 = 256; (I.27.3)$$

Now the fourth direct operation [4] is uniquely determined. And we'll be filling out the Table I.39 for it.:

[4]	1	2	3	b
1				
2				
3				
a				c

Table I.39

In this table we will put the formula:

$$a [4] b = c; (I.27.4)$$

$$1 [4] 1 = 1; (I.27.5)$$

To the right of [4] is 1, which means, that after the equal sign there is only one operand 1 and no operation [3].

$$2 [4] 1 = 2; (I.27.6)$$

$$3 [4] 1 = 3; (I.27.7)$$

$$1 [4] 2 = 1 [3] 1 = 1; (I.27.8)$$

$$1 [4] 3 = (1 [3] 1) [3] 1 = 1 [3] 1 = 1; (I.27.9)$$

$$2 [4] 2 = 2 [3] 2 = 4; (I.27.10)$$

$$2 [4] 3 = (2 [3] 2) [3] 2 = 4 [3] 2 = 16; (I.27.11)$$

$$3 [4] 2 = 3 [3] 3 = 27; (I.27.12)$$

$$3 [4] 3 = (3 [3] 3) [3] 3 = 27 [3] 3 = 19\ 683; (I.27.13)$$

[4]	1	2	3	b
1	1	1	1	
2	2	4	16	
3	3	27	19 683	
a				c

Table I.40

So we filled in the Table I.39. And now it is called the Table I.40.

And now we show a shortened method of recording the fourth direct operation [4] through the third direct operation [3]. Take the formula (I.27.4):

$$a [4] b = c;$$

And let $b = 5$. By definition of the fourth operation, we write this formula as:

$$c = a [4] 5 = (((a [3] a) [3] a) [3] a) [3] a; (I.27.14)$$

Since the properties [3] and [2] we have:

$$(a [3] a) [3] a = a [3] (a [2] a); (I.27.15)$$

Then (27.5) will look like this:

$$\begin{aligned} c &= a [4] 5 = (a [3] (a [2] a) [3] a) [3] a = \\ &= (a [3] (a [2] a [2] a)) [3] a = \\ &= a [3] (a [2] a [2] a [2] a) = a [3] (a [3] 4); (I.27.16) \end{aligned}$$

This example shows the following formula:

$$a [4] b = a [3] (a [3] (b - 1)); (I.27.17)$$

This conclusion is suitable for natural b . But perhaps the final formula (I.27.17) is also true for a wider class of numbers.

I.28 The first inverse operation [4]1] for the fourth direct operation [4].

Let's look at item I.8. The formula (I.8.1) for $n = 4$ has the form:

$$a [4] b = c; \text{ (I.28.1)}$$

And the formula (I.8.2) for $n = 4$ has the form:

$$c [4]1] b = a; \text{ (I.28.2)}$$

Let us rearrange in Table I.40 a and c, and we get Table I.41:

[4]1]	1	2	3	b
1	1	1	1	
2	2			
3	3			
4		2		
16			2	
27		3		
19 683			3	
c				a

Table I.41

Using (I.28.1) and (I.28.2) we will try to fill in the empty cells in Table I.41:

$$c = 4; b = 1; \text{ from (28.1) we obtain: } a = 4; \text{ (I.28.3)}$$

$$c = 16; b = 1; \text{ from (28.1) we obtain: } a = 16; \text{ (I.28.4)}$$

$$c = 27; b = 1; \text{ from (28.1) we obtain: } a = 27; \text{ (I.28.5)}$$

$$c = 19\,683; b = 1; \text{ from (28.1) we obtain: } a = 19\,683; \text{ (I.28.6)}$$

And then the equations for a are not included in the cells of the table, and we will indicate these equations by a number with a plus (we will use (I.27.17)):

$$1+ = a; c = 2; b = 2; a [4] 2 = a [3] a = c = 2; (I.28.7)$$

$$2+ = a; c = 2; b = 3; a [4] 3 = a [3] (a [3] 2) = c = 2; (I.28.8)$$

$$3+ = a; c = 3; b = 2; a [4] 2 = a [3] a = c = 3; (I.28.9)$$

$$4+ = a; c = 3; b = 3; a [4] 3 = a [3] (a [3] 2) = c = 3; (I.28.10)$$

$$5+ = a; c = 4; b = 3; a [4] 3 = a [3] (a [3] 2) = c = 4; (I.28.11)$$

$$6+ = a; c = 16; b = 2; a [4] 2 = a [3] a = c = 16; (I.28.12)$$

$$7+ = a; c = 27; b = 3; a [4] 3 = a [3] (a [3] 2) = c = 27; (I.28.13)$$

$$8+ = a; c = 19\ 683; b = 2; a [4] 2 = a [3] a = c = 19\ 683; (I.28.14)$$

The completed Table I.41 is now called Table I.42:

[4]1]	1	2	3	b
1	1	1	1	
2	2	1+	2+	
3	3	3+	4+	
4	4	2	5+	
16	16	6+	2	
27	27	3	7+	
19 683	19 683	8+	3	
c				a

Table I.42

I.29 The second inverse operation [4]2] for the fourth direct operation [4].

Formula (I. 8.1) at $n = 4$ gives:

a [4] b = c; (I.29.1)

Take the formula (I. 8.3) at n = 4:

a [4]2) c = b; (I.29.2)

Let's change b and c in the Table I.40, and get the Table. 43:

[4]2)	1	2	3	4	16	27	19 683	c
1	1, 2, 3							
2		1		2 3				
3			1			2	3	
a								b

Table I.43

Here we are again faced with a new type of numbers: numbers – strings. In cell b, where a = 1, c = 1 of Table I.43 are just 3 numbers: 1, 2, 3. And, if some formula will meet this number b, it is unclear which of its three values (1, 2, 3) to take for calculations. We have seen this in Table I.19 and in Table I.20. But there we were silent about this feature. The only thing we can do now is to develop a special designation for such numbers-strings, so that it is entirely placed in one cell. If the first value of the number-line denote - "m", the number of values in the number-line denote "N", and additive to move to the next value denote "n", and as a separator of these symbols to take a vertical line" | ", we get such a record:

m|N|n (I.29.3)

In our example (1, 2, 3) it will be written as:

1|3|1 (I.29.4)

Now fill in the empty cells in Table I.43 and get Table I.44:

[4]2)	1	2	3	4	16	27	19 683	c
1	1, 2, 3	3+	4+	5+	6+	7+	8+	
2	1+	1	9+	2	3	10+	11+	
3	2+	12+	1	13+	14+	2	3	
a								b

Table I.44

$$a = 2; c = 1; 2 [4] b = 1; 2 [4]2] 1 = b = 1+; (I.29.5)$$

$$a = 3; c = 1; 3 [4] b = 1; 3 [4]3] 1 = b = 2+; (I.29.6)$$

In equations 1+ and 2+, the values “b” reduce 2 and 3 to 1. That means they are infinitely small.

$$a = 1; c = 2; 1 [4] b = 2; 1 [4]2] 2 = b = 3+; (I.29.7)$$

$$a = 1; c = 3; 1 [4] b = 3; 1 [4]2] 3 = b = 4+; (I.29.8)$$

$$a = 1; c = 4; 1 [4] b = 4; 1 [4]2] 4 = b = 5+; (I.29.9)$$

$$a = 1; c = 16; 1 [4] b = 16; 1 [4]2] 16 = b = 6+; (I.29.10)$$

$$a = 1; c = 27; 1 [4] b = 27; 1 [4]2] 27 = b = 7+; (I.29.11)$$

$$a = 1; c = 19\ 683; 1 [4] b = 19\ 683; 1 [4]2] 19\ 683 = b = 8+; (I.29.12)$$

In equations from 3+ to 8+, the values “b” enlarge the 1 to 2, or to even larger numbers. So the values “b” are infinitely large.

$$a = 2; c = 3; 2 [4] b = 3; 2 [4]2] 3 = b = 9+; (I.29.13)$$

$$a = 2; c = 27; 2 [4] b = 27; 2 [4]2] 27 = b = 10+; (I.29.14)$$

$$a = 2; c = 19\ 683; 2 [4] b = 19\ 683; 2 [4]2] 19\ 683 = b = 11+; (I.29.15)$$

$$a = 3; c = 2; 3 [4] b = 2; 3 [4]2] 2 = b = 12+; (I.29.16)$$

$$a = 3; c = 4; 3 [4] b = 4; 3 [4]2] 4 = b = 13+; (I.29.17)$$

$$a = 3; c = 16; 3 [4] b = 16; 3 [4]2] 16 = b = 14+; (I.29.18)$$

In equations from 9+ to 14+ values “a” increase or decrease the original values, but neither the original nor the final values are equal to 1. So the values “b” in these equations are also finite.

I.30 Question numbers. Extension of the concept to new direct operations.

Remember the definition of question numbers in paragraph I.17:

- - - - -

$$1 [3] ?a = 1 \wedge ?a = a; (I.17.4)$$

Here a is any complex number. And the number $?a$ will be called a "question number".

- - - - -

Now that the fourth direct operation has appeared, this definition can be generalized to any direct operation with number n :

$$1 [n] n?a = a; (I.30.1)$$

Here a is any complex or question number. And number $n?a$ will be called a "question number". The connection of the old "question numbers" $?a$ and new "question numbers" $n?a$ is this:

$$3?a = ?a; (I.30.2)$$

$$1 [3] 3?a = a; (I.30.2)$$

And for the fourth direct operation [4] definition (I.30.1) gives:

$$1 [4] 4?a = a; (I.30.3)$$

For $a = 2$ we obtain the equation:

$$1 [4] 4?2 = 2; (I.30.4)$$

Here the number $4?2$ increases 1 to 2. So this is an infinitely large number. Number $4?3$ it turns out, too, endlessly large number. Also infinitely large numbers are $4?4$, $4?16$, $4?27$, $4?19683$, and so on.

For $a = 1/2$, we obtain the equation:

$$1 [4] 4?1/2 = 1/2; (I.30.5)$$

In this example, the number $4\frac{1}{2}$ reduce 1 to $\frac{1}{2}$. So it's an infinitely small number. Number $4\frac{1}{3}$ is also an infinitely small number. Also infinitesimal numbers are $4\frac{2}{5}$, $4\frac{3}{7}$, $4\frac{1}{12}$ – all, which have a <1 .

And number $4\frac{1}{1}$ gives the equation:

$$1 [4] 4\frac{1}{1} = 1; (I.30.6)$$

It stores 1 and thus is itself equal to 1:

$$4\frac{1}{1} = 1; (I.30.7)$$

I.31 Operation [4] for question numbers $?a (3?a)$.

From point I.27 we take the formula:

$$a [4] b = a [3] (a [3] (b-1)); (I.27.17)$$

For the brevity of the formulas, we will use the old notation. Recall that $?1 = 1$.

$$\begin{aligned} ?1 [4] ?1 &= ?1 [3] (?1 [3] (?1 - 1)) = 1 [3] (1 [3] (1 - 1)) = \\ &= 1 [3] (1 [3] 0) = 1 [3] 1 = 1; (I.31.1) \end{aligned}$$

$$\begin{aligned} ?1 [4] ?2 &= ?1 [3] (?1 [3] (?2 - 1)) = 1 [3] (1 [3] (?2 - 1)) = \\ &= 1 [3] ((1 [3] ?2) [2] (1 [3] (-1))) = 1 [3] (2 [2] 1) = \\ &= 1 [3] 2 = 1; (I.31.2) \end{aligned}$$

$$\begin{aligned} ?1 [4] ?3 &= 1 [3] (1 [3] (?3 - 1)) = 1 [3] ((1 [3] ?3) [2] (1 [3] (-1))) = \\ &= 1 [3] (3 [2] 1) = 1 [3] 3 = 1; (I.31.3) \end{aligned}$$

$$\begin{aligned} ?2 [4] ?1 &= ?2 [3] (?2 [3] (?1 - 1)) = ?2 [3] (?2 [3] (1 - 1)) = \\ &= ?2 [3] ((?2 [3] 1) / (?2 [3] 1)) = ?2 [3] (?2 / ?2) = ?2 [3] 1 = \\ &= ?2; (I.31.4) \end{aligned}$$

$$\begin{aligned} ?3 [4] ?1 &= ?3 [3] (?3 [3] (?1 - 1)) = ?3 [3] (?3 [3] (1 - 1)) = \\ &= ?3 [3] ((?3 [3] 1) / (?3 [3] 1)) = ?3 [3] (?3 / ?3) = \\ &= ?3 [3] 1 = ?3; (I.31.5) \end{aligned}$$

$$\begin{aligned} ?2 [4] ?2 &= ?2 [3] (?2 [3] (?2 - 1)) = \\ (\text{По Таблице I.24: } ?2 - 1 &= ?2 [1] 1) ?1 = ?2) \\ &= ?2 [3] (?2 [3] ?2); (I.31.6) \end{aligned}$$

$$\begin{aligned} ?2 [4] ?3 &= ?2 [3] (?2 [3] (?3 - 1)) = \\ &= ?2 [3] (?2 [3] ?3); (I.31.7) \end{aligned}$$

$$\begin{aligned} ?3 [4] ?2 &= ?3 [3] (?3 [3] (?2 - 1)) = \\ &= ?3 [3] (?3 [3] ?2); (I.31.8) \end{aligned}$$

$$\begin{aligned} ?3 [4] ?3 &= ?3 [3] (?3 [3] (?3 - 1)) = \\ &= ?3 [3] (?3 [3] ?3); (I.31.9) \end{aligned}$$

These results are summarized in Table 45:

[4]	?1	?2	?3	b
?1	1	1	1	
?2	?2	?2 [3] (?2 [3] ?2)	?2 [3] (?2 [3] ?3)	
?3	?3	?3 [3] (?3 [3] ?2)	?3 [3] (?3 [3] ?3)	
a				c

Table I.45

Table I.45 contains the formula: $a [4] b = c$; (I.31.10)

A little note. From (I.30.4) we have:

$$1 [4] 4?2 = 2;$$

And from (I.31.2) should (given $3?2 = ?2$):

$$1 [4] 3?2 = 1;$$

That means $4?2 > 3?2$; (I.31.11)

I.32 The first inverse operation [4]1 for the fourth direct operation [4] for question numbers ?a (3?a).

Let's look at item I.8. The formula (I.8.1) for $n = 4$ has the form:

$$a [4] b = c; (I.32.1)$$

And the formula (I.8.2) for $n = 4$ has the form:

$$c [4]1 b = a; (I.32.2)$$

We introduce the following notations:

$$?2 [3] (?2 [3] ?2) = "222"; ?2 [3] (?2 [3] ?3) = "223"; (I.32.3)$$

$$?3 [3] (?3 [3] ?2) = "332"; ?3 [3] (?3 [3] ?3) = "333"; (I.32.4)$$

If we rearrange "a" and "c" in Table I.45, we get Table I.46 for operation [4]1 applied to the numbers ?a (3?a):

[4]1	?1	?2	?3	b
1	1	1	1	
?2	?2			
?3	?3			
222		?2		
223			?2	
332		?3		
333			?3	
c				a

Table I.46

Using the formula (I.32.1) set $b = 1$ and obtain $a = c$. Table I.46 takes the form of a Table I.47:

[4]1	?1	?2	?3	b
1	1	1	1	
?2	?2			
?3	?3			
222	222	?2		
223	223		?2	
332	332	?3		
333	333		?3	
c				a

Table I.47

Fill the empty cells in the Table I.47 with icons – 1+, 2+, ..., 7+, 8+ – and get Table I.48:

[4]1	?1	?2	?3	b
1	1	1	1	
?2	?2	1+	2+	
?3	?3	3+	4+	
222	222	?2	5+	
223	223	6+	?2	
332	332	?3	7+	
333	333	8+	?3	
c				a

Table I.48

a [4] b = c;

Give the icons their corresponding values:

c = ?2; b = ?2; a [4] ?2 = ?2; 1+ = a; (I.32.5)

c = ?2; b = ?3; a [4] ?3 = ?2; 2+ = a; (I.32.6)

c = ?3; b = ?2; a [4] ?2 = ?3; 3+ = a; (I.32.7)

c = ?3; b = ?3; a [4] ?3 = ?3; 4+ = a; (I.32.8)

$c = 222; b = ?3; a [4] ?3 = 222; 5+ = a; (I.32.9)$

$c = 223; b = ?2; a [4] ?2 = 223; 6+ = a; (I.32.10)$

$c = 332; b = ?3; a [4] ?3 = 332; 7+ = a; (I.32.11)$

$c = 333; b = ?2; a [4] ?2 = 333; 8+ = a; (I.32.12)$

I.33 The second inverse operation [4]2 for the fourth direct operation [4] for question numbers ?a (3?a).

The formula (I.8.1) for $n = 4$ gives:

$a [4] b = c; (I.33.1)$

Take the formula (I.8.3) for $n = 4$:

$a [4]2 c = b; (I.33.2)$

We introduce the following notations:

$?2 [3] (?2 [3] ?2) = "222"; ?2 [3] (?2 [3] ?3) = "223"; (I.33.3)$

$?3 [3] (?3 [3] ?2) = "332"; ?3 [3] (?3 [3] ?3) = "333"; (I.33.4)$

Let's rearrange b and c in Table I.45 and get Table I.49:

[4]2	1	?2	?3	222	223	332	333	c
?1	1+							
?2		1		?2	?3			
?3			1			?2	?3	
a								b

Table I.49

Here 1+ is a number-string: $1+ = 1, ?2, ?3; (I.33.5)$

Fill the remaining empty cells of the Table I.49 with icons 2+, 3+, ..., 14+, 15+ and get the Table I.50:

[4]2	1	?2	?3	222	223	332	333	c
1	1+	4+	5+	6+	7+	8+	9+	
?2	2+	1	11+	?2	?3	14+	15+	
?3	3+	10+	1	12+	13+	?2	?3	
a								b

Table I.50

Give the values for the new icons in the Table I.50:

$$(a [4] b = c) (a [4]2] c = b)$$

$$a = ?2; c = 1; ?2 [4] 2+ = 1; (I.33.6)$$

$$a = ?3; c = 1; ?3 [4] 3+ = 1; (I.33.7)$$

$$a = 1; c = ?2; 1 [4] 4+ = ?2; (I.33.8)$$

$$a = 1; c = ?3; 1 [4] 5+ = ?3; (I.33.9)$$

$$a = 1; c = 222; 1 [4] 6+ = 222; (I.33.10)$$

$$a = 1; c = 223; 1 [4] 7+ = 223; (I.33.11)$$

$$a = 1; c = 332; 1 [4] 8+ = 332; (I.33.12)$$

$$a = 1; c = 333; 1 [4] 9+ = 333; (I.33.13)$$

$$a = ?3; c = ?2; ?3 [4] 10+ = ?2; (I.33.14)$$

$$a = ?2; c = ?3; ?2 [4] 11+ = ?3; (I.33.15)$$

$$a = ?3; c = 222; ?3 [4] 12+ = 222; (I.33.16)$$

$$a = ?3; c = 223; ?3 [4] 13+ = 223; (I.33.17)$$

$$a = ?2; c = 332; ?2 [4] 14+ = 332; (I.33.18)$$

$$a = ?2; c = 333; ?2 [4] 15+ = 333; (I.33.19)$$

I.34 Operation [4] for question numbers 4?a.

$$a [4] b = c; (I.34.1)$$

From (I.30.7) take the equation: $4?1 = 1$;

Then we will get:

$$4?1 [4] 4?1 = 1; (I.34.2)$$

$$4?1 [4] 4?2 = 2; (I.34.3)$$

$$4?1 [4] 4?3 = 3; (I.34.4)$$

$$4?2 [4] 4?1 = 4?2; (I.34.5)$$

$$4?3 [4] 4?1 = 4?3; (I.34.6)$$

For the formula (I.34.1) will be the Table I.51:

[4]	4?1	4?2	4?3	b
4?1	1	2	3	
4?2	4?2	4?2 [4] 4?2	4?2 [4] 4?3	
4?3	4?3	4?3 [4] 4?2	4?3 [4] 4?3	
a				c

Table I.51

I.35 The first inverse operation [4]1 to the fourth direct operation [4] on the set of question numbers 4?a.

To obtain the first inverse operation [4]1 to the fourth direct operation [4] it is necessary to rearrange "a" and "c" in Table I.51:

$$c [4]1 b = a; (I.35.1)$$

And then we get a Table I.52:

[4]1	4?1	4?2	4?3	b
1	1			
2		1		
3			1	
4?2	4?2			
4?3	4?3			
4?2 [4] 4?2		4?2		
4?2 [4] 4?3			4?2	
4?3 [4] 4?2		4?3		
4?3 [4] 4?3			4?3	
c				a

Table I.52

Using the formulas (I.32.1), (I.30.7) and the Table I.52 ($a [4] 1 = c$), we fill in a few more cells, and get Table I.53:

[4]1	4?1	4?2	4?3	b
1	1			
2	2	1		
3	3		1	
4?2	4?2			
4?3	4?3			
4?2 [4] 4?2	4?2 [4] 4?2	4?2		
4?2 [4] 4?3	4?2 [4] 4?3		4?2	
4?3 [4] 4?2	4?3 [4] 4?2	4?3		
4?3 [4] 4?3	4?3 [4] 4?3		4?3	
c				a

Table 53

In the remaining empty cells full expressions will not fit. So we'll fill them with short icons and then match the icons to the expressions. Table with icons will be called Table I.54:

[4]1	4?1	4?2	4?3	b
1	1	1+	2+	
2	2	1	3+	
3	3	4+	1	
4?2	4?2	5+	6+	
4?3	4?3	7+	8+	
4?2 [4] 4?2	4?2 [4] 4?2	4?2	9+	
4?2 [4] 4?3	4?2 [4] 4?3	10+	4?2	
4?3 [4] 4?2	4?3 [4] 4?2	4?3	11+	
4?3 [4] 4?3	4?3 [4] 4?3	12+	4?3	
c				a

Table I.54

1+ [4] 4?2 = 1; (I.35.2)

2+ [4] 4?3 = 1; (I.35.3)

3+ [4] 4?3 = 2; (I.35.4)

4+ [4] 4?2 = 3; (I.35.5)

5+ [4] 4?2 = 4?2; (I.35.6)

6+ [4] 4?3 = 4?2; (I.35.7)

7+ [4] 4?2 = 4?3; (I.35.8)

8+ [4] 4?3 = 4?3; (I.35.9)

9+ [4] 4?3 = 4?2 [4] 4?2; (I.35.10)

10+ [4] 4?2 = 4?2 [4] 4?3; (I.35.11)

11+ [4] 4?3 = 4?3 [4] 4?2; (I.35.12)

12+ [4] 4?2 = 4?3 [4] 4?3; (I.35.13)

I.36 The second inverse operation [4]2 to the fourth direct operation [4] on the set of question numbers 4?a.

To obtain the second inverse operation [4]2 to the fourth direct operation [4] it is necessary to rearrange "b" and "c" in Table I.51, and we get the Table I.55:

[4]2	1	2	3	4?2	4?3	22?	23?	32?	33?	c
4?1	1	4?2	4?3							
4?2				4?1		4?2	4?3			
4?3					4?1			4?2	4?3	
a										b

Table I.55

a [4] b = c; (I.36.1)

a [4]2 c = b; (I.36.2)

In Table 55 there are applied the reductions:

"22?" = "4?2 [4] 4?2"; "23?" = "4?2 [4] 4?3"; (I.36.3)

"32?" = "4?3 [4] 4?2"; "33?" = "4?3 [4] 4?3"; (I.36.4)

From (I.30.7) it is taken the equation: $4?1 = 1$;

Let's set "a" = $4?1 = 1$ and we will substitute in (I.36.1) different "c". We will get different equations for "b":

$c = 4?2$; $1 [4] b = 4?2$; $b = 4?(4?2)$; $b1 = b$; (I.36.5)

$c = 4?3$; $1 [4] b = 4?3$; $b = 4?(4?3)$; $b2 = b$; (I.36.6)

$c = 22?$; $1 [4] b = 22?$; $b = 4?(22?)$; $b3 = b$; (I.36.7)

$c = 23?$; $1 [4] b = 23?$; $b = 4?(23?)$; $b4 = b$; (I.36.8)

$c = 32?$; $1 [4] b = 32?$; $b = 4?(32?)$; $b5 = b$; (I.36.9)

$c = 33?$; $1 [4] b = 33?$; $b = 4?(33?)$; $b_6 = b$; (I.36.10)

Let's put into a Table I.55 the values: $b_1, b_2, b_3, b_4, b_5, b_6$ in appropriate places, and get a Table I.56:

[4]2	1	2	3	4?2	4?3	22?	23?	32?	33?	c
4?1	1	4?2	4?3	b_1	b_2	b_3	b_4	b_5	b_6	
4?2				1		4?2	4?3			
4?3					1			4?2	4?3	
a										b

Table I.56

Let's fill the remaining empty cells by indexes: $1+, 2+, \dots, 11+, 12+$ and get a Table I.57:

[4]2	1	2	3	4?2	4?3	22?	23?	32?	33?	c
4?1	1	4?2	4?3	b_1	b_2	b_3	b_4	b_5	b_6	
4?2	1+	3+	5+	1	8+	4?2	4?3	11+	12+	
4?3	2+	4+	6+	7+	1	9+	10+	4?2	4?3	
a										b

Table I.57

Now let's compare its value to each index:

$(a [4] b = c) (a [4]2) c = b$

$a = 4?2; c = 1; 4?2 [4] b = 1; 1+ = b$; (I.36.11)

$a = 4?3; c = 1; 4?3 [4] b = 1; 2+ = b$; (I.36.12)

$a = 4?2; c = 2; 4?2 [4] b = 2; 3+ = b$; (I.36.13)

$a = 4?3; c = 2; 4?3 [4] b = 2; 4+ = b$; (I.36.14)

$a = 4?2; c = 3; 4?2 [4] b = 3; 5+ = b$; (I.36.15)

$a = 4?3; c = 3; 4?3 [4] b = 3; 6+ = b$; (I.36.16)

In these six formulas the indexes 1+, 2+, 3+, 4+, 5+, 6+ are infinitely small.

$$a = 4?3; c = 4?2; 4?3 [4] b = 4?2; 7+ = b; (I.36.17)$$

$$a = 4?2; c = 4?3; 4?2 [4] b = 4?3; 8+ = b; (I.36.18)$$

$$a = 4?3; c = 22?; 4?3 [4] b = 22?; 9+ = b; (I.36.19)$$

$$a = 4?3; c = 23?; 4?3 [4] b = 23?; 10+ = b; (I.36.20)$$

$$a = 4?2; c = 32?; 4?2 [4] b = 32?; 11+ = b; (I.36.21)$$

$$a = 4?2; c = 33?; 4?2 [4] b = 33?; 12+ = b; (I.36.22)$$

In the last four formulas the indexes 9+, 10+, 11+, 12+ are infinitely large.

I.37 The construction of the fifth direct operation - [5].

Paragraph I.26 describes the general rule of constructing a $[n + 1]$ direct operation by $[n]$ direct operation. Here it is:

- - - - -

Operation $[n+1]$ is formed by operation $[n]$ as follows: the operand for operation $[n]$ is written to the left of $[n+1]$, and the number of these operands is written to the right of $[n+1]$.

- - - - -

You can look at paragraph I.27 - it describes the construction of the fourth direct operation $[4]$. It obtains a general rule of parenthesis arrangements – so, that the calculations are performed from left to right. In our case, $n = 4$. Let's look at some examples:

$a [5] b = c$; (I.37.1)

$2 [5] 2 = 2 [4] 2 = 4$; (I.37.2)

$2 [5] 3 = (2 [4] 2) [4] 2 = 4 [4] 2 = 4 [3] 4 = 256$; (I.37.3)

$3 [5] 2 = 3 [4] 3 = 19\ 683$; (I.37.4)

$3 [5] 3 = (3 [4] 3) [4] 3 = 19\ 683 [4] 3$; (I.37.5)

$1 [5] 1 = 1$; $1 [5] 2 = 1$; $1 [5] 3 = 1$; (I.37.6)

$2 [5] 1 = 2$; $3 [5] 1 = 3$; (I.37.7)

Let us summarize them in Table I.58:

[5]	1	2	3	b
1	1	1	1	
2	2	4	256	
3	3	19 683	19 683 [4] 3	
a				c

Table I.58

Then you should act as in the case of the fourth operation. And then build the sixth operation, and so on. This concludes Chapter I.

18.08.2018

CHAPTER II

II.1 Summary of Chapter I.

In 1984, I read in the journal "Science and life" №1 1984 article V. Ouspensky "non-Standard analysis". The hyperreal numbers were introduced in consideration in it. And, already somewhere in June 1984, I thought – " There was a time when they did not know about. And now learned. So maybe there are other numbers that we don't know now, and then we'll find out. And is it possible not to wait for this, but somehow to find out about these unknown numbers now?".

Dialectics says that the subject should be studied in development. In this case, the subject is numbers. And their development can be traced from my own experience-what numbers, and in what order I studied at school.

First, we studied the numbers: 1, 2, 3, ... - they are called natural. Then zero and negative numbers. Natural, zero and negative are called integers.

Then there were fractional numbers. Integers and fractional numbers are called rational numbers.

Then irrational numbers were studied. Rational numbers together with irrational numbers are called real numbers.

They were followed by imaginary numbers. Real numbers together with imaginary numbers are called complex numbers.

The operations, studied in school, are many. But the main ones are addition, multiplication, exponentiation. They will be called - "direct operations". We introduce the following notations: [1] – addition, [2] – multiplication, [3] – exponentiation. Also, we will introduce the conventional symbols of these three "direct operations": [1] – "+", [2] – "*", [3] – "^".

Let a direct operation n be defined on some set of numbers:

$$a [n] b = c; (I.8.1)$$

(Here a, b are numbers from this set. They are called operands.)

If in (8.1) we rearrange the first operand " a " with the result " c ", we get:

$$c [n]1] b = a; (I.8.2)$$

This is the definition of the first reverse operation $[n]1]$ to the direct operation n .

If in (8.1) we rearrange the second operand " b " with the result " c ", we get:

$$a [n]2] c = b; (I.8.3)$$

This is the definition of the second inverse operation $[n]2]$ for n direct operation.

So we traced the process of the emergence of new numbers, starting from natural to complex. And found that they occurred with the introduction of reverse operations to direct operations. The introduction of subtraction gave zero and negative numbers. The introduction of division gave the fractional numbers. The introduction of the root gave irrational and imaginary numbers. Their addition gave complex numbers. This pattern was confirmed for all numbers. And, if someone wants to build new numbers, you can use this pattern. But the fact is that all reverse operations have already been used. And all direct operations have already been used.

If we were given a new direct operation, we would build a pair of inverse operations for it, and then apply the found regularity to construct new numbers. Let us turn to dialectics again. It said that

the subject should be studied in development. In this case, the subject is the direct operations. And their development can be traced back to our own experience - what direct operations, and in what order studied at school.

At the very beginning, we have already mentioned all the currently known direct operations, and numbered them in the following order: addition – [1], multiplication – [2], exponentiation – [3]. In this order they have been studied in school. We are now interested in the rules of building new direct operations on the basis of previous direct operations.

The multiplication operation is formed as follows: to the left of the multiplication operation sign, the summand is written, and to the right of the sign the number of these summands is written.

The exponentiation operation is formed as follows: the cofactor is written to the left of the exponentiation operation sign, and the number of these cofactors is written to the right of this sign.

These two definitions can be summarized as follows:

Operation $[n+1]$ is formed by operation $[n]$ as follows: the operand for operation $[n]$ is written to the left of $[n+1]$, and the number of these operands is written to the right of $[n+1]$.

$$2 [4] 4 = 2 [3] 2 [3] 2 [3] 2 = X; (1.27.3)$$

In the last formula there are two options – the account in the right-hand side of the equation to lead a left-to-right (get 256), or from right to left (get 65 536). How to choose the right option?

We shall use the question number ?2. It is an infinitely large number $(1 [3] ?2 = 2)$:

$$2 [4] ?2 = 2 [3] 2 [3] 2 [3] \dots; (I.27.4)$$

In this formula there is only one option – from left to right. The second option (from right to left) is not possible because the number of operands on the right side of (I.27.4) is infinite and there is no right operand. Of course, this is a particular example, but the right option should be universal. Therefore, the equation (I.27.3) takes the form:

$$2 [4] 4 = ((2 [3] 2) [3] 2) [3] 2 = 256; (I.27.3)$$

Recall the definition of question numbers in paragraph I.17:

-- - - - -

$$1 [3] ?a = 1 \wedge ?a = a; (I.17.4)$$

Here a is any complex number. And the number $?a$ will be called a "question number".

-- - - - -

Now that the fourth direct operation has appeared, this definition can be generalized to any direct operation with number n :

$$1 [n] n?a = a; (I.30.1)$$

Here a is any complex or question number. And the number $n?a$ will be called a "question number". Connection of old "question numbers" $?a$ and the new "question numbers" $n?a$ is this:

$$3?a = ?a; (I.30.2)$$

$$1 [3] 3?a = a; (I.30.2)$$

And now we show a shortened method of recording the fourth direct operation [4] through the third direct operation [3].

$$a [4] b = a [3] (a [3] (b - 1)); (I.27.17)$$

This conclusion is suitable for natural b . But perhaps the final formula (I.27.17) is also true for a wider class of numbers.

[4]2	1	2	3	4	16	27	19 683	c
1	1, 2, 3							
2		1		2	3			
3			1			2	3	
a								b

Table I.43

Here we are again faced with a new type of numbers: numbers – strings. In cell b , where $a = 1$, $c = 1$ of Table I.43 are just 3 numbers: 1, 2, 3. And, if some formula will meet this number b , it is unclear which of its three values (1, 2, 3) to take for calculations. We have already seen this in Table I.19 and Table I.20. But there we were silent about this feature. The only thing we can do now is to develop a special designation for such numbers-strings, so that it fits entirely in one cell. If the first value of the number-line we denote " m ", the number of values in the number-line denote " N ", and additive to move to the next value denote " n ", and as a separator of these symbols take a vertical line "|", we get such a record:

$$m|N|n \text{ (I.29.3)}$$

In our example (1, 2, 3) it will be written as:

$$1|3|1 \text{ (I.29.4)}$$

II.2 Determination of direct zero [0] and direct negative [-m] operations.

Chapter I establishes a connection between direct operation [n] and direct operation [n + 1]. Here are a couple of examples of this connection:

$$3 [n + 1] 2 = 3 [n] 3; \text{ (II.2.1)}$$

$$2 [n + 1] 3 = (2 [n] 2) [n] 2; \text{ (II.2.2)}$$

If we take $n = 0$, we can determine by the known direct operation [1] (addition) a new direct operation [0]. And then, by direct operation [0] build a new direct operation [-1]. Thus we can construct all direct operations with negative numbers.

So the number of the direct operation can take the value of any integer.

II.3 Zero direct operation [0].

Take $n = 0$. Use also: $2 [1] 2 = 2 + 2 = 4$; (II.3.1)

In the sample (II.2.1) replace 3 with 2, and then we get the equation for 2 [0] 2:

$$4 = 2 [1] 2 = 2 [0] 2; \text{ (II.3.2)}$$

Further $2 [1] 3 = 2 + 3 = 5$; (II.3.3)

Apply the sample (II.2.2) and obtain the equation for 4 [0] 2:

$$5 = 2 [1] 3 = (2 [0] 2) [0] 2 = 4 [0] 2; \text{ (II.3.4)}$$

Now, considering (II.3.4), we construct the equation for 5 [0] 2:

$$6 = 2 [1] 4 = ((2 [0] 2) [0] 2) [0] 2 = (4 [0] 2) [0] 2 = 5 [0] 2; \text{ (II.3.5)}$$

And now, considering (II.3.5), the equation for 6 [0] 2:

$$7 = 2 [1] 5 = (5 [0] 2) [0] 2 = 6 [0] 2; (\text{II.3.6})$$

Next, we get a couple of equations:

$$8 = 2 [1] 6 = (6 [0] 2) [0] 2 = 7 [0] 2; (\text{II.3.7})$$

$$9 = 2 [1] 7 = (7 [0] 2) [0] 2 = 8 [0] 2; (\text{II. 3.8})$$

The equations obtained are generalized by the formula:

$$a [0] b = c; (\text{II.3.9})$$

Now let's build five more equations, but with $b = 3$:

$$5 = 3 [1] 2 = 3 [0] 3; (\text{II.3.10})$$

$$6 = 3 [1] 3 = (3 [0] 3) [0] 3 = 5 [0] 3; (\text{II.3.11})$$

$$7 = 3 [1] 4 = 6 [0] 3; (\text{II.3.12})$$

$$8 = 3 [1] 5 = 7 [0] 3; (\text{II.3.13})$$

$$9 = 3 [1] 6 = 8 [0] 3; (\text{II.3.14})$$

And now four more equations with $b = 4$:

$$6 = 4 [1] 2 = 4 [0] 4; (\text{II.3.15})$$

$$7 = 4 [1] 3 = (4 [0] 4) [0] 4 = 6 [0] 4; (\text{II.3.16})$$

$$8 = 4 [1] 4 = 7 [0] 4; (\text{II.3.17})$$

$$9 = 4 [1] 5 = 8 [0] 4; (\text{II.3.18})$$

We construct three equations with $b = 5$:

$$7 = 5 [1] 2 = 5 [0] 5; (\text{II.3.19})$$

$$8 = 5 [1] 3 = (5 [0] 5) [0] 5 = 7 [0] 5; (\text{II.3.20})$$

$$9 = 5 [1] 4 = 8 [0] 5; (\text{II.3.21})$$

Now let's build two equations with $b = 6$:

$$8 = 6 [1] 2 = 6 [0] 6; (\text{II.3.22})$$

$$9 = 6 [1] 3 = (6 [0] 6) [0] 6 = 8 [0] 6; (\text{II.3.23})$$

And finally, the last equation (with $b = 7$):

$$9 = 7 [1] 2 = 7 [0] 7; \text{ (II.3.24)}$$

$$a [0] b = c; \text{ (II.3.9)}$$

All these equations are presented in the form of Table II.1:

[0]	2	3	4	5	6	7	b
2	4						
3		5					
4	5		6				
5	6	6		7			
6	7	7	7		8		
7	8	8	8	8		9	
a							c

Table II.1

It is impossible to make formulas for empty cells. It remains only to speculate. If you try to make it logical, balanced and beautiful, you will get the Table II.2:

[0]	2	3	4	5	6	7	b
2	4	4.5	5	5	5	5	
3	4.5	5	5.5	6	6	6	
4	5	5.5	6	6.5	7	7	
5	6	6	6.5	7	7.5	8	
6	7	7	7	7.5	8	8.5	
7	8	8	8	8	8.5	9	
a							c

Table II.2

Here is an extended table for [0]:

[0]	-2	-1	0	1	2	3	4	5	6	7	b
-2	0	0.5	1	1	1	1	1	1	1	1	
-1	0.5	1	1.5	2	2	2	2	2	2	2	
0	1	1.5	2	2.5	3	3	3	3	3	3	
1	2	2	2.5	3	3.5	4	4	4	4	4	
2	3	3	3	3.5	4	4.5	5	5	5	5	
3	4	4	4	4	4.5	5	5.5	6	6	6	
4	5	5	5	5	5	5.5	6	6.5	7	7	
5	6	6	6	6	6	6	6.5	7	7.5	8	
6	7	7	7	7	7	7	7	7.5	8	8.5	
7	8	8	8	8	8	8	8	8	8.5	9	
a											c

Table II.2.1

II.4 The first inverse operation [0]1] to the zero direct operation [0].

Permute a and c in Table II.2 and get Table II.3 for the first inverse to zero:

c [0]1] b = a; (II.4.1)

[0]1]	2	3	4	5	6	7	b
4	2						
4.5	3	2					
5	4	3	2	2	2	2	
5.5		4	3				
6	5	5	4	3	3	3	
6.5			5	4			
7	6	6	6	5	4	4	
7.5				6	5		
8	7	7	7	7	6	5	
8.5					7	6	
9	8	8	8	8	8	9	
a							c

Table II.3

The contents of the empty cells conjectures, logical and simple.
The result is in the Table II.3.1:

[0]1]	2	3	4	5	6	7	b
4	2	1	1	1	1	1	
4.5	3	2	1.5	1.5	1.5	1.5	
5	4	3	2	2	2	2	
5.5	4.5	4	3	2.5	2.5	2.5	
6	5	5	4	3	3	3	
6.5	5.5	5.5	5	4	3.5	3.5	
7	6	6	6	5	4	4	
7.5	6.5	6.5	6.5	6	5	4.5	
8	7	7	7	7	6	5	
8.5	7.5	7.5	7.5	7.5	7	6	
9	8	8	8	8	8	9	
a							c

Table II.3.1

II.5 The second inverse operation [0]2] to the zero direct operation [0]. Part 1.

Swap the c and b in Table II.2 and obtain the Table II.4:

a [0]2] c = b; (II.5.1)

	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9	c
2	2	3	1+									
3		2	3	4	2+							
4			2	3	4	5	4+					
5					3+	4	5	6	7			
6							5+	5	6	7		
7									6+	6	7	
a												b

Table II.4 [0]2]

$1+ = 4, 5, 6, 7; 2+ = 5, 6, 7; 4+ = 6, 7; (II.5.2)$

$3+ = 2, 3; 5+ = 2, 3, 4; 6+ = 2, 3, 4, 5; (II.5.3)$

In this table we again met the number-lines: $1+, \dots, 6+$. We will write them down in the designations defined in paragraph 29:

If the first value of the number-line denote "m", the number of values in the number-line denote "N", and additive to move to the next value denote "n", and as a separator of these symbols take a vertical line "|", we get such a record:

$m|N|n (29.3)$

First, do this with the formulas (II.5.2):

$1+ = 4|4|1; 2+ = 5|3|1; 4+ = 6|2|1; (II.5.4)$

Let's look at Table II.2 again:

[0]	2	3	4	5	6	7	b
2	4	4.5	5	5	5	5	
3	4.5	5	5.5	6	6	6	
4	5	5.5	6	6.5	7	7	
5	6	6	6.5	7	7.5	8	
6	7	7	7	7.5	8	8.5	
7	8	8	8	8	8.5	9	
a							c

Table II.2

When $a = 2$ after the number of 4.5 and up to the border of the Table II.2 there is a continuous row of fives. The first of these is bordered by a number of 4.5 and has $b = 4$. So in $1+$ for m we get $b = 4$. The number of these fives from 4.5 to the right border of the table = 4. Therefore, for N we took 4. The first five has $b = 4$. The second five has $b = 5$. Additive = 1. The third five has $b = 6$.

Supplement, too, = 1. The fourth five has $b = 7$. Supplement, too, = 1. Therefore, $n = 1$. And we have for a series of four fives:

$$1+ = m | N | n = 4 | 4 | 1; \text{ (II.5.5)}$$

For a series of three sixes at $a = 3$ we have after the number 5.5 the first six with $b = 5$. So in $2+$ for m , we get $b = 5$. The number of these sixes from the number 5.5 to the right border of the table = 3. Therefore, for N we took 3. The first six has $b = 5$. The second six has $b = 6$. Addition to the first six = 1. The third six has $b = 7$. Additive to the second six = 1. So $n = 1$. And we have for a series of three sixes:

$$2+ = m | N | n = 5 | 3 | 1; \text{ (II.5.6)}$$

For a series of two sevens at $a = 4$ we have the number 6.5 on the left as the boundary. The first seven to the right of 6.5 has $b = 6$. Therefore, in $4+$ we took $m = 6$. The number of these sevens from 6.5 on the left, and to the right border of Table II.2 is equal to 2. Therefore, $N = 2$. The first seven has $b = 6$. The second seven has $b = 7$. Addition to the first seven = 1. Therefore, $n = 1$. And for a series of two sevens we have:

$$4+ = m | N | n = 6 | 2 | 1; \text{ (II.5.7)}$$

For Table II.2 we have taken for its right boundary $b = 7$. If it were not for the size limit of Table II.2 for the zero direct operation [0] of the page frames in this book, then from the general view of Table II.2 it is clear that the row of fives at $a = 2$ continues to the right endlessly. That is, $N = ?2$. (Question number ?2 coincides with the number of natural numbers.) And then for $1+$ we have:

$$1+ = 4 | ?2 | 1; \text{ (II.5.8)}$$

The same reasoning is true for the row of three sixes. And we have for $2+$:

$$2+ = 5 \mid ?2 \mid 1; \text{ (II.5.9)}$$

The same thing happens with a row of two sevens at $a = 4$:

$$4+ = 6 \mid ?2 \mid 1; \text{ (II.5.10)}$$

Now let's move on to the formulas (II.5.3). Let's start with $3+$ in Table II.4. There are two sixes at $a = 5$ in Table II.2 to the left of the number 6.5. The left border of the Table II.2 is located to the left of them. And to the left of it hides an endless series of sixes. Therefore, it is possible to start the numbering of sixes in the number-line $3+$ only from the right – with the number $b = 3$: $m = 3$. The number of sixes is infinite: $N = ?2$ (number of positive integers). The six to the left of the first has $b = 2$. So the addition is -1 , $n = -1$. And then we have:

$$3+ = m \mid N \mid n = 3 \mid ?2 \mid -1; \text{ (II.5.11)}$$

Continue $5+$ in Table II.4. There are three sevens at $a = 6$ in Table II.2 to the left of 7.5. The left border of the Table II.2 is located to the left of them. And to the left of this border hides an infinite number of sevens. Therefore, it is possible to start the numbering of sevens only from the right – with a seven having $b = 4$: $m = 4$. $N = ?2$. The second seven in the number-line $5+$ has $b = 3$. So the addition is -1 , $n = -1$. And then we have:

$$5+ = m \mid N \mid n = 4 \mid ?2 \mid -1; \text{ (II.5.12)}$$

Now is the turn to the $6+$ in Table II.4. There are four eights at $a = 7$ in Table II.2 to the left of 8.5. To the left of them out of a border of the Table II.2 is an infinite number of eights. Then $N = ?2$. And the rightmost eight has $b = 5$: $m = 5$. The next eight to the left of it has $b = 4$. Then increment = -1 , $n = -1$. And we now have:

$$6+ = m \mid N \mid n = 5 \mid ?2 \mid -1; \text{ (II.5.13)}$$

Let us introduce in Table II.4 four more numbers-strings:

$$7+ = 2 \mid ?2 \mid -1; 8+ = 1 \mid ?2 \mid -1; 9+ = 7 \mid ?2 \mid 1; 0+ = 8 \mid ?2 \mid 1; \text{ (II.5.14)}$$

And get the Table II.5:

	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9	c
2	2	3	1+									
3	8+	2	3	4	2+							
4			7+	3	4	5	4+					
5					3+	4	5	6	9+			
6							5+	5	6	7	0+	
7									6+	6	7	
a												b

Table II.5 [0]2]

Now let's explain - where from did these four numbers-lines appeared. The single five is placed in the Table II.2 with a = 4, at the left border. And on the other side of the border there is an endless row of fives. And the beginning of all these fives is the right five with b = 2: m = 2. N = ?2. To the left of it (just abroad) is a five with b = 1. Then increment = -1, n = -1. Here is the result:

$$7+ = m \mid \mid N = 2 \mid ?2 \mid -1; \text{ (II.5.15)}$$

In Table II.2 at a = 3 and b = 1 (just beyond the left border) is 4. To the right of it there is the number 4.5, and to the left of it there is an infinite number of fours. That is, m = b = 1 and N = ?2. The fours extend to the left in descending order of b. Additive = -1: n = -1. And we got:

$$8+ = m \mid N \mid n = 1 \mid ?2 \mid -1; \text{ (II.5.16)}$$

In Table II.2, with a = 5 and b = 7 is 8. To its left is the number 7.5. To its right is the right border of Table II.2, and just over the border - an endless series of figure eights. So m = b = 7. N = ?2. Eights are

numbered in ascending order of b. It means, that supplement = 1.
And we have:

$$9+ = m | N | n = 7 | ?2 | 1; \text{ (II.5.17)}$$

In Table II.2 for a = 6 and b = 8, 9 is located (to the right of 8.5 – just beyond the right border of Table II.2). It is followed to the right by an infinite number of nines with an increment of 1: n = 1. Also we have m = b = 8. N = ?2. And as a result:

$$0+ = m | N | n = 8 | ?2 | 1; \text{ (II.5.18)}$$

II.6 The second inverse operation [0]2] to the zero direct operation [0]. Part 2. Floor 1st.

Let us reproduce the final results obtained in Part 1:

	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9	c
2	2	3	1+									
3	8+	2	3	4	2+							
4			7+	3	4	5	4+					
5					3+	4	5	6	9+			
6							5+	5	6	7	0+	
7									6+	6	7	
a												b

Table II.5 [0]2]

$$1+ = 4 | ?2 | 1; \text{ (II.5.8)}$$

$$2+ = 5 | ?2 | 1; \text{ (II.5.9)}$$

$$4+ = 6 | ?2 | 1; \text{ (II.5.10)}$$

$$9+ = m | N | n = 7 | ?2 | 1; \text{ (II.5.17)}$$

$$0+ = m | N | n = 8 | ?2 | 1; \text{ (II.5.18)}$$

$$8+ = m|N|n = 1|?2|-1; (II.5.16)$$

$$7+ = m|N|n = 2|?2|-1; (II.5.15)$$

$$3+ = m|N|n = 3|?2|-1; (II.5.11)$$

$$5+ = m|N|n = 4|?2|-1; (II.5.12)$$

$$6+ = m|N|n = 5|?2|-1; (II.5.13)$$

The number-lines. What to do with them? In what order to extract numbers from them?

Let's assume that all initial values of the row numbers are in the plane of Table II.5. And all the others are above them. Each consecutive number from the number-line is on its floor. All the second numbers-rows from 1+ to 6+ - are on the second floor of the three-dimensional Table II.5. All third numbers of all row numbers are on the third floor of Table II.5. And so on.

Let's look at the first floor of Table II.5:

	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9	c
2	2	3	4									
3	1	2	3	4	5							
4			2	3	4	5	6					
5					3	4	5	6	7			
6							4	5	6	7	8	
7									5	6	7	
a												b

Table II.5 [0]2] 1-st floor

This turned out to be the "skeleton" of the table. This is what is obtained from formulas relating operations [0] and [1]. We must to dream (most likely) and build on this "skeleton" everything else.

If you look closely at the "skeleton", you can see this feature of the "skeleton": horizontal numbers are parts of integer series –

they can be extended to the left and to the right of the "skeletal" middle.

If you look at this "skeleton" again, you can see its second feature: vertical numbers are parts of integer series – they can be extended up and down from the "skeletal" middle.

There is also a third feature: the horizontal addition of integers does not interfere with the vertical addition of integers. And vertical additions of integers also do not interfere with horizontal additions of integers.

After these observations grow on the "skeleton" "meat" and get Table II.6:

	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9	c
2	2	3	4	5	6	7	8	9	10	11	12	
3	1	2	3	4	5	6	7	8	9	10	11	
4	0	1	2	3	4	5	6	7	8	9	10	
5	-1	0	1	2	3	4	5	6	7	8	9	
6	-2	-1	0	1	2	3	4	5	6	7	8	
7	-3	-2	-1	0	1	2	3	4	5	6	7	
a	-4	-3	-2	-1	0	1	2	3	4	5	6	b

Table II.6 **[0]2]** 1-st floor

II.7 The second inverse operation **[0]2]** to the zero direct operation **[0]**. Part 2. Floor 2nd.

Now let's go to the second floor. Insert in the cells of Table II.5 the second numbers of the corresponding numbers-rows and get the "skeleton" of Table II.7:

	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9	c
2	2	3	5									
3	0	2	3	4	6							
4			1	3	4	5	7					
5					2	4	5	6	8			
6							3	5	6	7	9	
7									4	6	7	
a												b

Table II.7 [0]2 Floor 2nd

If you look at this "skeleton", you can see that it has no horizontal ordering. At least there is no horizontal integer order.

A vertical ordering may be present. Where "c" is semi-integer, integer ordering is possible. And where "c" is an integer, an even or odd ordering is possible.

Nothing else comes to mind, so let's increase our vertical "meat" on this "skeleton" and get the Table II.8:

	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9	c
2	2	3	5	5	8	7	11	9	14	11	17	
3	0	2	3	4	6	6	9	8	12	10	15	
4	-2	1	1	3	4	5	7	7	10	9	13	
5	-4	0	-1	2	2	4	5	6	8	8	11	
6	-6	-1	-3	1	0	3	3	5	6	7	9	
7	-8	-2	-5	0	-2	2	1	4	4	6	7	
a												b

Table II.8 [0]2 Floor 2nd

II.8 The second inverse operation [0]2 to the zero direct operation [0]. Part 2. The 3rd floor.

Now let's go to the third floor. Insert in the cells of Table II.5 the third numbers of the corresponding numbers-rows and get the Table II.9:

	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9	c
2	2	3	6									
3	-1	2	3	4	7							
4			0	3	4	5	8					
5					1	4	5	6	9			
6							2	5	6	7	10	
7									3	6	7	
a												b

Table II.9 [0]2] The 3rd floor

There is no horizontal ordering in this "skeleton". While the vertical ordering is present. Where "c" is semi-integer, integer ordering is possible. And where "c" is an integer, the numbers are located with step 3.

Let's build on this "skeleton" "meat" and get Table II.10:

	4	4.5	5	5.5	6	6.5	7	7.5	8	8.5	9
2	2	3	6	5	10	7	14	9	18	11	22
3	-1	2	3	4	7	6	11	8	15	10	19
4	-4	1	0	3	4	5	8	7	12	9	16
5	-7	0	-3	2	1	4	5	6	9	8	13
6	-11	-1	-6	1	-2	3	2	5	6	7	10
7	-14	-2	-9	0	-5	2	-1	4	3	6	7
a	-17	-3	-12	-1	-8	1	-4	3	0	5	4

Table II.10 [0]2] The 3-rd floor

II.9 The second inverse operation [0]2] to the zero direct operation [0]. Part 2. Floor 4th and above.

Now it is clear what the ordering is on the 4th floor and above. Where "c" is half-integer, there the integer vertical alignment is possible. And where "c" is an integer, the numbers vertically go in increments equal to the floor number.

II.10 Zero direct operation [0] on the set of question numbers ?a.

We will now fill in Table II.11:

?a [0] ?b = c; (II.10.1)

[0]	?1	?2	?3	?4	?b
?1					
?2					
?3					
?4					
?a					c

Table II.11

?1 [0] ?1 = 1 [0] 1 = 1 [1] 2 = 1 + 2 = 3; (II.10.2)

?2 [0] ?2 = ?2 [1] 2 = ?2 [1] (1 [1] 1) = (?2 [1] 1) [1] 1 =
= ?2 [1] 1 = ?2; (II.10.3)

?3 [0] ?3 = ?3 [1] 2 = ?3 [1] (1 [1] 1) = (?3 [1] 1) [1] 1 =
= ?3 [1] 1 = ?3; (II.10.4)

?4 [0] ?4 = ?4 [1] 2 = ?4 [1] (1 [1] 1) = (?4 [1] 1) [1] 1 =
= ?4 [1] 1 = ?4; (II.10.5)

?2 [0] ?1 = ?2 [0] 1 = ?2; (II.10.6)

?3 [0] ?1 = ?3 [0] 1 = ?3; (II.10.7)

$$?4 [0] ?1 = ?4 [0] 1 = ?4; (\text{II.10.8})$$

$$\begin{aligned} ?1 [0] ?2 &= 1 [0] ?2 = (((1 [-1] 1) [-1] 1) [-1] 1) [-1] 1 \dots = \\ &= (((1 [0] 2) [-1] 1) [-1] 1) [-1] 1 \dots = \end{aligned}$$

Use Table II.2.1 (expansion of Table II.2)

$$= ((3.5 [-1] 1) [-1] 1) [-1] 1 \dots =$$

Use Table II.26:

$$= (4 [-1] 1) [-1] 1 \dots =$$

Use Table II.26 and (II.13.76) ?2 times:

$$= 4 [-1] 1 \dots = 4 [-1] 1 \dots = 4; (\text{II.10.9})$$

That is: ?1 [0] ?2 = 4; (II.10.9)

Similarly (?3 times):

$$\begin{aligned} ?1 [0] ?3 &= 1 [0] ?3 = (((1 [-1] 1) [-1] 1) [-1] 1) [-1] 1 \dots = \\ &= (4 [0] 1) [0] 1 \dots = 4 [0] 1 \dots = 4; (\text{II.10.10}) \end{aligned}$$

Also (?4 times):

$$?1 [0] ?4 = 4; (\text{II.10.11})$$

We did the Table II.12:

[0]	?1	?2	?3	?4	?b
?1	3	?2	?3	?4	
?2	?2	?2			
?3	?3		?3		
?4	?4			?4	
?a					c

Table II.12

Let's continue filling the cells in Table II.12:

$$?2 [0] ?3 = (((?2 [-1] ?2) [-1] ?2) [-1] ?2) [-1] ?2 \dots =$$

$$= (((?2 [0] 2) [-1] ?2) [-1] ?2) [-1] ?2 \dots =$$

Remember the definition of ?1:

$$1 [3] ?1 = 1;$$

Here as ?1 any natural number is good. Including 2. Advantage of this:

$$= (((?2 [0] ?1) [-1] ?2) [-1] ?2) [-1] ?2 \dots =$$

And now we have the right to replace ?1 on 1 since as ?1 any natural number is good. Including 1.

$$= (((?2 [0] 1) [-1] ?2) [-1] ?2) [-1] ?2 \dots =$$

$$= (((?2 [-1]) [-1] ?2) [-1] ?2) [-1] ?2 \dots =$$

$$= ((?2 [-1] ?2) [-1] ?2) [-1] ?2 \dots =$$

We took one step. Do ?3 steps and get:

$$= ?2; (II.10.12)$$

That way ?2 [0]?3 = ?2; (II.10.12)

In the same way – but making already ?4 steps – we will get:

$$?2 [0] ?4 = ?2; (II.10.13)$$

And, by doing ?2 and ?4 steps, we get a couple more formulas:

$$?3 [0] ?2 = ?3; (II.10.14)$$

$$?3 [0] ?4 = ?3; (II.10.15)$$

And the result is Table II.13:

[0]	?1	?2	?3	?b
?1	3	4	4	
?2	?2	?2	?2	
?3	?3	?3	?3	
?a				c

Table II.13

$?a [0] ?b = ?a$; ($a = 2, 3, 4, \dots$; $b = 1, 2, 3, 4, \dots$); (II.10.16)

Looking at the expanded Table II.2.1 for [0] we will see ($?1 = 1$):

$1 [0] 0 = 2.5$; (II.10.17)

$0 [0] 0 = 2$; (II.10.18)

$0 [0] 1 = 2.5$; (II.10.19)

$0 [0] 2 = 3$; (II.10.20)

$0 [0] 3 = 3$; (II.10.21)

$0 [0] 3 = (0 [-1] 0) [-1] 0 = (0 [0] 2) [-1] 0 = 3 [-1] 0$; (II.10.22)

(II.10.21) + (II.10.22) = (II.10.23):

$3 [-1] 0 = 3$; (II.10.23)

$0 [0] ?2 = ((0 [-1] 0) [-1] 0) [-1] 0 \dots = ((0 [0] 2) [-1] 0) [-1] 0 \dots =$
 $= (3 [-1] 0) [-1] 0 \dots = 3 [-1] 0 \dots = \dots ?2 \text{ times and } = 3$; (II.10.24)

That is $0 [0] ?2 = 3$; (II.10.24)

$0 [0] ?3 = \dots ?3 \text{ times and } = 3$; (II.10.25)

Insert these values into Table II.13 and get Table II.13.1:

[0]	0	?1	?2	?3	?b
0	2	2.5	3	3	
?1	2.5	3	4	4	
?2		?2	?2	?2	
?3		?3	?3	?3	
?a					c

Table II.13.1

Two cells are not clear how to fill, so let's limit ourselves to Table II.13.

II.11 The first inverse operation [0]1] to the zero direct operation [0] on the set of question numbers ?a.

Rearrange "?a" and "c" in Table II.13 and get the Table II.14:

[0]1]	?1	?2	?3	?b
3	?1			
4		?1	?1	
?2	?2	?2	?2	
?3	?3	?3	?3	
c				?a

Table II.14

?a [0] ?1 = ?a [0] 1 = 4; (II.11.1)

If you look at Table II.2.1 (expanded to Table II.2) then we'll see that ?a = 3. That is 4 [0]1]?1 = 3; (II.11.2)

?a [0]?2 = 3; according to Table II.13.1 see: ?a = 0; (II.11.3)

3 [0]1]?2 = 0; (II.11.4)

?a [0]?3 = 3; according to Table II.13.1 see: ?a = 0; (II.11.5)

3 [0]1]?3 = 0; (II.11.6)

Table II.14 will become Table II.15:

[0]1	?1	?2	?3	?b
3	1	0	0	
4	3	1	1	
?2	?2	?2	?2	
?3	?3	?3	?3	
c				?a

Table II.15

And as a result - for the first inverse operation [0]1 to the zero direct operation [0] for numbers ?a – we have Table II.15.

II.12 The second inverse operation [0]2 to the zero direct operation [0] on the set of question numbers ?a.

Let's see in detail how numbers-strings are born. Consider the first row in Table II.13 and leave it in Table II.16:

[0]	?1	?2	?3	?b
?1	3	4	4	c
?2				
?3				
?a				

Table II.16

Rearrange the "c" and "?b". Get Table II.16.1:

[0]2	3	4	4	c
?1	?1	?2	?3	?b
?2				
?3				
?a				

Table II.16.1

The column headings should be different, so let's combine the columns with the same headings in Table II.16.2:

[0]2]	3	4		c
?1	?1	?2, ?3		?b
?2				
?3				
?a				

Table II.16.2

Consider the second row in Table II.13 and leave it in Table II.16.3:

[0]	?1	?2	?3	?b
?1				
?2	?2	?2	?2	c
?3				
?a				

Table II.16.3

Rearrange the "c" and "?b". Get Table II.16.4:

[0]2]	?2	?2	?2	c
?1				
?2	?1	?2	?3	?b
?3				
?a				

Table II.16.4

Again, the same column headings. Again, combine the contents of these columns under a common name and get Table II.16.5:

[0]2]		?2		c
?1				
?2		?1, ?2, ?3		?b
?3				
?a				

Table II.16.5

Consider the third row in Table II.13 and leave it in Table II.16.6:

[0]	?1	?2	?3	?b
?1				
?2				
?3	?3	?3	?3	c
?a				

Table II.16.6

Rearrange the "c" and "?b". Get Table II.16.7:

[0]2]	?3	?3	?3	c
?1				
?2				
?3	?1	?2	?3	?b
?a				

Table II.16.7

Again, the same column headings. Again, combine the contents of these columns under a common name and get Table II.16.8:

[0]2]		?3	c
?1			
?2			
?3		?1, ?2, ?3	?b
?a			

Table II.16.8

Combine the Tables: II.16.2, II.16.5, II.16.8 and get Table II.20:

[0]2]	3	4	?2	?3	c
?1	?1	?2, ?3			
?2			?1, ?2, ?3		
?3				?1, ?2, ?3	
?a					?b

Table II.20

Numbers-lines denote as follows:

$1+ = ?2, ?3$; $2+ = ?1, ?2, ?3$; $3+ = ?1, ?2, ?3$; (II.12.1)

Table II.20 will take the form of Table II.20.1:

[0]2]	3	4	?2	?3	c
?1	?1	1+			
?2			2+		
?3				3+	
?a					?b

Table II.20.1

For numbers-lines, there is still a variant of record:

$1+ = ?2 | 2 | ?+1$; (II.12.2)

$2+ = ?1 | 3 | ?+1$; (II.12.3)

$3+ = ?1 | 3 | ?+1$; (II.12.4)

Before the first vertical line - "|" – the first number in the number-line is specified. It is followed by the number of numbers in the number-line. Then comes the second vertical line. It is followed by the step between the numbers in the number-line. In this case, for quantities beginning with the sign "?", specified index step following sign "?".

We introduce the floors above the table.

The very first value in all numbers-lines is located on the first floor in Table II.21:

[0]2]	3	4	?2	?3	c
?1	?1	?2			
?2			?1		
?3				?1	
?a					?b

Table II.21 1st floor

On the second floor of operation [0]2] for ?a each number-line contains its second value. This is shown in Table II.22:

[0]2]	3	4	?2	?3	c
?1	?1	?3			
?2			?2		
?3				?2	
?a					?b

Table II.22 2nd floor

The third floor contains the third value of numbers-lines. Once in a number-line 1+ only two values, then the last it's value - ?3 – is taken. The result is presented in Table II.23:

[0]2]	3	4	?2	?3	c
?1	?1	?3			
?2			?3		
?3				?3	
?a					?b

Table II.23 3rd floor and above

The fourth floor contains in each cell the fourth value of numbers-lines. In those cells, where the number of values in the number-line is less than four, the most recent values of these numbers-lines are located. Get again Table II.23. It is also true for higher floors.

We got three "skeletal" Tables: II.21, II.22, II.23. Let's try to increase them "meat". It is not clear how to fill in the first two Tables – II.21 and II.22, and here is the third Table - II.23, which forms itself in Table II.24:

[0]2]	3	4	?2	?3	c
?1	?1	?3	?5	?7	
?2	?(-1)	?1	?3	?5	
?3	?(-3)	?(-1)	?1	?3	
?a					?b

Table II.24 3rd floor and above

II.13 Direct operation [-1].

In Chapter II, paragraph 2, it is written, how to define an unknown operation [-1] by a known operation [0]:

$$3 [n + 1] 2 = 3 n] 3; \text{ (II.2.1)}$$

$$2 [n + 1] 3 = (2 [n] 2) [n] 2; \text{ (II.2.2)}$$

At $n = -1$ we obtain:

$$3 [0] 2 = 3 [-1] 3; \text{ (II.13.1)}$$

$$2 [0] 3 = (2 [-1] 2) [-1] 2; \text{ (II.13.2)}$$

Values for operation [0] will be taken from Table II.2. If the desired value lies outside the boundary of Table II.2, then you can push the boundaries of Table II.1 as far as you need, by entering new rows and new columns, and filling them with new values, and then on the principle of symmetry fill the remaining empty cells. In further calculations, the values for operation [0] are taken from such an extended Table II.2.1.

$$1 = -2 [0] 2 = -2 [-1] -2; \text{ (II.13.3)}$$

$$1 = -2 [0] 3 = (-2 [-1] -2) [-1] -2 = 1 [-1] -2; \text{ (II.13.4)}$$

$$1 = -2 [0] 4 = 1 [-1] -2; \text{ (II.13.5)}$$

...

The repeats are endless, so

$$1+ = 1 [?] 0; \text{ in cell } 1[-1] -2; \text{ (II.13.6)}$$

$$2 = -1 [0] 2 = -1 [-1] -1; \text{ (II.13.7)}$$

$$2 = -1 [0] 3 = (-1 [-1] -1) [-1] -1 = 2 [-1] -1; \text{ (II.13.8)}$$

$$2 = -1 [0] \ 4 = 2 [-1] -1; \text{ (II.13.9)}$$

...

The repeats are endless, so

$$2+ = 2 [?] 2 [0]; \text{ in cell 2 } [-1] -1; \text{ (II.13.10)}$$

$$3 = 0 [0] \ 2 = 0 [-1] 0; \text{ (II.13.11)}$$

$$3 = 0 [0] \ 3 = (0 [-1] 0) [-1] 0 = 3 [-1] 0; \text{ (II.13.12)}$$

$$3 = 0 [0] \ 4 = 3 [-1] 0; \text{ (II.13.13)}$$

...

The repeats are endless, so

$$3+ = 3 [?] 2 [0]; \text{ in cell 3 } [-1] 0; \text{ (II.13.14)}$$

$$3.5 = 1 [0] \ 2 = 1 [-1] 1; \text{ (II.13.15)}$$

$$4 = 1 [0] \ 3 = (1 [-1] 1) [-1] 1 = 3.5 [-1] 1; \text{ (II.13.16)}$$

$$4 = 1 [0] \ 4 = 4 [-1] 1; \text{ (II.13.17)}$$

$$4 = 1 [0] \ 5 = 4 [-1] 1; \text{ (II.13.18)}$$

$$4 = 1 [0] \ 6 = 4 [-1] 1; \text{ (II.13.19)}$$

...

The repeats are endless, so

$$4+ = 4 [?] 2 [0]; \text{ in cell 4 } [-1] 1; \text{ (II.13.20)}$$

$$4 = 2 [0] \ 2 = 2 [-1] 2; \text{ (II.13.21)}$$

$$4.5 = 2 [0] \ 3 = (2 [-1] 2) [-1] 2 = 4 [-1] 2; \text{ (II.13.22)}$$

$$5 = 2 [0] \ 4 = ((2 [-1] 2) [-1] 2) [-1] 2 = 4.5 [-1] 2; \text{ (II.13.23)}$$

$$5 = 2 [0] 5 = 5 [-1] 2; (\text{II.13.24})$$

$$5 = 2 [0] 6 = 5 [-1] 2; (\text{II.13.25})$$

$$5 = 2 [0] 7 = 5 [-1] 2; (\text{II.13.26})$$

...

The repeats are endless, so

$$5+ = 5 | ? 2 | 0; \text{ in cell } 5 [-1] 2; (\text{II.13.27})$$

$$4.5 = 3 [0] 2 = 3 [-1] 3; (\text{II.13.28})$$

$$5 = 3 [0] 3 = (3 [-1] 3) [-1] 3 = 4.5 [-1] 3; (\text{II.13.29})$$

$$5.5 = 3 [0] 4 = 5 [-1] 3; (\text{II.13.30})$$

$$6 = 3 [0] 5 = 5.5 [-1] 3; (\text{II.13.31})$$

$$6 = 3 [0] 6 = 6 [-1] 3; (\text{II.13.32})$$

$$6 = 3 [0] 7 = 6 [-1] 3; (\text{II.13.33})$$

...

The repeats are endless, so

$$6+ = 6 | ? 2 | 0; \text{ in cell } 6 [-1] 3; (\text{II.13.34})$$

$$5 = 4 [0] 2 = 4 [-1] 4; (\text{II.13.35})$$

$$5.5 = 4 [0] 3 = 5 [-1] 4; (\text{II.13.36})$$

$$6 = 4 [0] 4 = 5.5 [-1] 4; (\text{II.13.37})$$

$$6.5 = 4 [0] 5 = 6 [-1] 4; (\text{II.13.38})$$

$$7 = 4 [0] 6 = 6.5 [-1] 4; (\text{II.13.39})$$

$$7 = 4 [0] 7 = 7 [-1] 4; (\text{II.13.40})$$

$$7 = 4 [0] \ 8 = 7 [-1] \ 4; (\text{II.13.41})$$

...

The repeats are endless, so

$$7+ = 7 | ?2 | 0; \text{ in cell } 7 [-1] \ 4; (\text{II.13.42})$$

$$6 = 5 [0] \ 2 = 5 [-1] \ 5; (\text{II.13.43})$$

$$6 = 5 [0] \ 3 = 6 [-1] \ 5; (\text{II.13.44})$$

$$6.5 = 5 [0] \ 4 = 6 [-1] \ 5; (\text{II.13.45})$$

$$11+ = 6, 6.5; \text{ in cell } 6 [-1] \ 5; (\text{II.13.46})$$

$$7 = 5 [0] \ 5 = 6.5 [-1] \ 5; (\text{II.13.47})$$

$$7.5 = 5 [0] \ 6 = 7 [-1] \ 5; (\text{II.13.48})$$

$$8 = 5 [0] \ 7 = 7.5 [-1] \ 5; (\text{II.13.49})$$

$$8 = 5 [0] \ 8 = 8 [-1] \ 5; (\text{II.13.50})$$

$$8 = 5 [0] \ 9 = 8 [-1] \ 5; (\text{II.13.51})$$

...

The repeats are endless, so

$$8+ = 8 | ?2 | 0; \text{ in cell } 8 [-1] \ 5; (\text{II.13.52})$$

$$7 = 6 [0] \ 2 = 6 [-1] \ 6; (\text{II.13.53})$$

$$7 = 6 [0] \ 3 = 7 [-1] \ 6; (\text{II.13.54})$$

$$7 = 6 [0] \ 4 = 7 [-1] \ 6; (\text{II.13.55})$$

$$7.5 = 6 [0] \ 5 = 7 [-1] \ 6; (\text{II.13.56})$$

$$10+ = 7, 7, 7.5; \text{ in cell } 7 [-1] \ 6; (\text{II.13.57})$$

$$8 = 6 [0] 6 = 7.5 [-1] 6; (\text{II.13.58})$$

$$8.5 = 6 [0] 7 = 8 [-1] 6; (\text{II.13.59})$$

$$8 = 7 [0] 2 = 7 [-1] 7; (\text{II.13.60})$$

$$8 = 7 [0] 3 = 8 [-1] 7; (\text{II.13.61})$$

$$8 = 7 [0] 4 = 8 [-1] 7; (\text{II.13.62})$$

$$8 = 7 [0] 5 = 8 [-1] 7; (\text{II.13.63})$$

$$8.5 = 7 [0] 6 = 8 [-1] 7; (\text{II.13.64})$$

$$9+ = 8, 8, 8, 8.5; \text{ in cell } 8 [-1] 7; (\text{II.13.65})$$

All the above 65 equations are considered and the results are summarized in Table II.26 (a [-1] b = c):

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1										
-1		2									
0			3								
1	1+			3.5							
2		2+			4						
3			3+			4.5					
3.5				4							
4				4+	4.5		5				
4.5					5	5					
5					5+	5.5	5.5	6			
5.5						6	6				
6						6+	6.5	11+	7		
6.5							7	7			
7							7+	7.5	10+	8	
7.5								8	8		
8								8+	8.5	9+	
a											c

Table II.26

In the Table II.26 there are the following number-strings:

$$1+ = 1 \mid ?2 \mid 0; 2+ = 2 \mid ?2 \mid 0; (\text{II.13.75})$$

$$3+ = 3 \mid ?2 \mid 0; 4+ = 4 \mid ?2 \mid 0; (\text{II.13.76})$$

$$5+ = 5 \mid ?2 \mid 0; 6+ = 6 \mid ?2 \mid 0; 7+ = 7 \mid ?2 \mid 0; 8+ = 8 \mid ?2 \mid 0; (\text{II.13.77})$$

$$9+ = 8, 8, 8, 8.5; 10+ = 7, 7, 7.5; 11+ = 6, 6.5; (\text{II.13.78})$$

For simplicity, let us leave in the Table II.26 only lines with integer headings. Get Table II.27:

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1										
-1		2									
0			3								
1	1+			3.5							
2		2+			4						
3			3+			4.5					
4				4+	4.5		5				
5					5+	5.5	5.5	6			
6						6+	6.5	11+	7		
7							7+	7.5	10+	8	
8								8+	8.5	9+	
a											c

Table II.27

Now consider the "skeleton" for the first floor. To do this, take the first value from each number-line and place it in the appropriate cell. Instead of 11+ will be the number 6. Instead of 10+ will be 7. Instead of 9+ will stand 8. And for the rest of the numbers-lines it will be enough to remove the sign "+". And we get the Table II.28:

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1										
-1		2									
0			3								
1	1			3.5							
2		2			4						
3			3			4.5					
4				4	4.5		5				
5					5	5.5	5.5	6			
6						6	6.5	6	7		
7							7	7.5	7	8	
8								8	8.5	8	
a											c

Table II.28 1st floor "skeleton"

Now, after thinking, let's increase the "meat" on this "skeleton".

And we get the Table II.29:

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1	2	3	3.5	4	4.5	5	6	7	8	
-1	1	2	3	3.5	4	4.5	5	6	7	8	
0	1	2	3	3.5	4	4.5	5	6	7	8	
1	1	2	3	3.5	4	4.5	5	6	7	8	
2	2	2	3	3.75	4	4.5	5	6	7	8	
3	3	3	3	4	4.25	4.5	5	6	7	8	
4	4	4	4	4	4.5	4.75	5	6	7	8	
5	5	5	5	5	5	5.5	5.5	6	7	8	
6	6	6	6	6	6	6	6.5	6	7	8	
7	7	7	7	7	7	7	7	7.5	7	8	
8	8	8	8	8	8	8	8	8	8.5	8	
a											c

Table II.29 1st floor

Now consider the "skeleton" for the second floor. To do this, from each number-line take the second value from the left and put it in the appropriate cell. Instead of 11+ will stand the number 6.5.

Instead of 10+ will be 7. Instead of 9+ will stand 8. And for the rest of the numbers-lines, it will be enough to remove the sign"+". After all, they get the second value by adding 0 to the first value. And we get the Table II.30:

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1										
-1		2									
0			3								
1	1			3.5							
2		2			4						
3			3			4.5					
4				4	4.5		5				
5					5	5.5	5.5	6			
6						6	6.5	6.5	7		
7							7	7.5	7	8	
8								8	8.5	8	
a											c

Table II.30 2nd floor "skeleton"

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1	2	3	3.5	4	4.5	5	6	7	8	
-1	1	2	3	3.5	4	4.5	5	6	7	8	
0	1	2	3	3.5	4	4.5	5	6	7	8	
1	1	2	3	3.5	4	4.5	5	6	7	8	
2	2	2	3	3.75	4	4.5	5	6	7	8	
3	3	3	3	4	4.25	4.5	5	6	7	8	
4	4	4	4	4	4.5	4.75	5	6	7	8	
5	5	5	5	5	5	5.5	5.5	6	7	8	
6	6	6	6	6	6	6	6.5	6.5	7	8	
7	7	7	7	7	7	7	7	7.5	7	8	
8	8	8	8	8	8	8	8	8	8.5	8	
a											c

Table II.31 2nd floor

Now consider the "skeleton" for the third floor. It coincides with the skeleton for the second floor, but instead of 10+ is not 7, but 7.5 and we get the Table II.32 and Table II.33:

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1										
-1		2									
0			3								
1	1			3.5							
2		2			4						
3			3			4.5					
4				4	4.5		5				
5					5	5.5	5.5	6			
6						6	6.5	6.5	7		
7							7	7.5	7.5	8	
8								8	8.5	8	
a											c

Table II.32 3rd floor "skeleton"

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1	2	3	3.5	4	4.5	5	6	7	8	
-1	1	2	3	3.5	4	4.5	5	6	7	8	
0	1	2	3	3.5	4	4.5	5	6	7	8	
1	1	2	3	3.5	4	4.5	5	6	7	8	
2	2	2	3	3.75	4	4.5	5	6	7	8	
3	3	3	3	4	4.25	4.5	5	6	7	8	
4	4	4	4	4	4.5	4.75	5	6	7	8	
5	5	5	5	5	5	5.5	5.5	6	7	8	
6	6	6	6	6	6	6	6.5	6.5	7	8	
7	7	7	7	7	7	7	7	7.5	7.5	8	
8	8	8	8	8	8	8	8	8	8.5	8	
a											c

Table II.33 3rd floor

Now consider the "skeleton" for the 4th floor. It coincides with the skeleton for the 3rd floor, but instead of 9+ is not 8, but 8.5, and we get Table II.34 and Table II.35:

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1										
-1		2									
0			3								
1	1			3.5							
2		2			4						
3			3			4.5					
4				4	4.5		5				
5					5	5.5	5.5	6			
6						6	6.5	6.5	7		
7							7	7.5	7.5	8	
8								8	8.5	8.5	
a											c

Table II.34 4th floor and above "skeleton"

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1	2	3	3.5	4	4.5	5	6	7	8	
-1	1	2	3	3.5	4	4.5	5	6	7	8	
0	1	2	3	3.5	4	4.5	5	6	7	8	
1	1	2	3	3.5	4	4.5	5	6	7	8	
2	2	2	3	3.75	4	4.5	5	6	7	8	
3	3	3	3	4	4.25	4.5	5	6	7	8	
4	4	4	4	4	4.5	4.75	5	6	7	8	
5	5	5	5	5	5	5.5	5.5	6	7	8	
6	6	6	6	6	6	6	6.5	6.5	7	8	
7	7	7	7	7	7	7	7	7.5	7.5	8	
8	8	8	8	8	8	8	8	8	8.5	8.5	
a											c

Table II.35 4th floor and above

Table II.34 and II.35 are correct and for all overlying floors.

II.14 Direct operation [-1] for question numbers ?a.

Let's fill the diagonal in Table II.40:

[-1]	?1	?2	?3	?4	?b
?1					
?2					
?3					
?4					
?a					c

Table II.40

In the meantime, let's discuss one rule of computation with right 1 operand. We may do so in the course of filling in the Table II.40. Proceed:

$$?1 [-1] ?1 = 1 [-1] 1 =$$

There are two possible ways:

$$= 1 [0] 2 = 3.5; (\text{II.14.1})$$

$$= 1 [-2] = 1; (\text{II.14.2})$$

It is difficult to justify logically my choice, but from my practice of computing I choose the first way. Resume:

$$?2 [-1] ?2 = ?2 [0] 2 =$$

Remember the definition of ?1:

$$1 [3] ?1 = 1; (\text{II.14.3})$$

Here as ?1 any natural number is good. Including the number 2.

Let us use this and the Table II.13:

$$= ?2 [0] ?1 = ?2; (\text{II.14.4})$$

$$?3 [-1] ?3 = ?3 [0] 2 = ?3 [0] ?1 = ?3; (II.14.5)$$

$$?4 [-1] ?4 = ?4 [0] 2 = ?4 [0] ?1 = ?4; (II.14.6)$$

The diagonal is full, and we got a Table II.41:

[-1]	?1	?2	?3	?4	?b
?1	3.5				
?2		?2			
?3			?3		
?4				?4	
?a					c

Table II.41

$$?2 [-1] ?1 = ?2 [-1] 1 = ?2 [-2] = ?2; (II.14.7)$$

$$?3 [-1] ?1 = ?3 [-1] 1 = ?3 [-2] = ?3; (II.14.8)$$

$$?4 [-1] ?1 = ?4 [-1] 1 = ?4 [-2] = ?4; (II.14.9)$$

Now we have Table II.42:

[-1]	?1	?2	?3	?4	?b
?1	3.5				
?2	?2	?2			
?3	?3		?3		
?4	?4			?4	
?a					c

Table II.42

Let's continue our calculations:

$$?1 [-1] ?2 = 1 \{-1\} ?2 = (((1 [-1] 1) [-1] 1) [-1] 1) [-1] 1 \dots =$$

Apply the formula (II.14.1)

$$= (((3.5 [-1] 1) [-1] 1) [-1] 1) [-1] 1 \dots =$$

Apply Table II.26:

$$3.5 [-1] 1 = 4; (\text{II.14.10})$$

$$= ((\mathbf{4} [-1] \mathbf{1}) [-1] 1) [-1] 1 \dots =$$

Apply Table II.26 and (II.13.76):

$$4 [-1] 1 = 4; (\text{II.14.11})$$

$$= (\mathbf{4} [-1] 1) [-1] 1 \dots =$$

Again apply (II.14.11):

$$= 4 [-1] 1 \dots = \dots = \dots =$$

We will apply (II.14.11) ?2 times and get:

$$= 4; (\text{II.14.12})$$

That is: ?1 [-1] ?2 = 4; (II.14.12)

Further

$$?1 [-1] ?3 = 1 [-1] ?3 = (((\mathbf{1} [-1] \mathbf{1}) [-1] 1) [-1] 1) [-1] 1) [-1] 1 \dots =$$

Apply (II.14.1):

$$= (((\mathbf{3.5} [-1] 1) [-1] 1) [-1] 1) [-1] 1 \dots =$$

Apply (II.14.10):

$$= ((\mathbf{4} [-1] \mathbf{1}) [-1] 1) [-1] 1 \dots =$$

Apply (II.14.11):

$$= (\mathbf{4} [-1] 1) [-1] 1 \dots =$$

Again apply (II.14.11):

$$= 4 [-1] 1 \dots = \dots = \dots =$$

We will apply (II.14.11) ?3 times and get:

$$= 4; (\text{II.14.13})$$

That is: $?1 [-1] ?3 = 4; (\text{II.14.13})$

Similarly, we obtain:

$$?1 [-1] ?4 = 4; (\text{II.14.14})$$

And now we have filled three more boxes in the Table II.42, and got the Table II.43:

[-1]	?1	?2	?3	?4	?b
?1	3.5	4	4	4	
?2	?2	?2			
?3	?3		?3		
?4	?4			?4	
?a					c

Table II.43

Let's continue our calculations:

$$?2 [-1] ?3 = ((?2 [-2] ?2) [-2] ?2) [-2] ?2 \dots =$$

$$= ((?2 [-1] 2) [-2] ?2) [-2] ?2 \dots =$$

Apply the words following (II.14.3):

$$= ((?2 [-1] ?1) [-2] ?2) [-2] ?2 \dots =$$

Apply here (II.14.7):

$$?2 [-1] ?1 = ?2; (\text{II.14.7})$$

$$= (?2 [-2] ?2) [-2] ?2 \dots =$$

By doing ?3 such steps, we will get:

$$= ?2; (\text{II.14.15})$$

That is: $?2 [-1] ?3 = ?2; (\text{II.14.15})$

Similarly, we obtain:

$$?2 [-1] ?4 = ?2; (\text{II.14.16})$$

$$?3 [-1] ?2 = ((?3 [-2] ?3) [-2] ?3) [-2] ?3 \dots =$$

$$= ((?3 [-1] 2) [-2] ?3) [-2] ?3 \dots =$$

$$= ((?3 [-1] ?1) [-2] ?3) [-2] ?3 \dots =$$

Apply (II.14.8)

$$?3 [-1] ?1 = ?3 [-1] 1 = ?3 [-2] = ?3; \text{ (II.14.8)}$$

$$= (?3 [-2] ?3) [-2] ?3 \dots =$$

By doing ?2 such steps, we will get:

$$= ?3; \text{ (II.14.17)}$$

$$\text{That is: } ?3 [-1] ?2 = ?3; \text{ (II.14.17)}$$

Similarly, we obtain:

$$?3 [-1] ?4 = ?3; \text{ (II.14.18)}$$

Similarly, we obtain a couple of formulas:

$$?4 [-1] ?2 = ?4; \text{ (II.14.19)}$$

$$?4 [-1] ?3 = ?4; \text{ (II.14.20)}$$

So we filled in the empty cells in the Table II.43 and received the Table II.44:

[-1]	?1	?2	?3	?4	?b
?1	3.5	4	4	4	
?2	?2	?2	?2	?2	
?3	?3	?3	?3	?3	
?4	?4	?4	?4	?4	
?a					c

Table II.44

Recall that Table II.44 represents the direct operation [-1] for the question numbers ?a.

CHAPTER III

III.1 Summary of Chapter II.

In paragraph II.1 there is a summary of Chapter I.

In paragraph II.2 the principle of construction of zero and negative direct operations is described:

Chapter I establishes the connection between direct operation $[n]$ and direct operation $[n + 1]$. Here are a couple of examples of this connection:

$$3 [n + 1] 2 = 3 [n] 3; (II.2.1)$$

$$2 [n + 1] 3 = (2 [n] 2) [n] 2; (II.2.2)$$

If we take $n = 0$, we can determine a new direct operation $[0]$ by the known direct operation $[1]$ (addition). And then (at $n = -1$), by direct operation $[0]$ construct a new direct operation $[-1]$. Thus, we can construct all direct operations with negative numbers.

So the number of the direct operation can take the value of any integer.

In paragraph II.3 by the known operation $[1]$ – addition – built "skeleton" of the zero direct operation $[0]$. This is what was possible to calculate by the formulas of connection of these operations:

All these equations are presented in the form of Table II.1:

[0]	2	3	4	5	6	7	b
2	4						
3		5					
4	5		6				
5	6	6		7			
6	7	7	7		8		
7	8	8	8	8		9	
a							c

Table II.1

It is impossible to make formulas for empty cells. It remains only to speculate. If you try to make it logical, balanced and beautiful, you get Table II.2:

[0]	2	3	4	5	6	7	b
2	4	4.5	5	5	5	5	
3	4.5	5	5.5	6	6	6	
4	5	5.5	6	6.5	7	7	
5	6	6	6.5	7	7.5	8	
6	7	7	7	7.5	8	8.5	
7	8	8	8	8	8.5	9	
a							c

Table II.2

In **paragraph II.4** it is considered the reverse operation $[0]1]$ for the zero direct operation $[0]$.

Rearrange a and c in Table II.2 and get Table II.3 for the first reverse operation to zero operation:

$c [0]1] b = a$; (II.4.1)

[0]1]	2	3	4	5	6	7	b
4	2						
4.5	3	2					
5	4	3	2	2	2	2	
5.5		4	3				
6	5	5	4	3	3	3	
6.5			5	4			
7	6	6	6	5	4	4	
7.5				6	5		
8	7	7	7	7	6	5	
8.5					7	6	
9	8	8	8	8	8	9	
a							c

Table II.3

The contents of empty cells think logically and simply. Result in Table II.3.1:

[0]1	2	3	4	5	6	7	b
4	2	1	1	1	1	1	
4.5	3	2	1.5	1.5	1.5	1.5	
5	4	3	2	2	2	2	
5.5	4.5	4	3	2.5	2.5	2.5	
6	5	5	4	3	3	3	
6.5	5.5	5.5	5	4	3.5	3.5	
7	6	6	6	5	4	4	
7.5	6.5	6.5	6.5	6	5	4.5	
8	7	7	7	7	6	5	
8.5	7.5	7.5	7.5	7.5	7	6	
9	8	8	8	8	8	9	
a							c

Table II.3.1

In paragraphs from II.5 to II.9 the second inverse operation [0]2] to the zero direct operation [0] is considered. A significant part is the description of the work with numbers—strings. The concept of "floors" is introduced.

In paragraph II.10 we consider the zero direct operation [0] on the set of question numbers ?a.

And the result is Table II.13:

[0]	?1	?2	?3	?b
?1	3	4	4	
?2	?2	?2	?2	
?3	?3	?3	?3	
?a				c

Table II.13

In paragraph II.11 the first inverse operation [0]1] to the zero direct operation [0] on the set of question numbers ?a is considered.

Rearrange "?a" and "c" in Table II.13 and get Table II.14:

[0]1]	?1	?2	?3	?b
3	?1			
4		?1	?1	
?2	?2	?2	?2	
?3	?3	?3	?3	
c				?a

Table II.14

?a [0] ?1 = ?a [0] 1 = 4; (II.11.1)

If you look at Table II.2.1 (expanded to Table II.2) then we'll see that ?a = 3. That is 4 [0]1] ?1 = 3; (II.11.2)

?a [0] ?2 = 3; according to Table II.13.1 see: ?a = 0; (II.11.3)

3 [0]1] ?2 = 0; (II.11.4)

?a [0] ?3 = 3; according to Table II.13.1 see: ?a = 0; (II.11.5)

3 [0]1] ?3 = 0; (II.11.6)

Table II.14 will become Table II.15:

[0]1]	?1	?2	?3	?b
3	1	0	0	
4	3	1	1	
?2	?2	?2	?2	
?3	?3	?3	?3	
c				?a

Table II.15

And as a result - for the first inverse operation $[0]1$ to the zero direct operation $[0]$ for numbers $?a$ - we have Table II.15.

In paragraph II.12 the second inverse operation $[0]2$ to the zero direct operation $[0]$ on the set of question numbers $?a$ is considered. The birth of numbers-lines and floors is considered in detail.

In paragraph II.13 the direct operation $[-1]$ is considered.

All the above 65 equations are considered and the results are summarized in Table II.26 ($a[-1]b = c$):

[-1]	-2	-1	0	1	2	3	4	5	6	7	b
-2	1										
-1		2									
0			3								
1	1+			3.5							
2		2+			4						
3			3+			4.5					
3.5				4							
4				4+	4.5		5				
4.5					5	5					
5					5+	5.5	5.5	6			
5.5						6	6				
6						6+	6.5	11+	7		
6.5							7	7			
7							7+	7.5	10+	8	
7.5								8	8		
8								8+	8.5	9+	
a											c

Table II.26

There are numbers-strings in Table II.26:

$$1+ = 1|?2|0; 2+ = 2|?2|0; (\text{II.13.75})$$

$$3+ = 3 | ?2 | 0; 4+ = 4 | ?2 | 0; \text{ (II.13.76)}$$

$$5+ = 5 | ?2 | 0; 6+ = 6 | ?2 | 0; 7+ = 7 | ?2 | 0; 8+ = 8 | ?2 | 0; \text{ (II.13.77)}$$

$$9+ = 8, 8, 8, 8.5; 10+ = 7, 7, 7.5; 11+ = 6, 6.5; \text{ (II.13.78)}$$

For the first four floors of Table II.26 "skeletal" tables were built, and "meat" was built on them.

In paragraph II.14 the direct operation $[-1]$ on the set of question numbers $?a$ is considered:

So we filled in the empty cells in Table II.43 and derived Table II.44:

$[-1]$?1	?2	?3	?4	?b
?1	3.5	4	4	4	
?2	?2	?2	?2	?2	
?3	?3	?3	?3	?3	
?4	?4	?4	?4	?4	
?a					c

Table II.44

III.2 Numbers N1 and n1.

Back in 1984, I tried to extract new numbers from new equations. Also I tried the equation:

$$x + 1 = x; \text{ (III.2.1)}$$

But then I couldn't find a use for it. And it wasn't until a few years later, that I realized, that this equation describes an infinitely large number. And designated it as N1:

$$N1 + 1 = N1; \text{ (III.2.2)}$$

From this definition it follows:

$$N1 - N1 = 1; \text{ (III.2.3)}$$

$$N1 * (1 - 1) = 1; \text{ (III.2.4)}$$

We introduce a new value $n1$ as follows: $N1 * n1 = 1$; (III.2.5)

$$\text{It follows: } n1 = 1 / N1; \text{ (III.2.6)}$$

That is, $n1$ is an infinitely small number.

$$(III.2.5) + (III.2.4) = (III.2.7)$$

$$n1 = 1 - 1; \text{ (III.2.7)}$$

III.3 Numbers $N2$ and $n2$.

Using the number $N1$ we construct a new infinitely large number $N2$:

$$N2 = 2 \wedge N1 = 2 [3] N1; \text{ (III.3.1)}$$

$$N2 > N1; \text{ (III.3.2)}$$

We construct a new number $n2$ as follows: $n2 = 1 / N2$; (III.3.3)

It is an infinitesimal number, and it is less than $n1$

$$n2 < n1; \text{ (III.3.4)}$$

This follows from (III.2.6), (III.3.3), (III.3.2)

III.4 Numbers $N3$ and $n3$.

Using the number $N2$ we construct a new infinitely large number $N3$:

$$N3 = 2 \wedge N2 = 2 [3] N2; \text{ (III.4.1)}$$

$$N3 > N2; \text{ (III.4.2)}$$

We construct a new number n_3 as follows: $n_3 = 1 / N_3$; (III.4.3)

It is an infinitesimal number, and it is less than n_2

$$n_3 < n_2; \text{ (III.4.4)}$$

This follows from (III.4.3), (III.3.3), (III.4.2)

III.5 Numbers N_A and n_A .

Let's take a series of integers A (... -3, -2, -1, 0, 1, 2, 3, ...) and build a series of infinitely large numbers N_A (based on the already constructed numbers N_1, N_2, N_3):

$$N(A + 1) = 2 \uparrow N_A = 2 [3] N_A; \text{ (III.5.1)}$$

$$N(A + 1) > N_A; \text{ (III.5.2)}$$

When $A = 3$, we obtain a definition for N_4 :

$$N_4 = 2 [3] N_3; \text{ (III.5.3)}$$

For $A = 4$, we obtain a definition for N_5 through N_4 :

$$N_5 = 2 [3] N_4; \text{ (III.5.4)}$$

When taking increasing indices A , increasing numbers N_A are obtained.

Now let's take $A = 0$ and get:

$$N_1 = 2 [3] N_0; \text{ (III.5.5)}$$

This equation defines N_0 by N_1 .

Take $A = -1$ and get:

$$N_0 = 2 [3] N(-1); \text{ (III.5.6)}$$

Here $N(-1)$ is defined by N_0 .

When $A = -2$, we obtain the definition of $N(-2)$ by $N(-1)$:

$$N(-1) = 2 \quad [3] \quad N(-2); \text{ (III.5.7)}$$

Taking $A = -3, -4, -5, \dots$ will produce new positive infinitely large numbers $N(-3), N(-4), N(-5), \dots$ and all of them will fit in the interval from N_1 to 1.

Now let's build a series of infinitesimal numbers n_A :

$$n_A = 1 / N_A; \text{ (III.5.8)}$$

$$n(A+1) < n_A; \text{ (III.5.9)}$$

III.6 Topology of the numerical line for N-numbers.

The representation of a number line is usually given as follows:

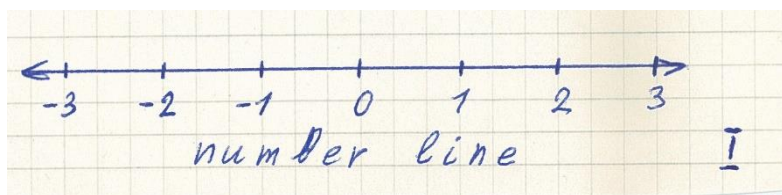


Figure 1

Given the infinitely large and the infinitely small N_A and n_A numbers give the following two figures:

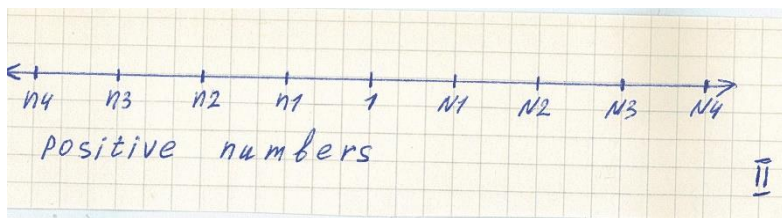


Figure 2

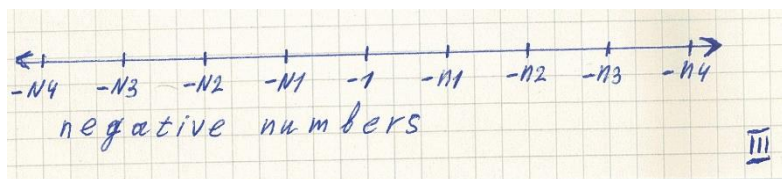


Figure 3

From (III.(2.7) should: $--n1 = -(1 - 1) = -1 + 1 = 1 - 1 = n1$; (III.6.1)
 $-n1 = n1$; (III.6.2)

Physically, $n1$ and $-n1$ lie on different number lines, but topologically they coincide. In other words, these two numeric lines intersect at this point. The result is in Figure 4.

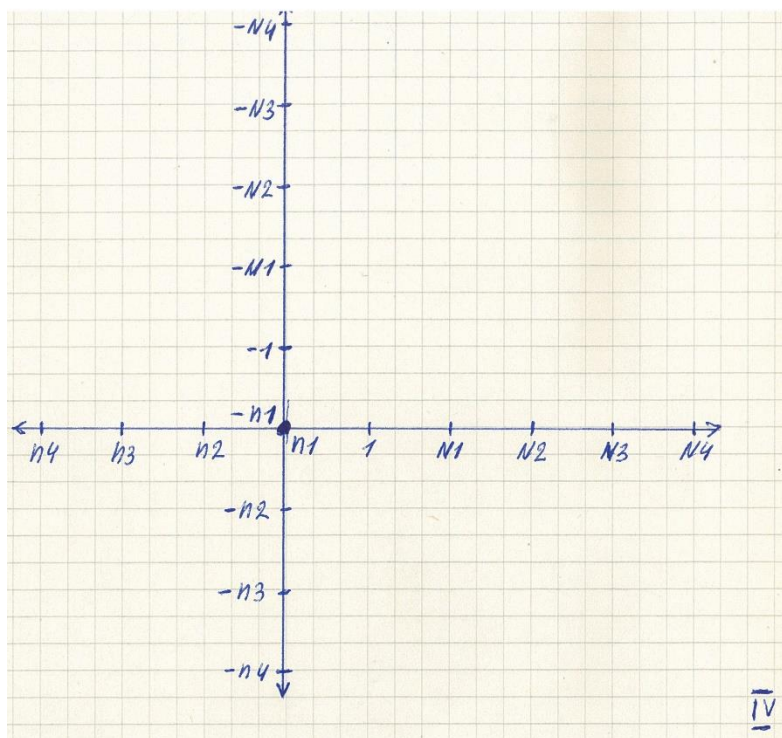


Figure 4

$$(III.2.5): N1 * n1 = 1;$$

$$(III.6.2): -n1 = n1;$$

$$(-N1) * n1 = -(N1 * n1) = N1 * (-n1) = N1 * n1; (III.6.3)$$

$$\text{It follows: } -N1 = N1; (III.6.4)$$

So these two very large numbers is also topologically identical. In some multidimensional space, these two numerical "lines" bend to intersect at this point. Result in figure 5:

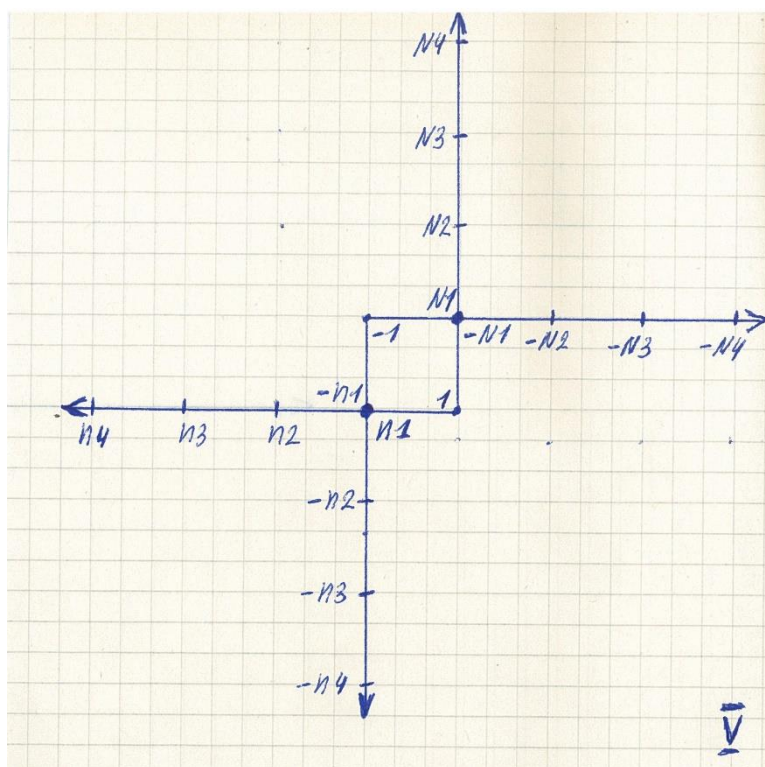


Figure 5

((III.3.1): $N2 = 2 \wedge N1$; (III.3.3): $n2 = 1 / N2$; (III.6.4): $- N1 = N1$;
 $N2 = 2 \wedge N1 = 2 \wedge (- N1) = 1 / (2 \wedge N1) = 1 / N2 = n2$; (III.6.5)
 $N2 = n2$; (III.6.6)

So an infinitely large number coincides with an infinitely small number! (opposites coincide - the main law of dialectics). The numbers $N2$ and $n2$ lie on the same straight line of positive numbers, but in different places. And so they coincide topologically, this "direct" line need to curve in some multidimensional space that they are superimposed on each other. The result is figure 6.

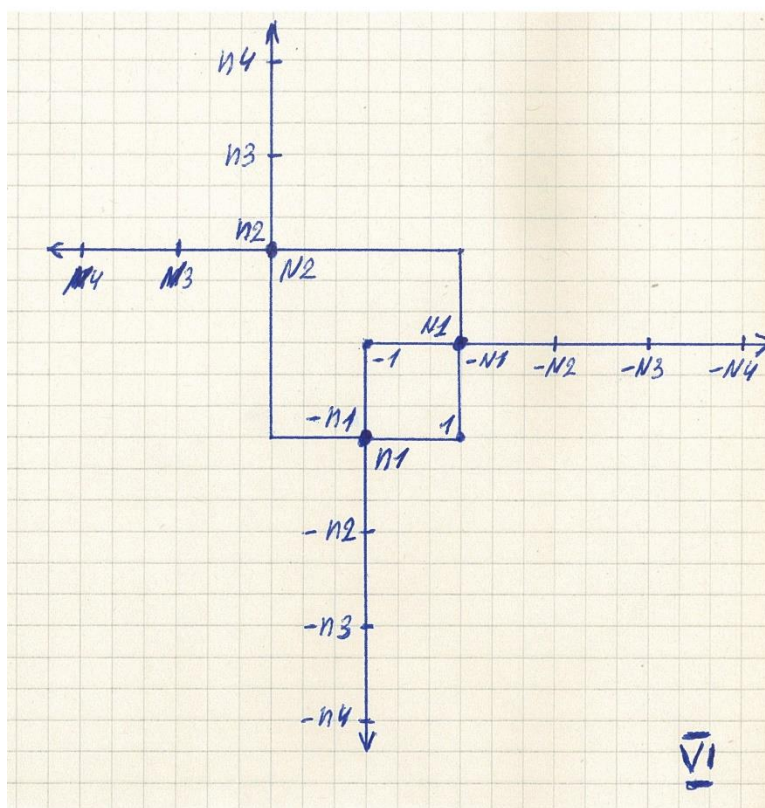
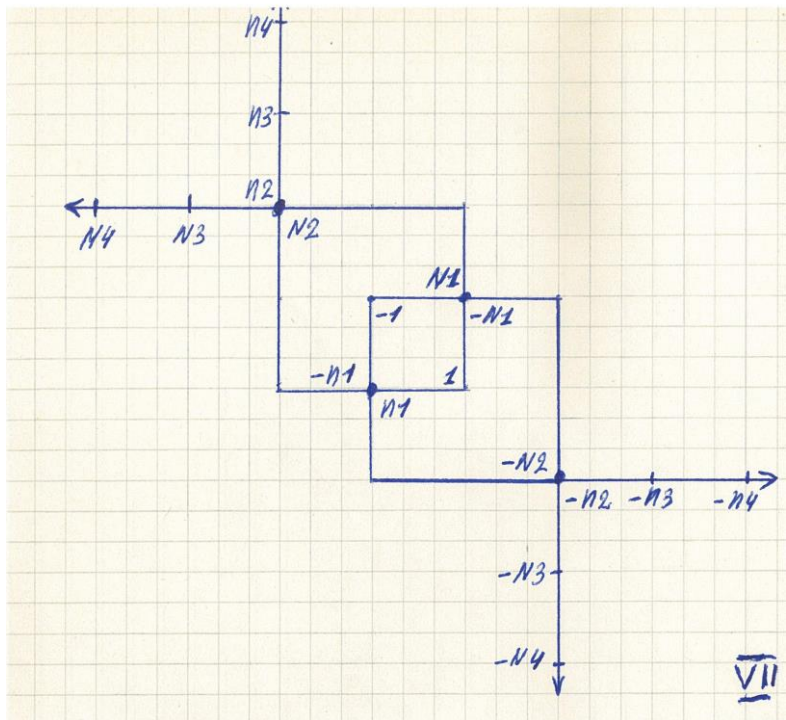


Figure 6

If the equation (III.6.6) is multiplied by -1, we get:

$$-N_2 = -n_2; \text{ (III.6.7)}$$

This pair of numbers (-N₂ and -n₂) lies on one "direct line" of negative numbers at its different ends. But they are topologically the same. To do this, the "direct line" of negative numbers should be curved in a special multidimensional space so that it has a self-intersection. The result is in figure 7.



intersections and self-intersections of numerical "direct lines" in flat drawings. Therefore, here are some pairs of topologically matching numbers without their graphical representation in two-dimensional space (three-dimensional space is required here):

$$N3 = 2 [3] \quad N2 = 2 [3] \quad n2 = e [3] \quad (n2 * \ln(2)) =$$

$$= 1 + n2 * \ln(2) + \dots ; \text{ (III.6.8)}$$

$$n3 = 1 / N3 = 1 - n2 * \ln(2) + \dots ; \text{ (III.6.9)}$$

$$- N3 = -1 - n2 * \ln(2) + \dots ; \text{ (III.6.10)}$$

$$- n3 = -1 + n2 * \ln(2) - \dots ; \text{ (III.6.11)}$$

$$N4 = 2 [3] \quad N3 = 2 [3] \quad (1 + n2 * \ln(2) + \dots) =$$

$$= 2 * 2 [3] \quad (n2 * \ln(2) + \dots) = 2 * e [3] \quad (\ln(2) * (n2 * \ln(2) + \dots)) =$$

$$= 2 * (1 + n2 * \ln(2) * \ln(2) + \dots) =$$

$$= 2 + n2 * 2 * \ln(2) * \ln(2) + \dots ; \text{ (III.6.12)}$$

$$N4 = 2 + n2 * 2 * \ln(2) * \ln(2) + \dots ; \text{ (III.6.12)}$$

$$n4 = 1 / N4 = 1/2 - n2 * 1/2 * \ln(2) * \ln(2) - \dots ; \text{ (III.6.13)}$$

$$-N4 = -2 - n2 * 2 * \ln(2) * \ln(2) - \dots ; \text{ (III.6.14)}$$

$$- n4 = -1/2 + n2 * 1/2 * \ln(2) * \ln(2) + \dots ; \text{ (III.6.15)}$$

We now see that such a simple numerical line has a very complex topological structure that requires the introduction of a multidimensional space for its complete representation.

III.7 how to travel on a numerical line using its topological structure.

The simplest journey (figure 5): move from point [1] to point [(N1) (- N1)]. Stop at this point. And turn your face to the left. Then move forward to [-1] and (non-stop) to [(- n 1) (n 1)]. Here you should stop and turn to the left. Then move straight to the point [1]. So you will make a simple journey and return home.

If at each intersection of the numeric line you choose the middle path, or just do not stop at all, then you will move along a one-dimensional straight line.

III.8 The connection of N-numbers with ?-numbers.

The construction of N-numbers begins with the definition of a positive infinitely large number N1 through the equation:

$$x + 1 = x; \text{ (III.8.1)}$$

$$x = N1; \text{ (III.8.2)}$$

Consider ?-number ?2. In the Table I.22 the operation [1] is given for ?-numbers'. From this table you can extract the equation:

$$?2 [1] ?1 = ?2; \text{ (III.8.3)}$$

Here ?1 - any natural number, we choose ?1 = 1.

We rewrite equation (III.8.3) in the form:

$$?2 + 1 = ?2; \text{ (III.8.4)}$$

It coincides with the equation (III.8.1). And the solutions of these two equations also coincide:

$$N1 = ?2; \text{ (III.8.5)}$$

The link of n1 with 0 is such:

$$n1 = 1 - 1 = 0; \text{ (III.8.6)}$$

Let's do a little check on the connection of N-numbers with ?-numbers. There is an equation for N-numbers:

$$N1 [2] n1 = N1 * n1 = 1; \text{ (III.2.5)}$$

For ?- the numbers it must match the equation:

$$?2 [2] 0 = ?2 * 0 = 1; \text{ (III.8.7)}$$

From the definition of ?- numbers such ratios follow :

$$1 [3] (?2 [2] 0) = 1 ^ { (?2 * 0) } = (1 ^ { ?2 }) ^ 0 = 2 ^ 0 = 1; \text{ (III.8.8)}$$

We raised 1 to a power equal to the left side of the equation (III.8.7) and received 1. Now raise 1 to the power equal to the right side of equation (III.8.7):

$$1 [3] 1 = 1 ^ 1 = 1; \text{ (III.8.9)}$$

The results matched ($1 = 1$). The test is completed.

All N-numbers constructed with N1 are now constructed with ?-number ?2. And the resulting topological structure of the numerical line is also obtained in ?-numbers.

But other than that ?-numbers have their own topological properties. Here's one example:

- ?2 this is a negative infinitely large number. From (III.8.5) should:

$$- ?2 = - N1 = N1 = ?2; \text{ (III.8.10)}$$

$$- ?2 = ?2; \text{ (III.8.11)}$$

This topological property of ?-number follows from the topological properties of the N-number. And now consider the following calculations:

$$1 [3] (-?2) = 1 \wedge (-?2) = 1 / (1 \wedge ?2) = 1 / 2 = 1 [3] ?(1/2); (III.8.12)$$

It follows:

$$- ?2 = ?(1/2); (III.8.13)$$

There is no such relation among N-numbers. Ratio (III.8.13) is easily generalized to the:

$$- ?n = ?(1/n); (III.8.14)$$

Where n is any positive integer.

III.9 When everything was written.

When all was written and sent in viXra and in samizdat, I thought back to the moment of introduction of the question numbers. In paragraph I.16 I marked these numbers with a sign "?" as having no solution in complex numbers. And now I doubt it. And I decided to try to solve the equation

$$1 [3] ?a = 1 \wedge ?a = a; (III.9.1)$$

in complex numbers. And that's what came out of it.

First I moved from raising to the power the 1, to raising to the power the e:

$$1 [3] x = e [3] ((\ln 1) [2] x); (III.9.2)$$

$$a = e [3] ((2 [2] \text{PI} [2] k [2] i) [2] (\ln a)); (III.9.3)$$

$$k = \dots -1, 0, 1, 2, \dots; (III.9.4)$$

$$(III.9.2) + (III.9.3) = (III.9.5):$$

$$(\ln 1) [2] x = (2 [2] \text{PI} [2] k [2] i) [2] (\ln a); (III.9.5)$$

Here $(\ln 1)$ is the logarithm 1 of base e ,

$e = 2.7$; $\pi = 3.14$; i - imaginary unit;

k - any integer;

$$x = ?a; \text{ (III.9.6)}$$

From

$$e^{[3] 0} = e^0 = 1; \text{ (III.9.7)}$$

it follows

$$\ln 1 = 0; \text{ (III.9.8)}$$

$$\text{(III.9.5)} + \text{(III.9.8)} + \text{(III.9.6)} = \text{(III.9.9)}$$

$$0 * ?a = 2 * \pi * k * i * (\ln a); \text{ (III.9.9)}$$

Now recall the N-numbers:

$$\text{(III.2.2): } N_1 + 1 = N_1;$$

$$\text{(III.2.7): } n_1 = 1 - 1 = 0;$$

$$\text{(III.2.5): } N_1 * n_1 = 1;$$

Let's multiply the equation (III.9.9) by N_1 :

$$N_1 * 0 * ?a = N_1 * 2 * \pi * k * i * (\ln a); \text{ (III.9.10)}$$

Applying (III.2.7), and (III.2.5) get:

$$?a = i * N_1 * 2 * \pi * k * (\ln a); \text{ (III.9.11)}$$

Taking $a = 1$ we obtain (considering (III.9.8)):

$$?1 = i * N_1 * 2 * \pi * k * (\ln 1) = i * N_1 * 2 * \pi * k * 0 =$$

$$= i * N_1 * 0 * 2 * \pi * k = i * 1 * 2 * \pi * k = i * 2 * \pi * k; \text{ (III.9.12)}$$

$$\varphi_1 = i * 2 * \pi * k; \text{ (III.9.13)}$$

From (III.8.3) we have:

$$\varphi_2 [1] \varphi_1 = \varphi_2; \text{ (III.8.3)}$$

$$\varphi_2 + \varphi_1 = \varphi_2 + i * 2 * \pi * k = \varphi_2; \text{ (III.9.14)}$$

This equation is different from

$$\text{(III.2.2): } N_1 + 1 = N_1;$$

By that way φ_2 is not equal to N_1 . From (III.9.11) we have:

$$\varphi_2 = i * N_1 * 2 * \pi * k * (\ln 2); \text{ (III.9.15)}$$

$$\text{Although, as in (III.8.11): } -\varphi_2 = \varphi_2; \text{ (III.9.16)}$$

Here is, that (III.9.1) has a solution in complex numbers. An explicit form of this solution is obtained in (III.9.11).

CHAPTER IV

IV.1 Summary of Chapter III.

In paragraph III.1 a summary of Chapter II is provided.

In paragraph III.2 the numbers N_1 and n_1 are determined.

In paragraph III.3 the numbers N_2 and n_2 are determined.

In paragraph III.4 the numbers N_3 and n_3 are defined.

In paragraph III.5 the numbers N_A and n_A are determined.

In paragraph III.6 the topology of the numerical line for N -numbers is described.

In paragraph III.7 an idea of how to travel along a numerical line using its topological structure is given.

In paragraph III.8 is the connection of N -numbers with φ -numbers.

In paragraph III.9 obtained a solution for φ_a in complex numbers, and a new connection of N -numbers with φ -numbers.

IV.2 Results.

In this paper, the birth of numbers from natural numbers to finite complex numbers was studied. The regularity of construction of new numbers on the basis of already known ones is found. On its basis, infinitely large and infinitesimal complex numbers (φ_a -numbers) were built.

By analogy with them were built $4\varphi_a$ -numbers.

In these constructions we had to enter numbers-strings (when there are several numbers separated by commas in one cell).

N-numbers (infinitely large and infinitely small numbers) are constructed in a completely different way. With the help of N-numbers the complex topological structure of a usual numerical line is shown.

An important result was the establishment of an explicit connection of N-numbers with ?-numbers.

Operations on numbers play a significant role. All operations are divided into direct, and reverse to direct. Addition, multiplication, exponentiation ([1], [2], [3]) - direct operations. Subtraction, division, root, logarithm - inverse operations to these direct operations. Several new direct operations were built: [4], [5], [0], [-1]. Two inverse operations were constructed for each of [4] and [0].

A significant drawback is the lack of equations for the construction of [0] operation by [1] operation (addition), as well as [-1] by [0]. This deficiency had to be compensated by imagination, and to complete tables of operations, being guided by any principle. Not the best way, but there's no other way.

Here I will make a small remark. In the course of work on the text each sign has a mental pronounce. With the appearance of question numbers, I was tormented by the mental pronunciation of "question mark". Over time, I replaced this phrase with one word: "question". And then spontaneously became to call "?" by the simple word 'Xi'. At first I resisted it because I knew it was the name of one Greek letter. But since I didn't have Greek letters in the text, I accepted it. Maybe someone in reading of my text will come in handy this replacement pronunciation.

06.11.2018

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